Spreading Properties of a City-Road Reaction-Diffusion Model on One-Dimensional Lattice

Grégory Faye*[1], Jean-Michel Roquejoffre[1], and Min Zhao[2]

[1]Univ Toulouse, CNRS, Institut de Mathématiques de Toulouse, Toulouse, France
[2]Aix Marseille Univ, CNRS, Institut de Mathématiques de Marseille, Marseille, France

November 19, 2025

Abstract

We propose and study a new model to describe biological invasions constrained on infinite homogeneous one dimensional metric graphs. Our model consists of an infinite PDE-ODE system where, at each vertex of the one-dimensional lattice $\mathbb{Z}$, we have a logistic equation, and connections between vertices are given by diffusion equations on the edges supplemented with Robin like boundary conditions at the vertices. We establish the main properties of the system and study the long time behavior of the solutions, especially by characterizing an asymptotic spreading speed for the system. In the fast diffusion regime, we derive a novel asymptotic model which exhibits similar propagation properties as the classical discrete Fisher-KPP on the one-dimensional lattice $\mathbb{Z}$.

Keywords: PDE-ODE model; Spreading speed; Discrete reaction-diffusion equations; Asymptotic behavior

MSC numbers: 35B40, 34D05, 35K55, 92D30

1 Introduction

Traveling waves in biology are ubiquitous and have been found in many contexts, such as the spread of cancer cells in healthy tissue, traveling bands of bacteria, the diffusion of genes within a population, or the spread of an epidemic, to name a few. One common feature of these biological spreading phenomena is that they are highly complex, network-driven dynamic processes. In many applications, the intrinsic heterogeneity of the underlying networks makes it very challenging to

*email: gregory.faye@math.univ-toulouse.fr

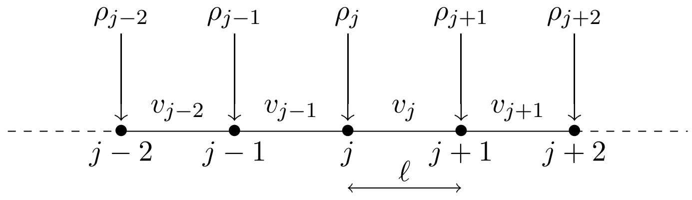


Figure 1: *Schematic spatial configuration of the system* (1.1)-(1.2) *where the unknowns* ρ_j *are indexed on the lattice* $\mathbb{Z}$ *while each* v_j *is locally defined on* $(0, \ell)$.

analyze these processes and assess the relevant factors that effectively drive the propagation. From a modeling perspective, it is also quite difficult, given the multiscale nature of the considered biological processes, to adopt a formalism that could combine intricate network structures and complex dynamics, and still provide comprehensive and valuable feedbacks to the biological community.

Our focus here will be on biological processes that can be well approximated by macroscopic models set on metric graphs. More precisely, given a metric graph, that is, a collection of interconnected vertices and edges with prescribed lengths, we will consider non classical reaction-diffusion models where, schematically, diffusion processes take place along the edges of the graph while reaction kinetics occur at the vertices with prescribed rules of exchanges between vertices and adjacent edges. Such a formalism has typically been proposed to study coupled membrane-bulk diffusion systems [11, 12, 18] and to analyze the effects of transportation networks such as roads, railways or waterways on the spread of epidemics among cities [3, 17] as reported for example for the spread of COVID-19, Chikungunya virus, Zika virus and HIV virus [9, 10, 14]. Other types of reaction-diffusion models on metric graphs have been proposed in the past decades. In population dynamics, the so-called river network models [7, 15] describe the dynamics of organisms living in a river system subject to a forced flow in the downstream direction. It typically consists of reaction-diffusion equations set on the vertices of a given prescribed network with a continuity condition at the edges, together with a Kirchoff law that translates the continuity of fluxes through the edges. In cellular physiology, models of cells coupled by gap junctions [6, 16, 21] are typically set on networks, and concentrations of diffusing particles follows a diffusion equation within each cell, idealized by an edge, and at the junction between two cells, that is, at each vertex of the network, specific boundary conditions are prescribed to account for the permeability properties of the cells membrane.

In the present work, we propose a model where the underlying metric graph is indexed by the one-dimensional lattice $\mathbb{Z}$, such that each vertex of the graph is exactly connected to two incident edges, and all edges have exactly the same length, denoted by $\ell > 0$. As previously emphasized, our framework is very general and relevant in a wide array of situations in biology. Nevertheless, for convenience, we will adopt a population dynamics point of view and for the sake of simplicity and illustration, we shall from now on refer to the vertices as “the cities” and the edges “the roads”. For $j \in \mathbb{Z}$, we denote by $\rho_j(t)$ the density of individuals which reside in city j while we denote by $v_j(t,x)$ the density of individuals diffusing along the road connecting city j to city $j+1$ where t

refers to time and $x \in [0, \ell]$ represents the local position on the road. We refer to Figure 1 for a schematic illustration of the spatial configuration of our so-called "city-road" model. Exchanges of populations take place between cities and adjacent roads. Namely, given a city, indexed by $j \in \mathbb{Z}$, a fraction $\alpha > 0$ of individuals from the two adjacent roads at the city, that are $v_{j-1}(t, \ell)$ and $v_j(t, 0)$, joins the city, while a fraction $\beta > 0$ of individuals from the city j transfers to each of the two adjacent roads. It is further assumed that the population in each city is subject to a logistic-type growth, resulting in a nonlinear reaction term $f(\rho)$ that models effective birth rate and intrinsic competition. On the other hand, we assume that no such reaction is relevant on the roads and consider solely a diffusion process, with diffusion coefficient $d > 0$, to describe the dynamics of each v_j. Transposing the above principles into equations, we are thus led to consider the following system of equations:

$$\forall t > 0, \ j \in \mathbb{Z}, \quad \begin{cases} \partial_t v_j(t, x) = d\partial_x^2 v_j(t, x), \quad x \in (0, \ell), \\ \rho_j'(t) = f(\rho_j(t)) + \alpha(v_j(t, 0) + v_{j-1}(t, \ell)) - 2\beta\rho_j(t), \end{cases} \tag{1.1}$$

with inhomogeneous Robin boundary conditions

$$\forall t > 0, \ j \in \mathbb{Z}, \quad \begin{cases} -d\partial_x v_j(t, 0) + \alpha v_j(t, 0) = \beta\rho_j(t), \\ d\partial_x v_j(t, \ell) + \alpha v_j(t, \ell) = \beta\rho_{j+1}(t). \end{cases} \tag{1.2}$$

The nonlinearity $f \in \mathscr{C}^1([0, 1])$ satisfies

$$f(0) = f(1) = 0, \quad 0 < f(u) \leq f'(0)u, \quad \forall u \in (0, 1).$$

We extend it to a negative function outside $[0, 1]$. Let us already remark that by performing the following rescaling

$$x' \longleftrightarrow \frac{x}{\ell}, \quad \widetilde{v}_j(t, x') \longleftrightarrow \ell v_j(t, x), \quad d' \longleftrightarrow \frac{d}{\ell^2} \quad \text{and} \quad \alpha' \longleftrightarrow \frac{\alpha}{\ell},$$

we may assume, for the rest of the paper, and without loss of generality, that $\ell = 1$.

Model (1.1)–(1.2) is largely inspired by the SIR model proposed by the first author and Besse in [3]. There are nevertheless three important differences between the two models. First of all, the intrinsic dynamics at each city are different, in our case it is given by a single logistic equation, while in [3] it was given by an SIR compartment model resulting in a system of equations. Second, the study [3] considered compact connected graphs, meaning that the number of cities and roads was finite, while here model (1.1)–(1.2) is indexed by the one-dimensional lattice $\mathbb{Z}$ and thus infinite. Finally, the model in [3] allowed a fraction of individuals to pass from one road to another one. This is not taken into account in the boundary conditions (1.2) and we refer to the last section of the present manuscript for a longer discussion about this possible extension into the model. One of the key feature of system (1.1)–(1.2) is the preservation of the total population in the absence of reaction kinetics at the cities. Indeed, assume that $(\mathbf{v}, \boldsymbol{\rho})$ with $\mathbf{v} = (v_j)_{j\in\mathbb{Z}}$ and $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{Z}}$ is a solution of (1.1)–(1.2) with $f = 0$ and such that the following quantity is well defined for all time

$t \geq 0$ for which the solution exists:

$$M(t) := \sum_{j \in \mathbb{Z}} \left[\rho_j(t) + \int_0^1 v_j(t,x)\mathrm{d}x \right].$$

Then, formally, integrating by parts in the first equation and using the boundary conditions, we obtain

$$\begin{aligned} \int_0^1 v_j(t,x)\mathrm{d}x - \int_0^1 v_j(0,x)\mathrm{d}x &= d \int_0^t \left(\partial_x v_j(s,1) - \partial_x v_j(s,0) \right) \mathrm{d}s \\ &= \beta \int_0^t \left(\rho_j(s) + \rho_{j+1}(s) \right) \mathrm{d}s - \alpha \int_0^t \left(v_j(s,1) + v_j(s,0) \right) \mathrm{d}s, \end{aligned}$$

while the second equation gives

$$\rho_j(t) - \rho_j(0) = \alpha \int_0^t (v_j(s,0) + v_{j-1}(s,1))\mathrm{d}s - 2\beta \int_0^t \rho_j(s)\mathrm{d}s.$$

Summing over $\mathbb{Z}$, we deduce that $M(t) = M(0)$ for all $t \geq 0$. As a consequence, we see that, in the absence of reaction kinetics, the exchanges between the cities and the roads exactly compensate each other.

Our aim here is to study the long time behavior of the solutions (1.1)–(1.2) as a function of the various parameters of the model: α, β, and d, the nonlinearity f and the chosen initial condition. We are especially interested in characterizing the spreading properties of the system. More precisely, given a compactly supported initial condition, that is, given an initial condition for which only finitely many cities and/or roads have a nonzero initial population, does the corresponding solution of the Cauchy problem converge to a unique positive stationary configuration? And if yes, at which speed does the convergence towards this eventual steady state take place? In a nutshell, our main results regarding our model (1.1)–(1.2) are as follows. At this stage of the presentation, we remain formal and refer to the following sections for precise statements and assumptions.

Existence and uniqueness of classical solutions. We prove in Theorem 1 below that for each well-prepared initial condition our model (1.1)–(1.2) admits a unique positive bounded classical solution which is global in time. The structure of (1.1)–(1.2) is non standard, and since the graph considered here is infinite, we cannot readily rely on the existing results of [3], which only apply for compact graphs. We adopt a similar approach and construct solutions via an iterative scheme. To obtain compactness and extract converging subsequences, we combine a priori estimates via comparison principle techniques and standard parabolic estimates for the heat equation with inhomogeneous Robin boundary conditions. This analysis is conducted in Section 2.

Long time behavior of the solutions. We fully characterize the long time behavior of the unique solution of our model. More precisely, in Theorem 2, we first prove that the only positive, bounded, stationary solution of (1.1)–(1.2) is the constant sequence $\left(\frac{\beta}{\alpha}, 1\right)_{j \in \mathbb{Z}}$. Interestingly enough, the proof relies on the fact that stationary solutions of (1.1)–(1.2) are in one-to-one correspondence with the stationary solutions of the discrete Fisher-KPP equation given by

$$\lambda\left(\rho_{j-1}-2\rho_j+\rho_{j+1}\right)+f(\rho_j)=0, \quad j\in\mathbb{Z},$$

for some $\lambda>0$ depending explicitly on α, β and d. Finally, we demonstrate that the positive bounded stationary solution $\left(\frac{\beta}{\alpha},1\right)_{j\in\mathbb{Z}}$ is the global attractor of the system (1.1)–(1.2) when initialized with nontrivial nonnegative bounded initial condition. We refer to Theorem 3 for a precise statement but we already emphasize that the convergence is locally uniform in $j\in\mathbb{Z}$ and uniform in $x\in[0,1]$. The aforementioned results are proved in Section 3 and rely on comparison principle techniques and the construction of adequate sub and super-solutions for the system (1.1)–(1.2).

Linear spreading speed. In Section 4, we analyze the linearized problem around the trivial constant state $(0,0)_{j\in\mathbb{Z}}$ and derive a theoretical formula for the linear spreading speed, denoted by c_*, and defined as the small possible speed $c>0$ for which there exists an exponential solution of the form

$$(v_j(t,x),\rho_j(t))=\left(\mathrm{e}^{-\mu(j-ct)}V(x),\mathrm{e}^{-\mu(j-ct)}\right),$$

for some prescribed positive profile V. The formula for c_* is given in equation (4.9) below and we refer to Figure 4 and Figure 5 for illustrations of the dependence of c_* as a function of the other parameters α, β, d and $f'(0)$. The characterization leading to the definition of c_* is quite intricate. Although we manage to prove that c_* is well-defined, it is yet a problem to prove that there exists a unique corresponding $\mu_*>0$ at which the spreading speed is attained, as it is usually the case for reaction-diffusion systems having a monotone structure. We conjecture that it is indeed the case based on our numerical computation of the linear spreading speed via its formula (4.9).

Asymptotic spreading. It turns out that the linear spreading speed c_* defined in formula (4.9) is precisely the asymptotic spreading speed of the nonlinear system (1.1)-(1.2) as proved in Theorem 4 in Section 5. More precisely, we show that solutions of system (1.1)-(1.2) starting from compactly supported initial conditions spread at speed c_*. Traduced mathematically, if $(\mathbf{v},\boldsymbol{\rho})$ is a corresponding solution, then we have the following dichotomy:

(i) for all $c>c_*$, we have

$$\lim_{t\to+\infty}\sup_{\substack{|j|\geq ct\\ x\in[0,1]}}(v_j(t,x),\rho_j(t))=(0,0);$$

(ii) for all $c\in(0,c_*)$, we have

$$\lim_{t\to+\infty}\inf_{\substack{|j|\leq ct\\ x\in[0,1]}}(v_j(t,x),\rho_j(t))=\left(\frac{\beta}{\alpha},1\right).$$

A key element of the proof is the construction of compactly supported generalized subsolutions for the nonlinear system (1.1)-(1.2).

Large diffusion limit. We finally investigate the large diffusion limit $d \to +\infty$ of the system in Section 6. Our first result, see Theorem 5 for a precise statement, ensures that for well-prepared initial conditions, the solution of system (1.1)-(1.2) converges[1] as $d \to +\infty$ towards $(\mathbf{V}, \mathbf{P})$, which is the solution of the asymptotic system

$$\forall t > 0, \; j \in \mathbb{Z}, \quad \begin{cases} V_j'(t) = -2\alpha V_j(t) + \beta(P_j(t) + P_{j+1}(t)), \\ P_j'(t) = f(P_j(t)) + \alpha(V_j(t) + V_{j-1}(t)) - 2\beta P_j(t). \end{cases}$$

For this asymptotic system, we also prove that $\left(\frac{\beta}{\alpha}, 1\right)_{j\in\mathbb{Z}}$ is the only positive bounded stationary solution and that solutions to the corresponding Cauchy problem starting from bounded nonnegative initial conditions asymptotically converge towards it, locally uniformly in $j \in \mathbb{Z}$. We further prove in Theorem 6 the existence of an asymptotic spreading speed, denoted by c_*^∞ (see formula 6.8), for the asymptotic system. We also conjecture[2] that

$$c_* \underset{d\to+\infty}{\longrightarrow} c_*^\infty,$$

where c_* is the spreading speed of the full system (1.1)-(1.2), and leave it to future work to rigorously demonstrate this asymptotic limit.

Our asymptotic spreading result echoes the ones obtained for standard reaction-diffusion equations set on graphs such as, for instance, the Fisher-KPP equation set on the lattice [22] or homogeneous trees [13], and where the linear spreading speed characterizes the long time behavior of the solutions of the Cauchy problem starting from compactly supported initial data. We also refer to [5] for the most recent results in the direction of the so-called logarithmic Bramson correction for the level sets of the solutions for the Fisher-KPP equations on the lattice. In our setting, as expected, the characterization of the spreading speed is less explicit and more intricate. Let us also remark that our framework is at the crossroad of the aforementioned standard discrete reaction-diffusion models and continuous models that take into account lines of transportation such as the so-called "field-road" model of Berestycki, Roquejoffre and Rossi [1]. Indeed, on a formal level, our proposed model can be thought of as being a one-dimensional version of the planar reaction-diffusion system of [1], if we consider only one city and one semi-infinite road.

2 The Cauchy problem

In this section, we focus on the well-posedness of the problem (1.1)–(1.2). As a consequence, we supplement the system with the initial condition

$$\forall j \in \mathbb{Z}, \quad \begin{cases} v_j(0,x) = h_j(x), \quad x \in (0,1), \\ \rho_j(0) = \Lambda_j. \end{cases} \tag{2.1}$$

[1] Locally uniformly in $(t,j) \in (0,+\infty) \times \mathbb{Z}$ and uniformly in $x \in [0,1]$.
[2] This conjecture is verified numerically in Figure 4(c).

We shall always assume that the initial sequences $\mathbf{h} = (h_j)_{j\in\mathbb{Z}}$ and $\boldsymbol{\Lambda} = (\Lambda_j)_{j\in\mathbb{Z}}$ satisfy the following compatibility condition

$$\forall j \in \mathbb{Z}, \quad \begin{cases} -d\partial_x h_j'(0) + \alpha h_j(0) = \beta\Lambda_j(0), \\ d\partial_x h_j'(1) + \alpha h_j(1) = \beta\Lambda_{j+1}(0). \end{cases} \tag{2.2}$$

Throughout the paper, we let $\ell^\infty(\mathbb{Z})$ denote the Banach space of bounded valued sequences indexed by $\mathbb{Z}$ and equipped with the norm:

$$\|\mathbf{u}\|_{\ell^\infty(\mathbb{Z})} := \max_{j\in\mathbb{Z}} |u_j|, \text{ for } \mathbf{u} = (u_j)_{j\in\mathbb{Z}},$$

and also define

$$\mathcal{X}^0 := \left\{\mathbf{u} = (u_j)_{j\in\mathbb{Z}} \mid \forall j \in \mathbb{Z},\ u_j \in \mathscr{C}^0([0,1],\mathbb{R}) \text{ and } \|\mathbf{u}\|_\infty < +\infty\right\},$$

with norm

$$\|u\|_\infty := \sup_{j\in\mathbb{Z}} \max_{x\in(0,1)} |u_j(x)|.$$

The main result of this section is the following.

Theorem 1. *The Cauchy problem (1.1)-(1.2)-(2.1) with nontrivial nonnegative bounded initial sequences $\mathbf{h} = (h_j)_{j\in\mathbb{Z}} \in \mathcal{X}^0$ and $\boldsymbol{\Lambda} = (\Lambda_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ satisfying the compatibility condition (2.2) admits a unique bounded positive global classical solution $(\mathbf{v},\boldsymbol{\rho}) = (v_j,\rho_j)_{j\in\mathbb{Z}}$ with $\rho_j \in \mathscr{C}^1([0,+\infty),\mathbb{R})$ and*

$$v_j \in \mathscr{C}^0([0,+\infty)\times[0,1],\mathbb{R}),\ \partial_t v_j, \partial_x^2 v_j \in \mathscr{C}^0((0,+\infty)\times(0,1),\mathbb{R}),\ \text{and } \partial_x v_j \in \mathscr{C}^0((0,+\infty)\times[0,1],\mathbb{R}),$$

for all $j \in \mathbb{Z}$. Furthermore, for all $t > 0$, one has

$$\forall j \in \mathbb{Z}, \quad 0 < v_j(t,x) \le \max\left\{\frac{\beta}{\alpha}, \|\mathbf{h}\|_\infty\right\},\ x \in [0,\ell],\ \text{and } 0 < \rho_j(t) \le \max\{\|\boldsymbol{\Lambda}\|_{\ell^\infty(\mathbb{Z})}, 1\}.$$

2.1 Uniqueness

In order to establish the uniqueness of the solution of the Cauchy problem (1.1)-(1.2)-(2.1), we shall rely on a comparison principle for (1.1)-(1.2). We first define the notion of super and subsolutions to (1.1)-(1.2). Let $(\overline{\mathbf{v}},\overline{\boldsymbol{\rho}}) = (\overline{v}_j,\overline{\rho}_j)_{j\in\mathbb{Z}}$ with $\overline{\rho}_j \in \mathscr{C}^1([0,+\infty),\mathbb{R})$ and

$$\overline{v}_j \in \mathscr{C}^0([0,+\infty)\times[0,1],\mathbb{R}),\ \partial_t\overline{v}_j, \partial_x^2\overline{v}_j \in \mathscr{C}^0((0,+\infty)\times(0,1),\mathbb{R}),\ \text{and } \partial_x\overline{v}_j \in \mathscr{C}^0((0,+\infty)\times[0,1],\mathbb{R}),$$

for all $j \in \mathbb{Z}$. We say that $(\overline{\mathbf{v}},\overline{\boldsymbol{\rho}})$ is a supersolution to (1.1)-(1.2) if it has the above regularity and satisfies

$$\begin{cases} \partial_t\overline{v}_j(t,x) \ge d\partial_{x^2}\overline{v}_j(t,x), \quad x \in (0,1), \\ \overline{\rho}_j{}'(t) \ge f(\overline{\rho}_j(t)) + \alpha(\overline{v}_j(t,0) + \overline{v}_{j-1}(t,1)) - 2\beta\overline{\rho}_j(t), \\ -d\partial_x\overline{v}_j(t,0) + \alpha\overline{v}_j(t,0) \ge \beta\overline{\rho}_j(t), \\ d\partial_x\overline{v}_j(t,1) + \alpha\overline{v}_j(t,1) \ge \beta\overline{\rho}_{j+1}(t), \end{cases}$$

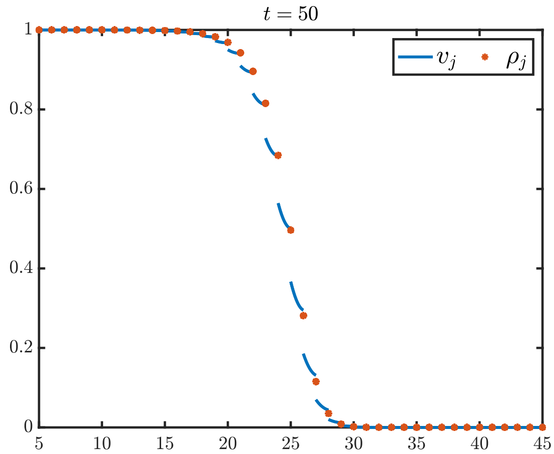


Figure 2: *Numerically computed solution of system* (1.1)–(1.2) *at time* $t = 50$ *for* $f(u) = u(1-u)$ *and* $(\alpha, \beta, d) = (1, 1, 1)$ *starting from an initial condition where* $h_j \equiv 0$ *for all* $j \in \mathbb{Z}$ *and* $\Lambda_j = 1$ *for* $j \leq 0$ *and* $\Lambda_j = 0$ *for* $j \geq 1$*. The red dots represent* ρ_j *located at position* j *while each blue curve represents* v_j *located on the interval* $[j, j+1]$.

for all $t > 0$. We define similarly a subsolution $(\underline{\mathbf{v}}, \underline{\boldsymbol{\rho}})$ to (1.1)–(1.2) with the same regularity and all above inequalities being reversed.

Proposition 2.1. *Let* $(\underline{\mathbf{v}}, \underline{\boldsymbol{\rho}})$ *and* $(\overline{\mathbf{v}}, \overline{\boldsymbol{\rho}})$ *be respectively a subsolution and supersolution to* (1.1)–(1.2)*. If we assume that* $(\underline{\mathbf{v}}, \underline{\boldsymbol{\rho}})$ *and* $(\overline{\mathbf{v}}, \overline{\boldsymbol{\rho}})$ *are locally bounded in time and satisfy for all* $j \in \mathbb{Z}$ *that* $\underline{v}_j(0,x) \leq \overline{v}_j(0,x)$ *for all* $x \in [0,1]$ *and* $\underline{\rho}_j(0) \leq \overline{\rho}_j(0)$*, then we have* $\underline{v}_j(t,x) \leq \overline{v}_j(t,x)$ *and* $\underline{\rho}_j(t) \leq \overline{\rho}_j(t)$ *for all* $t > 0$*,* $x \in [0,1]$ *and* $j \in \mathbb{Z}$*. Furthermore, if* $\underline{\mathbf{v}}(0) \not\equiv \overline{\mathbf{v}}(0)$ *or* $\underline{\boldsymbol{\rho}}(0) \not\equiv \overline{\boldsymbol{\rho}}(0)$*, then we have* $\underline{v}_j(t,x) < \overline{v}_j(t,x)$ *and* $\underline{\rho}_j(t) < \overline{\rho}_j(t)$ *for all* $t > 0$*,* $x \in [0,1]$ *and* $j \in \mathbb{Z}$*.*

The above comparison principle immediately extends to generalized sub and supersolutions, given by the supremum of subsolutions and the infimum of supersolutions respectively.

Proof. We start by defining the following sequences $\mathbf{w} = \overline{\mathbf{v}} - \underline{\mathbf{v}}$ and $\mathbf{z} := \overline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}$ which share the same regularity as the super and subsolutions, and satisfy the following system of equations

$$\forall j \in \mathbb{Z}, \quad \begin{cases} \partial_t w_j(t,x) \geq d\partial_{x^2} w_j(t,x), \quad x \in (0,1), \\ z_j{}'(t) \geq (g_j(t) - 2\beta)\, z_j(t) + \alpha(w_j(t,0) + w_{j-1}(t,1)), \\ -d\partial_x w_j(t,0) + \alpha w_j^n(t,0) \geq \beta z_j(t), \\ d\partial_x w_j(t,1) + \alpha z_j^n(t,1) \geq \beta z_{j+1}(t), \end{cases}$$

for all $t > 0$, together with

$$\forall j \in \mathbb{Z}, \quad \begin{cases} w_j(0,x) \geq 0, \quad x \in [0,1], \\ z_j(0) \geq 0. \end{cases}$$

In the above system, we have also defined

$$\forall t>0,\ j\in\mathbb{Z},\quad g_j(t):=\begin{cases}\dfrac{f(\overline{\rho}_j(t))-f(\underline{\rho}_j(t))}{\overline{\rho}_j(t)-\underline{\rho}_j(t)}, & \overline{\rho}_j(t)\neq\underline{\rho}_j(t),\\ f'(\overline{\rho}_j(t)), & \overline{\rho}_j(t)=\underline{\rho}_j(t).\end{cases}$$

Since $(\underline{\mathbf{v}},\underline{\boldsymbol{\rho}})$ and $(\overline{\mathbf{v}},\overline{\boldsymbol{\rho}})$ are locally bounded in time, we have that for all $T>0$, there exists a constant $C>0$ such that $|g_j(t)|\leq C$ for all $t\in(0,T]$ and $j\in\mathbb{Z}$. As a consequence, we can rely on Proposition B.1 to infer that $w_j(t,x)\geq 0$ and $z_j(t)\geq 0$ for all $t>0$, $x\in[0,1]$ and $j\in\mathbb{Z}$. The same Proposition B.1 also ensures that if furthermore $\mathbf{w}(0)\not\equiv 0$ or $\mathbf{z}(0)\not\equiv 0$, then $w_j(t,x)>0$ and $z_j(t)>0$ for all $t>0$, $x\in[0,1]$ and $j\in\mathbb{Z}$. This concludes the proof. ∎

Proof of uniqueness of Theorem 1. Assume that $(\mathbf{v}_1,\boldsymbol{\rho}_1)$ and $(\mathbf{v}_2,\boldsymbol{\rho}_2)$ are two bounded positive global classical solutions to (1.1)-(1.2)-(2.1) starting from the same initial condition $(\mathbf{h},\boldsymbol{\Lambda})$. Applying twice the comparison principle of Proposition 2.1, we readily obtain that $(\mathbf{v}_1,\boldsymbol{\rho}_1)\equiv(\mathbf{v}_2,\boldsymbol{\rho}_2)$. ∎

2.2 Existence

Throughout this section, we shall always assume that the nontrivial nonnegative bounded initial sequences $\mathbf{h}=(h_j)_{j\in\mathbb{Z}}\in\mathcal{X}^0$ and $\boldsymbol{\Lambda}=(\Lambda_j)_{j\in\mathbb{Z}}\in\ell^\infty(\mathbb{Z})$ satisfy the compatibility condition (2.2).

To establish the existence of a solution to system (1.1)-(1.2)-(2.1), we construct an iterative sequence. More precisely, we obtain a solution to (1.1)-(1.2)-(2.1) as the limit of the sequence of solutions $(\mathbf{v}^n,\boldsymbol{\rho}^n)_{n\in\mathbb{N}}$ starting from $(\mathbf{v}^0,\boldsymbol{\rho}^0)=(\mathbf{h},\boldsymbol{\Lambda})$, and where for each $n\geq 1$, the sequences $\mathbf{v}^n=(v_j^n)_{j\in\mathbb{Z}}$ and $\boldsymbol{\rho}^n=(\rho_j^n)_{j\in\mathbb{Z}}$ are solutions to the following problem:

$$\forall t>0,\quad \begin{cases}\partial_t v_j^n(t,x)=d\partial_x^2 v_j^n(t,x), \quad x\in(0,1),\\ \dfrac{\mathrm{d}\rho_j^n(t)}{\mathrm{d}t}=f(\rho_j^n(t))+\alpha(v_j^{n-1}(t,0)+v_{j-1}^{n-1}(t,1))-2\beta\rho_j^n(t),\end{cases}\tag{2.3}$$

with Robin boundary conditions

$$\forall t>0,\quad \begin{cases}-d\partial_x v_j^n(t,0)+\alpha v_j^n(t,0)=\beta\rho_j^n(t),\\ d\partial_x v_j^n(t,1)+\alpha v_j^n(t,1)=\beta\rho_{j+1}^n(t),\end{cases}\tag{2.4}$$

and initial datum

$$\begin{cases}v_j^n(0,x)=h_j(x), \quad x\in[0,1],\\ \rho_j^n(0)=\Lambda_j,\end{cases}\tag{2.5}$$

for all $j\in\mathbb{Z}$.

We say that $(\overline{\mathbf{v}}^n,\overline{\boldsymbol{\rho}}^n)_{n\in\mathbb{N}}$ is a supersolution to (2.3)-(2.4), if for all $n\geq 1$ one has $\overline{\rho}_j^n\in\mathcal{C}^1([0,+\infty),\mathbb{R})$ and each $\overline{v}_j^n$ has the following regularity

$$\overline{v}_j^n\in\mathscr{C}^0([0,+\infty)\times[0,1],\mathbb{R}),\ \partial_t\overline{v}_j^n,\partial_x^2\overline{v}_j^n\in\mathscr{C}^0((0,+\infty)\times(0,1),\mathbb{R}),\ \text{and}\ \partial_x\overline{v}_j^n\in\mathscr{C}^0((0,+\infty)\times[0,1],\mathbb{R}),$$

and satisfy

$$\forall t>0,\ j\in\mathbb{Z},\quad \begin{cases} \partial_t\overline{v}_j^n(t,x)\geq d\partial_{x^2}\overline{v}_j^n(t,x), \quad x\in(0,1),\\ \overline{\rho}_j^{n\prime}(t)\geq f(\overline{\rho}_j^n(t))+\alpha(\overline{v}_j^{n-1}(t,0)+\overline{v}_{j-1}^{n-1}(t,1))-2\beta\overline{\rho}_j^n(t),\\ -d\partial_x\overline{v}_j^n(t,0)+\alpha\overline{v}_j^n(t,0)\geq\beta\overline{\rho}_j^n(t),\\ d\partial_x\overline{v}_j^n(t,1)+\alpha\overline{v}_j^n(t,1)\geq\beta\overline{\rho}_{j+1}^n(t). \end{cases}$$

We define similarly a subsolution $(\underline{\mathbf{v}}^n,\underline{\boldsymbol{\rho}}^n)_{n\in\mathbb{N}}$ to (2.3)-(2.4) with all above inequalities being reversed and the same notion of regularity. For our purposes, we present a comparison principle which will play an important role in the forthcoming proof of existence of solutions to (2.3)-(2.4)-(2.5) and which is very similar to the one already proved in Proposition 2.1.

Proposition 2.2 (Comparison principle). *Assume that $(\overline{\mathbf{v}}^n,\overline{\boldsymbol{\rho}}^n)_{n\in\mathbb{N}}$ and $(\underline{\mathbf{v}}^n,\underline{\boldsymbol{\rho}}^n)_{n\in\mathbb{N}}$ are respectively supersolution and subsolution to (2.3)-(2.4). If $(\underline{\mathbf{v}}^0,\underline{\boldsymbol{\rho}}^0)\leq(\overline{\mathbf{v}}^0,\overline{\boldsymbol{\rho}}^0)$[3] and for all $n\geq 1$ one has $\underline{v}_j^n(0,x)\leq\overline{v}_j^n(0,x)$ and $\underline{\rho}_j^n(0)\leq\overline{\rho}_j^n(0)$, for $x\in[0,1]$ and $j\in\mathbb{Z}$, then for all $t>0$*

$$\underline{v}_j^n(t,x)\leq\overline{v}_j^n(t,x),\ x\in[0,1],\qquad \textit{and}\qquad \underline{\rho}_j^n(t)\leq\overline{\rho}_j^n(t),$$

for any $n\geq 1$ and $j\in\mathbb{Z}$.

Proof. We start by defining for all $n\in\mathbb{N}$ the following sequences $\mathbf{w}^n:=\overline{\mathbf{v}}^n-\underline{\mathbf{v}}^n$ and $\mathbf{z}^n:=\overline{\boldsymbol{\rho}}^n-\underline{\boldsymbol{\rho}}^n$ which satisfies for all $n\geq 1$ the following system of equations

$$\begin{cases} \partial_t w_j^n(t,x)\geq d\partial_{x^2}w_j^n(t,x), \quad x\in(0,1),\\ z_j^{n\prime}(t)\geq\left(g_j^n(t)-2\beta\right)z_j^n(t)+\alpha(w_j^{n-1}(t,0)+w_{j-1}^{n-1}(t,1)),\\ -d\partial_x w_j^n(t,0)+\alpha w_j^n(t,0)\geq\beta z_j^n(t),\\ d\partial_x w_j^n(t,1)+\alpha z_j^n(t,1)\geq\beta z_{j+1}^n(t), \end{cases}$$

together with

$$\begin{cases} w_j^0(t,x)\geq 0, \quad x\in[0,1],\\ z_j^0(t)\geq 0,\\ w_j^n(0,x)\geq 0, \quad x\in[0,1],\\ z_j^n(0)\geq 0, \end{cases}$$

for all $t>0$ and $j\in\mathbb{Z}$. In the above system, we have also defined

$$g_j^n(t)=\begin{cases} \dfrac{f(\overline{\rho}_j^n(t))-f(\underline{\rho}_j^n(t))}{\overline{\rho}_j^n(t)-\underline{\rho}_j^n(t)}, & \overline{\rho}_j^n(t)\neq\underline{\rho}_j^n(t),\\ f'(\overline{\rho}_j^n(t)), & \overline{\rho}_j^n(t)=\underline{\rho}_j^n(t), \end{cases}$$

[3]We use the notation $(\mathbf{u},\boldsymbol{\rho})\leq(\mathbf{v},\boldsymbol{\lambda})$ whenever $u_j(t,x)\leq v_j(t,x)$ and $\rho_j(t)\leq\lambda_j(t)$ for all $t\geq 0$, $j\in\mathbb{Z}$ and $x\in[0,1]$.

which is well defined by the regularity of f. We shall now complete the proof of the proposition by induction.

For $n=1$, integrating the second inequality of the above system, we find

$$z_j^1(t) \geq z_j^1(0)\mathrm{e}^{\int_0^t (g_j^1(s)-2\beta)\mathrm{d}s} + \alpha \int_0^t (w_j^0(s,0)+w_{j-1}^0(s,1))\mathrm{e}^{\int_s^t (g_j^1(\tau)-2\beta)\mathrm{d}\tau}\mathrm{d}s \geq 0, \quad t>0, \quad j\in\mathbb{Z},$$

since $z_j^1(0)\geq 0$ and $w_j^0(t,x)\geq 0$ for all $x\in[0,1]$, $t>0$ and $j\in\mathbb{Z}$. Now, for each $j\in\mathbb{Z}$ we have

$$\begin{cases} \partial_t w_j^1(t,x) \geq d\partial_x^2 w_j^1(t,x), & x\in(0,1), \\ -d\partial_x w_j^1(t,0)+\alpha w_j^1(t,0)\geq 0, \\ d\partial_x w_j^1(t,1)+\alpha z_j^1(t,1) \geq 0, \\ w_j(0,x)\geq 0, & x\in[0,1], \end{cases}$$

then the weak maximum principle for parabolic equation with Robin boundary condition [20] ensures that

$$w_j^1(t,x)\geq 0, \quad x\in[0,1],$$

for all $t>0$.

Finally, assume that the property holds for $n-1$, that is, $w_j^{n-1}(t,x)\geq 0$ for $x\in(0,1)$ and $z_j^{n-1}(t)\geq 0$ for all $t>0$ and $j\in\mathbb{Z}$. Once again, using the assumption that $z_j^n(0)\geq 0$ and the variation of constants formula, we derive

$$z_j^n(t) \geq z_j^n(0)\mathrm{e}^{\int_0^t (g_j^n(s)-2\beta)\mathrm{d}s} + \alpha \int_0^t (w_j^{n-1}(s,0)+w_{j-1}^{n-1}(s,1))\mathrm{e}^{\int_s^t (g_j^n(\tau)-2\beta)\mathrm{d}\tau}\mathrm{d}s \geq 0, \quad t>0, \quad j\in\mathbb{Z},$$

from which we deduce, applying again the weak maximum principle, that

$$w_j^n(t,x)\geq 0, \quad x\in[0,1],$$

for all $t>0$. This completes the proof of the proposition. ∎

As already emphasized, we shall construct a classical solution $(\mathbf{v},\boldsymbol{\rho})$ to (1.1)-(1.2)-(2.1) as the limit of the sequence $(\mathbf{v}^n,\boldsymbol{\rho}^n)_{n\in\mathbb{N}}$ initialized with $(\mathbf{v}^0,\boldsymbol{\rho}^0)=(\mathbf{h},\boldsymbol{\Lambda})$, where each $(\mathbf{v}^n,\boldsymbol{\rho}^n)$ is the solution of (2.3)-(2.4)-(2.5). We divide the proof into several steps.

Step 1: solvability of (2.3)-(2.4)-(2.5) on $[0,T_n)$ for some $T_n>0$. We use induction to show that (2.3)-(2.4)-(2.5) has a unique solution. For $n=1$, we have that for each $j\in\mathbb{Z}$, the function ρ_j^1 are solutions of the following Cauchy problem

$$\begin{cases} {\rho_j^1}'(t) = f(\rho_j^1(t)) + \alpha(v_j^0(t,0)+v_{j-1}^0(t,1)) - 2\beta\rho_j^1(t), & t>0, \\ \rho_j^1(0)=\Lambda_j. \end{cases} \tag{2.6}$$

Since f is Lipschitz continuous, and by definition $v_j^0(t,0)=h_j(0)$ and $v_{j-1}^0(t,1)=h_{j-1}(1)$, the Cauchy-Lipschitz theorem ensures the existence of $0<T_1<+\infty$, the maximal time of existence, such that the Cauchy problem (2.6) has a unique solution $\rho_j^1 \in \mathscr{C}^1([0,T_1),\mathbb{R})$. Next, we for each $j\in\mathbb{Z}$, we look at the following evolutionary problem on $(0,T_1)$

$$\begin{cases} \partial_t v_j^1(t,x) = d\partial_x^2 v_j^1(t,x), \quad x\in(0,1), \\ -d\partial_x v_j^1(t,0) + \alpha v_j^1(t,0) = \beta\rho_j^1(t), \\ d\partial_x v_j^1(t,1) + \alpha v_j^1(t,1) = \beta \rho_{j+1}^1(t), \end{cases} \tag{2.7}$$

with initial data

$$v_j^1(0,x) = h_j(x), \quad x\in[0,1].$$

Since $\rho_j^1 \in \mathscr{C}^1([0,T_1),\mathbb{R})$ for all $j\in\mathbb{Z}$, there exists a unique classical solution v_j^1, that is

$$v_j^1 \in \mathscr{C}^0([0,T_1)\times[0,1],\mathbb{R}),\ \partial_t v_j^1, \partial_x^2 v_j^1 \in \mathscr{C}^0((0,T_1)\times(0,1),\mathbb{R}),\ \text{and}\ \partial_x v_j^1 \in \mathscr{C}^0((0,T_1)\times[0,1],\mathbb{R}).$$

We remark that $t\longmapsto v_j^1(t,0)$ and $t\longmapsto v_j^1(t,1)$ are continuous on $[0,T_1)$. As a consequence, we can apply an induction argument to obtain the existence of a nonincreasing sequence of times $T_n>0$ such that $0<T_n\le T_{n-1}\le\cdots\le T_1\le+\infty$ system (2.3)-(2.4)-(2.5) admits a unique couple of solution $\rho_j^n \in \mathcal{C}^1([0,T_n),\mathbb{R})$ and v_j^n having the following regularity

$$v_j^n \in \mathscr{C}^0([0,T_n)\times[0,1],\mathbb{R}),\ \partial_t v_j^n, \partial_x^2 v_j^n \in \mathscr{C}^0((0,T_n)\times(0,1),\mathbb{R}),\ \text{and}\ \partial_x v_j^n \in \mathscr{C}^0((0,T_n)\times[0,1],\mathbb{R}),$$

for each $j\in\mathbb{Z}$.

Step 2: solvability of (2.3)-(2.4)-(2.5) on $[0,+\infty)$. We give some *a priori* estimates to extend the solution constructed in the previous step to $T_n=+\infty$. We claim that for each $n\ge1$ and $j\in\mathbb{Z}$ one has

$$0\le v_j^n(t,x)\le \max\left\{\frac{\beta}{\alpha},\|\mathbf{h}\|_\infty\right\},\ x\in[0,1],\ \text{and}\ 0\le\rho_j^n(t)\le\max\{\|\mathbf{\Lambda}\|_{\ell^\infty(\mathbb{Z})},1\}, \tag{2.8}$$

for all $t\in[0,T_n)$. First, since both $(\mathbf{v}^0,\boldsymbol{\rho}^0)=(\mathbf{h},\mathbf{\Lambda})\ge(0,0)$ and $(\mathbf{v}^n(t=0),\boldsymbol{\rho}^n(t=0)) = (\mathbf{h},\mathbf{\Lambda})\ge(0,0)$, and $(\underline{\mathbf{v}}^n,\underline{\boldsymbol{\rho}}^n)_{n\in\mathbb{N}}\equiv(0,0)$ is a trivial subsolution, the comparison principle from Proposition 2.2 ensures that for all $t\in[0,T_n)$ one has

$$0\le v_j^n(t,x),\ x\in[0,1], \quad \text{and} \quad \rho_j^n(t)\le\overline{\rho}_j^n(t),$$

for any $n\ge1$ and $j\in\mathbb{Z}$. On the other hand if we define for each $n\in\mathbb{N}$

$$(\overline{\mathbf{v}}^n,\overline{\boldsymbol{\rho}}^n)\equiv\left(\max\left\{\frac{\beta}{\alpha},\|\mathbf{h}\|_\infty\right\},\max\{\|\mathbf{\Lambda}\|_{\ell^\infty(\mathbb{Z})},1\}\right),$$

then we can readily check that $(\mathbf{v}^0,\boldsymbol{\rho}^0)=(\mathbf{h},\mathbf{\Lambda})\le(\overline{\mathbf{v}}^0,\overline{\boldsymbol{\rho}}^0)$ and also $(\mathbf{v}^n(t=0),\boldsymbol{\rho}^n(t=0)) = (\mathbf{h},\mathbf{\Lambda})\le(\overline{\mathbf{v}}^n(t=0),\overline{\boldsymbol{\rho}}^n(t=0))$. It is also easy to check that $(\overline{\mathbf{v}}^n,\overline{\boldsymbol{\rho}}^n)_{n\in\mathbb{N}}$ is a supersolution to (2.3)-(2.4) since f is assumed to be negative outside the interval $[0,1]$. Applying Proposition 2.2, we obtain

$$v_j^n(t,x)\le \max\left\{\frac{\beta}{\alpha},\|\mathbf{h}\|_\infty\right\},\ x\in[0,1],\ \text{and}\ \rho_j^n(t)\le\max\{\|\mathbf{\Lambda}\|_{\ell^\infty(\mathbb{Z})},1\},$$

for all $n \geq 1$ and $j \in \mathbb{Z}$. This uniform bound implies that $T_n = +\infty$ for all $n \geq 1$. As a complementary remark, let us observe that thanks to our assumption on f and the uniform bound (2.8), one gets the following uniform bound for the time derivative of ρ_j^n, namely

$$\forall t > 0, \quad \left|\rho_j^{n\prime}(t)\right| \leq \left(f'(0) + 2\beta\right) \max\{\|\mathbf{\Lambda}\|_{\ell^\infty(\mathbb{Z})}, 1\} + 2\alpha \max\left\{\frac{\beta}{\alpha}, \|\mathbf{h}\|_\infty\right\},$$

for any $n \geq 1$ and $j \in \mathbb{Z}$.

Step 3: existence of a solution. Let $T > 0$ be fixed and $(\mathbf{v}^n, \boldsymbol{\rho}^n)_{n\in\mathbb{N}}$ be the solution of (2.3)-(2.4)-(2.5) constructed in the previous step. We already know that for each $n \geq 1$ and $j \in \mathbb{Z}$ the function ρ_j^n is globally Lipschitz continuous. As a consequence, since each v_j^n is a solution of

$$\partial_t v_j^n(t,x) = d\partial_x^2 v_j^n(t,x), \quad t > 0, \ x \in (0,1),$$

with Robin boundary conditions

$$\begin{cases} -d\partial_x v_j^n(t,0) + \alpha v_j^n(t,0) = \beta \rho_j^n(t), \\ \ \ d\partial_x v_j^n(t,1) + \alpha v_j^n(t,1) = \beta \rho_{j+1}^n(t), \end{cases} \quad \forall t > 0,$$

and initial datum

$$v_j^n(0,x) = h_j(x), \quad x \in [0,1],$$

we have, by standard parabolic estimates for the heat equation on bounded domain with Robin boundary conditions [19], that there exists $0 < \nu < 1$ such that for any $\tau \in (0,T)$ and

$$\begin{aligned} \forall n \geq 1, \ j \in \mathbb{Z}, \quad & \|v_j^n\|_{\mathscr{C}^{0,\nu}([\tau,T]\times[0,1])} + \|\partial_x v_j^n\|_{\mathscr{C}^{0,\nu}([\tau,T]\times[0,1])} \\ & \leq C\left(\|\rho_j^n\|_{L^\infty([0,T+1])} + \|\rho_{j+1}^n\|_{L^\infty([0,T+1])} + \|v_j^n\|_{L^\infty([0,T+1]\times[0,1])}\right) \\ & \leq C\left(2\max\{\|\mathbf{\Lambda}\|_{\ell^\infty(\mathbb{Z})}, 1\} + \max\left\{\frac{\beta}{\alpha}, \|\mathbf{h}\|_\infty\right\}\right), \end{aligned}$$

where the constant $C > 0$ only depends on $\nu, \tau, T, d, \alpha, \beta$. Then, by Schauder estimates [19], we also have for all $n \geq 1$ and $j \in \mathbb{Z}$

$$\begin{aligned} & \|\partial_t v_j^n\|_{\mathscr{C}^{0,\nu}([\tau,T]\times[0,1])} + \|\partial_x^2 v_j^n\|_{\mathscr{C}^{0,\nu}([\tau,T]\times[0,1])} \\ & \leq C'\left(\|\rho_j^n\|_{\mathscr{C}^{0,\nu}([\tau/2,T])} + \|\rho_{j+1}^n\|_{\mathscr{C}^{0,\nu}([\tau/2,T])} + \|v_j^n\|_{L^\infty([0,T+1]\times[0,1])}\right) \\ & \leq C'\left(2\left(f'(0) + 2\beta\right)\max\{\|\mathbf{\Lambda}\|_{\ell^\infty(\mathbb{Z})}, 1\} + (4\alpha + 1)\max\left\{\frac{\beta}{\alpha}, \|\mathbf{h}\|_\infty\right\}\right), \end{aligned}$$

for some constant $C' > 0$ independent of n and j. Finally, returning to the equation satisfied by ρ_j^n, we also deduce that $\rho_j^{n\prime} \in \mathscr{C}^{0,\nu}([\tau,T])$ with the following uniform estimate

$$\|\rho_j^{n\prime}\|_{\mathscr{C}^{0,\nu}([\tau,T])} \leq (f'(0) + 2\beta)\|\rho_j^n\|_{\mathscr{C}^{0,\nu}([\tau,T])} + 2\alpha\|v_j^n\|_{\mathscr{C}^{0,\nu}([\tau,T]\times[0,1])} \leq C''(\|\mathbf{\Lambda}\|_{\ell^\infty(\mathbb{Z})} + \|\mathbf{h}\|_\infty),$$

for some $C'' > 0$ independent of n and j.

As a consequence, for any $(N,M)\in\mathbb{Z}^2$ such that $N<M$, the sequence

$$((v_j^n)_{j=N,\dots,M},(\rho_j^n)_{j=N,\dots,M})_{n\in\mathbb{N}},$$

together with its respective time derivatives and space derivatives for v_j^n up to order 2, is uniformly bounded in $\mathscr{C}^{0,\nu}$ norm on the compact set $[\tau,T]\times[0,1]$. By Arzela-Ascoli's theorem, up to a subsequence, there exists a limit sequence $(\mathbf{v},\boldsymbol{\rho})=(v_j,\rho_j)_{j\in\mathbb{Z}}$ such that $(\mathbf{v}^n,\boldsymbol{\rho}^n)$ converges to $(\mathbf{v},\boldsymbol{\rho})$ as $n\to+\infty$ on any compact of $(0,+\infty)\times[0,1]\times\mathbb{Z}$, but also its respective time derivative and space derivatives (up to order 2).

From Proposition A.1, one has

$$\begin{aligned}v_j^n(t,x)=&\int_0^1\mathcal{K}(t,x-y)h(y)\mathrm{d}y+\int_0^t\left[\mathcal{K}(t-s,x-1)\rho_{j+1}^n(s)+\mathcal{K}(t-s,x)\rho_j^n(s)\right]\mathrm{d}s\\&+\int_0^t\left[-\alpha\mathcal{K}(t-s,x-1)+d\partial_x\mathcal{K}(t-s,x-1)\right]v_j^n(s,1)\mathrm{d}s\\&-\int_0^t\left[\alpha\mathcal{K}(t-s,x)+d\partial_x\mathcal{K}(t-s,x)\right]v_j^n(s,0)\mathrm{d}s,\end{aligned}$$

where $\mathcal{K}(t,x):=\frac{1}{\sqrt{4\pi dt}}\exp\left(-\frac{x^2}{4dt}\right)$, and passing to the limit as $n\to+\infty$ for $t\in[\tau,T]$, $x\in[0,1]$ and $j\in[\![N,M]\!]$ for $N<M$, we end up with

$$\begin{aligned}v_j(t,x)=&\int_0^1\mathcal{K}(t,x-y)h_j(y)\mathrm{d}y+\int_0^t\left[\mathcal{K}(t-s,x-1)\rho_{j+1}(s)+\mathcal{K}(t-s,x)\rho_j(s)\right]\mathrm{d}s\\&+\int_0^t\left[-\alpha\mathcal{K}(t-s,x-1)+d\partial_x\mathcal{K}(t-s,x-1)\right]v_j(s,1)\mathrm{d}s\\&-\int_0^t\left[\alpha\mathcal{K}(t-s,x)+d\partial_x\mathcal{K}(t-s,x)\right]v_j(s,0)\mathrm{d}s.\end{aligned}$$

Taking $t=\tau\to0$, we recover

$$v_j(t,x)\underset{t\to0}{\longrightarrow}h_j(x),\quad x\in[0,1].$$

Integrating the second equation in (2.3) from 0 to t, we get that

$$\rho_j^n(t)=\Lambda_j+\int_0^t(f(\rho_j^n(s))-2\beta\rho_j^n(s))\mathrm{d}s+\alpha\int_0^t(v_j^{n-1}(s,0)+v_{j-1}^{n-1}(s,1))\mathrm{d}s,$$

and passing to the limit as $n\to+\infty$, we get

$$\rho_j(t)=\Lambda_j+\int_0^t(f(\rho_j(s))-2\beta\rho_j(s))\mathrm{d}s+\alpha\int_0^t(v_j(s,0)+v_{j-1}(s,1))\mathrm{d}s,$$

from which we also recover that

$$\rho_j(t)\underset{t\to0}{\longrightarrow}\Lambda_j.$$

Thanks to the regularity of $\partial_x v_j^n$ up to the boundary at $x=0$ and $x=1$, we can also pass to the limit as $n\to+\infty$ in the boundary condition. As a consequence, $(\mathbf{v},\boldsymbol{\rho})$ is a classical solution to (2.3)-(2.4)-(2.5). By uniqueness of the problem (2.3)-(2.4)-(2.5), we remark that the convergence of $(\mathbf{v}^n, \boldsymbol{\rho}^n)$ towards $(\mathbf{v}, \boldsymbol{\rho})$ holds for all n, and not only up to a subsequence. Finally, the a priori bound (2.8) gives

$$\forall t > 0,\ j \in \mathbb{Z}, \quad 0 \le v_j(t,x) \le \max\left\{\frac{\beta}{\alpha}, \|\mathbf{h}\|_\infty\right\},\ x \in [0,1],\ \text{and } 0 \le \rho_j(t) \le \max\{\|\boldsymbol{\Lambda}\|_{\ell^\infty(\mathbb{Z})}, 1\}.$$

Since $(\mathbf{h}, \boldsymbol{\Lambda}) \not\equiv (0,0)$, the comparison principle from Proposition 2.1 ensures that

$$\forall t > 0,\ j \in \mathbb{Z}, \quad 0 < v_j(t,x),\ x \in [0,1],\ \text{and } 0 < \rho_j(t).$$

This concludes the proof of Theorem 1.

3 Long time behavior

We now turn to the study of the long time behavior of (1.1)–(1.2).

Theorem 2. *The unique non-negative, bounded stationary solutions for equation* (1.1)*-*(1.2) *are* $(v_j^\infty, \rho_j^\infty)_{j\in\mathbb{Z}} \equiv (0,0)$ *and* $(v_j^\infty, \rho_j^\infty)_{j\in\mathbb{Z}} \equiv (\frac{\beta}{\alpha}, 1)$.

Bounded nonnegative stationary solutions of system (1.1)-(1.2) are solutions to

$$\begin{cases} 0 = dv_j''(x), \quad x \in (0,1), j \in \mathbb{Z}, \\ 0 = f(\rho_j) + \alpha(v_j(0) + v_{j-1}(1)) - 2\beta\rho_j, \quad j \in \mathbb{Z}, \end{cases} \tag{3.1}$$

together with the boundary conditions

$$\begin{cases} -dv_j'(0) + \alpha v_j(0) = \beta\rho_j, \quad j \in \mathbb{Z}, \\ dv_j'(1) + \alpha v_j(1) = \beta\rho_{j+1}, \quad j \in \mathbb{Z}. \end{cases} \tag{3.2}$$

It follows from the v_j-equation of (3.1) that there exist two sequences $(a_j)_{j\in\mathbb{Z}}$ and $(b_j)_{j\in\mathbb{Z}}$ of real numbers such that

$$v_j(x) = a_j x + b_j, \quad x \in (0,1), \quad j \in \mathbb{Z}. \tag{3.3}$$

Then using (3.3), we have that

$$v_j'(0) = v_j'(1) = v_j(1) - v_j(0). \tag{3.4}$$

Substituting (3.4) into (3.2) we get that

$$\begin{cases} (d+\alpha)v_j(0) - dv_j(1) = \beta\ \rho_j, \quad j \in \mathbb{Z}, \\ -dv_j(0) + (d+\alpha)v_j(1) = \beta\rho_{j+1}, \quad j \in \mathbb{Z}. \end{cases}$$

Solving the above system, we obtain that

$$v_j(0) = \frac{\beta d}{\alpha(2d+\alpha)}\left(\frac{d+\alpha}{d}\rho_j + \rho_{j+1}\right), \quad \forall j \in \mathbb{Z}, \tag{3.5}$$

and

$$v_j(1) = \frac{\beta d}{\alpha(2d+\alpha)}\left(\rho_j + \frac{d+\alpha}{d}\rho_{j+1}\right), \quad \forall j \in \mathbb{Z}. \tag{3.6}$$

By applying the second equation of the system (3.1), and combining it with (3.5) and (3.6), we derive that

$$\frac{\beta d}{2d+\alpha}(\rho_{j-1} - 2\rho_j + \rho_{j+1}) + f(\rho_j) = 0, \quad \forall j \in \mathbb{Z}. \tag{3.7}$$

As a consequence, the existence and uniqueness of bounded nonnegative stationary solutions to system (3.1)-(3.2) is equivalent to the existence and uniqueness of bounded nonnegative stationary solutions to (3.7).

Proof of Theorem 2. Since $f(0) = f(1) = 0$, it is clear that $\rho_j \equiv 0, \forall j \in \mathbb{Z}$ and $\rho_j \equiv 1, \forall j \in \mathbb{Z}$ are always solutions of (3.7). Let us prove that they are actually the only bounded nonnegative stationary solutions of (3.7).

Let $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{Z}} \neq 0$ be a non zero, bounded, nonnegative stationary solution (3.7). Then, necessarily, one has $\rho_j > 0$ for all $j \in \mathbb{Z}$. Indeed, if $\rho_{j_0} = 0$ for some $j_0 \in \mathbb{Z}$, then equation (3.7) implies that $\rho_{j_0+1} = \rho_{j_0-1} = 0$, and by induction, $\rho_j = 0$ for all $j \in \mathbb{Z}$. So from now on, we assume that $\boldsymbol{\rho}$ satisfies $\rho_j > 0$ for all $j \in \mathbb{Z}$. We let $N \geq 2$ be an integer which satisfies

$$\frac{2\beta d}{2d+\alpha}\left(1 - \cos\left(\frac{\pi}{N+1}\right)\right) < f'(0). \tag{3.8}$$

Consider the eigenvalue problem

$$-(\rho_{j-1} - 2\rho_j + \rho_{j+1}) = \mu\rho_j, \quad j = 1, \dots, N,$$

and $\rho_j = 0$ for all $j \leq 0$ and $j \geq N+1$. One easily finds that the eigenvalues are given by

$$\mu_p = 2\left(1 - \cos\left(\frac{p\pi}{N+1}\right)\right), \quad p = 1, \dots, N,$$

with corresponding eigenfunctions $\boldsymbol{\phi}^p = (\phi_j^p)_{j\in\mathbb{Z}}$ defined as

$$\phi_j^p = \begin{cases} \sin\left(\frac{pj\pi}{N+1}\right), & j = 1, \dots, N, \\ 0, & \text{otherwise.} \end{cases}$$

We consider the principal eigenfunction $\boldsymbol{\phi}^1$ with eigenvalue $\mu_1 > 0$. Thanks to condition (3.8) on N which ensures that

$$\frac{\beta d}{2d+\alpha}\mu_1 < f'(0),$$

and thanks to the KPP assumption on the function f, there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ one has

$$-\mathscr{L}(\epsilon\boldsymbol{\phi}^1)_j < f(\epsilon\phi_j^1), \quad j = 1, \dots, N$$

where the operator $\mathscr{L} : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ is defined as

$$\mathscr{L}(\boldsymbol{\rho})_j = \frac{\beta d}{2d+\alpha}(\rho_{j-1} - 2\rho_j + \rho_{j+1}), \quad j \in \mathbb{Z},$$

for any $\boldsymbol{\rho} \in \ell^\infty(\mathbb{Z})$. From the discrete comparison principle (see Proposition B.4), we deduce that

$$\epsilon\phi_j^1 < \rho_j, \quad j = 1, \dots, N.$$

By the discrete translation invariance of the problem, we deduce that

$$m := \inf_{j\in\mathbb{Z}} \ \rho_j > 0.$$

Assume that $m < 1$. We let $(j_k)_{k\in\mathbb{N}}$ such that

$$\rho_{j_k} \underset{k\to+\infty}{\longrightarrow} m.$$

For each $j \in \mathbb{Z}$, we denote

$$\hat{\rho}_j := \lim_{k\to+\infty} \rho_{j+j_k},$$

and we remark that $\hat{\rho}_j$ also satisfies (3.7). We also note that $\hat{\rho}_0 = m$ and by construction

$$\hat{\rho}_0 = m = \inf_{j\in\mathbb{Z}} \ \hat{\rho}_j.$$

It is also satisfies

$$\frac{\beta d}{2d+\alpha}\left(\underbrace{\hat{\rho}_{-1} - \hat{\rho}_0}_{\geq 0} + \underbrace{\hat{\rho}_1 - \hat{\rho}_0}_{\geq 0}\right) = -f(\hat{\rho}_0) = -f(m) < 0,$$

which is impossible. As a consequence, one has $m \geq 1$. By a similar argument, this time with $M = \sup_{j\in\mathbb{Z}} \ \rho_j > 0$, one gets that necessarily $M \leq 1$. This implies that $\rho_j = 1$ for all $j \in \mathbb{Z}$ and concludes the proof of the theorem. ∎

Next, we demonstrate that the positive stationary solution $(v_j^\infty, \rho_j^\infty)_{j\in\mathbb{Z}} \equiv (\frac{\beta}{\alpha}, 1)$ is the global attractor of the system (1.1)–(1.2) starting from nontrivial nonnegative bounded initial condition. More precisely, we shall prove the following result.

Theorem 3. *Let $(\mathbf{v}, \boldsymbol{\rho})$ be the unique global classical solution of (1.1)-(1.2)-(2.1) starting from a nontrivial nonnegative bounded initial sequence $(\mathbf{h}, \boldsymbol{\Lambda}) \in \mathcal{X}^0 \times \ell^\infty(\mathbb{Z})$ satisfying the compatibility condition (2.2). Then,*

$$\lim_{t\to+\infty} (v_j(t,x), \rho_j(t)) = \left(\frac{\beta}{\alpha}, 1\right), \quad \forall x \in [0,1],$$

locally uniformly $j \in \mathbb{Z}$.

Proof of Theorem 3. The first part of the proof consists of constructing a nonnegative, compactly supported, stationary subsolution to (1.1)-(1.2). Actually, following the proof of Theorem 2, there exists $N_0 > 1$ large enough such that condition (3.8) is satisfied for all $N \geq N_0$. Next, with $N \geq N_0$, we define $\underline{\boldsymbol{\rho}} = \left(\underline{\rho}_j\right)_{j\in\mathbb{Z}}$ as

$$\underline{\rho}_j := \begin{cases} \sin\left(\dfrac{j\pi}{N+1}\right), & j = 1,\dots,N, \\ 0, & \text{otherwise}, \end{cases}$$

and set

$$\underline{v}_j(x) := a_j x + b_j, \quad x \in [0,1], \quad j \in \mathbb{Z},$$

with

$$a_j := \frac{\beta}{2d+\alpha}\left(\underline{\rho}_{j+1} - \underline{\rho}_j\right) \quad \text{and} \quad b_j := \frac{\beta d}{\alpha(2d+\alpha)}\left(\frac{d+\alpha}{d}\underline{\rho}_j + \underline{\rho}_{j+1}\right), \quad \forall j \in \mathbb{Z}.$$

By construction, $\underline{\mathbf{v}} = (\underline{v}_j)_{j\in\mathbb{Z}}$ is compactly supported, and we have

$$d\partial_x^2 \underline{v}_j(x) = 0, \quad x \in (0,1),$$

together with

$$\begin{cases} -d\partial_x \underline{v}_j(0) + \alpha \underline{v}_j(0) = \beta \underline{\rho}_j, \\ d\partial_x \underline{v}_j(1) + \alpha \underline{v}_j(1) = \beta \underline{\rho}_{j+1}, \end{cases}$$

for all $j \in \mathbb{Z}$. Finally, there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ one has:

$$\frac{\epsilon\beta d}{2d+\alpha}\left(\underline{\rho}_{j-1} - 2\underline{\rho}_j + \underline{\rho}_{j+1}\right) + f(\epsilon\underline{\rho}_j) > 0, \quad ,j = 1,\dots,N.$$

As a consequence $\left(\epsilon\underline{\mathbf{v}}, \epsilon\underline{\boldsymbol{\rho}}\right)$ is a stationary, compactly supported, subsolution for all $N \geq N_0$ and $\epsilon \in (0, \epsilon_0]$.

We now use the method of super- and subsolutions to prove the theorem. First, we consider $(\overline{\mathbf{v}}, \overline{\boldsymbol{\rho}})$ defined by

$$\overline{v}_j(x) := \max\left\{\|\mathbf{h}\|_\infty, \frac{\beta}{\alpha}\right\}, \quad \overline{\rho}_j = \max\left\{\|\boldsymbol{\Lambda}\|_{\ell^\infty(\mathbb{Z})}, 1\right\}, \quad x \in [0,1], \ j \in \mathbb{Z}.$$

Let $t \mapsto (\widehat{\mathbf{v}}(t), \widehat{\boldsymbol{\rho}}(t))$ be the global solution of (1.1)-(1.2) with initial condition $(\widehat{\mathbf{v}}(0), \widehat{\boldsymbol{\rho}}(0)) = (\overline{\mathbf{v}}, \overline{\boldsymbol{\rho}})$. It follows from the comparison principle from Proposition 2.1 that $t \mapsto (\widehat{\mathbf{v}}(t), \widehat{\boldsymbol{\rho}}(t))$ is non increasing in time t and satisfies $\left(\frac{\beta}{\alpha}, 1\right) \leq (\widehat{\mathbf{v}}(t), \widehat{\boldsymbol{\rho}}(t))$ for all $t > 0$. Thus, owing to Theorem 2, as $t \to +\infty$, it convergences locally uniformly in j to the unique positive solution of (3.1)-(3.2), namely $\left(v_j^\infty, \rho_j^\infty\right) \equiv \left(\frac{\beta}{\alpha}, 1\right)$, that is

$$\forall x \in [0,1], \quad \widehat{v}_j(t,x) \underset{t\to+\infty}{\longrightarrow} \frac{\beta}{\alpha} \quad \text{and} \quad \widehat{\rho}_j(t) \underset{t\to+\infty}{\longrightarrow} 1,$$

locally uniformly in $j \in \mathbb{Z}$. Now, let $t \mapsto (\mathbf{v}(t), \boldsymbol{\rho}(t))$ be the solution of (1.1)-(1.2)-(2.1) starting from the nonnegative, not identically equal to zero, bounded initial datum $(\mathbf{h}, \boldsymbol{\Lambda})$. Since $(\mathbf{h}, \boldsymbol{\Lambda}) \leq (\overline{\mathbf{v}}, \overline{\boldsymbol{\rho}})$, we have $(\mathbf{v}(t), \boldsymbol{\rho}(t)) \leq (\widehat{\mathbf{v}}(t), \widehat{\boldsymbol{\rho}}(t))$ for all $t > 0$ and thus

$$\forall x \in [0,1], \quad \limsup_{t \to +\infty} \ (v_j(t,x), \rho_j(t)) \leq \left(\frac{\beta}{\alpha}, 1\right),$$

locally uniformly in $j \in \mathbb{Z}$. Furthermore, since $0 \lneqq (\mathbf{h}, \boldsymbol{\Lambda})$, by the comparison principle from Proposition 2.1, we have that $0 < (\mathbf{v}(t), \boldsymbol{\rho}(t))$ for all $t > 0$. As a consequence, upon reducing further the size of ϵ, we can always ensure that $\left(\epsilon \underline{\mathbf{v}}, \epsilon \underline{\boldsymbol{\rho}}\right) \leq (\mathbf{v}(1), \boldsymbol{\rho}(1))$. We now let $t \mapsto \left(\underline{\mathbf{v}}(t), \underline{\boldsymbol{\rho}}(t)\right)$ be the global solution of (1.1)-(1.2) with initial condition $\left(\epsilon \underline{\mathbf{v}}, \epsilon \underline{\boldsymbol{\rho}}\right)$, which by the comparison principle, is nondecreasing in t. As a consequence, it also converges locally uniformly j to $\left(v_j^\infty, \rho_j^\infty\right) \equiv \left(\frac{\beta}{\alpha}, 1\right)$, the unique positive solution of (3.1)-(3.2). Thus, we have that

$$\forall x \in [0,1], \quad \left(\frac{\beta}{\alpha}, 1\right) = \lim_{t \to +\infty} (\underline{v}_j(t,x), \underline{\rho}_j(t)) \leq \liminf_{t \to +\infty} (v_j(t+1,x), \rho_j(t+1)),$$

locally uniformly j. This completes the proof of Theorem 3. ∎

4 Exponential solutions and linear spreading speed

In order to study the spreading properties of system (1.1)-(1.2), we consider the existence of exponential solutions for the linearized problem around the trivial state which writes:

$$\forall t > 0, \ j \in \mathbb{Z}, \quad \begin{cases} \partial_t v_j(t,x) = d \partial_x^2 v_j(t,x), & x \in (0,1), \\ \rho_j'(t) = f'(0)\rho_j(t) + \alpha(v_j(t,0) + v_{j-1}(t,1)) - 2\beta \rho_j(t), \end{cases} \tag{4.1}$$

with the boundary condition

$$\forall t > 0, \ j \in \mathbb{Z}, \quad \begin{cases} -d\partial_x v_j(t,0) + \alpha v_j(t,0) = \beta \rho_j(t), \\ d\partial_x v_j(t,1) + \alpha v_j(t,1) = \beta \rho_{j+1}(t). \end{cases} \tag{4.2}$$

We will be looking for solutions of the form

$$(v_j(t,x), \rho_j(t)) = \left(\mathrm{e}^{-\mu(j-ct)} V(x), \mathrm{e}^{-\mu(j-ct)}\right), \tag{4.3}$$

where

$$V(x) = a \cosh\left(\sqrt{\frac{\lambda}{d}} x\right) + b \sinh\left(\sqrt{\frac{\lambda}{d}} x\right), \quad \forall x \in [0,1],$$

for some $\lambda > 0$, $\mu > 0$, $c > 0$ and $(a,b) \in \mathbb{R}^2$ that will be determined later. We substitute ansatz (5.1) into (4.1), (4.2) and get that

$$\begin{cases} \mu c = \lambda, \\ \mu c = f'(0) + \alpha(V(0) + \mathrm{e}^{\mu} V(1)) - 2\beta, \\ -dV'(0) + \alpha V(0) = \beta, \\ dV'(1) + \alpha V(1) = \beta \mathrm{e}^{-\mu}. \end{cases} \tag{4.4}$$

We first express (a,b) as a function of $(V(0),V(1))$, that is

$$a = V(0), \quad b = \frac{V(1)}{\sinh\left(\sqrt{\frac{\lambda}{d}}\right)} - \frac{V(0)}{\tanh\left(\sqrt{\frac{\lambda}{d}}\right)}.$$

We also deduce that

$$V'(0) = \sqrt{\frac{\lambda}{d}}\left[-\frac{1}{\tanh\left(\sqrt{\frac{\lambda}{d}}\right)}V(0) + \frac{1}{\sinh\left(\sqrt{\frac{\lambda}{d}}\right)}V(1)\right],$$

$$V'(1) = \sqrt{\frac{\lambda}{d}}\left[-\frac{1}{\sinh\left(\sqrt{\frac{\lambda}{d}}\right)}V(0) + \frac{1}{\tanh\left(\sqrt{\frac{\lambda}{d}}\right)}V(1)\right].$$

As a consequence, using the Robin type boundary in (4.4), we deduce that

$$\begin{pmatrix} \alpha + \frac{\sqrt{d\lambda}}{\tanh\left(\sqrt{\frac{\lambda}{d}}\right)} & -\frac{\sqrt{d\lambda}}{\sinh\left(\sqrt{\frac{\lambda}{d}}\right)} \\ -\frac{\sqrt{d\lambda}}{\sinh\left(\sqrt{\frac{\lambda}{d}}\right)} & \alpha + \frac{\sqrt{d\lambda}}{\tanh\left(\sqrt{\frac{\lambda}{d}}\right)} \end{pmatrix} \begin{pmatrix} V(0) \\ V(1) \end{pmatrix} = \beta \begin{pmatrix} 1 \\ \mathrm{e}^{-\mu} \end{pmatrix}. \tag{4.5}$$

Define

$$\Delta(\lambda) := \alpha^2 + d\lambda + \frac{2\sqrt{d\lambda}\alpha}{\tanh\left(\sqrt{\frac{\lambda}{d}}\right)},$$

and let us remark that when $\lambda > 0$ we have $\Delta(\lambda) > 0$ and Δ is well-defined up to $\lambda = 0$ with $\Delta(0) = \alpha^2 + 2d\alpha > 0$. Thus, we can invert the above system (4.5) and deduce that

$$V(0) = \frac{\beta}{\Delta(\lambda)}\left(\frac{\sqrt{d\lambda}}{\tanh\left(\sqrt{\frac{\lambda}{d}}\right)} + \alpha + \frac{\sqrt{d\lambda}}{\sinh\left(\sqrt{\frac{\lambda}{d}}\right)}\mathrm{e}^{-\mu}\right),$$

and

$$V(1) = \frac{\beta}{\Delta(\lambda)}\left(\frac{\sqrt{d\lambda}}{\tanh\left(\sqrt{\frac{\lambda}{d}}\right)}\mathrm{e}^{-\mu} + \alpha\mathrm{e}^{-\mu} + \frac{\sqrt{d\lambda}}{\sinh\left(\sqrt{\frac{\lambda}{d}}\right)}\right).$$

Therefore, we substitute the above two formulas into (4.4) to obtain

$$\begin{cases} \mu c = \lambda, \\ \mu c = f'(0) - 2\beta + \frac{2\alpha\beta}{\Delta(\lambda)}\left[\alpha + \frac{\sqrt{d\lambda}}{\tanh\left(\sqrt{\frac{\lambda}{d}}\right)} + \frac{\sqrt{d\lambda}}{\sinh\left(\sqrt{\frac{\lambda}{d}}\right)}\cosh(\mu)\right]. \end{cases} \tag{4.6}$$

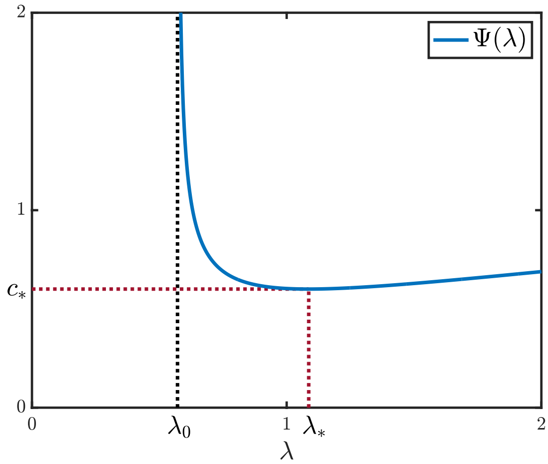


Figure 3: *Typical representation of the map* Ψ *on* $(\lambda_0, +\infty)$ *with a unique global minimum at* $\lambda = \lambda_*$. *Here, parameters values are set to* $(\alpha, \beta, d, f'(0)) = (1, 1, 1, 1)$.

As a consequence, we have that

$$\cosh\left(\frac{\lambda}{c}\right) = y(\lambda),$$

where

$$y(\lambda) := \left[\frac{\Delta(\lambda)}{2\alpha\beta}(\lambda + 2\beta - f'(0)) - \left(\alpha + \frac{\sqrt{d\lambda}}{\tanh\left(\sqrt{\frac{\lambda}{d}}\right)}\right)\right] \frac{\sinh\left(\sqrt{\frac{\lambda}{d}}\right)}{\sqrt{d\lambda}}. \tag{4.7}$$

Solving the above equation, one gets that

$$c = \frac{\lambda}{\mu(\lambda)}, \tag{4.8}$$

where

$$\mu(\lambda) := \ln\left(y(\lambda) + \sqrt{y^2(\lambda) - 1}\right),$$

and $y(\lambda)$ is defined in (4.7). Now, to justify all the above computations, one needs to show that $\lambda \mapsto \mu(\lambda)$ is well-defined which is equivalent to proving that $y(\lambda) > 1$ for some range of λ. First of all, one can actually check that $y(\lambda) \to +\infty$ as $\lambda \to +\infty$. By a direct computation, we also have that

$$y(0^+) = 1 - \frac{f'(0)}{\beta}\left(\frac{\alpha}{2d} + 1\right) < 1.$$

Thus, there exists $\lambda_0 > 0$ such that $y(\lambda_0) = 1$ and

$$y(\lambda) > 1, \quad \forall \lambda > \lambda_0.$$

We claim that such a $\lambda_0 > 0$ is unique. First, we observe that the equality $y(\lambda_0) = 1$ is equivalent to

$$\frac{2\alpha\beta}{\Delta(\lambda_0)}\left[\alpha + \frac{\sqrt{d\lambda_0}}{\tanh\left(\sqrt{\frac{\lambda_0}{d}}\right)} + \frac{\sqrt{d\lambda_0}}{\sinh\left(\sqrt{\frac{\lambda_0}{d}}\right)}\right] = \lambda_0 + 2\beta - f'(0).$$

On the one hand, the map $g : \lambda \mapsto g(\lambda) = \lambda + 2\beta - f'(0)$ is strictly increasing on $\mathbb{R}_+$ with $g(0) = 2\beta - f'(0)$ and $g(\lambda) \sim \lambda$ as $\lambda \to +\infty$. On the other hand, the map G defined as

$$G : \lambda \mapsto G(\lambda) = \frac{2\alpha\beta}{\Delta(\lambda)} \left[\alpha + \frac{\sqrt{d\lambda}}{\tanh\left(\sqrt{\frac{\lambda}{d}}\right)} + \frac{\sqrt{d\lambda}}{\sinh\left(\sqrt{\frac{\lambda}{d}}\right)} \right],$$

is decreasing on $\mathbb{R}_+$ with $G(0) = 2\beta$ and $G(\lambda) \sim \frac{2\alpha\beta}{\sqrt{d\lambda}}$ as $\lambda \to +\infty$. The fact that G is decreasing on $\mathbb{R}_+$ comes from the direct computation:

$$\forall \lambda > 0, \quad G'(\lambda) = -\frac{\alpha\beta\left(\lambda + \sqrt{\frac{\lambda}{d}}\sinh\left(\sqrt{\frac{\lambda}{d}}\right)\right)}{\lambda \sinh\left(\sqrt{\frac{\lambda}{d}}\right)^2 \Delta(\lambda)^2}\Theta(\lambda) < 0,$$

where

$$\Theta(\lambda) := d\lambda\left(\cosh\left(\sqrt{\frac{\lambda}{d}}\right) - 1\right) + \alpha^2\left(\cosh\left(\sqrt{\frac{\lambda}{d}}\right) + 1\right) + 2\sqrt{d\lambda}\sinh\left(\sqrt{\frac{\lambda}{d}}\right) > 0.$$

As a consequence since $g(0) = 2\beta - f'(0) < 2\beta = G(0)$ and g is strictly increasing with $g(\lambda) \to +\infty$ as $\lambda \to +\infty$ while G is strictly decreasing $G(\lambda) \to 0$ as $\lambda \to +\infty$, we obtain the existence of a unique $\lambda_0 > 0$ such that $g(\lambda_0) = G(\lambda_0)$ and $g(\lambda) < G(\lambda)$ for all $\lambda \in [0, \lambda_0)$ together with $g(\lambda) > G(\lambda)$ for all $\lambda > \lambda_0$. This proves the claim.

The uniqueness of λ_0 implies that $y(\lambda) < 1$ for all $\lambda \in (0, \lambda_0)$ such that $\mu(\lambda)$ is only well-defined for all $\lambda > \lambda_0$. Coming back to (4.8), we define $\Psi : (\lambda_0, +\infty) \to \mathbb{R}_+$ as

$$\Psi(\lambda) := \frac{\lambda}{\mu(\lambda)}.$$

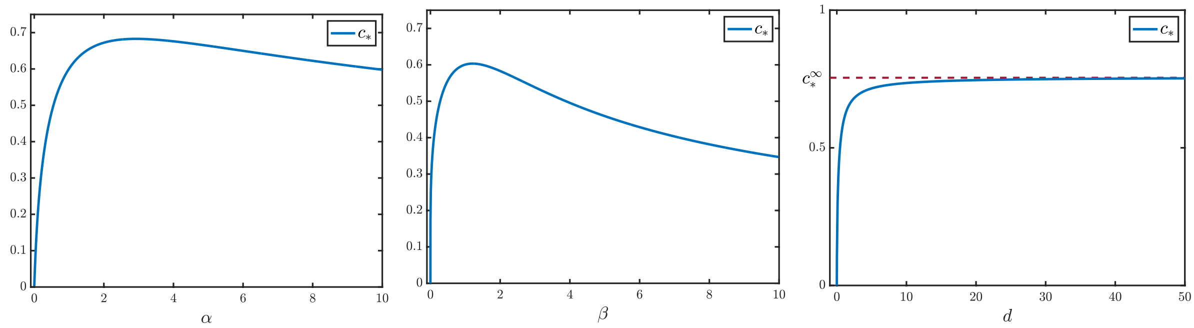
We readily note that $\Psi(\lambda) \to +\infty$ as $\lambda \to \lambda_0^+$ and as $\lambda \to +\infty$. Since Ψ is smooth on $(\lambda_0, +\infty)$, it achieves a minimum on $(\lambda_0, +\infty)$, and we can define

$$c_* := \min_{\lambda > \lambda_0} \frac{\lambda}{\mu(\lambda)}. \tag{4.9}$$

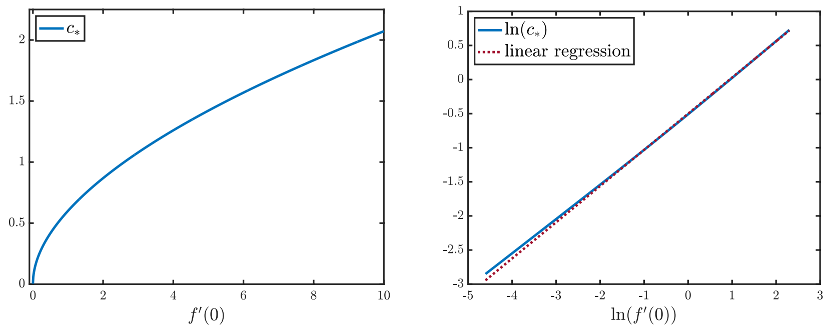
We present in Figure 3 a typical representation of the map Ψ on $(\lambda_0, +\infty)$. It exhibits a unique global minimum at some $\lambda_* > \lambda_0$, values at which one has

$$c_* = \frac{\lambda_*}{\mu(\lambda_*)}.$$

We numerically computed the linear spreading speed c_* by systematically evaluating the global minimum of the function Ψ as given by formula (4.9) as a function of the various parameters of the system. We reported the corresponding results in Figure 4. For the chosen parameter values, variations of the linear spreading speed c_* as a function of α and β show a similar pattern with, in both cases, the existence of a maximal spreading speed (see panels (a) and (b) of Figure 4) at some



(a) $(\beta, d, f'(0)) = (1, 1, 1)$. (b) $(\alpha, d, f'(0)) = (1, 1, 1)$. (c) $(\alpha, \beta, f'(0)) = (1, 1, 1)$.



(d) $(\alpha, \beta, d) = (1, 1, 1)$. (e) $(\alpha, \beta, d) = (1, 1, 1)$.

Figure 4: *Plots of the linear spreading speed* (1.1) *as parameters are varied.*

optimal value of the parameters α or β. More precisely, as either α or β is varied, while all other parameters are kept fixed, the linear spreading speed is first increasing from zero towards a global maximum value and then decreasing. On the other hand, when varying the parameter d, we clearly observe a monotone convergence towards a limiting asymptotic value. The limiting value exactly matches the asymptotic spreading speed c_*^∞ defined in formula (6.8) of Section 6 below. Finally, as it is the case for spreading speeds for scalar continuous Fisher-KPP equations, we see that the spreading speed c_* is a strictly monotone function of the parameter $f'(0)$, and we conjecture that c_* is proportional to $\sqrt{f'(0)}$. This is numerically confirmed (see panel (e) of Figure 4) by performing a linear regression of $\ln(c_*)$ as a function of $\ln(f'(0))$. We find that $\ln(c_*) \sim a_1 \ln(f'(0)) + a_0$ with $(a_1, a_0) \simeq (0.5306, -0.5012)$, where the relative error of the coefficient a_1 compared to the predicted value of $1/2$ is approximately 0.0613.

We have also further explored the dependence of the spreading speed as a function of α and β by showing in Figure 5 the color plot of the map $(\alpha, \beta) \mapsto c_*(\alpha, \beta)$ and several of its isolines (red curves). It shows that the spreading speed seems to converge towards a limiting value as $\alpha = \beta \to +\infty$. We numerically confirmed this behavior by plotting the linear spreading speed c_* as a function of $\alpha = \beta$ where we observe a monotone convergence towards a limiting asymptotic value, see the right panel of Figure 5. We leave it as future work to theoretically investigate this asymptotic limit.

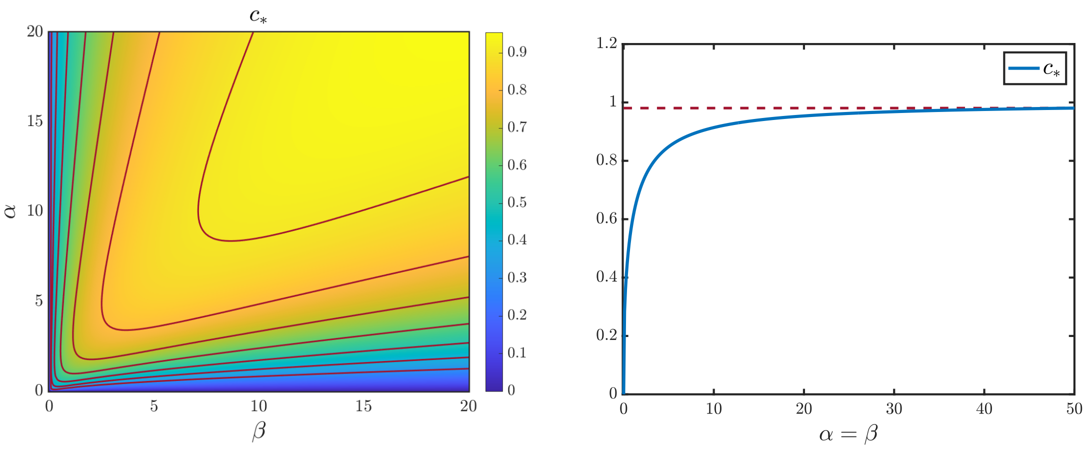


Figure 5: *Left: Amplitude of the spreading speed c_* as a function of (α, β) in the square $[0, 20] \times [0, 20]$. Several isolines (red curves) are also reported. Right: Linear spreading speed c_* as a function of $\alpha = \beta$. Other values of the parameters are set to $(d, f'(0)) = (1, 1)$.*

5 Asymptotic spreading

In this section, we investigate the asymptotic spreading properties of system (1.1)-(1.2) starting from compactly supported initial conditions. We anticipate that the linear spreading speed c_* defined in the previous section via formula (4.9) is precisely the asymptotic spreading speed of the nonlinear system (1.1)-(1.2) as stated in the following theorem.

Theorem 4. *Let $(\mathbf{v}, \boldsymbol{\rho})$ be the unique bounded classical solution of the Cauchy problem (1.1)-(1.2)-(2.1) starting from a nontrivial bounded compactly supported initial datum $(0,0) \not\equiv (\mathbf{h}, \boldsymbol{\Lambda}) \leq \left(\frac{\beta}{\alpha}, 1\right)$. Let $c_* > 0$ be defined in (4.9). Then:*

(i) for all $c > c_*$, we have

$$\lim_{t \to +\infty} \sup_{\substack{|j| \geq ct \\ x \in [0,1]}} (v_j(t,x), \rho_j(t)) = (0,0),$$

(ii) for all $c \in (0, c_*)$, we have

$$\lim_{t \to +\infty} \inf_{\substack{|j| \leq ct \\ x \in [0,1]}} (v_j(t,x), \rho_j(t)) = \left(\frac{\beta}{\alpha}, 1\right).$$

We illustrate the above result in Figure 6 by directly comparing the theoretical spreading speed c_* given by formula (4.9) and numerically computed spreading speed (dark red circles) obtained by numerically solving system (1.1)-(1.2) from compactly supported initial conditions using the numerical scheme proposed in [3].

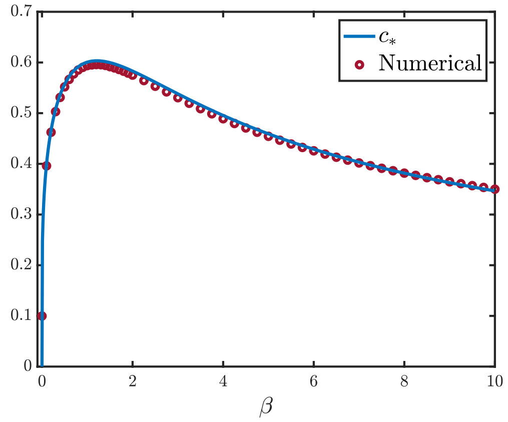


Figure 6: *Comparison between the theoretical spreading speed c_* given by formula (4.9) and numerically computed spreading speed (dark red circles) obtained by numerically solving system (1.1)-(1.2) from compactly supported initial conditions.*

5.1 Upper estimate

We first prove item (i) of Theorem 4, which is a direct consequence of the analysis conducted in the previous section. Let $c_* > 0$ be given by formula (4.9) and let $\lambda_* > \lambda_0$ be such that

$$c_* = \min_{\lambda > \lambda_0} \frac{\lambda}{\mu(\lambda)} = \frac{\lambda_*}{\mu_*},$$

where we have set $\mu_* := \mu(\lambda_*) > 0$. Then the following sequence

$$\forall t \geq 0, \quad j \in \mathbb{Z}, \quad x \in [0,1], \quad (v_j(t,x), \rho_j(t)) = \left(\mathrm{e}^{-\mu_*(j - c_* t)} V_*(x), \mathrm{e}^{-\mu_*(j-c_*t)} \right), \tag{5.1}$$

where

$$\forall x \in [0,1], \quad V_*(x) = \frac{\beta}{\sinh\left(\sqrt{\frac{\lambda_*}{d}}\right) \Delta(\lambda_*)} \left[\sqrt{\lambda_* d} \cosh\left(\sqrt{\frac{\lambda_*}{d}}(1-x) \right) + \alpha \sinh\left(\sqrt{\frac{\lambda_*}{d}}(1-x) \right) \right]$$
$$+ \frac{\beta \mathrm{e}^{-\mu_*}}{\sinh\left(\sqrt{\frac{\lambda_*}{d}}\right) \Delta(\lambda_*)} \left[\sqrt{\lambda_* d} \cosh\left(\sqrt{\frac{\lambda_*}{d}} x \right) + \alpha \sinh\left(\sqrt{\frac{\lambda_*}{d}} x \right) \right],$$

is a solution of the linearized problem (4.1). We readily remark that $V_*(x) > 0$ for all $x \in [0,1]$. We can then introduce the sequences

$$\forall t \geq 0, \quad j \in \mathbb{Z}, \quad x \in [0,1], \quad \overline{v}_j(t,x) = \min\left(\vartheta \mathrm{e}^{-\mu_*(j-c_*t)} V_*(x), \frac{\beta}{\alpha} \right),$$

and

$$\forall t \geq 0, \quad j \in \mathbb{Z}, \quad \overline{\rho}_j(t) = \min\left(\vartheta \mathrm{e}^{-\mu_*(j-c_*t)}, 1 \right),$$

for some $\vartheta > 0$ to be fixed. Since the initial datum $(0,0) \not\equiv (\mathbf{h}, \mathbf{\Lambda}) \leq \left(\frac{\beta}{\alpha}, 1\right)$ is assumed to be compactly supported, we can always find $\vartheta > 0$ sufficiently large such that

$$\forall j \in \mathbb{Z}, \quad x \in [0,1], \quad h_j(x) \leq \overline{v}_j(0,x) \quad \text{and} \quad \Lambda_j \leq \overline{\rho}_j(0).$$

From the comparison principle of Proposition 2.1, we deduce that

$$\forall t \geq 0, \quad j \in \mathbb{Z}, \quad x \in [0,1], \quad v_j(t,x) \leq \overline{v}_j(t,x) \quad \text{and} \quad \rho_j(t) \leq \overline{\rho}_j(t),$$

and thus for all $c > c_*$ one has

$$\lim_{t\to+\infty} \sup_{\substack{j\geq ct \\ x\in[0,1]}} (v_j(t,x), \rho_j(t)) \leq \lim_{t\to+\infty} \sup_{\substack{j\geq ct \\ x\in[0,1]}} (\overline{v}_j(t,x), \overline{\rho}_j(t)) = (0,0)\,.$$

By symmetry, that is using $\overline{v}_{-j}(t,x)$ and $\overline{\rho}_{-j}(t)$ instead, we deduce item (i) of Theorem 4.

5.2 Lower estimate

Our aim is now to prove the lower estimate of item (ii) of Theorem 4. For that purpose, we shall construct compactly supported subsolutions of the linear system penalized by $\delta > 0$ which reads

$$\forall t > 0,\ j \in \mathbb{Z}, \quad \begin{cases} \partial_t v_j(t,x) = d\partial_x^2 v_j(t,x), \quad x \in (0,1), \\ \rho_j'(t) = (f'(0) - \delta)\, \rho_j(t) + \alpha(v_j(t,0) + v_{j-1}(t,1)) - 2\beta\rho_j(t), \end{cases} \tag{5.2}$$

together with the usual boundary conditions

$$\forall t > 0,\ j \in \mathbb{Z}, \quad \begin{cases} -d\partial_x v_j(t,0) + \alpha v_j(t,0) = \beta\rho_j(t), \\ d\partial_x v_j(t,1) + \alpha v_j(t,1) = \beta\rho_{j+1}(t). \end{cases} \tag{5.3}$$

The main result of this section is the following.

Proposition 5.1. *Let c_* be given by formula (4.9). For all $c \in (0, c_*)$ close enough to c_* there exists $\delta > 0$ such that the penalized linear system (5.2)-(5.3) admits a nonnegative, compactly supported, generalized subsolution $(\underline{\mathbf{v}}, \underline{\boldsymbol{\rho}}) \not\equiv (0,0)$.*

Proof. In order to keep the presentation as light as possible, we will proceed with $f'(0)$ instead of $f'(0) - \delta$ in (5.2) since our arguments naturally perturb for $\delta > 0$ small enough.

We set $c \in (0, c_*)$ and consider once again exponential solutions of (5.2)-(5.3) of the form

$$(v_j(t,x), \rho_j(t)) = \left(\mathrm{e}^{-\mu(j-ct)} V(x), \mathrm{e}^{-\mu(j-ct)}\right),$$

where

$$V(x) = a\cosh\left(\sqrt{\frac{\lambda}{d}}x\right) + b\sinh\left(\sqrt{\frac{\lambda}{d}}x\right), \quad \forall x \in [0,1],$$

this time with eventual complex parameters $(\lambda, \mu, a, b) \in \mathbb{C}^4$ that will be fixed along the proof. Performing similar computations as in the previous section, we readily obtain that, given $c \in (0, c_*)$, the couple $(\lambda, \mu) \in \mathbb{C}^2$ is a solution of the system (4.6) from which one obtains equation (4.8) which we rewrite as

$$\Phi(c, \lambda) = 0,$$

with $\Phi(c,\lambda) := \lambda - c\mu(\lambda)$ where $\mu(\lambda) = \ln\left(y(\lambda) + \sqrt{y(\lambda)^2 - 1}\right)$ and $y(\lambda)$ is given in (4.7). By definition of c_* and analyticity of the map Φ on its domain of definition we have that there exists a positive integer $p \geq 1$ such that

$$\Phi(c_*, \lambda_*) = 0, \quad \partial_\lambda^k \Phi(c_*, \lambda_*) = 0 \quad \text{for} \quad k = 1, \dots, 2p-1 \quad \text{and} \quad \partial_\lambda^{2p} \Phi(c_*, \lambda_*) > 0.$$

Next, introducing the auxiliary variables

$$\xi := c_* - c > 0 \quad \text{and} \quad z := \lambda - \lambda_* \in \mathbb{C},$$

we see that $\Phi(c,\lambda) = 0$ is equivalent, in a neighborhood of $(c,\lambda) = (c_*, \lambda_*)$, to

$$\mu_* \xi + \mu'(\lambda_*) \xi z + a_* z^{2p} = \phi(z, \xi), \quad a_* := \frac{\partial_\lambda^{2p} \Phi(c_*, \lambda_*)}{(2p)!} = -\frac{c_*}{(2p)!} \mu^{(2p)}(\lambda_*) > 0,$$

where ϕ is analytic in a neighborhood of $(0,0)$ and $\phi(z,\xi) = O\left(|z|^{2p+1} + \xi |z|^2\right)$ as $(z,\xi) \to (0,0)$. For small $\xi > 0$, the polynomial equation $\mu_* \xi + \mu'(\lambda_*) \xi z + a_* z^{2p} = 0$ has $2p$ complex conjugate roots which writes

$$z_\pm^k(\xi) = \left(\frac{\mu_*}{a_*} \xi\right)^{\frac{1}{2p}} \mathrm{e}^{\mathbf{i}\left[\pm \frac{\pi}{2p} + \frac{2k\pi}{p}\right]} + O\left(\xi^{\frac{1}{p}}\right), \text{ for } k = 0, \dots, p-1.$$

Applying Rouché's theorem, we get that the algebraic equation $\mu_* \xi + \mu'(\lambda_*) \xi z + a_* z^{2p} = \phi(z, \xi)$ has also $2p$ complex roots which we denote by $\widetilde{z}_\pm^k(\xi)$ and these roots still satisfy

$$\widetilde{z}_\pm^k(\xi) = \left(\frac{\mu_*}{a_*} \xi\right)^{\frac{1}{2p}} \mathrm{e}^{\mathbf{i}\left[\pm \frac{\pi}{2p} + \frac{2k\pi}{p}\right]} + O\left(\xi^{\frac{1}{p}}\right), \text{ for } k = 0, \dots, p-1.$$

As a consequence, reverting to the full notation, we observe that for c strictly less than and sufficiently close to c_*, the equation $\Phi(c,\lambda) = 0$ admits a solution of the form

$$\lambda = \lambda_* + \widetilde{z}_+^0(\xi),$$

with the following properties:

$$\mathrm{Re}(\lambda) = \lambda_* + O\left(\xi^{\frac{1}{2p}}\right) > 0, \quad \mathrm{Im}(\lambda) = \left(\frac{\mu_*}{a_*} \xi\right)^{\frac{1}{2p}} \sin\left(\frac{\pi}{2p}\right) + O\left(\xi^{\frac{1}{p}}\right) > 0.$$

The corresponding profile V, given by

$$\forall x \in [0,1], \quad V(x) = \frac{\beta}{\sinh\left(\sqrt{\frac{\lambda}{d}}\right) \Delta(\lambda)} \left[\sqrt{\lambda d} \cosh\left(\sqrt{\frac{\lambda}{d}}(1-x)\right) + \alpha \sinh\left(\sqrt{\frac{\lambda}{d}}(1-x)\right)\right]$$
$$+ \frac{\beta \mathrm{e}^{-\mu(\lambda)}}{\sinh\left(\sqrt{\frac{\lambda}{d}}\right) \Delta(\lambda)} \left[\sqrt{\lambda d} \cosh\left(\sqrt{\frac{\lambda}{d}} x\right) + \alpha \sinh\left(\sqrt{\frac{\lambda}{d}} x\right)\right],$$

satisfies

$$\forall x\in[0,1],\quad \mathrm{Re}\left(V(x)\right)=\mathrm{Re}\left(V_*(x)\right)+O\left(\xi^{\frac{1}{2p}}\right)>0,\quad \mathrm{Im}\left(V(x)\right)=O\left(\xi^{\frac{1}{2p}}\right)\neq 0$$

and

$$\forall x\in[0,1],\quad \mathrm{Arg}(V(x))=O\left(\xi^{\frac{1}{2p}}\right),$$

where we denoted by $\mathrm{Arg}(V(x))\in(-\pi,\pi]$ the principal argument of $V(x)$. Taking the real parts of the just constructed exponential solutions, we set

$$\widetilde{v}_j(t,x)=|V(x)|\mathrm{e}^{-\frac{\lambda_*+\mathrm{Re}(\widetilde{z}_+^0(\xi))}{c}(j-ct)}\cos\left(\frac{\mathrm{Im}(\widetilde{z}_+^0(\xi))}{c}(j-ct)-\mathrm{Arg}(V(x))\right),$$

and

$$\widetilde{\rho}_j(t)=\mathrm{e}^{-\frac{\lambda_*+\mathrm{Re}(\widetilde{z}_+^0(\xi))}{c}(j-ct)}\cos\left(\frac{\mathrm{Im}(\widetilde{z}_+^0(\xi))}{c}(j-ct)\right),$$

for all $t\geq 0$, $j\in\mathbb{Z}$ and $x\in[0,1]$. In order to obtain compactly supported subsolutions, we truncate the above solutions as follows. We define the sets

$$\Omega_v(t,x):=\left\{y\in\mathbb{R}\mid ct-\frac{c\pi}{2\mathrm{Im}(\widetilde{z}_+^0(\xi))}<y-\frac{c\mathrm{Arg}(V(x))}{\mathrm{Im}(\widetilde{z}_+^0(\xi))}<ct+\frac{c\pi}{2\mathrm{Im}(\widetilde{z}_+^0(\xi))}\right\},$$

and

$$\Omega_\rho(t):=\left\{y\in\mathbb{R}\mid ct-\frac{c\pi}{2\mathrm{Im}(\widetilde{z}_+^0(\xi))}<y<ct+\frac{c\pi}{2\mathrm{Im}(\widetilde{z}_+^0(\xi))}\right\},$$

and we let

$$\forall t\geq 0,\ j\in\mathbb{Z},\ x\in[0,1],\quad \underline{v}_j(t,x):=\begin{cases}\widetilde{v}_j(t,x), & j\in\Omega_v(t,x),\\ 0, & \text{otherwise},\end{cases}\qquad \underline{\rho}_j(t):=\begin{cases}\widetilde{\rho}_j(t), & j\in\Omega_\rho(t),\\ 0, & \text{otherwise}.\end{cases}$$

Let us quickly check that $(\underline{v}_j(t,x),\underline{\rho}_j(t))$ provides a generalized subsolution to the linear system (5.2)-(5.3). Fix $t\geq 0$ and $x\in[0,1]$ and consider $j\in\Omega_v(t,x)\cap\Omega_\rho(t)$, then by construction and definition, we have that $(\underline{v}_j(t,x),\underline{\rho}_j(t))=(\widetilde{v}_j(t,x),\widetilde{\rho}_j(t))$ is a solution of (5.2)-(5.3). Let us now consider $j\in\Omega_v(t,x)\backslash\Omega_\rho(t)$ such that $\underline{v}_j(t,x)=\widetilde{v}_j(t,x)>0$ and $\underline{\rho}_j(t)=0$, then

$$\partial_t\underline{v}_j(t,x)=d\partial_x^2\underline{v}_j(t,x),$$

and

$$\underbrace{\underline{\rho}_j'(t)}_{\leq 0}+(2\beta-f'(0))\underbrace{\underline{\rho}_j(t)}_{=0}-\alpha\left(\underbrace{\underline{v}_j(t,0)+\underline{v}_{j-1}(t,1)}_{\geq 0}\right)\leq 0,$$

while

$$\begin{cases}-d\partial_x\underline{v}_j(t,0)+\alpha\underline{v}_j(t,0)-\beta\underline{\rho}_j(t)\leq 0,\\ d\partial_x\underline{v}_j(t,1)+\alpha\underline{v}_j(t,1)-\beta\underline{\rho}_{j+1}(t)\leq 0,\end{cases}$$

since $\underline{\rho}_{j+1}(t)\geq 0$ and by the Hopf lemma, one has $\partial_x\underline{v}_j(t,0)\leq 0$ and $\partial_x\underline{v}_j(t,1)\geq 0$. On the other hand, if $j\in\Omega_\rho(t)\backslash\Omega_v(t,x)$ then one has

$$\underline{v}_j(t,x)=0,\quad \partial_t\underline{v}_j(t,x)\leq 0 \text{ and } \partial_x^2\underline{v}_j(t,x)\geq 0,$$

such that

$$\partial_t \underline{v}_j(t,x) - d\partial_x^2 \underline{v}_j(t,x) \leq 0.$$

Next, since $\underline{\rho}_j(t) = \widetilde{\rho}_j(t)$, we have

$$\begin{aligned}\underline{\rho}_j'(t) + (2\beta - f'(0))\underline{\rho}_j(t) - \alpha\left(\underline{v}_j(t,0) + \underline{v}_{j-1}(t,1)\right) &\leq \widetilde{\rho}_j'(t) + (2\beta - f'(0))\widetilde{\rho}_j(t)\\ &= \alpha\left(\widetilde{v}_j(t,0) + \widetilde{v}_{j-1}(t,1)\right),\end{aligned}$$

and be choosing ξ even smaller, we can always ensure that both $\widetilde{v}_j(t,0) \leq 0$ and $\widetilde{v}_{j-1}(t,1) \leq 0$. And for the boundary conditions, we once again have

$$\begin{cases} -d\partial_x \underline{v}_j(t,0) + \alpha \underline{v}_j(t,0) - \beta \underline{\rho}_j(t) < 0, \\ d\partial_x \underline{v}_j(t,1) + \alpha \underline{v}_j(t,1) - \beta \underline{\rho}_{j+1}(t) \leq 0. \end{cases}$$

Finally, using similar arguments, it is not difficult to check that in the remaining regime with $j \in \mathbb{Z}\backslash\Omega_v(t,x) \cup \Omega_\rho(t)$ where $(\underline{v}_j(t,x), \underline{\rho}_j(t)) = (0,0)$ that $(\underline{v}_j(t,x), \underline{\rho}_j(t))$ is a subsolution. This concludes the proof of the proposition. ∎

Proof of item (ii) of Theorem 4. Let $c \in (0, c_*)$ and choose $c' \in (c, c_*)$ very close to c_* such that, from the previous Proposition 5.1, we get the existence of $\delta > 0$ such that the penalized linear system (5.2)-(5.3) admits a nonnegative, compactly supported, generalized subsolution that we denote $(\underline{\mathbf{v}}^{c',\delta}, \underline{\boldsymbol{\rho}}^{c',\delta}) \not\equiv (0,0)$. By regularity of the nonlinearity f, there exists $\iota > 0$ such that

$$(f'(0) - \delta)u \leq f(u), \quad 0 \leq u \leq \iota.$$

Then, one can find $\eta > 0$, small enough, such that $\eta \underline{\rho}_j^{c',\delta}(t) \leq \iota$ for all $t \geq 0$ and $j \in \mathbb{Z}$. As a consequence $(\eta \underline{\mathbf{v}}^{c',\delta}, \eta \underline{\boldsymbol{\rho}}^{c',\delta}) \not\equiv (0,0)$ is a nonnegative compactly supported subsolution to the full nonlinear system (1.1)-(1.2). By positivity of the solution of the nonlinear system (1.1)-(1.2) ensured by Theorem 1 and upon eventually reducing the size of $\eta > 0$, we can always ensure that at time $t = 1$ the unique solution $(\mathbf{v}, \boldsymbol{\rho})$ of the Cauchy problem (1.1)-(1.2)-(2.1) starting from the nontrivial bounded compactly supported initial datum $(0,0) \not\equiv (\mathbf{h}, \boldsymbol{\Lambda}) \leq \left(\frac{\beta}{\alpha}, 1\right)$ satisfies

$$(\eta \underline{\mathbf{v}}^{c',\delta}(0), \eta \underline{\boldsymbol{\rho}}^{c',\delta}(0)) \leq (\mathbf{v}(1), \boldsymbol{\rho}(1)).$$

From the comparison principle of Proposition 2.1 we obtain that

$$\forall t \geq 1, \quad (\eta \underline{\mathbf{v}}^{c',\delta}(t-1), \eta \underline{\boldsymbol{\rho}}^{c',\delta}(t-1)) \leq (\mathbf{v}(t), \boldsymbol{\rho}(t)).$$

As a consequence, there exists $\nu \in (0,1)$ small such that

$$v_{\lfloor c't \rfloor}(t,x) \geq \eta \underline{v}_{\lfloor c't \rfloor}^{c',\delta}(t-1,x) \geq \frac{\beta}{\alpha}\nu \quad \text{and} \quad v_{\lfloor c't \rfloor + 1}(t,x) \geq \eta \underline{v}_{\lfloor c't \rfloor + 1}^{c',\delta}(t-1,x) \geq \frac{\beta}{\alpha}\nu,$$

with

$$\rho_{\lfloor c't \rfloor}(t) \geq \eta \underline{\rho}_{\lfloor c't \rfloor}^{c',\delta}(t-1) \geq \nu \quad \text{and} \quad \rho_{\lfloor c't \rfloor + 1}(t) \geq \eta \underline{\rho}_{\lfloor c't \rfloor + 1}^{c',\delta}(t-1) \geq \nu,$$

for all $t \geq 1$ and $x \in [0,1]$. Here, we denote by $\lfloor x \rfloor$ the integer part of $x \in \mathbb{R}$. By a symmetry argument, we also obtain

$$v_{-\lfloor c't \rfloor}(t,x) \geq \frac{\beta}{\alpha}\nu \quad \text{and} \quad v_{-\lfloor c't \rfloor -1}(t,x) \geq \frac{\beta}{\alpha}\nu,$$

and

$$\rho_{-\lfloor c't \rfloor}(t) \geq \nu \quad \text{and} \quad \rho_{-\lfloor c't \rfloor -1}(t) \geq \nu,$$

for all $t \geq 1$ and $x \in [0,1]$. Upon eventually reducing the size of ν and by positivity of the solution $(\mathbf{v}, \boldsymbol{\rho})$ we can always ensure that

$$v_j(1,x) \geq \frac{\beta}{\alpha}\nu \quad \text{and} \quad \rho_j(1) \geq \nu \quad \text{for all} \quad x \in [0,1] \quad \text{and} \quad -c'-1 \leq j \leq c'+1.$$

Since $\left(\frac{\beta}{\alpha}\nu, \nu\right)_{j\in\mathbb{Z}}$ is a homogeneous subsolution of (1.1)-(1.2), we can apply a variant of the comparison principle, Proposition 2.1, but with two boundaries as stated in Proposition B.2 of the Appendix. More precisely, we set $\zeta(t) = -c't$ and $\xi(t) = c't$, from the previous analysis, we have $v_j(t,x) \geq \frac{\beta}{\alpha}\nu$ and $\rho_j(t) \geq \nu$ for all $t \geq 1$, $j \in [\zeta(t)-1, \zeta(t)) \cup (\xi(t), \xi(t)+1]$ and $x \in [0,1]$. Furthermore, at time $t=1$, we also have $v_j(1,x) \geq \frac{\beta}{\alpha}\nu$ and $\rho_j(1) \geq \nu$ for all $j \in [\zeta(1)-1, \xi(1)+1]$. As a consequence, the comparison principle with two boundaries ensures that

$$\forall t \geq 1, \ x \in [0,1], \ |j| \leq c't, \quad v_j(t,x) \geq \frac{\beta}{\alpha}\nu \quad \text{and} \quad \rho_j(t) \geq \nu,$$

from which we deduce, from Theorem 3, that

$$\forall x \in [0,1], \quad \liminf_{t\to+\infty} \inf_{|j|\leq ct} (v_j(t,x), \rho_j(t)) \geq \liminf_{t\to+\infty} \inf_{|j|\leq c't} (v_j(t,x), \rho_j(t)) \geq \left(\frac{\beta}{\alpha}, 1\right).$$

Since, we trivially have

$$\forall x \in [0,1], \quad \limsup_{t\to+\infty} \inf_{|j|\leq ct} (v_j(t,x), \rho_j(t)) \leq \left(\frac{\beta}{\alpha}, 1\right),$$

this concludes the proof of the theorem. ∎

6 Large diffusion limit

Motivated by our numerical finding (see panel (c) of Figure 4), in this section, we study the asymptotic regime when $d \to +\infty$. For that purpose, we first set $\epsilon := 1/d > 0$ such that system (1.1)-(1.2) rewrites

$$\forall t > 0, \ j \in \mathbb{Z}, \quad \begin{cases} \epsilon \partial_t v_j(t,x) = \partial_x^2 v_j(t,x), \quad x \in (0,1), \\ \rho_j'(t) = f(\rho_j(t)) + \alpha(v_j(t,0) + v_{j-1}(t,1)) - 2\beta\rho_j(t), \end{cases} \tag{6.1}$$

with associated Robin boundary conditions

$$\forall t > 0, \ j \in \mathbb{Z}, \quad \begin{cases} -\partial_x v_j(t,0) + \alpha\epsilon v_j(t,0) = \beta\epsilon\rho_j(t), \\ \partial_x v_j(t,1) + \alpha\epsilon v_j(t,1) = \beta\epsilon\rho_{j+1}(t). \end{cases} \tag{6.2}$$

6.1 Derivation of the asymptotic limiting system

Fix $\epsilon_0 > 0$. For each $\epsilon \in (0, \epsilon_0]$, we consider $\mathbf{h}^\epsilon = (h_j^\epsilon)_{j\in\mathbb{Z}} \in \mathscr{X}^2$ and $\mathbf{\Lambda}^\epsilon = (\Lambda_j^\epsilon)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$, satisfying the compatibility condition (2.2), with

$$\mathscr{X}^2 := \left\{\mathbf{u} = (u_j)_{j\in\mathbb{Z}} \mid \forall j \in \mathbb{Z},\ u_j \in \mathscr{C}^2([0,1],\mathbb{R}) \text{ and } \|\mathbf{u}\|_\infty < +\infty\right\}.$$

We further suppose that there exists some positive constant $\kappa > 0$ such that

$$\forall \epsilon \in (0, \epsilon_0], \quad 0 < \|\mathbf{h}^\epsilon\|_\infty + \|\mathbf{\Lambda}^\epsilon\|_{\ell^\infty(\mathbb{Z})} \leq \kappa, \quad \text{and} \quad \left\|\mathbf{h}^{\epsilon\prime\prime}\right\|_\infty \leq \epsilon\kappa, \tag{6.3}$$

and we also assume that the sequences $\mathbf{V}^0$ and $\mathbf{P}^0$ defined as the following limits

$$\forall j \in \mathbb{Z}, \quad V_j^0 = \lim_{\epsilon\to 0} \int_0^1 h_j^\epsilon(x)\mathrm{d}x, \text{ and } P_j^0 = \lim_{\epsilon\to 0} \Lambda_j^\epsilon, \tag{6.4}$$

satisfy $(0,0) \not\equiv (\mathbf{V}^0, \mathbf{P}^0) \in \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$.

For each $\epsilon \in (0, \epsilon_0]$, we shall denote by $(\mathbf{v}^\epsilon, \boldsymbol{\rho}^\epsilon)$ the solution of (6.1)-(6.2) given by Theorem 1 with initial condition

$$\forall j \in \mathbb{Z}, \quad \begin{cases} v_j^\epsilon(0,x) = h_j^\epsilon(x), & x \in (0,1), \\ \rho_j^\epsilon(0) = \Lambda_j^\epsilon, \end{cases} \tag{6.5}$$

with $(\mathbf{h}^\epsilon, \mathbf{\Lambda}^\epsilon)$ satisfying the above conditions. For all $t > 0$, one has

$$\forall t > 0,\ \forall j \in \mathbb{Z}, \quad 0 < v_j^\epsilon(t,x) \leq \max\left\{\frac{\beta}{\alpha}, \kappa\right\},\ x \in [0,1], \text{ and } 0 < \rho_j^\epsilon(t) \leq \max\{1,\kappa\},$$

from which we deduce that

$$\forall t > 0,\ \forall j \in \mathbb{Z}, \quad \left|\rho_j^{\epsilon\prime}(t)\right| \leq \left(f'(0) + 2\beta\right)\max\{1,\kappa\} + 2\alpha \max\left\{\frac{\beta}{\alpha}, \kappa\right\}.$$

Let us set

$$\forall t > 0, \quad \forall j \in \mathbb{Z}, \quad \forall x \in [0,1], \quad w_j^\epsilon(t,x) := \partial_t v_j^\epsilon(t,x).$$

It follows from (6.1) and (6.2) that $\mathbf{w}^\epsilon = (w_j^\epsilon)_{j\in\mathbb{Z}}$ is a solution of

$$\forall t > 0,\ j \in \mathbb{Z}, \quad \epsilon\partial_t w_j^\epsilon(t,x) = \partial_x^2 w_j^\epsilon(t,x), \quad x \in (0,1),$$

and

$$\forall t > 0,\ j \in \mathbb{Z}, \quad \begin{cases} -\partial_x w_j^\epsilon(t,0) + \alpha\epsilon w_j^\epsilon(t,0) = \beta\epsilon\rho_j^{\epsilon\prime}(t), \\ \partial_x w_j^\epsilon(t,1) + \alpha\epsilon w_j^\epsilon(t,1) = \beta\epsilon{\rho_{j+1}^\epsilon}'(t), \end{cases}$$

with initial condition given by

$$w_j^\epsilon(0,x) = \frac{1}{\epsilon} h_j^{\epsilon\prime\prime}(x), \quad x \in [0,1].$$

Thanks to our condition on $\mathbf{h}^\epsilon$, we have that

$$\forall \epsilon \in (0, \epsilon_0], \quad \|\mathbf{w}^\epsilon\|_\infty \leq \kappa.$$

From the parabolic comparison principle, we have for each $t > 0$ that

$$\forall j \in \mathbb{Z}, \qquad \sup_{(s,x)\in[0,t]\times[0,1]} w_j^\epsilon(s,x) \leq \sup_{s\in[0,t]} w_j^\epsilon(s,0) + \sup_{s\in[0,t]} w_j^\epsilon(s,1) + \sup_{x\in[0,1]} w_j^\epsilon(0,x),$$

and the Hopf Lemma [20] ensures that

$$\forall j \in \mathbb{Z}, \quad \sup_{s\in[0,t]} w_j^\epsilon(s,0) \leq \frac{\beta}{\alpha} \sup_{s\in[0,t]} {\rho_j^\epsilon}'(s) \leq \frac{\beta}{\alpha}\left(\big(f'(0)+2\beta\big)\max\{1,\kappa\} + 2\alpha \max\left\{\frac{\beta}{\alpha},\kappa\right\}\right),$$

and

$$\forall j \in \mathbb{Z}, \quad \sup_{s\in[0,t]} w_j^\epsilon(s,1) \leq \frac{\beta}{\alpha} \sup_{s\in[0,t]} {\rho_{j+1}^\epsilon}'(s) \leq \frac{\beta}{\alpha}\left(\big(f'(0)+2\beta\big)\max\{1,\kappa\} + 2\alpha \max\left\{\frac{\beta}{\alpha},\kappa\right\}\right).$$

Applying a similar argument to $-w_j^\epsilon$, we obtain for all $t > 0$ that

$$\forall j \in \mathbb{Z}, \qquad \sup_{(s,x)\in[0,t]\times[0,1]} \left|w_j^\epsilon(s,x)\right| \leq \kappa + \frac{2\beta}{\alpha}\left(\big(f'(0)+2\beta\big)\max\{1,\kappa\} + 2\alpha \max\left\{\frac{\beta}{\alpha},\kappa\right\}\right).$$

This implies that for each $\epsilon \in (0,\epsilon_0]$ one has

$$\forall t > 0, \quad \forall j \in \mathbb{Z}, \quad |\partial_t v_j^\epsilon(t,x)| \leq \kappa + \frac{2\beta}{\alpha}\left(\big(f'(0)+2\beta\big)\max\{1,\kappa\} + 2\alpha \max\left\{\frac{\beta}{\alpha},\kappa\right\}\right),$$

and

$$\forall t > 0, \quad \forall j \in \mathbb{Z}, \quad |\partial_x^2 v_j^\epsilon(t,x)| \leq \epsilon_0\kappa + \frac{2\beta\epsilon_0}{\alpha}\left(\big(f'(0)+2\beta\big)\max\{1,\kappa\} + 2\alpha \max\left\{\frac{\beta}{\alpha},\kappa\right\}\right),$$

for all $x \in [0,1]$. On the other hand, we also have

$$\partial_x v_j^\epsilon(t,x) = \partial_x v_j^\epsilon(t,0) + \epsilon\int_0^1 w_j(t,y)\mathrm{d}y = \alpha\epsilon v_j^\epsilon(t,0) - \beta\epsilon\rho_j^\epsilon(t) + \epsilon\int_0^1 w_j(t,y)\mathrm{d}y$$

such that

$$\left|\partial_x v_j^\epsilon(t,x)\right| \leq C(\epsilon_0,\alpha,\beta,\kappa,f'(0)),$$

for all $t > 0$, $j \in \mathbb{Z}$, $x \in [0,1]$ and $\epsilon \in (0,\epsilon_0]$.

Based on the above estimates, we apply Arzela-Ascoli's theorem, together with a diagonal extraction argument, to obtain, up to a subsequence, the existence of a limit $(\mathbf{U},\mathbf{P})$ with

$$\forall x \in [0,1], \quad \lim_{\epsilon\to 0} v_j^\epsilon(t,x) = U_j(t,x), \quad \lim_{\epsilon\to 0} \rho_j^\epsilon(t) = P_j(t),$$

locally uniformly in $(t,j) \in (0,+\infty)\times\mathbb{Z}$. The convergence also holds for the respective time and space derivatives. At the limit, one has

$$\forall t > 0,\ j \in \mathbb{Z}, \qquad \begin{cases} 0 = \partial_x^2 U_j(t,x), \quad x \in (0,1), \\ P_j'(t) = f(P_j(t)) + \alpha(U_j(t,0) + U_{j-1}(t,1)) - 2\beta P_j(t), \\ \partial_x U_j(t,0) = \partial_x U_j(t,1) = 0. \end{cases}$$

As a consequence, one necessarily has

$$\forall t>0,\ j\in\mathbb{Z},\quad U_j(t,x)=V_j(t).$$

Integrating (6.1) from $x=0$ to $x=1$ and using the Robin boundary conditions (6.2), one also finds

$$\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 v_j^\epsilon(t,x)\mathrm{d}x=\partial_x v_j^\epsilon(t,1)-\partial_x v_j^\epsilon(t,0)=\epsilon\beta(\rho_j^\epsilon(t)+\rho_{j+1}^\epsilon(t))-\alpha\epsilon(v_j^\epsilon(t,0)+v_j^\epsilon(t,1)),$$

from which we get

$$\forall t>0,\ j\in\mathbb{Z},\quad V_j'(t)=-2\alpha V_j(t)+\beta(P_j(t)+P_{j+1}(t)).$$

By definition of the sequences $\mathbf{V}^0$ and $\mathbf{P}^0$, we also have

$$\forall j\in\mathbb{Z},\quad V_j(0)=V_j^0\quad\text{and}\quad P_j(0)=P_j^0.$$

As a consequence, we have obtained the following result.

Theorem 5. *Let $\epsilon_0>0$. For any initial sequences $(\mathbf{h}^\epsilon)_{0<\epsilon\leq\epsilon_0}$ and $(\mathbf{\Lambda}^\epsilon)_{0<\epsilon\leq\epsilon_0}$, with $\mathbf{h}^\epsilon\in\mathcal{X}^2$ and $\mathbf{h}^\epsilon\in\ell^\infty(\mathbb{Z})$ for all $\epsilon\in(0,\epsilon_0]$ and satisfying the compatibility condition (2.2) together with the assumptions (6.3) and (6.4), the corresponding unique global classical positive solution $(\mathbf{v}^\epsilon,\boldsymbol{\rho}^\epsilon)$ satisfies*

$$\forall x\in[0,1],\quad \lim_{\epsilon\to0} v_j^\epsilon(t,x)=V_j(t),\quad \lim_{\epsilon\to0}\rho_j^\epsilon(t)=P_j(t),$$

locally uniformly in $(t,j)\in(0,+\infty)\times\mathbb{Z}$, wherein $(\mathbf{V},\mathbf{P})$ is solution of the asymptotic system

$$\forall t>0,\ j\in\mathbb{Z},\quad \begin{cases}V_j'(t)=-2\alpha V_j(t)+\beta(P_j(t)+P_{j+1}(t)),\\ P_j'(t)=f(P_j(t))+\alpha(V_j(t)+V_{j-1}(t))-2\beta P_j(t),\end{cases}\tag{6.6}$$

with initial condition $V_j(0)=V_j^0$ and $P_j(0)=P_j^0$, $j\in\mathbb{Z}$, defined in (6.4).

6.2 Spreading properties of the asymptotic limiting system

In the following, we focus on the study of the long time behavior of system (6.6) and its spreading properties. For that purpose, we first start by giving the notion of super and sub-solutions and prove a comparison principle.

We say that $(\overline{\mathbf{V}},\overline{\mathbf{P}})$ is a supersolution to (6.6) if for all $j\in\mathbb{Z}$ one has $\overline{V}_j,\overline{P}_j\in\mathscr{C}^1([0,+\infty),\mathbb{R})$ which satisfy

$$\forall t>0,\ j\in\mathbb{Z},\quad \begin{cases}\overline{V}_j'(t)\geq-2\alpha\overline{V}_j(t)+\beta(\overline{P}_j(t)+\overline{P}_{j+1}(t)),\\ \overline{P}_j'(t)\geq f(\overline{P}_j(t))+\alpha(\overline{V}_j(t)+\overline{V}_{j-1}(t))-2\beta\overline{P}_j(t).\end{cases}$$

We similarly define a subsolution $(\underline{\mathbf{V}},\underline{\mathbf{P}})$ to (6.6) with the same regularity and all the above inequalities being reversed. We can now state a comparison principle for (6.6) whose proof is a direct consequence of Proposition B.5.

Proposition 6.1. *Let $(\underline{\mathbf{V}},\underline{\mathbf{P}})$ and $(\overline{\mathbf{V}},\overline{\mathbf{P}})$ be respectively a subsolution and supersolution to (6.6). If we assume that $(\underline{\mathbf{V}},\underline{\mathbf{P}})$ and $(\overline{\mathbf{V}},\overline{\mathbf{P}})$ are locally bounded in time and satisfy for all $j\in\mathbb{Z}$ that $\underline{V}_j(0)\leq\overline{V}_j(0)$ and $\underline{P}_j(0)\leq\overline{P}_j(0)$, then we have $\underline{V}_j(t)\leq\overline{V}_j(t)$ and $\underline{P}_j(t)\leq\overline{P}_j(t)$ for all $t>0$ and $j\in\mathbb{Z}$. Furthermore, if $(\underline{\mathbf{V}}(0),\underline{\mathbf{P}}(0))\not\equiv(\overline{\mathbf{V}}(0),\overline{\mathbf{P}}(0))$, then we have $\underline{V}_j(t)<\overline{V}_j(t)$ and $\underline{P}_j(t)<\overline{P}_j(t)$ for all $t>0$ and $j\in\mathbb{Z}$.*

A direct consequence of the above comparison principle is the uniqueness of bounded solutions of system (6.6). More generally, for each nontrivial nonnegative initial condition $(\mathbf{V}^0,\mathbf{P}^0)\in\ell^\infty(\mathbb{Z})\times\ell^\infty(\mathbb{Z})$, there exists a unique classical global solution $(\mathbf{V},\mathbf{P})$ of system (6.6) with $(\mathbf{V}(0),\mathbf{P}(0))=(\mathbf{V}^0,\mathbf{P}^0)$ such that $V_j,P_j\in\mathscr{C}^1([0,+\infty),\mathbb{R})$ for all $j\in\mathbb{Z}$, together with uniform bounds

$$\forall t>0,\quad \forall j\in\mathbb{Z},\quad 0<V_j(t)\leq\max\left(\left\|\mathbf{V}^0\right\|_{\ell^\infty(\mathbb{Z})},\frac{\beta}{\alpha}\right)\quad\text{and}\quad 0<P_j(t)\leq\max\left(\left\|\mathbf{P}^0\right\|_{\ell^\infty(\mathbb{Z})},1\right).$$

Regarding the long time behavior of the solutions of system (6.6), we have the following result which mirrors Theorem 3.

Proposition 6.2. *Let $(\mathbf{V},\mathbf{P})$ be the unique global classical solution of (6.6) starting from a nontrivial nonnegative bounded initial sequence $(\mathbf{V}^0,\mathbf{P}^0)\in\ell^\infty(\mathbb{Z})\times\ell^\infty(\mathbb{Z})$. Then,*

$$\lim_{t\to+\infty}(V_j(t),P_j(t))=\left(\frac{\beta}{\alpha},1\right),$$

locally uniformly $j\in\mathbb{Z}$.

Proof. Stationary solutions of system (6.6) satisfy

$$\forall j\in\mathbb{Z},\quad \begin{cases}0=-2\alpha V_j+\beta(P_j+P_{j+1}),\\ 0=f(P_j)+\alpha(V_j+V_{j-1})-2\beta P_j,\end{cases}$$

from which we deduce that $V_j=\frac{\beta}{2\alpha}(P_j+P_{j+1})$ and

$$0=f(P_j)+\frac{\beta}{2}\left(P_{j-1}-2P_j+P_{j+1}\right),$$

for all $j\in\mathbb{Z}$. As a consequence, from the proof of Theorem 2, we deduce that $(V_j,P_j)=(\beta/\alpha,1)$, for all $j\in\mathbb{Z}$, is the only positive stationary solution to (6.6).

We now let $N_0>1$ be large enough such that

$$\beta\left(1-\cos\left(\frac{\pi}{N+1}\right)\right)<f'(0),$$

for all $N\geq N_0$. We then define $\underline{\mathbf{P}}=(\underline{P}_j)_{j\in\mathbb{Z}}$ as

$$\underline{P}_j:=\begin{cases}\sin\left(\dfrac{j\pi}{N+1}\right), & j=1,\dots,N,\\ 0, & \text{otherwise},\end{cases}$$

and set $\underline{\mathbf{V}} = (\underline{V}_j)_{j\in\mathbb{Z}}$ with

$$\underline{V}_j := \begin{cases} \dfrac{\beta}{2\alpha}\left(\sin\left(\dfrac{j\pi}{N+1}\right) + \sin\left(\dfrac{(j+1)\pi}{N+1}\right)\right), & j = 1,\dots,N-1, \\ \dfrac{\beta}{2\alpha}\sin\left(\dfrac{\pi}{N+1}\right), & j = 0, \\ \dfrac{\beta}{2\alpha}\sin\left(\dfrac{N\pi}{N+1}\right), & j = N, \\ 0, & \text{otherwise.} \end{cases}$$

By definition, one has $\underline{V}_j = \frac{\beta}{2\alpha}(\underline{P}_j + \underline{P}_{j+1})$ for all $j \in \mathbb{Z}$. As a consequence, there exists $\nu_0 > 0$ such that $(\nu\underline{\mathbf{V}}, \nu\underline{\mathbf{P}})$ is a compactly supported stationary subsolution for all $N \geq N_0$ and $\nu \in (0, \nu_0]$. One can also easily check that

$$\forall j \in \mathbb{Z}, \quad \overline{V}_j = \max\left(\left\|\mathbf{V}^0\right\|_{\ell^\infty(\mathbb{Z})}, \frac{\beta}{\alpha}\right) \quad \text{ and } \quad \overline{P}_j = \max\left(\left\|\mathbf{P}^0\right\|_{\ell^\infty(\mathbb{Z})}, 1\right),$$

gives a stationary supersolution. One can then adapt the arguments of the proof of Theorem 3 to obtain the local uniform asymptotic convergence of the solutions towards the unique positive stationary solution. ∎

Linearizing (6.6) around the trivial steady state, we obtain the following linear system

$$\forall t > 0,\ j \in \mathbb{Z}, \quad \begin{cases} V_j'(t) = -2\alpha V_j(t) + \beta(P_j(t) + P_{j+1}(t)), \\ P_j'(t) = (f'(0) - 2\beta)P_j(t) + \alpha(V_j(t) + V_{j-1}(t)). \end{cases} \tag{6.7}$$

We look for exponential solutions of the form

$$(V_j(t), P_j(t)) = \mathrm{e}^{-\mu(j-ct)}(v_0, p_0),$$

where $v_0 > 0, p_0 > 0$, $c > 0$ and $\mu > 0$ to be determined later. Substituting this ansatz into the linear system, we obtain that

$$\begin{cases} (\mu c + 2\alpha)v_0 - \beta(1 + \mathrm{e}^{-\mu})p_0 = 0, \\ (\mu c + 2\beta - f'(0))p_0 - \alpha(1 + \mathrm{e}^{\mu})v_0 = 0, \end{cases}$$

which implies that

$$\begin{pmatrix} \mu c + 2\alpha & -\beta(1 + \mathrm{e}^{-\mu}) \\ -\alpha(1 + \mathrm{e}^{\mu}) & \mu c + 2\beta - f'(0) \end{pmatrix} \begin{pmatrix} v_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since we are interested in nontrivial solutions, we must have that

$$\det \begin{pmatrix} \mu c + 2\alpha & -\beta(1 + \mathrm{e}^{-\mu}) \\ -\alpha(1 + \mathrm{e}^{\mu}) & \mu c + 2\beta - f'(0) \end{pmatrix} = 0,$$

which also reads

$$(\mu c)^2 + (2\beta - f'(0) + 2\alpha)\mu c + 2\alpha(2\beta - f'(0)) - 2\alpha\beta(1 + \cosh(\mu)) = 0.$$

It follows from the above equation that

$$c_{\pm}(\mu) = \frac{-(2\alpha + 2\beta - f'(0)) \pm \sqrt{\Delta(\mu)}}{2\mu},$$

where

$$\Delta(\mu) := (2\alpha - 2\beta + f'(0))^2 + 8\alpha\beta(1 + \cosh(\mu)) > 0.$$

Only retaining the positive root, we define

$$c_*^\infty := \min_{\mu>0} c_+(\mu) = \min_{\mu>0} \frac{-(2\alpha + 2\beta - f'(0)) + \sqrt{\Delta(\mu)}}{2\mu}. \tag{6.8}$$

Let us show that c_*^∞ is well-defined. Indeed, we consider the following function

$$\Psi(c,\mu) := -(2\alpha + (2\beta - f'(0))) + \sqrt{\Delta(\mu)} - 2\mu c.$$

By easy calculations, we have

$$\Psi(c,0) = -(2\alpha + 2\beta - f'(0)) + \sqrt{\Delta(0)} > 0,$$

$$\forall \mu > 0, \quad \left.\frac{\partial \Psi(c,\mu)}{\partial \mu}\right|_{\mu=0} = -2c < 0, \quad \frac{\partial \Psi(c,\mu)}{\partial c} = -2\mu < 0,$$

and

$$\frac{\partial^2 \Psi(c,\mu)}{\partial \mu^2} = \frac{4\alpha\beta}{\sqrt{\Delta(\mu)}\Delta(\mu)} \Big(\cosh(\mu)\left(2\alpha - 2\beta + f'(0)\right)^2 + 8\alpha\beta\cosh(\mu) + 4\alpha\beta(1 + \cosh^2(\mu))\Big)$$
$$> 0.$$

In view of the above properties of the function $\Psi(c,\mu)$, there exists $c_*^\infty > 0$ and $\mu_* > 0$ such that

$$\left.\frac{\partial \Psi(c,\mu)}{\partial \mu}\right|_{(c_*^\infty,\mu_*)} = 0 \text{ and } \Psi(c_*^\infty,\mu_*) = 0.$$

Furthermore,

(i) if $0 < c < c_*^\infty$, then $\Psi(c,\mu) > 0, \forall \mu > 0$,

(ii) if $c > c_*^\infty$, then the equation $\Psi(c,\mu) = 0$ has two positive real roots $\mu_1(c), \mu_2(c)$ with $0 < \mu_1(c) < \mu_* < \mu_2(c) < +\infty$, such that $\Psi(c,\cdot) < 0$ in $(\mu_1(c),\mu_2(c))$ and $\Psi(c,\cdot) > 0$ in $(0,\mu_1(c)) \cup (\mu_2(c),+\infty)$.

Therefore, c_*^∞ is well-defined. It actually characterizes the spreading speed of (6.6) as stated in the theorem below. Let us remark that our numerical evaluation of the linear spreading speed c_*, given by formula c_*, as a function of d while all other parameters being kept fixed suggests that

$$c_* \underset{d\to+\infty}{\longrightarrow} c_*^\infty.$$

We refer to Figure 4(c) for an illustration. We leave for future work to rigorously prove such a limit.

Theorem 6. *Let* $(\mathbf{V},\mathbf{P})$ *be the unique global classical solution of* (6.6) *starting from a nontrivial nonnegative compactly supported initial sequence* $(\mathbf{V}^0,\mathbf{P}^0)$ *satisfying* $(0,0)\lneqq(\mathbf{V}^0,\mathbf{P}^0)\leq(\beta/\alpha,1)$. *Then,*

(i) for all $c>c_*^\infty$*, we have*

$$\lim_{t\to+\infty}\sup_{|j|\geq ct}(V_j(t),P_j(t))=(0,0);$$

(ii) for all $c\in(0,c_*^\infty)$*, we have*

$$\lim_{t\to+\infty}\inf_{|j|\leq ct}(V_j(t),P_j(t))=\left(\frac{\beta}{\alpha},1\right).$$

Proof. Let us first prove item (i) of the theorem. We introduce the sequences

$$\forall t\geq 0,\quad j\in\mathbb{Z},\quad \overline{V_j}(t)=\min\left(\vartheta v_0\mathrm{e}^{-\mu_*(j-c_*^\infty t)},\frac{\beta}{\alpha}\right)\quad\text{ and }\quad \overline{P_j}(t)=\min\left(\vartheta\mathrm{e}^{-\mu_*(j-c_*^\infty t)},1\right),$$

with

$$v_0=\beta\frac{1+\mathrm{e}^{-\mu_*}}{\mu_*c_*^\infty+2\alpha}>0.$$

Here $\vartheta>0$ is chosen large enough such that $V_j^0\leq\overline{V_j}(0)$ and $P_j^0\leq\overline{P_j}(0)$ for all $j\in\mathbb{Z}$ which is always possible since $(\mathbf{V}^0,\mathbf{P}^0)$ is assumed to be compactly supported. By construction $(\overline{\mathbf{V}},\overline{\mathbf{P}})$ is a supersolution of system (6.6). Thus if $(\mathbf{V},\mathbf{P})$ is the unique global classical solution of (6.6) starting from the nontrivial nonnegative compactly supported initial sequence $(\mathbf{V}^0,\mathbf{P}^0)$ then one has

$$\forall t\geq 0,\quad j\in\mathbb{Z},\quad V_j(t)\leq\overline{V_j}(t)\quad\text{ and }\quad P_j(t)\leq\overline{P_j}(t),$$

from which we readily deduce that for all $c>c_*^\infty$

$$\lim_{t\to+\infty}\sup_{j\geq ct}V_j(t)\leq\lim_{t\to+\infty}\sup_{j\geq ct}\overline{V_j}(t)=0,$$

and

$$\lim_{t\to+\infty}\sup_{j\geq ct}P_j(t)\leq\lim_{t\to+\infty}\sup_{j\geq ct}\overline{P_j}(t)=0.$$

By symmetry, we obtain a similar result for all $j\leq -ct$ which concludes the proof of the first part of the theorem.

The second step of the proof is to devise a compactly supported subsolution whose support moves with speed c close to c_*^∞. So let $c\in(0,c_*^\infty)$ and consider the linear system

$$\forall t>0,\ j\in\mathbb{Z},\quad\begin{cases}V_j'(t)=-2\alpha V_j(t)+\beta(P_j(t)+P_{j+1}(t)),\\ P_j'(t)=(f'(0)-2\beta)P_j(t)+\alpha(V_j(t)+V_{j-1}(t)).\end{cases}$$

Looking once again at exponential solutions of the form

$$(V_j(t),P_j(t))=\mathrm{e}^{-\mu(j-ct)}(v_0,p_0),$$

with $\mu, v_0, p_0 \in \mathbb{C}$ and $c > 0$, we see, from the above discussion, that μ and c should satisfy $\Psi(c,\mu) = 0$. We recall that

$$\Psi(c_*^\infty, \mu_*) = 0, \quad \partial_\mu \Psi(c_*^\infty, \mu_*) = 0, \text{ and } 2a := \partial_{\mu\mu}\Psi(c_*^\infty, \mu_*) > 0.$$

In addition, we also have

$$\partial_c \Psi(c_*^\infty, \mu_*) = -2\mu_* < 0, \quad \partial_{c\mu}\Psi(c_*^\infty, \mu_*) = -2 < 0.$$

We then consider a neighborhood of (c_*^∞, μ_*), thus we set

$$\xi := c_*^\infty - c, \quad \tau := \mu - \mu_*.$$

The equation $\Psi(c,\mu) = 0$ becomes, for (c,μ) in a neighborhood of (c_*^∞, μ_*):

$$a\tau^2 + 2\xi\tau + 2\mu_*\xi = \phi(\tau, \xi), \tag{6.9}$$

where $\phi(\tau,\xi)$ is analytic in τ in a neighborhood of 0, vanishing at $(0,0)$ like $|\tau|^3 + |\xi|^2$. For small $\xi > 0$, the equation $a\tau^2 + 2\xi\tau + 2\mu_*\xi = 0$ has two complex roots

$$\tau_\pm(\xi) = \pm i\sqrt{\frac{2\mu_*}{a}\xi} + O(\xi).$$

By applying Rouché's theorem, we find that equation (6.9) has also two complex roots, which are complex conjugates up to order ξ, and are denoted by $\widetilde{\tau}_\pm$. These roots satisfy $\widetilde{\tau}_\pm(\xi) = \pm i\sqrt{\frac{2\mu_*}{a}\xi} + O(\xi)$. Reverting to the full notation, we observe that for c strictly less than and sufficiently close to c_*^∞, the equation $\Psi(c,\mu) = 0$ admits a solution μ with the following properties: its real part is ξ-close to μ_*, and hence positive; moreover, it has a nonzero imaginary part of order $\xi^{1/2}$. Setting $p_0 = 1$, we get that

$$v_0 = \frac{\beta(1 + \mathrm{e}^{-\mu})}{\mu c + 2\alpha},$$

and since $\mu = \mu_* + \widetilde{\tau}_\pm(\xi)$, we infer that

$$\mathrm{Re}(v_0) > 0, \quad \mathrm{Im}(v_0) < 0, \quad \text{and} \quad |\mathrm{Arg}(v_0)| = \mathcal{O}(\sqrt{\xi}),$$

where we denoted by $\mathrm{Arg}(v_0) \in (-\pi, \pi]$ the principal argument of v_0. Taking the real parts of the constructed exponential solutions, we set

$$\forall t \geq 0, \quad j \in \mathbb{Z}, \quad \begin{cases} \widetilde{V}_j(t) := \mathrm{Re}(V_j(t)) = |v_0|\mathrm{e}^{-\mathrm{Re}(\mu)(j-ct)} \cos\left(\mathrm{Im}(\mu)(j-ct) - \mathrm{Arg}(v_0)\right), \\ \widetilde{P}_j(t) := \mathrm{Re}(P_j(t)) = \mathrm{e}^{-\mathrm{Re}(\mu)(j-ct)} \cos\left(\mathrm{Im}(\mu)(j-ct)\right). \end{cases}$$

In order to obtain compactly supported subsolutions, we truncate the above solutions as follows. We define

$$\Omega_V(t) := \left\{ x \in \mathbb{R} \mid ct - \frac{\pi}{2\mathrm{Im}(\mu)} + \frac{\mathrm{Arg}(v_0)}{\mathrm{Im}(\mu)} \leq x \leq ct + \frac{\pi}{2\mathrm{Im}(\mu)} + \frac{\mathrm{Arg}(v_0)}{\mathrm{Im}(\mu)} \right\},$$

and

$$\Omega_P(t) := \left\{ x \in \mathbb{R} \mid ct - \frac{\pi}{2\mathrm{Im}(\mu)} \leq x \leq ct + \frac{\pi}{2\mathrm{Im}(\mu)} \right\},$$

and set

$$\forall t \geq 0, \quad \underline{V}_j(t) := \begin{cases} \widetilde{V}_j(t), & j \in \Omega_V(t), \\ 0, & \text{otherwise}, \end{cases} \qquad \underline{P}_j(t) := \begin{cases} \widetilde{P}_j(t), & j \in \Omega_P(t), \\ 0, & \text{otherwise}. \end{cases} \tag{6.10}$$

Since both $\mathrm{Im}(\mu) = O(\sqrt{\xi})$ and $|\mathrm{Arg}(v_0)| = O(\sqrt{\xi})$, we thus have that $\frac{|\mathrm{Arg}(v_0)|}{\mathrm{Im}(\mu)} = O(1)$. We readily have that when $j \in \Omega_V(t) \cap \Omega_P(t)$ or $j \in \mathbb{Z}\backslash\Omega_V(t) \cap \Omega_P(t)$, then $(\underline{V}_j(t), \underline{P}_j(t))$ is a solution of the linear system (6.7). On the other hand, if $j \in \Omega_V(t)\backslash\Omega_P(t)$, then

$$\begin{aligned} \underline{V}_j'(t) + 2\alpha\underline{V}_j(t) - \beta(\underline{P}_j(t) + \underline{P}_{j+1}(t)) = \widetilde{V}_j'(t) + 2\alpha\widetilde{V}_j(t) = \beta(\widetilde{P}_j(t) + \widetilde{P}_{j+1}(t)) \leq 0, \\ \underline{P}_j'(t) + (2\beta - f'(0))\underline{P}_j(t) - \alpha(\underline{V}_j(t) + \underline{V}_{j-1}(t)) = -\alpha(\widetilde{V}_j(t) + \widetilde{V}_{j-1}(t)) \leq 0, \end{aligned}$$

upon taking ξ small enough to ensure that both $\widetilde{P}_j(t) < 0$ and $\widetilde{P}_{j+1}(t) \leq 0$. A similar argument with $j \in \Omega_P(t)\backslash\Omega_V(t)$ shows that $(\underline{V}_j(t), \underline{P}_j(t))$ is a subsolution of (6.7) for all $t \geq 0$ and $j \in \mathbb{Z}$.

At the moment, we have only constructed a compactly supported subsolution for the linear system (6.7). It is not difficult to check that all the above arguments naturally perturb if instead we consider the modified linear system

$$\forall t > 0, \; j \in \mathbb{Z}, \quad \begin{cases} V_j'(t) = -2\alpha V_j(t) + \beta(P_j(t) + P_{j+1}(t)), \\ P_j'(t) = (f'(0) - 2\beta - \delta)P_j(t) + \alpha(V_j(t) + V_{j-1}(t)), \end{cases} \tag{6.11}$$

for some small $\delta > 0$. More precisely, there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$, one can construct a compactly supported subsolution $(\underline{\mathbf{V}}^\delta, \underline{\mathbf{P}}^\delta) = (\underline{V}_j^\delta, \underline{P}_j^\delta)_{j\in\mathbb{Z}}$ in the form of (6.10). Furthermore, let $\iota_0 > 0$ be such that

$$(f'(0) - \delta_0)u \leq f(u), \quad 0 \leq u \leq \iota_0.$$

Then one can find $\eta_0 > 0$ small enough such that $\eta_0 \underline{P}_j^{\delta_0}(t) \leq \iota_0$ for all $t > 0$ and $j \in \mathbb{Z}$. As a consequence $(\eta_0\underline{\mathbf{V}}^{\delta_0}, \eta_0\underline{\mathbf{P}}^{\delta_0})$ is a compactly supported subsolution to the nonlinear system (6.6).

We can now prove item (ii) of the theorem. Let $c \in (0, c_*^\infty)$ and choose $c' \in (c, c_*^\infty)$ very close to c_*^∞. From the positivity of the solution of the nonlinear system (6.6), upon eventually decreasing the size of $\eta_0 > 0$, we can always ensure that at time $t = 1$ one has

$$(\mathbf{V}(1), \mathbf{P}(1)) \geq (\eta_0\underline{\mathbf{V}}^{\delta_0}(0), \eta_0\underline{\mathbf{P}}^{\delta_0}(0)),$$

where $(\eta_0\underline{\mathbf{V}}^{\delta_0}, \eta_0\underline{\mathbf{P}}^{\delta_0})$ is the compactly supported subsolution associated to the speed c' constructed in the previous step. From the comparison principle of Proposition 6.1 we obtain that

$$\forall t \geq 1, \quad (\mathbf{V}(t), \mathbf{P}(t)) \geq (\eta_0\underline{\mathbf{V}}^{\delta_0}(t-1), \eta_0\underline{\mathbf{P}}^{\delta_0}(t-1)).$$

There exists $\nu \in (0,1)$ small, depending on c', such that

$$V_{\lfloor c't \rfloor}(t) \geq \eta_0 \underline{V}^\delta_{\lfloor c't \rfloor}(t-1) \geq \frac{\beta}{\alpha}\nu \quad \text{and} \quad V_{\lfloor c't \rfloor+1}(t) \geq \eta_0 \underline{V}^\delta_{\lfloor c't \rfloor+1}(t-1) \geq \frac{\beta}{\alpha}\nu,$$

with

$$P_{\lfloor c't \rfloor}(t) \geq \eta_0 \underline{P}^{\delta}_{\lfloor c't \rfloor}(t-1) \geq \nu \quad \text{and} \quad P_{\lfloor c't \rfloor+1}(t) \geq \eta_0 \underline{P}^{\delta}_{\lfloor c't \rfloor+1}(t-1) \geq \nu,$$

for all $t \geq 1$. By a symmetry argument, we also obtain that

$$V_{-\lfloor c't \rfloor}(t) \geq \frac{\beta}{\alpha}\nu \quad \text{and} \quad V_{-\lfloor c't \rfloor-1}(t) \geq \frac{\beta}{\alpha}\nu,$$

with

$$P_{-\lfloor c't \rfloor}(t) \geq \nu \quad \text{and} \quad P_{-\lfloor c't \rfloor-1}(t) \geq \nu,$$

for all $t \geq 1$. Upon even reducing the size of ν, by positivity of the solution $(\mathbf{V}, \mathbf{P})$, we can always ensure that

$$V_j(1) \geq \frac{\beta}{\alpha}\nu \quad \text{and} \quad P_j(1) \geq \nu \quad \text{for all} \quad -c'-1 \leq j \leq c'+1$$

Since $\left(\frac{\beta}{\alpha}\nu, \nu\right)_{j\in\mathbb{Z}}$ is a homogeneous subsolution of (6.6), we are thus in a position to apply the comparison principle of Proposition B.7 with two moving boundaries given by $\zeta(t) = -c't$ and $\xi(t) = c't$. It implies that

$$\forall t \geq 1, \quad \inf_{|j|\leq ct} V_j(t) \geq \frac{\beta}{\alpha}\nu \quad \text{and} \quad \inf_{|j|\leq ct} P_j(t) \geq \nu,$$

from which we deduce that

$$\liminf_{t\to+\infty} \inf_{|j|\leq ct} (V_j(t), P_j(t)) \geq \liminf_{t\to+\infty} \inf_{|j|\leq c't} (V_j(t), P_j(t)) \geq \left(\frac{\beta}{\alpha}, 1\right).$$

But from Proposition 6.2, we have

$$\limsup_{t\to+\infty} \inf_{|j|\leq ct} (V_j(t), P_j(t)) \leq \left(\frac{\beta}{\alpha}, 1\right).$$

As a conclusion, we have proved that

$$\lim_{t\to+\infty} \inf_{|j|\leq ct} (V_j(t), P_j(t)) = \left(\frac{\beta}{\alpha}, 1\right),$$

for all $c \in (0, c_*^\infty)$. This concludes the proof of the theorem. ■

7 Discussion

Summary of main results. In this work, we have proposed a new model to describe biological invasions constrained on infinite homogeneous one dimensional metric graphs. Our model consists of an infinite PDE-ODE system where, at each vertex of the one-dimensional lattice $\mathbb{Z}$, we have a standard logistic equation and connections between vertices are given by diffusion equations on the edges supplemented with Robin like boundary conditions at the vertices. Our first main result is the existence and uniqueness of classical, global in time, positive bounded solutions of our PDE-ODE model. Our second main result is the characterization of the long time behavior of the unique solution of our model, where we prove local uniform convergence towards the unique positive bounded stationary solution of the system. Next, we analyzed the linearized problem around the trivial constant state and derived a theoretical formula for the linear spreading speed of our model, defined as the smallest possible speed for which there exist exponential solutions with prescribed form. We then proved that this linear spreading speed is actually the asymptotic spreading speed of the full nonlinear model, which constitutes the key result of our present study. Finally, we investigated the large diffusion limit of the model and established the convergence towards an asymptotic system for which we also managed to fully characterize its asymptotic spreading properties. We also illustrated our theoretical findings with a selection of numerical simulations.

Natural extensions. From a biological point of view, it could be interesting to consider several extensions of the model. First of all, roads could be modeled as a hostile environment such that the diffusion equation of (1.1) could be replaced by

$$\partial_t v_j = d\partial_x^2 v_j - \lambda v_j,$$

for some $\lambda > 0$ representing a death rate on the road. Such a modeling assumption has already been proposed for other reaction-diffusion models [2]. We expect that the presence of a hostile environment will have a direct effect on the stationary solutions of the model and thus on the long time behavior of the solutions. More precisely, we anticipate a threshold effect and the existence of a critical value for λ (depending on all other parameters of the model), above which the only stationary solution is the trivial constant steady state, and below which there exists a unique bounded positive stationary solution. For values of λ above this critical parameter, solutions of the Cauchy problem are expected to uniformly converge to the trivial constant steady state, and thus go extinct, reflecting the fact that the road is too hostile for the population to survive.

As explained in the introduction, for simplicity, our model neglects the possibility that individuals could pass from one road to an adjacent one. Assuming that such exchanges are homogeneous and symmetric, and if $\nu > 0$ denotes the corresponding exchange, then the Robin boundary conditions (1.2) should be modified according to

$$\begin{cases} -d\partial_x v_j(t,0) + \alpha v_j(t,0) = \beta \rho_j(t) + \nu \left(v_{j-1}(t,\ell) - v_j(t,0) \right), \\ d\partial_x v_j(t,\ell) + \alpha v_j(t,\ell) = \beta \rho_{j+1}(t) + \nu \left(v_{j+1}(t,0) - v_j(t,\ell) \right). \end{cases}$$

These new exchange terms typically account for the permeability of cell membranes in gap junction models [16]. We anticipate a similar threshold behavior as in the case of a hostile environment described above with the existence of a critical value for ν above which the populations on the roads and the cities should go extinct and below which we observe similar spreading properties as the one presented in our work. A possible interpretation is that for large ν, the exchange terms act as a dilution mechanism preventing the reaction kinetics at the cities to take over the diffusion on the roads. We leave the analysis of these natural extensions for future work.

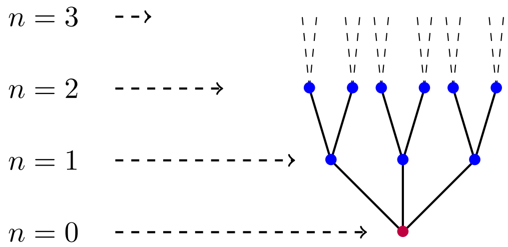


Figure 7: *Homogeneous rooted tree of degree $k = 2$. The red node represents the root of the tree. Each generation within the tree is labelled by an integer $n \in \mathbb{N}$.*

Beyond the lattice case. Our model (1.1)-(1.2) considers the simplest connected metric graph possible: the one dimensional lattice $\mathbb{Z}$. It would be very relevant to extend our framework to other classes of metric graphs. It seems natural to start by considering homogeneous trees, and we already refer to recent developments regarding spreading properties of reaction-diffusion equations on homogeneous trees [4, 8, 13]. In order to better explain the class of models we have in mind, we introduce some notations. We let $k \in \mathbb{N}$ with $k \geq 1$ and shall denote by $\mathbb{T}_k$ a homogeneous tree of degree k with the convention that $\mathbb{T}_1 = \mathbb{Z}$. We recall that a homogeneous tree of degree k is an infinite graph where each vertex has precisely $k+1$ adjacent vertices, and we refer to Figure 7 for an illustration in the case $k = 2$. As in our original model, we suppose that all edges of the tree have the same length $\ell > 0$. To simplify the presentation, we will identify one vertex as being the root of the tree and by convention we will label this vertex as $n = 0$ with associated population density $\rho_0(t)$. We will also assume that all populations at some fixed distance away from the root are equal. As a consequence, it will be convenient to denote by $\rho_n(t)$ as a representative population at distance $n \geq 1$ from the root. Similarly, we shall also denote by $v_n(t,x)$ a representative population leaving on the edge at distance n from the root. We readily remark that the new model is now indexed by the natural integers $\mathbb{N}$. Only the dynamics for each $\rho_n(t)$ has to be modified according to

$$\begin{cases} \rho_0'(t) = f(\rho_0(t)) + (k+1)\left(\alpha v_0(t,0) - \beta\rho_0(t)\right), \\ \rho_n'(t) = f(\rho_n(t)) + \alpha\left(v_{n-1}(t,\ell) + k v_n(t,0)\right) - (k+1)\beta\rho_n(t), \quad n \geq 1. \end{cases}$$

Let us already remark that the new exchange terms can also be written as

$$\begin{aligned} \alpha\left(v_{n-1}(t,\ell) + k v_n(t,0)\right) - (k+1)\beta\rho_n(t) = \alpha\left(v_{n-1}(t,\ell) + v_n(t,0)\right) - 2\beta\rho_n(t) \\ + (k-1)\left(\alpha v_n(t,0) - \beta\rho_n(t)\right), \end{aligned}$$

where we see the presence of a new term $(k-1)\left(\alpha v_n(t,0) - \beta\rho_n(t)\right)$ which can be interpreted as a drift that may or may not block the propagation within the tree depending on the other parameters of the model. For reaction-diffusion equations set on homogeneous trees [4, 13], the presence of such a term typically prevents propagation within the tree for k large enough and we expect a similar threshold to also happen here. We shall investigate the extension to homogeneous trees in a forthcoming work.

Acknowledgment

Authors acknowledge support from ANR project Indyana under grant agreement ANR-21-CE40-0008 and ANR project ReaCh under grant agreement ANR-23-CE40-0023.

Data availability statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

A Representation formula

We consider the heat equation

$$\partial_t v(t,x) = d\partial_x^2 v(t,x), \quad t>0,\ x\in(0,1), \tag{A.1}$$

with inhomogeneous Robin boundary conditions

$$\begin{cases} -d\partial_x v(t,0) + \alpha v(t,0) = g(t), \\ \ \ d\partial_x v(t,1) + \alpha v(t,1) = h(t), \end{cases} \quad t>0, \tag{A.2}$$

and initial condition

$$v(0,x) = v_0(x), \quad x\in[0,1]. \tag{A.3}$$

We also define

$$\mathcal{K}(t,x) := \frac{1}{\sqrt{4\pi dt}} \mathrm{e}^{-\frac{x^2}{4dt}}, \quad t>0, x\in\mathbb{R}.$$

Proposition A.1. *Assume that $h,g \in \mathscr{C}^0(\mathbb{R}_+,\mathbb{R})$ and $v_0 \in \mathcal{C}^0([0,1],\mathbb{R})$, then the solution to (A.1)-(A.2)-(A.3) can be represented as follows*

$$\begin{aligned} u(t,x) = &\int_0^1 \mathcal{K}(t,x-y)v_0(y)\mathrm{d}y + \int_0^t \left[\mathcal{K}(t-s,x-1)h(s) + \mathcal{K}(t-s,x)g(s)\right]\mathrm{d}s \\ &+ \int_0^t \left[-\alpha\mathcal{K}(t-s,x-1) + d\partial_x\mathcal{K}(t-s,x-1)\right]u(s,1)\mathrm{d}s \\ &- \int_0^t \left[\alpha\mathcal{K}(t-s,x) + d\partial_x\mathcal{K}(t-s,x)\right]u(s,0)\mathrm{d}s, \end{aligned} \tag{A.4}$$

for all $t>0$ and $x\in[0,1]$.

Proof. Let $w \in \mathscr{C}^2((0,+\infty)\times[0,1],\mathbb{R})$ and for any $t>0$, from (A.1) we have

$$\begin{aligned} 0 &= \int_0^t\int_0^1 w(s,y)\left(\partial_s v(s,y) - d\partial_y^2 v(s,y)\right)\mathrm{d}y\mathrm{d}s \\ &= \int_0^1 \left(w(t,y)v(t,y) - w(0,y)v(0,y)\right)\mathrm{d}y - \int_0^t\int_0^1 \left(\partial_s w(s,y) + d\partial_y^2 w(s,y)\right)v(s,y)\mathrm{d}y\mathrm{d}s \\ &\quad - d\int_0^t \left(w(s,1)\partial_y v(s,1) - w(s,0)\partial_y v(s,0)\right)\mathrm{d}s + d\int_0^t \left(\partial_y w(s,1)v(s,1) - \partial_y w(s,0)v(s,0)\right)\mathrm{d}s, \end{aligned}$$

for all $t > 0$ and $x \in [0,1]$. We now specify w to

$$w(s,y) = \mathcal{K}(t+\epsilon-s, x-y),$$

for $\epsilon > 0$. Note that for all $s \in [0,t]$ and $y \in [0,1]$ it satisfies $\partial_s w + d\partial_y^2 w = 0$. We also note that

$$\int_0^1 w(t,y)v(t,y)\mathrm{d}y = \int_0^1 \mathcal{K}(\epsilon, x-y)v(t,y)\mathrm{d}y \underset{\epsilon\to 0}{\longrightarrow} u(t,x),$$

while

$$\int_0^1 w(0,y)v(0,y)\mathrm{d}y = \int_0^1 \mathcal{K}(t+\epsilon, x-y)v_0(y)\mathrm{d}y \underset{\epsilon\to 0}{\longrightarrow} \int_0^1 \mathcal{K}(t, x-y)v_0(y)\mathrm{d}y.$$

Finally, we simply note that $\partial_y w(s,y) = -\partial_x \mathcal{K}(t+\epsilon-s, x-y)$, and using the Robin boundary condition (A.2), we eventually derive (A.4). ■

B Comparison principles

Proposition B.1. *Let* $\mathbf{v}$ *and* $\boldsymbol{\rho}$ *with* $\rho_j \in \mathscr{C}^1([0,+\infty),\mathbb{R})$ *and*

$$v_j \in \mathscr{C}^0([0,+\infty)\times[0,1],\mathbb{R}),\ \partial_t v_j, \partial_x^2 v_j \in \mathscr{C}^0((0,+\infty)\times(0,1),\mathbb{R}),\ \text{and } \partial_x v_j \in \mathscr{C}^0((0,+\infty)\times[0,1],\mathbb{R}),$$

for all $j \in \mathbb{Z}$, *which satisfy*

$$\begin{cases} \partial_t v_j(t,x) - d\partial_x^2 v_j(t,x) \geq 0, & x \in (0,1), \\ \rho_j'(t) - c_j(t)\rho_j(t) \geq \alpha\left[v_j(t,0) + v_{j-1}(t,1)\right], \\ -d\partial_x v_j(t,0) + \alpha v_j(t,0) \geq \beta\rho_j(t), \\ d\partial_x v_j(t,1) + \alpha v_j(t,1) \geq \beta\rho_{j+1}(t), \end{cases} \tag{B.1}$$

for all $t > 0$ *and* $j \in \mathbb{Z}$ *with some* $\mathbf{c} = (c_j)_{j\in\mathbb{Z}} \in L^\infty(\mathbb{R}_+, \ell^\infty(\mathbb{Z}))$. *Assume that* $v_j(0,x) \geq 0$ *and* $\rho_j(0) \geq 0$ *for all* $x \in [0,1]$ *and* $j \in \mathbb{Z}$, *then* $v_j(t,x) \geq 0$ *and* $\rho_j(t) \geq 0$ *for all* $t > 0$, $x \in [0,1]$ *and* $j \in \mathbb{Z}$. *If furthermore* $\mathbf{v}(0) \not\equiv 0$ *or* $\boldsymbol{\rho}(0) \not\equiv 0$, *then* $v_j(t,x) > 0$ *and* $\rho_j(t) > 0$ *for all* $t > 0$, $x \in [0,1]$ *and* $j \in \mathbb{Z}$.

Proof. Fix $T > 0$. By assumption on the sequence $\mathbf{c}$, there exists $K > 0$ such that

$$K - c_j(t) > 0, \text{ for } t \in (0,T] \text{ and } j \in \mathbb{Z}.$$

For any $\gamma > 0$, we define

$$\begin{cases} w_j(t,x) := \mathrm{e}^{-\gamma|j|-Kt} v_j(t,x), \\ z_j(t) := \mathrm{e}^{-\gamma|j|-Kt}\rho_j(t). \end{cases}$$

Since $\mathbf{v}$ and $\boldsymbol{\rho}$ are assumed to be locally bounded, we have for each $t \in (0,T]$, $j \in \mathbb{Z}$ and $x \in [0,1]$ that

$$\begin{cases} w_j(t,x) \underset{j\to\pm\infty}{\longrightarrow} 0, \\ z_j(t) \underset{j\to\pm\infty}{\longrightarrow} 0. \end{cases}$$

The sequences $\mathbf{w}$ and $\mathbf{z}$ now satisfy

$$\begin{cases} \partial_t w_j(t,x) - d\partial_x^2 w_j(t,x) + K w_j(t,x) \geq 0, \quad x \in (0,1), \\ z_j'(t) + (K - c_j(t)) z_j(t) \geq \alpha \left[w_j(t,0) + C_j^\gamma w_{j-1}(t,1) \right], \\ -d\partial_x w_j(t,0) + \alpha w_j(t,0) \geq \beta z_j(t), \\ d\partial_x w_j(t,1) + \alpha w_j(t,1) \geq \beta C_{j+1}^{-\gamma} z_{j+1}(t), \end{cases} \tag{B.2}$$

where the sequence C_j^γ is defined as follows

$$C_j^\gamma = \begin{cases} \mathrm{e}^{-\gamma}, & j \geq 1, \\ \mathrm{e}^{\gamma}, & j \leq 0. \end{cases} \tag{B.3}$$

We now let $\epsilon > 0$ and define

$$\begin{cases} w_j^\epsilon(t,x) := w_j(t,x) + \epsilon \mathrm{e}^{\varrho t + \delta \left(x - \frac{1}{2}\right)^2}, \\ z_j^\epsilon(t) := z_j(t) + \epsilon \mathrm{e}^{\varrho t + \frac{\delta}{4}}, \end{cases}$$

for two constants $\varrho > 0$ and $\delta > 0$ that will be fixed later in the proof. Elementary computations give

$$\begin{aligned} &\partial_t w_j^\epsilon(t,x) - d\partial_x^2 w_j^\epsilon(t,x) + K w_j^\epsilon(t,x) \\ &= \partial_t w_j(t,x) - d\partial_x^2 w_j(t,x) + K w_j(t,x) + \epsilon \left(\varrho + K - 2d\delta - 4d\delta^2 \left(x - \frac{1}{2} \right)^2 \right) \mathrm{e}^{\varrho t + \delta \left(x - \frac{1}{2}\right)^2}, \end{aligned}$$

and

$$\begin{aligned} &z_j^{\epsilon\prime}(t) + (K - c_j(t)) z_j^\epsilon(t) - \alpha \left[w_j^\epsilon(t,0) + C_j^\gamma w_{j-1}^\epsilon(t,1) \right] \\ &= z_j'(t) + (K - c_j(t)) z_j(t) - \alpha \left[w_j(t,0) + C_j^\gamma w_{j-1}(t,1) \right] + \epsilon \left(\varrho + K - c_j(t) - \alpha(1 + C_j^\gamma) \right) \mathrm{e}^{\varrho t + \frac{\delta}{4}}, \end{aligned}$$

together with

$$\begin{aligned} -d\partial_x w_j^\epsilon(t,0) + \alpha w_j^\epsilon(t,0) - \beta z_j^\epsilon(t) &= -d\partial_x w_j(t,0) + \alpha w_j(t,0) - \beta z_j(t) + \epsilon(d\delta + \alpha - \beta) \mathrm{e}^{\varrho t + \frac{\delta}{4}}, \\ d\partial_x w_j^\epsilon(t,1) + \alpha w_j^\epsilon(t,1) - C_{j+1}^{-\gamma} \beta z_{j+1}^\epsilon(t) &= d\partial_x w_j(t,1) + \alpha w_j(t,1) - C_{j+1}^{-\gamma} \beta z_{j+1}(t) \\ &\quad + \epsilon(d\delta + \alpha - \beta C_{j+1}^{-\gamma}) \mathrm{e}^{\varrho t + \frac{\delta}{4}}. \end{aligned}$$

As a consequence, we first fix $\delta > 0$ such that

$$\delta > \frac{\beta \mathrm{e}^\gamma - \alpha}{d},$$

and then select $\varrho > 0$ large enough such that

$$\varrho + K - 2d\delta - d\delta^2 > 0 \text{ and } \varrho - \alpha(1 + \mathrm{e}^\gamma) > 0,$$

that is

$$\varrho > \max\left(2d\delta + d\delta^2 - K, \alpha(1 + \mathrm{e}^\gamma)\right).$$

With such a choice, the sequences $\mathbf{w}^\epsilon$ and $\mathbf{z}^\epsilon$ now satisfy

$$\begin{cases} \partial_t w_j^\epsilon(t,x) - d\partial_x^2 w_j^\epsilon(t,x) + K w_j^\epsilon(t,x) > 0, \quad x \in (0,1), \\ z_j^{\epsilon\prime}(t) + (K - c_j(t)) z_j^\epsilon(t) - \alpha \left[w_j^\epsilon(t,0) + C_j^\gamma w_{j-1}^\epsilon(t,1) \right] > 0, \\ -d\partial_x w_j^\epsilon(t,0) + \alpha w_j^\epsilon(t,0) - \beta z_j^\epsilon(t) > 0, \\ d\partial_x w_j^\epsilon(t,1) + \alpha w_j^\epsilon(t,1) - \beta C_{j+1}^{-\gamma} z_{j+1}^\epsilon(t) > 0, \end{cases} \tag{B.4}$$

for all $t \in (0,T]$, $x \in [0,1]$ and $j \in \mathbb{Z}$ with

$$\begin{cases} w_j^\epsilon(0,x) := w_j(0,x) + \epsilon \mathrm{e}^{\delta\left(x-\frac{1}{2}\right)^2} > 0, \\ z_j^\epsilon(0) := z_j(0) + \epsilon \mathrm{e}^{\frac{\delta}{4}} > 0, \end{cases}$$

and for each $t \in (0,T]$, $x \in [0,1]$ and $j \in \mathbb{Z}$

$$\begin{cases} w_j^\epsilon(t,x) \underset{j\to\pm\infty}{\longrightarrow} \epsilon \mathrm{e}^{\varrho t + \delta\left(x-\frac{1}{2}\right)^2} > 0, \\ z_j^\epsilon(t) \underset{j\to\pm\infty}{\longrightarrow} \epsilon \mathrm{e}^{\varrho t + \frac{\delta}{4}} > 0. \end{cases}$$

As a consequence, there exists some $J > 0$ such that $w_j^\epsilon(t,x) > 0$ and $z_j^\epsilon(t) > 0$ for all $t \in (0,T]$, $x \in [0,1]$ and $|j| \geq J$. Our aim is to show that this is also true for all $|j| \leq J$. By contradiction, assume that there exists $t_0 \in (0,T]$, $j_0 \in [\![-J,J]\!]$ and $x_0 \in [0,1]$ such that $w_{j_0}^\epsilon(t_0,x_0) = 0$ while $w_j^\epsilon(t,x) \geq 0$ and $z_j^\epsilon(t) \geq 0$ for all $t \in (0,t_0]$, $x \in [0,1]$ and $j \in [\![-J,J]\!]$. If $x_0 \in (0,1)$, then by definition we have $\partial_t w_{j_0}^\epsilon(t_0,x_0) \leq 0$ and $\partial_x^2 w_{j_0}^\epsilon(t_0,x_0) \geq 0$ such that

$$0 \geq \partial_t w_{j_0}^\epsilon(t_0,x_0) - d\partial_x^2 w_{j_0}^\epsilon(t_0,x_0) + K w_{j_0}^\epsilon(t_0,x_0) > 0,$$

which is a contradiction. If $x_0 = 0$ then the Hopf Lemma ensures that $\partial_x w_{j_0}^\epsilon(t_0,0) > 0$ and the boundary condition gives

$$0 > -d\partial_x w_{j_0}^\epsilon(t_0,0) + \alpha \underbrace{w_{j_0}^\epsilon(t_0,0)}_{=0} > \beta z_{j_0}^\epsilon(t_0) \geq 0,$$

which is impossible. A similar argument shows that if $x_0 = 1$ one also reaches a contradiction. Finally, if on the other hand we had assumed that $z_{j_0}(t_0) = 0$ while $w_j^\epsilon(t,x) \geq 0$ and $z_j^\epsilon(t) \geq 0$ for all $t \in (0,t_0]$, $x \in [0,1]$ and $j \in [\![-J,J]\!]$, then using the equation satisfied by z_{j_0} we find

$$0 \geq \underbrace{z_{j_0}^{\epsilon}{}'(t_0)}_{\leq 0} + (K - c_j(t)) \underbrace{z_{j_0}^\epsilon(t_0)}_{=0} > \alpha \left[w_{j_0}^\epsilon(t_0,0) + C_j^\gamma w_{j_0-1}^\epsilon(t_0,1) \right] \geq 0,$$

which is a contradiction. Let us remark that in the above inequality we have used that $w_{j_0-1}^\epsilon(t_0,1) \geq 0$. This holds by definition of (t_0,x_0,j_0) if $j_0 \in [\![-J+1,J]\!]$, and if $j_0 = -J$, the fact that $w_{-J-1}^\epsilon(t_0,1) \geq 0$ holds thanks to the definition of J and the fact that $w_j^\epsilon(t,x) > 0$ for all $t \in (0,T]$, $x \in [0,1]$ and $|j| \geq J$.

As a conclusion, we have proved that $w_j^\epsilon(t,x) > 0$ and $z_j^\epsilon(t) > 0$ for all $t \in (0,T]$, $x \in [0,1]$ and $j \in \mathbb{Z}$. Since $\epsilon > 0$ was left arbitrary by passing to the limit $\epsilon \to 0$ we obtain that $w_j(t,x) \geq 0$ and $z_j(t) \geq 0$ for all $t \in (0,T]$, $x \in [0,1]$ and $j \in \mathbb{Z}$, which concludes the first part of the proof.

In order to prove the last part of the proposition, we shall instead prove that if there exists $t_0 \in (0,T]$ and $j_0 \in \mathbb{Z}$ such that $\rho_{j_0}(t_0) = 0$ or if there exists $t_0 \in (0,T]$, $j_0 \in \mathbb{Z}$ and $x_0 \in [0,1]$ such that $v_{j_0}(t_0,x_0) = 0$ then $\rho_j(t) = 0$ and $v_j(t,x) = 0$ for all $t \in [0,t_0]$, $x \in [0,1]$ and $j \in \mathbb{Z}$. If $\rho_{j_0}(t_0) = 0$, then integrating the equation for ρ_{j_0} from $t = 0$ to t_0, we obtain that

$$0 = \rho_{j_0}(t_0) \geq \mathrm{e}^{\int_0^{t_0} c_{j_0}(s)\mathrm{d}s}\rho_{j_0}(0) + \alpha \int_0^{t_0} \mathrm{e}^{\int_s^{t_0} c_{j_0}(\tau)\mathrm{d}\tau} \left(v_{j_0}(s,0) + v_{j_0-1}(s,1)\right) \mathrm{d}s \geq 0.$$

As a consequence, we deduce that $\rho_{j_0}(0) = 0$, $v_{j_0}(t,0) = v_{j_0-1}(t,1) = 0$ for all $t \in [0,t_0]$ and thus $\rho_{j_0}(t) = 0$ for all $t \in [0,t_0]$. But then, the strong maximum principle applied to v_{j_0} and v_{j_0-1} gives that $v_{j_0}(t,x) = v_{j_0-1}(t,x) = 0$ for all $t \in [0,t_0]$ and $x \in [0,1]$. Now, by contradiction, if $\mathbf{v} \neq 0$ or $\boldsymbol{\rho} \neq 0$ on $[0,t_0]$, without loss of generality, we may assume that there exists $p \in \mathbb{Z}$ with $p > j_0$ and $x_* \in [0,1]$ such that $v_p(0,x_*) > 0$. By continuity of v_p, there exists $r > 0$ such that $v_p(0,x) > 0$ for all $x \in [0,1] \cap B_r(x_*)$. Recalling that v_p satisfies

$$\begin{cases} \partial_t v_p(t,x) - d\partial_x^2 v_p(t,x) \geq 0, \quad x \in (0,1), \\ -d\partial_x v_p(t,0) + \alpha v_p(t,0) \geq 0, \\ d\partial_x v_p(t,1) + \alpha v_p(t,1) \geq 0, \end{cases}$$

since $\rho_j(t) \geq 0$ for all $j \in \mathbb{Z}$, the strong maximum principle implies that $v_p(t,x) > 0$ for all $t \in (0,t_0]$ and $x \in [0,1]$. This, in turn, also implies that

$$\rho_p(t) \geq \mathrm{e}^{\int_0^{t} c_p(s)\mathrm{d}s}\rho_p(0) + \alpha \int_0^{t} \mathrm{e}^{\int_s^{t} c_p(\tau)\mathrm{d}\tau} \left(v_p(s,0) + v_{p-1}(s,1)\right) \mathrm{d}s > 0,$$

for all $t \in (0,t_0]$. Now, inspecting the equation satisfied by v_{p-1}, we have for all $t \in (0,t_0]$ that

$$\begin{cases} \partial_t v_{p-1}(t,x) - d\partial_x^2 v_{p-1}(t,x) \geq 0, \quad x \in (0,1), \\ -d\partial_x v_{p-1}(t,0) + \alpha v_{p-1}(t,0) \geq 0, \\ d\partial_x v_{p-1}(t,1) + \alpha v_{p-1}(t,1) > 0, \\ v_{p-1}(0,x) \geq 0, \quad x \in [0,1]. \end{cases}$$

Once again, the strong maximum principle implies that $v_{p-1}(t,x) > 0$ for all $t \in (0,t_0]$ and $x \in [0,1]$. By induction, we reach a contradiction since we eventually end up proving that $v_{j_0}(t,x) > 0$ for $t \in (0,t_0]$ and $x \in [0,1]$, which is impossible. ∎

Proposition B.2. *Let* $\mathbf{v}$ *and* $\boldsymbol{\rho}$ *with* $\rho_j \in \mathscr{C}^1([0,+\infty),\mathbb{R})$ *and*

$v_j \in \mathscr{C}^0([0,+\infty)\times[0,1],\mathbb{R})$, $\partial_t v_j, \partial_x^2 v_j \in \mathscr{C}^0((0,+\infty)\times(0,1),\mathbb{R})$, *and* $\partial_x v_j \in \mathscr{C}^0((0,+\infty)\times[0,1],\mathbb{R})$, *for all $j \in \mathbb{Z}$, which satisfy*

$$\forall t > 0,\ \zeta(t) \le j \le \xi(t), \quad \begin{cases} \partial_t v_j(t,x) - d\partial_x^2 v_j(t,x) \ge 0, \quad x \in (0,1), \\ \rho_j'(t) - c_j(t)\rho_j(t) \ge \alpha\left[v_j(t,0) + v_{j-1}(t,1)\right], \\ -d\partial_x v_j(t,0) + \alpha v_j(t,0) \ge \beta \rho_j(t), \\ d\partial_x v_j(t,1) + \alpha v_j(t,1) \ge \beta \rho_{j+1}(t), \end{cases} \tag{B.5}$$

for some $\mathbf{c} = (c_j)_{j\in\mathbb{Z}} \in L^\infty(\mathbb{R}_+, \ell^\infty(\mathbb{Z}))$ and continuous functions $\zeta : \mathbb{R}_+ \to \mathbb{R}$ and $\xi : \mathbb{R}_+ \to \mathbb{R}$. Assume that $v_j(0,x) \ge 0$ and $\rho_j(0) \ge 0$ for all $x \in [0,1]$ and $\zeta(0) - 1 \le j \le \xi(0) + 1$ together with $v_j(t,x) \ge 0$ and $\rho_j(t) \ge 0$ for all $t > 0$, $x \in [0,1]$ and $j \in [\zeta(t) - 1, \zeta(t)) \cup (\xi(t), \xi(t) + 1]$, then $v_j(t,x) \ge 0$ and $\rho_j(t) \ge 0$ for all $t > 0$, $x \in [0,1]$ and $\zeta(t) \le j \le \xi(t)$.

Proof. The proof is a direct adaptation of the proof of the previous proposition. Fix $T > 0$. By assumption on the sequence $\mathbf{c}$, there exists $K > 0$ such that

$$K - c_j(t) > 0, \text{ for } t \in (0,T] \text{ and } j \in \mathbb{Z}.$$

For any $\epsilon > 0$, we define

$$\begin{cases} w_j^\epsilon(t,x) := \mathrm{e}^{-Kt} v_j(t,x) + \epsilon \mathrm{e}^{\varrho t + \delta\left(x - \frac{1}{2}\right)^2}, \\ z_j^\epsilon(t) := \mathrm{e}^{-Kt}\rho_j(t) + \epsilon \mathrm{e}^{\varrho t + \frac{\delta}{4}}, \end{cases}$$

where $\varrho > 0$ and $\delta > 0$ are taken large enough to ensure that $w_j^\epsilon(t,x)$ and $z_j^\epsilon(t)$ satisfy

$$\forall t > 0,\ \zeta(t) \le j \le \xi(t), \quad \begin{cases} \partial_t w_j^\epsilon(t,x) - d\partial_x^2 w_j^\epsilon(t,x) + K w_j^\epsilon(t,x) > 0, \quad x \in (0,1), \\ z_j^{\epsilon\prime}(t) + (K - c_j(t)) z_j^\epsilon(t) - \alpha\left[w_j^\epsilon(t,0) + w_{j-1}^\epsilon(t,1)\right] > 0, \\ -d\partial_x w_j^\epsilon(t,0) + \alpha w_j^\epsilon(t,0) - \beta z_j^\epsilon(t) > 0, \\ d\partial_x w_j^\epsilon(t,1) + \alpha w_j^\epsilon(t,1) - \beta z_{j+1}^\epsilon(t) > 0, \end{cases}$$

Furthermore, one also has $w_j^\epsilon(0,x) > 0$ and $z_j^\epsilon(0) > 0$ for all $x \in [0,1]$ and $\zeta(0) - 1 \le j \le \xi(0) + 1$ together with $w_j^\epsilon(t,x) > 0$ and $z_j^\epsilon(t) > 0$ for all $t > 0$, $x \in [0,1]$ and $j \in [\zeta(t)-1, \zeta(t)) \cup (\xi(t), \xi(t)+1]$.

By contradiction, assume that there exists $t_0 \in (0,T]$, $j_0 \in [\![\zeta(t_0), \xi(t_0)]\!]$ and $x_0 \in [0,1]$ such that $w_{j_0}^\epsilon(t_0,x_0) = 0$ while $w_j^\epsilon(t,x) \ge 0$ and $z_j^\epsilon(t) \ge 0$ for all $t \in (0,t_0]$, $x \in [0,1]$ and $j \in [\![\zeta(t), \xi(t)]\!]$. If $x_0 \in (0,1)$, then by definition we have $\partial_t w_{j_0}^\epsilon(t_0,x_0) \le 0$ and $\partial_x^2 w_{j_0}^\epsilon(t_0,x_0) \ge 0$ such that

$$0 \ge \partial_t w_{j_0}^\epsilon(t_0,x_0) - d\partial_x^2 w_{j_0}^\epsilon(t_0,x_0) + K w_{j_0}^\epsilon(t_0,x_0) > 0,$$

which is a contradiction. If $x_0 = 0$ then the Hopf Lemma ensures that $\partial_x w_{j_0}^\epsilon(t_0,0) > 0$ and the boundary condition gives

$$0 > -d\partial_x w_{j_0}^\epsilon(t_0,0) + \alpha \underbrace{w_{j_0}^\epsilon(t_0,0)}_{=0} > \beta z_{j_0}^\epsilon(t_0) \ge 0,$$

which is impossible. On the other hand, if $x_0 = 1$, the boundary condition gives

$$0 > d\,\underbrace{\partial_x w^\epsilon_{j_0}(t_0,1)}_{<0} + \alpha w^\epsilon_{j_0}(t_0,0) > \beta z^\epsilon_{j_0+1}(t_0) \geq 0,$$

which is also impossible. The fact that $z^\epsilon_{j_0+1}(t_0) \geq 0$ even if $j_0 = \xi(t_0)$ is ensured by the assumption that $z^\epsilon_j(t) > 0$ for $j \in [\zeta(t)-1, \zeta(t)) \cup (\xi(t), \xi(t)+1]$. Finally, if we had assumed that $z_{j_0}(t_0) = 0$ while $w^\epsilon_j(t,x) \geq 0$ and $z^\epsilon_j(t) \geq 0$ for all $t \in (0, t_0]$, $x \in [0,1]$ and $j \in [\![\zeta(t), \xi(t)]\!]$, then using the equation satisfied by z_{j_0} we find

$$0 \geq \underbrace{z^{\epsilon}_{j_0}{}'(t_0)}_{\leq 0} + (K - c_j(t))\underbrace{z^\epsilon_{j_0}(t_0)}_{=0} > \alpha\left[w^\epsilon_{j_0}(t_0,0) + w^\epsilon_{j_0-1}(t_0,1)\right] \geq 0.$$

Once again, the fact that $w^\epsilon_{j_0-1}(t_0,1) \geq 0$ even if $j_0 = \zeta(t_0)$ comes from the assumption that $w^\epsilon_j(t,x) > 0$ for $j \in [\zeta(t)-1, \zeta(t)) \cup (\xi(t), \xi(t)+1]$ and all $x \in [0,1]$.

To conclude the proof one just passes to the limit $\epsilon \to 0$. ■

Proposition B.3. *Let $\lambda > 0$ and $(N, M) \in \mathbb{Z}^2$ such that $N < M$. Consider a sequence $\mathbf{w} = (w_j)_{j=N-1,\dots,M+1}$ satisfying*

$$\lambda(w_{j+1} - 2w_j + w_{j-1}) - c_j w_j \leq 0, \quad j = N, \dots, M,$$

for some sequence $\mathbf{c} = (c_j)_{j=N,\dots,M}$ satisfying $c_j \geq 0$ for all $j \in [\![N, M]\!]$. If $w_j \geq 0$ for $j \in \{N-1, M+1\}$ then $w_j \geq 0$ for all $j \in [\![N, M]\!]$.

Proof. Let us assume first that $\mathbf{c} = 0$ and that $\lambda(w_{j+1} - 2w_j + w_{j-1}) < 0$ for $j = N, \dots, M$. We claim that $\mathbf{w}$ cannot have a minimum on $[\![N, M]\!]$. Indeed if $j_0 \in [\![N, M]\!]$ is such a minimum then one has

$$0 \leq \lambda(w_{j_0+1} - 2w_{j_0} + w_{j_0-1}) < 0,$$

which is impossible. Assume now that $\lambda(w_{j+1} - 2w_j + w_{j-1}) \leq 0$ for $j = N, \dots, M$, then we can define $w^\epsilon_j = w_j - \epsilon e^{\gamma j}$ for $\epsilon > 0$ and $\gamma > 0$. A direct computation shows that

$$\lambda(w^\epsilon_{j+1} - 2w^\epsilon_j + w^\epsilon_{j-1}) = \lambda(w_{j+1} - 2w_j + w_{j-1}) - 2\epsilon(\cosh(\gamma) - 1)e^{\gamma j} < 0,$$

from which we deduce that

$$\inf_{j=N-1,\dots,M+1} w^\epsilon_j = \inf_{j \in \{N-1, M+1\}} w^\epsilon_j,$$

and thus by sending ϵ to 0 we deduce that

$$\inf_{j=N-1,\dots,M+1} w_j = \inf_{j \in \{N-1, M+1\}} w_j.$$

As a conclusion, if we further assume that $w_j \geq 0$ for $j \in \{N-1, M+1\}$ then $w_j \geq 0$ for all $j \in [\![N, M]\!]$.

Let us now assume that $c_j \geq 0$ for all $j \in [\![N, M]\!]$. We denote by $\Omega_- := \{j \in [\![N, M]\!] \mid w_j < 0\}$ and $\Omega_+ := \{j \in [\![N-1, M+1]\!] \mid w_j \geq 0\}$. We also let

$$\partial\Omega_- := \{j \in \Omega_+ \mid j+1 \in \Omega_- \text{ or } j-1 \in \Omega_-\}.$$

If $\Omega_- = \emptyset$ then we are done, so we assume that $\Omega_- \neq \emptyset$. By assumption, for any $j \in \Omega_-$ one has

$$\lambda(w_{j+1} - 2w_j + w_{j-1}) \leq c_j w_j \leq 0,$$

and we can use the previous step to infer that

$$\inf_{j \in \Omega_- \cup \partial\Omega_-} w_j = \inf_{j \in \partial\Omega_-} w_j,$$

which is impossible. Indeed, on the one hand we have that

$$\inf_{j \in \Omega_- \cup \partial\Omega_-} w_j \leq \inf_{j \in \Omega_-} w_j < 0,$$

and on the other hand

$$0 \leq \inf_{j \in \Omega_+} w_j \leq \inf_{j \in \partial\Omega_-} w_j.$$

As a conclusion $\Omega_- = \emptyset$ and this concludes the proof of the proposition. ∎

Proposition B.4. *Let $\lambda > 0$ and $(N, M) \in \mathbb{Z}^2$ such that $N < M$. Consider two bounded sequence $\underline{\boldsymbol{\rho}} = (\underline{\rho}_j)_{j=N-1,\ldots,M+1}$ and $\overline{\boldsymbol{\rho}} = (\overline{\rho}_j)_{j=N-1,\ldots,M+1}$ satisfying*

$$\lambda(\underline{\rho}_{j+1} - 2\underline{\rho}_j + \underline{\rho}_{j-1}) + f(\underline{\rho}_j) \geq 0,$$
$$\lambda(\overline{\rho}_{j+1} - 2\overline{\rho}_j + \overline{\rho}_{j-1}) + f(\overline{\rho}_j) \leq 0,$$

for each $j = N, \ldots, M$. If $\overline{\rho}_j \geq \underline{\rho}_j$ for $j \in \{N-1, M+1\}$, then $\overline{\rho}_j \geq \underline{\rho}_j$ for all $j = N, \ldots, M$.

Proof. We set $A := \max\left(\|\underline{\boldsymbol{\rho}}\|_{\ell^\infty}, \|\overline{\boldsymbol{\rho}}\|_{\ell^\infty}\right)$ and let $K_A > 0$ be the Lipschitz constant of f on the interval $[-A, A]$. We define $\tilde{f}(u) := f(u) + K_A u$ which is nondecreasing on $[-A, A]$ by construction. Upon setting $w_j := \overline{\rho}_j - \underline{\rho}_j$, we obtain

$$\lambda(w_{j+1} - 2w_j + w_{j-1}) - K_A w_j + \tilde{f}(\overline{\rho}_j) - \tilde{f}(\underline{\rho}_j) \leq 0, \quad j = N, \ldots, M,$$

with $w_j \geq 0$ for $j \in \{N-1, M+1\}$. Once again, we define $\Omega_- = \{j \in [\![N, M]\!] \mid w_j < 0\}$, $\Omega_+ = \{j \in [\![N-1, M+1]\!] \mid w_j \geq 0\}$ and $\partial\Omega_- = \{j \in \Omega_+ \mid j+1 \in \Omega_- \text{ or } j-1 \in \Omega_-\}$. Let us assume that $\Omega_- \neq \emptyset$. For $j \in \Omega_-$, we obtain

$$\lambda(w_{j+1} - 2w_j + w_{j-1}) - K_A w_j \leq \tilde{f}(\overline{\rho}_j - w_j) - \tilde{f}(\overline{\rho}_j) \leq 0,$$

which gives a contradiction thanks to the previous proposition. As a consequence, we necessarily have $\Omega_- = \emptyset$ which concludes the proof. ∎

Proposition B.5. *Let* $\mathbf{W} = (W_j)_{j\in\mathbb{Z}}$ *and* $\mathbf{Q} = (Q_j)_{j\in\mathbb{Z}}$ *with* $W_j, Q_j \in \mathscr{C}^1([0,+\infty),\mathbb{R})$ *for all* $j \in \mathbb{Z}$ *which satisfy*

$$\forall t > 0,\ j \in \mathbb{Z}, \quad \begin{cases} W_j'(t) \geq -2\alpha W_j(t) + \beta(Q_j(t) + Q_{j+1}(t)), \\ Q_j'(t) \geq c_j(t) Q_j(t) + \alpha(W_j(t) + W_{j-1}(t)), \end{cases}$$

with some $\mathbf{c} = (c_j)_{j\in\mathbb{Z}} \in L^\infty(\mathbb{R}_+, \ell^\infty(\mathbb{Z}))$. *Assume that* $W_j(0) \geq 0$ *and* $Q_j(0) \geq 0$ *for all* $j \in \mathbb{Z}$, *then* $W_j(t) \geq 0$ *and* $Q_j(t) \geq 0$ *for all* $t > 0$ *and* $j \in \mathbb{Z}$. *If furthermore* $\mathbf{W}(0) \not\equiv 0$ *or* $\mathbf{Q}(0) \not\equiv 0$, *then* $W_j(t) > 0$ *and* $Q_j(t) > 0$ *for all* $t > 0$ *and* $j \in \mathbb{Z}$.

Proof. Fix $T > 0$. By assumption on the sequence $\mathbf{c}$, there exists $K > 0$ such that

$$K - c_j(t) > 0, \text{ for } t \in (0,T] \text{ and } j \in \mathbb{Z}.$$

For any $\gamma > 0$, we define

$$\begin{cases} w_j(t) := \mathrm{e}^{-\gamma|j| - Kt} W_j(t), \\ q_j(t) := \mathrm{e}^{-\gamma|j| - Kt} Q_j(t). \end{cases}$$

Since $\mathbf{W}$ and $\mathbf{Q}$ are assumed to be locally bounded, we have for each $t \in (0,T]$ and $j \in \mathbb{Z}$ that

$$\begin{cases} w_j(t) \underset{j\to\pm\infty}{\longrightarrow} 0, \\ q_j(t) \underset{j\to\pm\infty}{\longrightarrow} 0 \end{cases}$$

The sequences $\mathbf{w}$ and $\mathbf{q}$ now satisfy

$$\forall t \in (0,T], \quad j \in \mathbb{Z}, \quad \begin{cases} w_j'(t) \geq -2\alpha w_j(t) + \beta(q_j(t) + C_{j+1}^{-\gamma} q_{j+1}(t)), \\ q_j'(t) \geq -(K - c_j(t)) q_j(t) + \alpha(w_j(t) + C_j^{\gamma} w_{j-1}(t)), \end{cases} \tag{B.6}$$

with C_j^γ defined in (B.3). As in the proof of Proposition B.1, we define

$$\begin{cases} w_j^\epsilon(t) := w_j(t) + \epsilon \mathrm{e}^{\rho t}, \\ q_j^\epsilon(t) := q_j(t) + \epsilon \mathrm{e}^{\rho t}, \end{cases}$$

with $\epsilon > 0$ and $\rho > 0$ chosen such that

$$\rho > \max\left(\alpha(1 + \mathrm{e}^\gamma), \beta(1 + \mathrm{e}^\gamma) - 2\alpha\right).$$

With such a choice, we readily have that

$$\forall t \in (0,T], \quad j \in \mathbb{Z}, \quad \begin{cases} w_j^{\epsilon\prime}(t) > -2\alpha w_j^\epsilon(t) + \beta(q_j^\epsilon(t) + C_{j+1}^{-\gamma} q_{j+1}^\epsilon(t)), \\ q_j^{\epsilon\prime}(t) > -(K - c_j(t)) q_j^\epsilon(t) + \alpha(w_j^\epsilon(t) + C_j^{\gamma} w_{j-1}^\epsilon(t)), \end{cases} \tag{B.7}$$

with

$$\begin{cases} w_j^\epsilon(0) := w_j(0) + \epsilon > 0, \\ q_j^\epsilon(0) := q_j(0) + \epsilon > 0, \end{cases}$$

and for each $t \in (0,T]$ and $j \in \mathbb{Z}$

$$\begin{cases} w_j^\epsilon(t) \underset{j\to\pm\infty}{\longrightarrow} \epsilon \mathrm{e}^{\varrho t} > 0, \\ q_j^\epsilon(t) \underset{j\to\pm\infty}{\longrightarrow} \epsilon \mathrm{e}^{\varrho t} > 0. \end{cases}$$

As a consequence, there exists some $J > 0$ such that $w_j^\epsilon(t) > 0$ and $q_j^\epsilon(t) > 0$ for all $t \in (0,T]$ and $|j| \geq J$. Our aim is to show that this is also true for all $|j| \leq J$. By contradiction, without loss of generality, assume that there exists $t_0 \in (0,T]$ and $j_0 \in [\![-J,J]\!]$ such that $w_{j_0}^\epsilon(t_0) = 0$ while $w_j^\epsilon(t) \geq 0$ and $q_j^\epsilon(t) \geq 0$ for all $t \in (0,t_0]$ and $j \in [\![-J,J]\!]$. Then by definition, we also have that ${w_{j_0}^\epsilon}'(t_0) \leq 0$ such that

$$0 \geq {w_{j_0}^\epsilon}'(t_0) + 2\alpha w_{j_0}^\epsilon(t_0) - \beta(q_{j_0}^\epsilon(t_0) + C_{j_0+1}^{-\gamma} q_{j_0+1}^\epsilon(t_0)) > 0,$$

which is a contradiction. As a consequence, one has $w_j^\epsilon(t) > 0$ and $q_j^\epsilon(t) > 0$ for all $t \in (0,T]$ and $j \in \mathbb{Z}$. Since $\epsilon > 0$ was left arbitrary by passing to the limit $\epsilon \to 0$, we obtain that $w_j(t) \geq 0$ and $q_j(t) \geq 0$ for all $t \in (0,T]$ and $j \in \mathbb{Z}$, which concludes the first part of the proof.

In order to prove the last part of the proposition, we shall instead prove that if there exists $t_0 \in (0,T]$ and $j_0 \in \mathbb{Z}$ such that $W_{j_0}(t_0) = 0$ or $Q_{j_0}(t_0) = 0$ then $W_j(t) = 0$ and $Q_j(t) = 0$ for all $t \in [0,t_0]$ and $j \in \mathbb{Z}$. Without loss of generality, suppose that $W_{j_0}(t_0) = 0$, then we obtain

$$0 = W_{j_0}(t_0) \geq \mathrm{e}^{-2\alpha t_0} W_{j_0}(0) + \beta \int_0^{t_0} \mathrm{e}^{-2\alpha(t_0-s)} \left(Q_{j_0}(s) + Q_{j_0+1}(s)\right) \mathrm{d}s \geq 0,$$

from which we infer that $W_{j_0}(0) = 0$ and $Q_{j_0}(s) = Q_{j_0+1}(s) = 0$ for all $s \in [0,t_0]$. As a consequence, we deduce that $W_{j_0}(s) = 0$ for all $s \in [0,t_0]$. Now, using the equation satisfied by Q_{j_0} and Q_{j_0+1}, we infer that $W_{j_0-1}(s) = 0$ and $W_{j_0+1}(s) = 0$ for all $s \in [0,t_0]$. As a consequence, by induction, we get that $W_j(t) = 0$ and $Q_j(t) = 0$ for all $t \in [0,t_0]$ and $j \in \mathbb{Z}$, which concludes the proof. ∎

Proposition B.6. *Let $\mathbf{W} = (W_j)_{j\in\mathbb{Z}}$ and $\mathbf{Q} = (Q_j)_{j\in\mathbb{Z}}$ with $W_j, Q_j \in \mathscr{C}^1([0,+\infty),\mathbb{R})$ for all $j \in \mathbb{Z}$ which satisfy*

$$\forall t > 0,\ \zeta(t) \leq j \leq \xi(t), \quad \begin{cases} W_j'(t) \geq -2\alpha W_j(t) + \beta(Q_j(t) + Q_{j+1}(t)), \\ Q_j'(t) \geq c_j(t) Q_j(t) + \alpha(W_j(t) + W_{j-1}(t)), \end{cases}$$

with some $\mathbf{c} = (c_j)_{j\in\mathbb{Z}} \in L^\infty(\mathbb{R}_+, \ell^\infty(\mathbb{Z}))$ and continuous functions $\zeta : \mathbb{R}_+ \to \mathbb{R}$ and $\xi : \mathbb{R}_+ \to \mathbb{R}$. Assume that $W_j(0) \geq 0$ and $Q_j(0) \geq 0$ for all $\zeta(0) - 1 \leq j \leq \xi(0) + 1$ together with $W_j(t) \geq 0$ and $Q_j(t) \geq 0$ for all $t > 0$ and $j \in [\zeta(t) - 1, \zeta(t)) \cup (\xi(t), \xi(t) + 1]$, then we have $W_j(t) \geq 0$ and $Q_j(t) \geq 0$ for all $t > 0$ and $\zeta(t) \leq j \leq \xi(t)$.

Proof. Fix $T > 0$. By assumption on the sequence $\mathbf{c}$, there exists $K > 0$ such that

$$K - c_j(t) > 0, \text{ for } t \in (0,T] \text{ and } j \in \mathbb{Z}.$$

Next, let us define

$$\begin{cases} w_j^\epsilon(t) := \mathrm{e}^{-Kt} W_j(t) + \epsilon \mathrm{e}^{\rho t}, \\ q_j^\epsilon(t) := \mathrm{e}^{-Kt} Q_j(t) + \epsilon \mathrm{e}^{\rho t}, \end{cases}$$

for $\epsilon > 0$ and

$$\rho > 2\max(\alpha, \beta - \alpha).$$

As a consequence $(w_j^\epsilon, q_j^\epsilon)$ satisfies all the assumptions with strict inequalities and we can argue as in the previous proof. Without loss of generality, we assume by contradiction that there exists $t_0 \in (0,T]$ and $j_0 \in [\zeta(t_0), \xi(t_0)] \cap \mathbb{Z}$ such that $w_{j_0}^\epsilon(t_0) = 0$ while $w_j^\epsilon(t) \geq 0$ and $q_j^\epsilon(t) \geq 0$ for all $t \in (0, t_0]$ and $\zeta(t) \leq j \leq \xi(t)$. Then by definition, we also have that ${w_{j_0}^\epsilon}'(t_0) \leq 0$ such that

$$0 \geq {w_{j_0}^\epsilon}'(t_0) + 2\alpha w_{j_0}^\epsilon(t_0) - \beta(q_{j_0}^\epsilon(t_0) + q_{j_0+1}^\epsilon(t_0)) > 0,$$

which is a contradiction. In the above inequality, we crucially used our assumption that $w_j^\epsilon(t) > 0$ for $j \in (\xi(t), \xi(t)+1]$. We can then pass to the limit $\epsilon \to 0$ and conclude the proof. ∎

Proposition B.7. *Let* $(\underline{\mathbf{W}}, \underline{\mathbf{Q}})$ *and* $(\overline{\mathbf{W}}, \overline{\mathbf{Q}})$ *with* $\underline{W}_j, \underline{Q}_j, \overline{W}_j, \overline{Q}_j \in \mathscr{C}^1([0,+\infty), \mathbb{R})$ *for all* $j \in \mathbb{Z}$ *be respectively subsolution and supersolution of the asymptotic system* (6.6) *for* $t > 0$ *and* $j \in [\zeta(t), \xi(t)] \cap \mathbb{Z}$ *for some continuous functions* $\zeta : \mathbb{R}_+ \to \mathbb{R}$ *and* $\xi : \mathbb{R}_+ \to \mathbb{R}$*. Assume that* $(\underline{W}_j(0), \underline{Q}_j(0)) \leq (\overline{W}_j(0), \overline{Q}_j(0))$ *for all* $\zeta(0) - 1 \leq j \leq \xi(0) + 1$ *together with* $(\underline{W}_j(t), \underline{Q}_j(t)) \leq (\overline{W}_j(t), \overline{Q}_j(t))$ *for all* $t > 0$ *and* $j \in [\zeta(t) - 1, \zeta(t)) \cup (\xi(t), \xi(t) + 1]$*, then we have* $(\underline{W}_j(t), \underline{Q}_j(t)) \leq (\overline{W}_j(t), \overline{Q}_j(t))$ *for all* $t > 0$ *and* $\zeta(t) \leq j \leq \xi(t)$*.*

Proof. The proof is a direct consequence of the previous proposition. ∎

References

[1] H. Berestycki, J.-M. Roquejoffre and L. Rossi. The influence of a line of fast diffusion in Fisher-KPP propagation. *J Math Biol*, 66 (2013), 743–766.

[2] H. Berestycki, J. M. Roquejoffre and L. Rossi. Travelling waves, spreading and extinction for Fisher-KPP propagation driven by a line with fast diffusion. *Nonlinear Analysis*, 137, (2016) 171–189.

[3] C. Besse and G. Faye. Dynamics of epidemic spreading on connected graphs. *J. Math. Biol*, 82 (2021), 1–52.

[4] C. Besse and G. Faye. Spreading properties for SIR models on homogeneous trees. *Bull. Math. Biol*, 83:114 (2021) , pp. 1–27 .

[5] C. Besse, G. Faye, J.-M. Roquejoffre and M. Zhang. The logarithmic Bramson correction for Fisher-KPP equations on the lattice $\mathbb{Z}$. *Trans. Am. Math. Soc.* 376 (2023), pp. 8553-8619.

[6] P.C. Bressloff. Local accumulation time for diffusion in cells with gap junction coupling. *Phys. Rev. E*, 105(3), (2022) 034404.

[7] Y. Du, B. Lou, R. Peng and M. Zhou. The Fisher-KPP equation over simple graphs: varied persistence states in river networks. *J. Math. Biol.*, 80(5), (2020) 1559-1616.

[8] W. T. L. Fan, W. Hu and G. Terlov. Wave propagation for reaction-diffusion equations on infinite random trees. *Commun. Math. Phys.*, 384(1), (2021), 109–163.

[9] N. Faria, A. Rambaut, M. Suchard, G. Baele, T. Bedford, M. Ward, A. Tatem, J. Sousa, N. Arinaminpathy, J. Pepin, D. Posada, M. Peeters, O. Pybus and P. Lemey. HIV epidemiology. the early spread and epidemic ignition of hiv-1 in human populations. *Science*, 346 (2014), 56–61.

[10] M. Gatto, E. Bertuzzo, L. Mari, S. Miccoli, L. Carraro, R. Casagrandi and A. Rinaldo. Spread and dynamics of the COVID-19 epidemic in Italy: effects of emergency containment measures. *Proc. Nat. Acad. Sci.*, (2020), 10484–10491.

[11] J. Gou and M.J. Ward. Oscillatory dynamics for a coupled membrane-bulk diffusion model with Fitzhugh-Nagumo membrane kinetics. *SIAM J Appl Math*, 76(2) (2016), 776–804.

[12] J. Gou, Y.X. Li, W. Nagata and M.J. Ward. Synchronized oscillatory dynamics for a 1-Dmodel ofmembrane kinetics coupled by linear bulk diffusion. *SIAM J Appl Dyn Syst*, 14(4) (2015), 2096–2137.

[13] A. Hoffman and M. Holzer. Invasion fronts on graphs: the Fisher-KPP equation on homogeneous trees and Erdős–Réyni graphs. *Discrete Contin Dyn Syst B*, 24(2) (2019), 671.

[14] C.M. Hale, C. Thomas, et al.. Spatiotemporal heterogeneity in the distribution of chikungunya and Zika virus case incidences during their 2014 to 2016 epidemics in Barranquilla, Colombia. *Int. J. Environ. Res. Public Health*, 16 (2019), 1759.

[15] Y. Jin, R. Peng and J. Shi. Population dynamics in river networks. *J. Nonlinear Sci.*, 29(6), (2019), 2501-2545.

[16] J. P. Keener and J. Sneyd. Mathematical Physiology I: Cellular Physiology. *Springer, New York,* (2009) 2nd edition.

[17] H. Kravitz, C. Durón and M. Brio. A Coupled Spatial-Network Model: A Mathematical Framework for Applications in Epidemiology. *Bull. Math. Biol.* 86.11 (2024): 132.

[18] F. Paquin-Lefebvre, W. Nagata and M.J. Ward. Weakly nonlinear theory for oscillatory dynamics in a one-dimensional PDE-ODE model of membrane dynamics coupled by a bulk diffusion field. *SIAM J Appl Math*, 80(3) (2020), 1520–1545.

[19] O. A Ladyzhenskaja, V.A. Solonnikov and N.N. Ural'ceva. Linear and Quasi-linear Equations of Parabolic Type. *American Mathematical Soc.*, vol 23, (1968).

[20] M. H. Protter and H.F. Weinberge. Maximum principles in differential equations. *Springer Science & Business Media* (2012).

[21] S. V. Ramanan and P. R. Brink. Exact solution of a model of diffusion in an infinite chain or monlolayer of cells coupled by gap junctions. *Biophys. J.* 58, 631 (1990).

[22] H. Weinberger. Long-time behavior of a class of biological models. *SIAM J Math Anal*, 13(3) (1982), 353–396.