Bifurcation to locked fronts in two component reaction-diffusion systems

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Abstract

We study invasion fronts and spreading speeds in two component reaction-diffusion systems. Using Lin's method, we construct traveling front solutions and show the existence of a bifurcation to locked fronts where both components invade at the same speed. Expansions of the wave speed as a function of the diffusion constant of one species are obtained. The bifurcation can be sub or super-critical depending on whether the locked fronts exist for parameter values above or below the bifurcation value. Interestingly, in the sub-critical case the spreading speed of the system does not depend continuously on the coefficient of diffusion.

Keywords: invasion fronts, spreading speeds, Lin's method

1 Introduction

We study invasion fronts for general systems of reaction-diffusion equations,

$$\begin{aligned} u_t &= u_{xx} + F(u,v), \\ v_t &= \sigma v_{xx} + G(u,v), \end{aligned} \tag{1.1}$$

where $\sigma > 0$ and $x \in \mathbb{R}$. More specifically, we are interested in traveling wave solutions of the form $(u(x-st), v(x-st))$ which satisfy

$$\begin{aligned} -su' &= u'' + F(u,v), \\ -sv' &= \sigma v'' + G(u,v), \end{aligned}$$

where we have set $\xi = x - st$ and used the notation u' for $\dfrac{\mathrm{d}u}{\mathrm{d}\xi}$ and u'' for $\dfrac{\mathrm{d}^2u}{\mathrm{d}\xi^2}$. It will be more convenient to write this system as a first-order system

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= -su_2 - F(u_1,v_1), \\ v_1' &= v_2, \\ \sigma v_2' &= -sv_2 - G(u_1,v_1). \end{aligned} \tag{1.2}$$

Throughout this paper, the reaction terms are assumed to have the form,

$$F(u,v) = uf(u,v), \quad G(u,v) = vg(u,v), \quad \text{ with } f(0,0) > 0 \text{ and } g(0,0) > 0. \tag{1.3}$$

Precise assumptions regarding the functions $F(u,v)$ and $G(u,v)$ are listed in Section 2. We sketch those assumptions now to better set the stage:

(H1) System (1.1) has three nonnegative homogeneous steady states: $\mathbf{p}_0 = (0,0)$, $\mathbf{p}_1 = (u^+,0)$ and $\mathbf{p}_2 = (u^*,v^*)$ and the associated traveling wave equation (1.2) has three corresponding fixed points $\mathbf{P}_0 = (0,0,0,0)$, $\mathbf{P}_1 = (u^+,0,0,0)$ and $\mathbf{P}_2 = (u^*,0,v^*,0)$.

(H2) We assume an ordering of the eigenvalues for the linearization of the traveling wave equation near $\mathbf{P}_0$ and $\mathbf{P}_1$ together with a non-resonance condition.

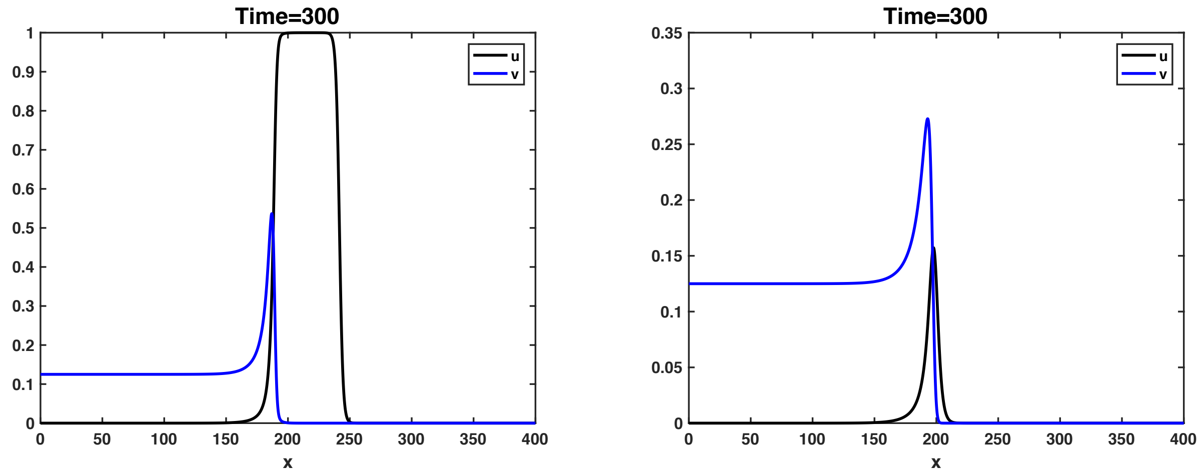
(H3) There exists a pushed front $(U_p(x-s^*t),0)$ that propagates to the right with speed s^* and leaves the homogeneous state $\mathbf{p}_1$ in its wake.

(H4) There exists a $\sigma^* > 0$ such that the linearization of the v component about the pushed front has marginally stable spectrum at $\sigma = \sigma^*$. If $\sigma < \sigma^*$, then small perturbations of the front $(U_p(x-s^*t),0)$ in the v component propagate slower than s^* whereas for $\sigma > \sigma^*$ these perturbations spread faster than s^*.

(H5) There is a family of traveling front solutions connecting $\mathbf{p}_2$ to $\mathbf{p}_1$ for all wave speeds s near s^*. These fronts have weak exponential decay representing the fact that the invasion speed of $\mathbf{p}_2$ into $\mathbf{p}_1$ is slower than s^*.

One can think of u and v as representing independent species that diffuse through space and interact through the reaction terms $F(u,v)$ and $G(u,v)$. When σ is small, we expect the spreading speed of the u component to exceed that of the v component. The dynamics in this regime is that of a staged invasion process: the zero state is first invaded by the u component, then at some later time is subsequently invaded by the v component, see Figure 1(a). As σ is increased, the speed of this secondary front will increase until eventually the two fronts lock and form a coherent coexistence front where the zero state is invaded by the stable state $\mathbf{p}_2$, see Figure 1(b). Broadly speaking, this transition to locking is the phenomena that we are concerned with in this article. Our primary goal is to determine parameter values for which this onset to locking is to be expected and whether the speed of the combined front is faster or slower than the speed of the individual fronts.

Our main result is the existence of a bifurcation leading to locked fronts occurring at the parameter values $(\sigma,s) = (\sigma^*,s^*)$. Depending on properties of the reaction terms the bifurcation will occur either for $\sigma > \sigma^*$ (super-critical) or for $\sigma < \sigma^*$ (sub-critical). In the super-critical case, the coexistence front does not appear until after the bifurcation at σ^* and the speed of the locked front changes continuously following the bifurcation – varying quadratically in a neighborhood of the bifurcation point (see Figure 2 for an illustration). The dynamics of the system in the sub-critical case are much different. In this scenario, the system transitions from a staged invasion process to locked fronts at a value of σ strictly less than the critical value σ^* and the spreading speed at this point is not continuous as a function of σ and we refer to Figure 3 for an illustration.



(a) Staged invasion fronts.

(b) Locked fronts.

Figure 1: *Profiles of the solutions of ([1.1](#)), evaluated at time* $t = 300$*, with nonlinear terms* $f(u, v) = (1 - u)(u + 1/16) - v$ *and* $g(u, v) = 2u(1 - u) + 1/8 - v$ *for different values of* σ*. (a) We observe a staged invasion process where the zero state is first invaded by the* u *component, then at some later time is subsequently invaded by the* v *component. Here we have set* $\sigma = 0.25$*. (b) We observe locked fronts with both components traveling at the same wave speed. Here we have set* $\sigma = 0.3$*.*

We employ a dynamical systems approach and construct these traveling fronts as heteroclinic orbits of the corresponding traveling wave equation (1.2). The traveling front solutions that we are interested in lie near a concatenation of traveling front solutions: the first being the pushed front connecting $\mathbf{P}_0$ to $\mathbf{P}_1$ (see **(H3)**) and the second connecting this intermediate state $\mathbf{P}_1$ to the stable coexistence state $\mathbf{P}_2$ (see **(H5)**). A powerful technique to constructing solutions near heteroclinic chains is Lin's method [14, 16, 17]. In this approach, perturbed solutions are obtained by variation of constants and these perturbed solutions are matched via Liapunov-Schmidt reduction leading to a system of bifurcation equations. Two common assumptions when using these techniques are a) that the dimensions of the stable and unstable manifolds of each fixed point in the chain are equal and b) the sum of tangent spaces of the intersecting unstable and stable manifolds have co-dimension one. Neither of these assumptions hold in our case. As fixed points of the traveling wave equation the stable coexistence state $\mathbf{P}_2$ has two unstable eigenvalues and two stable eigenvalues, the intermediate saddle state $\mathbf{P}_1$ has three stable eigenvalues and one unstable eigenvalues and the unstable zero state $\mathbf{P}_0$ has four stable eigenvalues. Restricting to fronts with strong exponential decay, the zero state can be thought of as having a two-two splitting of eigenvalues, but no such reduction is possible for the intermediate state.

One interesting phenomena that we observe is a discontinuity of the spreading speed as a function of σ in the sub-critical regime. The discontinuous nature of spreading speeds with respect to system parameters has been observed previously, see for example [6, 8, 9, 11]. However, the discontinuity in those cases is typically observed as a parameter is altered from zero to some non-zero value representing the onset of coupling of some previously uncoupled modes. The mechanism here appears to be different.

There is a large literature pertaining to traveling fronts in systems of reaction-diffusion equations. Directly related to the work here is [10], where system (1.1) is studied under the assumption that the second component is decoupled from the first, i.e. that $g(u,v) = g(v)$. Further assuming that the system obeys a comparison principle, precise statements regarding the evolution of compactly supported initial data can be made; see also [2]. Here, we do not assume monotonicity and therefore a dynamical system approach is required. A similar approach is used in [10], however, the decoupling of the v component reduces the traveling wave equation to a three dimensional system.

The present work is also partially motivated by recent studies of bacterial invasion fronts similar to [13]. In this context, the u component can be thought of as a bacterial population of cooperators while the v component are defectors. In a well mixed population the defectors out compete the cooperators. However, in a spatially extended system the cooperators may persist via spatial movement by outrunning the defectors. This depends on the relative diffusivities, where for σ small the cooperators are able to escape. However, for σ sufficiently large the defector front is sufficiently fast to lock with the cooperator front and slow its invasion. Our result characterizes how this locking may take place. See also [22, 23] for similar systems of equations.

Finally, we comment on the relationship between the spreading speed of the system and the existence of invasion fronts. Spreading speed is typically used to refer to the asymptotic speed of invasion of compactly supported perturbations of an unstable state, see [1]. For monotone systems, it is often the case that spreading speeds can be established rigorously. These properties have been used to great effect; primarily in scalar equations but also for monotone systems of equations. Many systems, including the ones considered here, lack a comparison structure and consequently it becomes extremely difficult to rigorously establish spreading speeds in the traditional sense. In such cases, marginal stability is typically used as a selection criterion, see [4, 21]. Marginal stability requires the existence of fronts with steep exponential decay, which are the fronts that we construct here.

We now proceed to outline our assumptions in more detail and state our main result.

2 Set up and statement of main results

In this section, we specify the precise assumptions required of (1.1) and state our main result. We first make some assumptions on the reaction terms $F(u,v)$ and $G(u,v)$ that have the specific form defined in (1.3).

Hypothesis (H1) *Assume that that system*

$$\begin{aligned} u_t &= F(u,v), \\ v_t &= G(u,v), \end{aligned}$$

has three non-negative equilibrium points which we denote by $\mathbf{p}_0 = (0,0)$, $\mathbf{p}_1 = (u^+,0)$ and $\mathbf{p}_2 = (u^*,v^*)$ for some $u^* \geq 0$ and $v^* > 0$. Assume that $\mathbf{p}_0$ is an unstable node, $\mathbf{p}_1$ is a saddle with one stable direction in the $v = 0$ coordinate axis and an unstable direction transverse to this axis, while p_2 is a stable node.

The traveling wave equation (1.2) naturally inherits equilibrium points from the homogeneous equation which we denote as $\mathbf{P}_0 = (0,0,0,0)$, $\mathbf{P}_1 = (u^+,0,0,0)$ and $\mathbf{P}_2 = (u^*,0,v^*,0)$. At either the fixed point $\mathbf{P}_0$

or $\mathbf{P}_1$, the linearization is block triangular and eigenvalues can be computed explicitly. At $\mathbf{P}_0$, the four eigenvalues are

$$
\begin{aligned}
\mu_u^{\pm}(s) &= -\frac{s}{2} \pm \frac{1}{2}\sqrt{s^2 - 4f(\mathbf{p}_0)}, \\
\mu_v^{\pm}(s,\sigma) &= -\frac{s}{2\sigma} \pm \frac{1}{2\sigma}\sqrt{s^2 - 4\sigma g(\mathbf{p}_0)},
\end{aligned}
$$

where we used the fact that $F_u(\mathbf{p}_0) = f(\mathbf{p}_0)$ and $G_u(\mathbf{p}_0) = g(\mathbf{p}_0)$. Similarly, at $\mathbf{P}_1$, the linearization has eigenvalues

$$
\begin{aligned}
\nu_u^{\pm}(s) &= -\frac{s}{2} \pm \frac{1}{2}\sqrt{s^2 - 4F_u(\mathbf{p}_1)}, \\
\nu_v^{\pm}(s,\sigma) &= -\frac{s}{2\sigma} \pm \frac{1}{2\sigma}\sqrt{s^2 - 4\sigma g(\mathbf{p}_1)},
\end{aligned}
$$

where once again we used the fact that $G_v(\mathbf{p}_1) = g(\mathbf{p}_1)$.

Hypothesis (H2) *The eigenvalues of the linearization of (1.2) at $\mathbf{P}_0$ has four unstable eigenvalues and we assume that there exists an $\alpha > 0$ such that*

$$\mu_u^-(s) < -\alpha < \mu_u^+(s), \quad \mu_v^-(s,\sigma) < -\alpha < \mu_v^+(s,\sigma). \tag{2.1}$$

For the eigenvalues of the linearization at $\mathbf{P}_1$ we assume that they can be ordered

$$\nu_v^-(s,\sigma) < \nu_u^-(s) < \nu_v^+(s,\sigma) < 0 < \nu_u^+(s). \tag{2.2}$$

In addition, we assume the following non-resonance condition:

$$\nu_u^-(s) < 2\nu_v^+(s,\sigma). \tag{2.3}$$

When the v component is identically zero, system (1.1) reduces to a scalar reaction-diffusion equation

$$u_t = u_{xx} + F(u,0), \tag{2.4}$$

and the traveling wave equation (1.2) reduces to the planar system

$$
\begin{aligned}
u_1' &= u_2, \\
u_2' &= -su_2 - F(u_1,0).
\end{aligned}
$$

We now list assumptions related to traveling front solutions of (2.4).

Hypothesis (H3) *We assume that there exists $s^* > 2\sqrt{f(\mathbf{p}_0)}$ for which (2.4) has a pushed front solution $U_p(x-s^*t)$ moving to the right with speed s^*. By pushed front, we mean that the solution as steep exponential decay $U_p(\xi) \sim Ce^{\mu_u^-(s^*)\xi}$ as $\xi \to \infty$ and has stable spectrum in the weighted space $L^2_\alpha(\mathbb{R})$ with the exception of an eigenvalue at zero due to translational invariance. There is, in fact, a one parameter family of translates of these fronts and we therefore impose that $U_p''(0) = 0$ and restrict to one element of the family.*

Now consider the linearization of the v component of (1.1) around the traveling front solution $U_p(x - s^*t)$,

$$\mathcal{L}_v := \sigma\partial_{\xi\xi} + s^*\partial_\xi + g(U_p(\xi), 0).$$

The spectrum of this operator posed on $L^2(\mathbb{R})$ is unstable due to the instability of the asymptotic rest states. However, this spectrum may be stable when $\mathcal{L}_v$ is viewed as an operator on

$$L^2_d(\mathbb{R}) = \left\{ \phi(\xi) \in L^2(\mathbb{R}) \mid \phi(\xi)e^{d\xi} \in L^2(\mathbb{R}) \right\}.$$

Let $d = \frac{s^*}{2\sigma}$. The operator $\mathcal{L}_v = \sigma\partial_{\xi\xi} + s^*\partial_\xi + g(U_p(\xi), 0)$ restricted to L^2_d is isomorphic to the operator $H_\sigma : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, where

$$H_\sigma := \sigma\partial_{\xi\xi} + \left(-\frac{(s^*)^2}{4\sigma} + g(U_p(\xi), 0) \right).$$

We now state our assumptions on the spectrum of H_σ.

Hypothesis (H4) *We suppose that the most unstable spectra of H_σ is point spectra and define*

$$\lambda(\sigma) = \sup_{\omega \in \mathrm{spec}(H_\sigma)} \omega.$$

Let σ^* be defined such that $\lambda(\sigma^*) = 0$. Associated to this eigenvalue is a bounded eigenfunction which we denote $\tilde{\phi}(\xi)$. In the unweighted space, this eigenfunction becomes $\phi(\xi) = e^{-\frac{s^*}{2\sigma^*}\xi}\tilde{\phi}(\xi)$ which is unbounded as $\xi \to -\infty$.

The final set of assumptions pertain to the existence and character of traveling front solutions connecting $\mathbf{P}_2$ to $\mathbf{P}_1$.

Hypothesis (H5) *We assume that there exists a family of traveling front solutions of (1.2) that connect the steady state at $\mathbf{P}_2$ to the steady state at $\mathbf{P}_1$. We further assume that these fronts have generic decay and the traveling front approaches the steady state $\mathbf{P}_1$ along the weak-stable eigendirection. Furthermore, for each fixed s and σ, the two dimensional tangent space to $W^u(\mathbf{P}_0)$ approaches the origin tangent to the unstable/weak-stable eigenspace. We note that these fronts therefore have exponential decay rate $Ce^{\nu_v^+(s,\sigma)\xi}$ as $\xi \to \infty$.*

Remarks on assumptions (H1) – (H5). The assumptions listed above form the setting of a staged invasion process. Localized perturbations of the homogeneous zero state will grow and form a traveling front. We are interested in the speed of that front. Assumption (**H3**) implies that that u component in isolation will form a traveling front propagating with speed s^* and will leave in its wake the steady state $\mathbf{p}_1$. Assumption (**H5**) implies that localized perturbations of $\mathbf{p}_1$ will form a traveling front with secondary speed strictly less than s^*. We are interested in detailing situations where the interaction of these two fronts could lead to a locked front where the zero state is invaded by a single traveling front replacing $\mathbf{p}_0$ with the stable state $\mathbf{p}_2$. This is where assumption (**H4**) enters the picture. The location of $\lambda(\sigma)$ determines the pointwise exponential behavior of solutions of the v component in the linearization about the primary front. If for some value of σ, we have that $\lambda(\sigma) < 0$, then perturbations are decaying pointwise exponentially fast; whereas if for some value of $\lambda(\sigma) > 0$, then perturbations are growing pointwise at an exponential rate. The critical value of σ^* is exactly where perturbations in the v component are traveling at exactly the speed of the pushed front itself.

We remark that $(\mathbf{H1}) - (\mathbf{H2})$ are straightforward to verify for a specific choice of $F(u,v)$ and $G(u,v)$. Assumption $(\mathbf{H3})$ is more challenging, but due to the planar nature of the traveling wave equation it is plausible that such a condition could be checked in practice. We refer the reader to [15] for a general variational method suited to such problems. Assumption $(\mathbf{H4})$ is yet more challenging to verify, however as a Sturm-Liouville operator there are many results in the literature pertaining to qualitative features of the spectrum of these operators. Finally, assumption $(\mathbf{H5})$ is the most difficult to verify in practice, as it requires a rather complete analysis of a fully four dimensional system of differential equations (1.2). Nonetheless, our assumptions there simply state that the traveling front solutions have the most generic behavior possible as heteroclinic orbits between $\mathbf{P}_2$ and $\mathbf{P}_1$. In this sense, we argue that assumption $(\mathbf{H5})$ is not so extreme, in spite of the challenge presented in actually verifying that it would hold in specific examples.

We also remark that the precise ordering of the eigenvalues assumed in $(\mathbf{H2})$ are technical assumptions and could likely be relaxed in some cases.

Main Result. We can now state our main result.

Theorem 1. *Consider (1.1) and assume that Hypotheses* **(H1)-(H5)** *hold. Then there exists a constant* $M_\rho \neq 0$ *such that:*

- *(sub-critical) if* $M_\rho < 0$ *then there exists* $\delta > 0$ *such that there exists positive traveling front solutions* $(U(x - s(\sigma)t), V(x - s(\sigma)t))$ *for any* $\sigma^* - \delta < \sigma < \sigma^*$ *with speed*

$$s(\sigma) = s^* + M_s(\sigma - \sigma^*)^2 + \mathcal{O}(3);$$

- *(super-critical) if* $M_\rho > 0$ *then there exists* $\delta > 0$ *such that there exists positive traveling front solutions* $(U(x - s(\sigma)t), V(x - s(\sigma)t))$ *for any* $\sigma^* < \sigma < \sigma^* + \delta$ *with speed*

$$s(\sigma) = s^* + M_s(\sigma - \sigma^*)^2 + \mathcal{O}(3).$$

We make several remarks.

Remark 2. *As part of the proof of Theorem 1 we obtain expressions for* M_ρ *and* M_σ*. In particular,*

$$\begin{aligned}\operatorname{sign}(M_\rho) &= \operatorname{sign}\left(-r_2 \int_{\xi_0}^{\infty} e^{\frac{s^*}{\sigma^*}\xi}\left(\frac{G_{uv}(U_p(\xi),0)}{\sigma^*} a_1(\xi)\phi(\xi)^2 + \frac{G_{vv}(U_p(\xi),0)}{2\sigma^*}\phi^3(\xi)\right) \mathrm{d}\xi \right. \\ &- \left. r_1\left(\tilde{\phi}''(\xi_0)\tilde{\phi}(\xi_0) - (\tilde{\phi}'(\xi_0))^2\right) + \frac{1}{r_2} e^{\frac{s^*}{\sigma^*}\xi_0}\gamma^{(2)}(s^*,\sigma^*)\left(\nu_v^-(s^*,\sigma^*)\phi(\xi_0) - \phi'(\xi_0)\right)\right),\end{aligned}$$

where $r_{1,2}$*,* $a_1(\xi)$ *and* $\gamma^{(2)}(s^*,\sigma^*)$ *are all defined below. A similar expression holds for* M_σ*, but is quite complicated.*

Remark 3. *We comment on the sub-critical case. Our analysis holds only in a neighborhood of the bifurcation point. However, we expect that this curve could be followed in* (σ, s) *parameter space to a* *saddle-node bifurcation where the curve would subsequently reverse direction with respect to σ. This curve can be found numerically using numerical continuation methods, see Figure 3. These numerics reveal two branches of fronts that appear via a saddle node bifurcation. It is the lower branch of solutions that we expect to be marginally stable and reflects the invasion speed of the system.*

For systems of equations without a comparison principle, the selected front is classified as the marginal stable front, see [4, 21]. It is interesting to note that in these examples there appear to be two marginally (spectrally) stable fronts – the original front $(U_p(x-s^*t),0)$ and the coexistence front – and the full system selects the slower of these two fronts.

We now comment on the strategy of the proof that employs a variation of Lin's method. The traveling fronts that we seek are heteroclinic orbits in the traveling wave equations connecting $\mathbf{P}_2$ to $\mathbf{P}_0$. We further require that these fronts have strong exponential decay in a neighborhood of $\mathbf{P}_0$. As such, these traveling fronts belong to the intersection of the unstable manifold $W^u(\mathbf{P}_2)$ and the strong stable manifold $W^{ss}(\mathbf{P}_0)$. Therefore, the goal is to track $W^{ss}(\mathbf{P}_0)$ backwards along the pushed front heteroclinic $(U_p(\xi),U_p'(\xi),0,0)^T$ to a neighborhood of $\mathbf{P}_1$. The dependence of this manifold on the parameters s and σ can be characterized using Melnikov type integrals and the manifold can be expressed as a graph over the strong stable tangent space. To track $W^u(\mathbf{P}_0)$ forwards we use **(H5)** to get an expression for this manifold as it enters a neighborhood of $\mathbf{P}_1$. To track this manifold past the fixed point requires a Shilnikov type analysis near $\mathbf{P}_1$. Finally, we compare the two manifolds near a common point on the heteroclinic $(U_p(\xi),U_p'(\xi),0,0)^T$ and following a Liapunov-Schmidt reduction we obtain the required expansions of s as a function of σ.

Numerical illustration of the main result. Before proceeding to the proof of Theorem 1, we illustrate the result on an example. We consider the following nonlinear functions $f_\epsilon(u,v)$ and $g(u,v)$ that lead to a supercritical bifurcation when $\epsilon=1$ and exhibit a sub-critical bifurcation for $\epsilon=-1$:

$$f_\epsilon(u,v)=(1-u)(u+a)+\epsilon v, \text{ and } \quad g(u,v)=2u(1-u)+2a-v, \tag{2.5}$$

where $\epsilon\in\{\pm1\}$. In both cases, when v is set to zero the system reduces to the scalar Nagumo's equation

$$u_t=u_{xx}+u(1-u)(u-a). \tag{2.6}$$

The dynamics of (2.6) are well understood, see for example [7]. For $a<1/2$, the system forms a pushed front propagating with speed $s^*=\sqrt{2}\left(\frac{1}{2}+a\right)$. For the numerical computations presented in both Figures 2 and 3, we have discretized (1.1) by the method of finite differences and used a semi-implicit scheme with time step $\delta t=0.05$ and space discretization $\delta x=0.05$ with $x\in[0,400]$ and imposed Neumann boundary conditions. All simulations are done from compactly initial data and the speed of each component was calculated by computing how much time elapsed between the solution passing a threshold at two separate points in the spatial domain. In Figure 2, we present the case of a super-critical bifurcation where locked fronts are shown to exist past the bifurcation point $\sigma=\sigma^*$. In Figure 3, we illustrate the case of a sub-critical bifurcation where locked fronts are shown to exist before the bifurcation point $\sigma=\sigma^*$. We observe a discontinuity of the wave speed as σ is increased. We then implemented a numerical continuation scheme to continue the wave speed of these locked fronts back to the bifurcation point $\sigma=\sigma^*$. In the process, we see a turning point for some value of σ near 0.273. We expect that locked fronts on this branch to be unstable as solutions of (1.1) which explains why one observes the lower branch of the bifurcation curve. It is interesting to note the relative good agreement between the wave speed obtained by numerical continuation and the wave speed obtained by direct numerical simulation of the system (1.1).

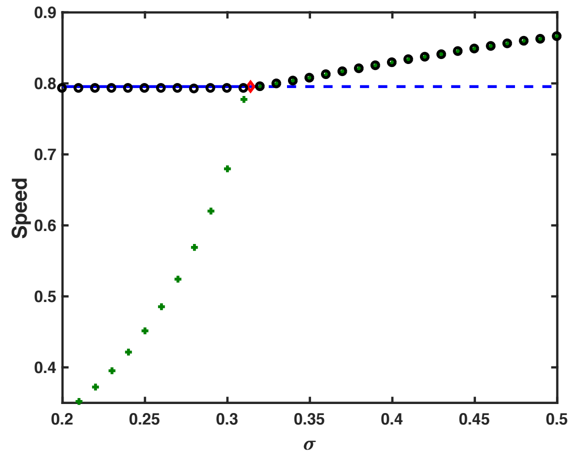


Figure 2: *Numerically computed wave speeds of the u-component, black circles, and of the v-component, green plus sign for* $\epsilon = 1$ *in* (2.5)*. The horizontal blue line* $s = s^* = \sqrt{2}(a + 1/2)$ *represents the sign of the associated principal eigenvalue of the operator* H_σ *and the red diamond indicates the critical value* σ^* *at which this principal eigenvalue vanishes. The solid part of the line indicates a negative principal eigenvalue while the dashed part indicates a positive one. Here,* $\sigma^* \simeq 0.314$*. For all numerical simulations we have set* $a = 1/16$*.*

Outline of the paper. In Section 3, we track the strong stable manifold $W^{ss}(\mathbf{P}_0)$ backwards and provide useful information on its tangent space. In the following Section 4, we track the unstable manifold $W^u(\mathbf{P}_2)$ forwards using Shilnikov Theorem get precise asymptotics passed the saddle point $\mathbf{P}_1$. Finally, in the last Section 5 we prove our main Theorem 1 by resolving the bifurcation equation when matching the strong stable manifold $W^{ss}(\mathbf{P}_0)$ with the unstable manifold $W^u(\mathbf{P}_2)$ in a neighborhood of $\mathbf{P}_1$. Some proofs and calculations are provided in the Appendix.

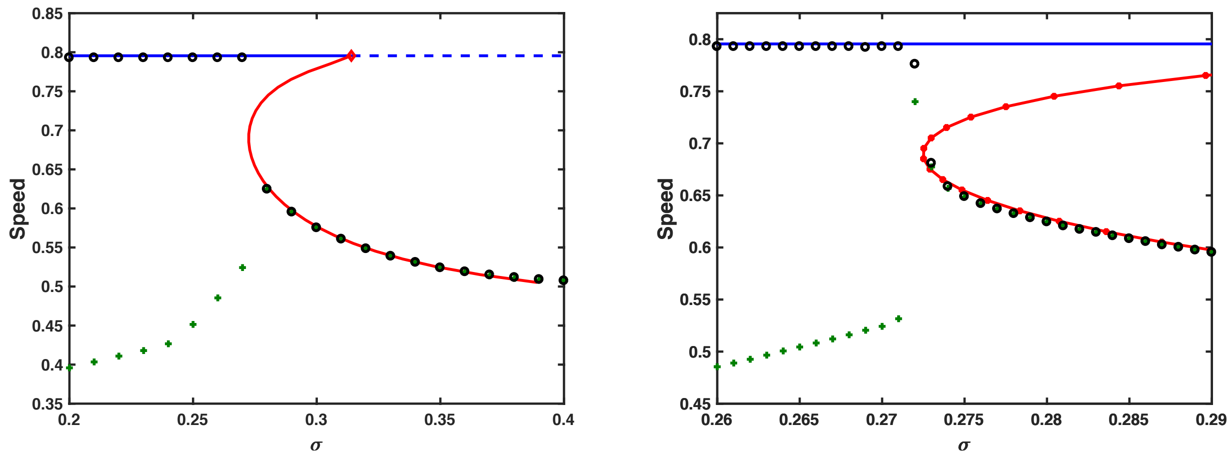
3 Tracking the strong stable manifold $W^{ss}(\mathbf{P}_0)$ backwards

In this section, we derive an expression for the strong stable manifold of the fixed point $\mathbf{P}_0$ near the fixed point $\mathbf{P}_1$. Recall that for $(s, \sigma) = (s^*, \sigma^*)$, there exists a heteroclinic orbit given by $(U_p(\xi), U_p'(\xi), 0, 0)^T$ that connects $\mathbf{P}_1$ to $\mathbf{P}_0$. By assumption $(H3)$, this orbit lies in the strong stable manifold. We will use this orbit to track the strong stable manifold back to a neighborhood of $\mathbf{P}_1$. Change variables via

$$\begin{aligned} u_1 &= U_p(\xi) + p_1 \\ u_2 &= U_p'(\xi) + p_2 \\ v_1 &= q_1 \\ v_2 &= q_2. \end{aligned}$$

Writing $z = (p_1, p_2, q_1, q_2)^T$, then we can express (1.2) as the non-autonomous system in compact form,

$$z' = A(\xi, s^*, \sigma^*)z + g(\xi, s) + N(\xi, z, s, \sigma), \tag{3.1}$$



(a) $\epsilon = -1$.

(b) Zoom of Figure (a) near $\sigma \sim 0.27$.

Figure 3: *(a) Numerically computed wave speeds of the u-component, black circles, and of the v-component, green plus sign for* $\epsilon = -1$ *in* (2.5). *We observe a discontinuity in the value of the measured wave speed as* σ *is varied indicating a sub-critical bifurcation of the locked fronts. The horizontal blue line* $s = s^* = \sqrt{2}(a + 1/2)$ *represents the sign of the associated principal eigenvalue of the operator* H_σ *and the red diamond indicates the critical value* σ^* *at which this principal eigenvalue vanishes. The solid part of the line indicates a negative principal eigenvalue while the dashed part indicates a positive one. Here,* $\sigma^* \simeq 0.314$. *The red curve is a continuation of the wave speed of locked fronts up to the bifurcation point* $\sigma = \sigma^*$. *(b) Refinement of Figure (a) near the fold point. Here, the red stars are wave speeds obtained by numerical continuation. For all numerical simulations we have set* $a = 1/16$.

where

$$A(\xi, s^*, \sigma^*) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -F_u(U_p(\xi), 0) & -s^* & -F_v(U_p(\xi), 0) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{G_v(U_p(\xi),0)}{\sigma^*} & -\frac{s^*}{\sigma^*} \end{pmatrix}, \tag{3.2}$$

and

$$g(\xi, s) = \begin{pmatrix} 0 \\ -(s - s^*)U_p'(\xi) \\ 0 \\ 0 \end{pmatrix}, \quad N(z, \xi, s, \sigma) = \begin{pmatrix} 0 \\ N_p(z, \xi, s, \sigma) \\ 0 \\ N_q(z, \xi, s, \sigma) \end{pmatrix} \tag{3.3}$$

with

$$\begin{aligned} N_p(z, \xi, s, \sigma) &= -(s - s^*)p_2 - \frac{F_{uu}(U_p(\xi), 0)}{2}p_1^2 - F_{uv}(U_p(\xi), 0)p_1 q_1 - \frac{F_{vv}(U_p(\xi), 0)}{2}q_1^2 + \mathcal{O}(3) \\ N_q(z, \xi, s, \sigma) &= -\frac{1}{\sigma^*}(s - s^*)q_2 + \frac{s^*}{(\sigma^*)^2}(\sigma - \sigma^*)q_2 - \frac{G_{uv}(U_p(\xi), 0)}{\sigma^*}p_1 q_1 \\ &\quad - \frac{G_{vv}(U_p(\xi), 0)}{2\sigma^*}q_1^2 + \frac{G_v(U_p(\xi), 0)}{(\sigma^*)^2}q_1(\sigma - \sigma^*) + \mathcal{O}(3). \end{aligned}$$

These expressions have been simplified by noting that $G_u(U_p(\xi), 0) = 0$ and $G_{uu}(U_p(\xi), 0) = 0$.

Lemma 4. *Let $\Phi(\xi,\tilde{\xi})$ be the fundamental matrix solution of*

$$z' = A(\xi, s^*, \sigma^*)z. \tag{3.4}$$

Then (3.4) has a generalized exponential dichotomy on $[\xi_0,\infty)$ with strong stable projection $\mathrm{P}^{ss}(\xi)$ and there exists a $K>0$ and $0<\gamma<\alpha$ for which

$$\begin{aligned} \left\|\Phi(\xi,\tilde{\xi})\mathrm{P}^{ss}(\tilde{\xi})\right\| &\leq Ke^{-\alpha(\xi-\tilde{\xi})} \text{ for } \xi>\tilde{\xi} \\ \left\|\Phi(\xi,\tilde{\xi})\left(\mathrm{Id}-\mathrm{P}^{ss}(\tilde{\xi})\right)\right\| &\leq Ke^{\gamma(\tilde{\xi}-\xi)} \text{ for } \xi<\tilde{\xi}. \end{aligned}$$

Proof. This is a standard result on exponential dichotomies, see for example [3]. Define $A_\infty(s^*,\sigma^*) = \lim_{\xi\to\infty} A(\xi,s^*,\sigma^*)$. Since the convergence is exponential and there is a gap between the strong stable and weak stable eigenvalues, see **(H2)**, the constant-coefficient asymptotic system has an exponential dichotomy and the non-autonomous system inherits one with the same decay rates . ∎

With the existence of an exponential dichotomy, we can express the strong stable manifold in the usual way as the fixed point of a variation-of-constants formula. In the following, we use the the notation

$$\Phi^{ss}(\xi,\xi_0) = \Phi(\xi,\xi_0)\mathrm{P}^{ss}(\xi_0), \quad \Phi^{ws}(\xi,\xi_0) = \Phi(\xi,\xi_0)\left(\mathrm{Id}-\mathrm{P}^{ss}(\xi_0)\right).$$

Lemma 5. *Let $\xi_0<0$ be arbitrary. Define*

$$S = \left\{\phi \in C^0_\beta([\xi_0,\infty),\mathbb{R}^4)\right\},$$

with norm $\|\phi\|_S = \sup_{\xi\in[\xi_0,\infty)} e^{\beta(\xi-\xi_0)}\|\phi(\xi)\|$ for $\gamma<\beta<\alpha$. Given $Y\in \mathrm{Rg}\left(\mathrm{P}^{ss}(\xi_0)\right)$, consider the operator T defined for all $\xi\geq\xi_0$ as

$$\begin{aligned} TQ(\xi) := \Phi^{ss}(\xi,\xi_0)Y &+ \int_{\xi_0}^{\xi} \Phi^{ss}(\xi,\tau)\left(g(\tau,s)+N(Q(\tau),\tau,s,\sigma)\right)\mathrm{d}\tau \\ &- \int_{\xi}^{\infty} \Phi^{ws}(\xi,\tau)\left(g(\tau,s)+N(Q(\tau),\tau,s,\sigma)\right)\mathrm{d}\tau. \end{aligned} \tag{3.5}$$

There exists an $r>0$ and a $c>0$ such that for any small $Y\in\mathrm{Rg}\left(\mathrm{P}^{ss}(\xi_0)\right)$ and all $(|s-s^*|+|\sigma-\sigma^*|)<c$ the operator T is a contraction mapping on $\mathrm{B}_r(0)\subset S$, where $\mathrm{B}_r(0)$ stands for the ball of radius r centered at $Q=0$ in S.

Proof. The proof is standard, but we include it since we will require some information regarding the value of the contraction constant. Note first that $\|g(\tau,s,\sigma)\| < C|s-s^*|e^{-\alpha\tau}$. Also, for r sufficiently small there exists positive constants $l(r), l_s$ and l_σ such that for any $\tau\in[\xi_0,\infty)$,

$$\|N(Q_1(\tau),\tau,s,\sigma)-N(Q_2(\tau),\tau,s,\sigma)\| \leq \left(l(r)+l_s|s-s^*|+l_\sigma|\sigma-\sigma^*|\right)\|Q_1(\tau)-Q_2(\tau)\|.$$

Note that $l(r)\to 0$ as $r\to 0$. For brevity, let

$$L(r,s,\sigma) = l(r)+l_s|s-s^*|+l_\sigma|\sigma-\sigma^*|.$$

Then

$$e^{\beta(\xi-\xi_0)}\|TQ(\xi)\| \leq Ke^{(\beta-\alpha)(\xi-\xi_0)}\|Y\| + e^{(\beta-\alpha)\xi}\int_{\xi_0}^{\xi} Ke^{-\alpha\tau-\beta\xi_0}\left(C|s-s^*|e^{-\alpha\tau} + L(r,s,\sigma)\|Q(\tau)\|\right)\mathrm{d}\tau$$
$$+ e^{(\beta-\gamma)\xi}\int_{\xi}^{\infty} Ke^{\gamma\tau-\beta\xi_0}\left(C|s-s^*|e^{-\alpha\tau} + L(r,s,\sigma)\|Q(\tau)\|\right)\mathrm{d}\tau.$$

Since $\beta-\alpha<0$ we obtain constants C_g, C_N such that

$$\|TQ\|_S \leq K\|Y\| + C_g|s-s^*| + L(r,s,\sigma)C_N\|Q\|_S, \tag{3.6}$$

and we observe that for $|s-s^*|$, $|\sigma-\sigma^*|$ and $\|Y\|$ sufficiently small the operator maps $T:\mathrm{B}_r(0)\to\mathrm{B}_r(0)$. For any fixed Y, we have

$$\begin{aligned}
e^{\beta(\xi-\xi_0)}\|TQ_1(\xi)-TQ_2(\xi)\| &\leq e^{\beta(\xi-\xi_0)}\int_{\xi_0}^{\xi}\Phi^{ss}(\xi,\tau)\|N(Q_1(\tau),\tau,s,\sigma)-N(Q_2(\tau),\tau,s,\sigma)\|\mathrm{d}\tau\\
&\quad+ e^{\beta(\xi-\xi_0)}\int_{\xi}^{\infty}\Phi^{ws}(\xi,\tau)\|N(Q_1(\tau),\tau,s,\sigma)-N(Q_2(\tau),\tau,s,\sigma)\|\mathrm{d}\tau\\
&\leq e^{\beta(\xi-\xi_0)}e^{-\alpha\xi}KL(r,s,\sigma)||Q_1-Q_2||_S\int_{\xi_0}^{\xi}e^{(\alpha-\beta)\tau}\mathrm{d}\tau\\
&\quad+ e^{\beta(\xi-\xi_0)}e^{-\gamma\xi}KL(r,s,\sigma)||Q_1-Q_2||_S\int_{\xi}^{\infty}e^{(\gamma-\beta)\tau}\mathrm{d}\tau.
\end{aligned}$$

Since $\gamma<\beta<\alpha$ the integrals converge and we obtain that T is a contraction for L sufficiently small, or equivalently for $r>0$ and $c>0$ sufficiently small. And for future reference, we denote by $\kappa(r,s,\sigma)$ the associated contraction constant so that

$$\|TQ_1-TQ_2\|_S \leq \kappa(r,s,\sigma)||Q_1-Q_2||_S.$$

∎

The strong stable manifold is therefore given as the fixed point of (3.5) and at ξ_0 this manifold can be expressed as a graph from $\mathrm{Rg}(\mathrm{P}^{ss}(\xi_0))$ to $\mathrm{Rg}(\mathrm{Id}-\mathrm{P}^{ss}(\xi_0))$. We now select coordinates. The range of the strong stable projection is spanned by the vectors

$$\theta_1 = \begin{pmatrix} U_p'(\xi_0)\\ U_p''(\xi_0)\\ 0\\ 0\end{pmatrix}, \quad \theta_2 = \begin{pmatrix} a_1(\xi_0)\\ a_2(\xi_0)\\ \phi(\xi_0)\\ \phi'(\xi_0)\end{pmatrix}, \tag{3.7}$$

where $\phi(\xi)$ is defined in $(H4)$ and $a_1(\xi)$ and $a_2(\xi)$ are solutions of

$$\begin{aligned}
a_1'(\xi) &= a_2(\xi)\\
a_2'(\xi) &= -F_u(U_p(\xi),0)a_1(\xi) - s^*a_2(\xi) - F_v(U_p(\xi),0)\phi(\xi).
\end{aligned}$$

The homogeneous equation has a pair of linearly independent solutions,

$$A_1(\xi) = U_p'(\xi), \quad A_2(\xi) = U_p'(\xi)\int_{\xi_0}^{\xi}\frac{e^{-s^*\tau}}{(U_p(\tau)')^2}\mathrm{d}\tau. \tag{3.8}$$

Note that $A_1(\xi) < 0$ and $A_2(\xi) < 0$ for $\xi > \xi_0$. A family of solutions with strong exponential decay as $\xi \to \infty$ is given by

$$\begin{aligned} a_1(\xi) = c_1 A_1(\xi) + A_1(\xi) \int_{\xi_0}^{\xi} e^{s^*\tau} A_2(\tau) F_v(U_p(\tau), 0)\phi(\tau) \mathrm{d}\tau \\ + A_2(\xi) \int_{\xi}^{\infty} e^{s^*\tau} A_1(\tau) F_v(U_p(\tau), 0)\phi(\tau) \mathrm{d}\tau. \end{aligned} \tag{3.9}$$

Then

$$\begin{aligned} a_2(\xi) = c_1 A_1'(\xi) + A_1'(\xi) \int_{\xi_0}^{\xi} e^{s^*\tau} A_2(\tau) F_v(U_p(\tau), 0)\phi(\tau) \mathrm{d}\tau \\ + A_2'(\xi) \int_{\xi}^{\infty} e^{s^*\tau} A_1(\tau) F_v(U_p(\tau), 0)\phi(\tau) \mathrm{d}\tau. \end{aligned}$$

We select c_1 so that θ_1 and θ_2 are orthogonal at ξ_0. This implies

$$c_1 \left((U_p'(\xi_0))^2 + (U_p''(\xi_0)^2) \right) = -\frac{U_p''(\xi_0)}{U_p'(\xi_0)} e^{-s^*\xi_0} \int_{\xi_0}^{\infty} e^{s^*\tau} A_1(\tau) F_v(U_p(\tau), 0)\phi(\tau) \mathrm{d}\tau.$$

We make several observations here that will be of importance later. First, the sign of c_1 depends on the value of $F_v(U_p(\xi), 0)$. If $F_v(U_p(\xi), 0)$ has one sign, then c_1 shares that sign. Second, if we set $\xi = \xi_0$ we observe that the integrand $e^{s^*\tau} A_1(\tau) F_v(U_p(\tau), 0)\phi(\tau)$ converges exponentially as $\tau \to -\infty$. Finally, we note that $a_1(\xi)$ and $a_2(\xi)$ share the same decay rate as $\phi(\xi)$ as $\xi \to -\infty$ while their decay rate exceeds that of $\phi(\xi)$ as $\xi \to \infty$.

The range of $(\mathrm{Id} - \mathrm{P}^{ss}(\xi_0))$ can be expressed in terms of solutions to the adjoint equation,

$$\psi' = -A(\xi, s^*, \sigma^*)^T \psi. \tag{3.10}$$

Note that the adjoint equation also admits a generalized exponential dichotomy with fundamental matrix solution $\tilde{\Phi}(\xi, \xi_0) = \left(\Phi(\xi, \xi_0)^{-1} \right)^T$. The generalized exponential dichotomy distinguishes between solutions with weak and strong unstable dynamics. The weak unstable projection for the adjoint equation has two dimensional range spanned by,

$$\psi_1 = \begin{pmatrix} -e^{s^*\xi_0} U_p''(\xi_0) \\ e^{s^*\xi_0} U_p'(\xi_0) \\ b_1(\xi_0) \\ b_2(\xi) \end{pmatrix}, \quad \psi_2 = e^{\frac{s^*}{\sigma^*}\xi_0} \begin{pmatrix} 0 \\ 0 \\ -\phi'(\xi_0) \\ \phi(\xi_0) \end{pmatrix}, \tag{3.11}$$

where $b_1(\xi)$ and $b_2(\xi)$ satisfy

$$\begin{aligned} b_1'(\xi) &= \frac{G_v(U_p(\xi), 0)}{\sigma^*} b_2(\xi) + F_v(U_p(\xi), 0) e^{s^*\xi} U_p'(\xi) \\ b_2'(\xi) &= -b_1(\xi) + \frac{s^*}{\sigma^*} b_2(\xi). \end{aligned}$$

This system can be re-expressed as the second order equation,

$$\sigma^* b_2''(\xi) - s^* b_2'(\xi) + G_v(U_p(\xi), 0) b_2(\xi) = -\sigma^* F_v(U_p(\xi), 0) e^{s^*\xi} U_p'(\xi). \tag{3.12}$$

The homogeneous system has a pair of linearly independent solutions,

$$B_1(\xi) = e^{\frac{s^*}{\sigma^*}\xi}\phi(\xi), \quad B_2(\xi) = e^{\frac{s^*}{\sigma^*}\xi}\phi(\xi)\int_{\xi_0}^{\xi} \frac{e^{-\frac{s^*}{\sigma^*}\tau}}{\phi^2(\tau)}\mathrm{d}\tau.$$

Note that $B_1(\xi)$ possesses weak unstable growth as $\xi \to \infty$ and $B_2(\xi)$ has strong unstable growth. For ξ tending to $-\infty$, we have that $B_1(\xi)$ and $B_2(\xi)$ both converge exponentially to zero.

Variation of parameters yields a solution to the inhomogeneous equation (3.12) with weak-unstable growth as $\xi \to \infty$,

$$\begin{aligned} b_2(\xi) = \tilde{c}_1 B_1(\xi) + \sigma^* B_1(\xi) \int_{\xi_0}^{\xi} e^{-\frac{s^*}{\sigma^*}\tau} B_2(\tau) F_v(U_p(\tau),0) e^{s^*\tau} U_p'(\tau)\mathrm{d}\tau \\ + \sigma^* B_2(\xi) \int_{\xi}^{\infty} e^{-\frac{s^*}{\sigma^*}\tau} B_1(\tau) F_v(U_p(\tau),0) e^{s^*\tau} U_p'(\tau)\mathrm{d}\tau, \end{aligned} \tag{3.13}$$

where we note that the integrand converges exponentially as $\tau \to \infty$ and, hence, the integral converges. Finally, we select $\tilde{c}_1$ so that ψ_1 and ψ_2 are orthogonal. Orthogonality requires that

$$-\phi'(\xi_0)\left(\frac{s^*}{\sigma^*}b_2(\xi_0) - b_2'(\xi_0)\right) + b_2(\xi_0)\phi(\xi_0) = 0,$$

from which

$$\tilde{c}_1\left((\phi(\xi_0))^2 + (\phi'(\xi_0))^2\right) = -\sigma^* \frac{\phi'(\xi_0)}{\phi(\xi_0)} \int_{\xi_0}^{\infty} e^{-\frac{s^*}{\sigma^*}\tau} B_1(\tau) F_v(U_p(\tau),0) e^{s^*\tau} U_p'(\tau)\mathrm{d}\tau.$$

We introduce the notation,

$$\Omega_1 = \langle \psi_1(\xi_0), \psi_1(\xi_0)\rangle, \quad \Omega_2 = \langle \psi_2(\xi_0), \psi_2(\xi_0)\rangle. \tag{3.14}$$

Lemma 6. *There exists functions h_1 and h_2 such that the manifold $W^{ss}(\mathbf{P}_0)$ can be expressed as*

$$\begin{pmatrix} U_p(\xi_0) \\ U_p'(\xi_0) \\ 0 \\ 0 \end{pmatrix} + \eta_1\theta_1 + \eta_2\theta_2 + (s - s^*)\Gamma_0\psi_1 + h_1(\eta_1,\eta_2,s,\sigma)\psi_1 + h_2(\eta_1,\eta_2,s,\sigma)\psi_2, \tag{3.15}$$

where $h_{1,2}$ are quadratic or higher order in all their arguments. Expansions of $h_{1,2}$ are obtained in Appendix A.

Proof. Given $Y = \eta_1\theta_1 + \eta_2\theta_2 \in \mathrm{Rg}\,(\mathrm{P}^{ss}(\xi_0))$ and $|\eta_1| + |\eta_2| + |s - s^*| + |\sigma - \sigma^*|$ small enough, let $Q^*(\cdot,\eta_1,\eta_2,s,\sigma) \in S$ be the unique fixed point solution to $TQ^* = Q^*$ in $\mathrm{B}_r(0)$ from which evaluating (3.5) at $\xi = \xi_0$ we obtain

$$Q^*(\xi_0,\eta_1,\eta_2,s,\sigma) = \eta_1\theta_1 + \eta_2\theta_2 - \int_{\xi_0}^{\infty} \Phi^{ws}(\xi_0,\tau)\left(g(\tau,s) + N(Q^*(\tau,\eta_1,\eta_2,s,\sigma),\tau,s,\sigma)\right)\mathrm{d}\tau. \tag{3.16}$$

Using the fact that $\mathrm{P}^{ss}(\xi)\Phi(\xi,\tau) = \Phi(\xi,\tau)\mathrm{P}^{ss}(\tau)$ we have that

$$\Phi^{ws}(\xi_0,\tau) = \Phi(\xi_0,\tau)\left(\mathrm{Id} - \mathrm{P}^{ss}(\tau)\right) = \left(\mathrm{Id} - \mathrm{P}^{ss}(\xi_0)\right)\Phi(\xi_0,\tau),$$

which shows that the second term in (3.16) belongs to $\mathrm{Rg}\left(\mathrm{Id}-\mathrm{P}^{ss}(\xi_0)\right)$ and thus

$$\begin{aligned}\mathrm{P}^{ss}(\xi_0)Q^*(\xi_0,\eta_1,\eta_2,s,\sigma) &= \eta_1\theta_1+\eta_2\theta_2,\\ \left(\mathrm{Id}-\mathrm{P}^{ss}(\xi_0)\right)Q^*(\xi_0,\eta_1,\eta_2,s,\sigma) &= -\int_{\xi_0}^{\infty}\Phi^{ws}(\xi_0,\tau)\left(g(\tau,s)+N(Q^*(\tau,\eta_1,\eta_2,s,\sigma),\tau,s,\sigma)\right)\mathrm{d}\tau.\end{aligned}$$

It is first easy to check, using the specific form of $g(\tau,s)$ that

$$\int_{\xi_0}^{\infty}\langle\psi_1(\xi_0),\Phi^{ws}(\xi_0,\tau)g(\tau,s)\rangle\,\mathrm{d}\tau=\int_{\xi_0}^{\infty}\big\langle\psi_1(\tau),g(\tau,s)\big\rangle\mathrm{d}\tau=-(s-s^*)\int_{\xi_0}^{\infty}e^{s^*\tau}\left(U_p'(\tau)\right)^2\mathrm{d}\tau,$$

together with

$$\int_{\xi_0}^{\infty}\langle\psi_2(\xi_0),\Phi^{ws}(\xi_0,\tau)g(\tau,s)\mathrm{d}\tau\rangle=\int_{\xi_0}^{\infty}\big\langle\psi_2(\tau),g(\tau,s)\big\rangle\mathrm{d}\tau=0.$$

As a consequence, there exists $h_{1,2}(\eta_1,\eta_2,s,\sigma)$ so that equation (3.16) can be written as

$$Q^*(\xi_0,\eta_1,\eta_2,s,\sigma)=\eta_1\theta_1+\eta_2\theta_2+(s-s^*)\Gamma_0\psi_1+h_1(\eta_1,\eta_2,s,\sigma)\psi_1+h_2(\eta_1,\eta_2,s,\sigma)\psi_2,$$

where

$$\begin{aligned}\Gamma_0 &:= \frac{1}{\Omega_1}\int_{\xi_0}^{\infty}e^{s^*\tau}\left(U_p'(\tau)\right)^2\mathrm{d}\tau,\\ h_{1,2}(\eta_1,\eta_2,s,\sigma) &:= -\frac{1}{\Omega_{1,2}}\int_{\xi_0}^{\infty}\langle\psi_{1,2}(\xi_0),\Phi^{ws}(\xi_0,\tau)N(Q^*(\tau,\eta_1,\eta_2,s,\sigma),\tau,s,\sigma)\rangle\,\mathrm{d}\tau,\end{aligned}$$

and $\Omega_{1,2}$ have been introduced in (3.14).

In the remaining of the proof, we show that the maps $h_{1,2}(\eta_1,\eta_2,s,\sigma)$ are at least quadratic in their arguments and present a procedure which allows one to compute the leading order terms in their expansions, the explicit formulae being provided in Appendix A. Let $Q^0(\xi):=\eta_1\theta_1(\xi)+\eta_2\theta_2(\xi)+(s-s^*)\theta_s(\xi)$ where

$$(s-s^*)\theta_s(\xi)=\int_{\xi_0}^{\xi}\Phi^{ss}(\xi,\tau)g(\tau,s)\mathrm{d}\tau-\int_{\xi}^{\infty}\Phi^{ws}(\xi,\tau)g(\tau,s)\mathrm{d}\tau,$$

with $\theta_s(\xi_0)=\Gamma_0\psi_1$. Define now $Q^1:=TQ^0$, that is

$$\begin{aligned}Q^1(\xi)=\Phi^{ss}(\xi,\xi_0)Y&+\int_{\xi_0}^{\xi}\Phi^{ss}(\xi,\tau)\left(g(\tau,s)+N(Q^0(\tau),\tau,s,\sigma)\right)\mathrm{d}\tau\\ &-\int_{\xi}^{\infty}\Phi^{ws}(\xi,\tau)\left(g(\tau,s)+N(Q^0(\tau),\tau,s,\sigma)\right)\mathrm{d}\tau.\end{aligned}\tag{3.17}$$

Let us remark that $\Phi^{ss}(\xi,\xi_0)Y=\Phi^{ss}(\xi,\xi_0)\left(\eta_1\theta_1+\eta_2\theta_2\right)=\eta_1\theta_1(\xi)+\eta_2\theta_2(\xi)$ for any $\xi\geq\xi_0$ such that (3.17) can be written in a condensed form

$$Q^1(\xi)=Q^0(\xi)+\int_{\xi_0}^{\xi}\Phi^{ss}(\xi,\tau)N(Q^0(\tau),\tau,s,\sigma)\mathrm{d}\tau-\int_{\xi}^{\infty}\Phi^{ws}(\xi,\tau)N(Q^0(\tau),\tau,s,\sigma)\mathrm{d}\tau.$$

From the contraction mapping theorem, we find that $\|Q^1-Q^*\|_S<\frac{\kappa}{1-\kappa}\|Q^1-Q^0\|_S$ where $\kappa(r,s,\sigma)$ is the contraction constant from Lemma 5. Essentially repeating the estimate in (3.6), we also find that there exists a constant $C_L>0$ for which

$$\|Q^1-Q^0\|_S\leq C_L L(r,s,\sigma)\|Q^0\|_S.\tag{3.18}$$

Let $\xi = \xi_0$ in (3.17) to obtain

$$Q^1(\xi_0) = \eta_1\theta_1 + \eta_2\theta_2 + (s - s^*)\Gamma_0\psi_1 - \int_{\xi_0}^{\infty} \Phi^{ws}(\xi_0, \tau)N(\eta_1\theta_1(\tau) + \eta_2\theta_2(\tau) + (s - s^*)\theta_s(\tau), \tau, s, \sigma)\mathrm{d}\tau.$$

The inequality (3.18) implies that $h_{1,2}$ are at least quadratic in their arguments and that we can compute terms up to quadratic order in $h_{1,2}$ by projecting $Q^1(\xi_0)$ onto $\psi_{1,2}$. We now refer to Appendix A for the quadratic expansions of the maps $h_{1,2}$. ∎

Remark 7. *An explicit expression for θ_s can be obtained in a fashion analogous to that of the terms $a_{1,2}(\xi)$. Namely, we find that $\theta_s(\xi) = (\theta_s^1(\xi), \theta_s^2(\xi), 0, 0)^T$ solves*

$$\begin{aligned}
\frac{\mathrm{d}\theta_s^1}{\mathrm{d}\xi} &= \theta_s^2 \\
\frac{\mathrm{d}\theta_s^1}{\mathrm{d}\xi} &= -s^*\theta_s^2 - F_u(U_p(\xi), 0)\theta_s^1 - U_p'(\xi).
\end{aligned}$$

Then a solution with strong exponential decay as $\xi \to \infty$ is given by

$$\theta_s^1(\xi) = \hat{c}_1 A_1(\xi) + A_1(\xi)\int_{\xi_0}^{\xi} e^{s^*\tau} A_2(\tau)U_p'(\tau)\mathrm{d}\tau + A_2(\xi)\int_{\xi}^{\infty} e^{s^*\tau} A_1(\tau)U_p'(\tau)\mathrm{d}\tau, \tag{3.19}$$

where $A_{1,2}(\xi)$ are defined in (3.8) and $\hat{c}_1$ is chosen so that $\theta_s(\xi_0)$ is orthogonal to θ_1.

3.1 The tangent space of $W^{ss}(\mathbf{P}_0)$

Before proceeding to a local analysis of the dynamics near $\mathbf{P}_1$, we pause to comment on the behavior of the tangent space of $W^{ss}(\mathbf{P}_0)$ in the limit as $\xi \to -\infty$. This is most easily accomplished in the coordinates of (3.1), where we focus on the system $z' = A(\xi, s^*, \sigma^*)z$ with $z = (p_1, p_2, q_1, q_2)^T$.

We track two dimensional subspaces using the coordinates, see [18]

$$\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ 0 & z_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

wherein,

$$\begin{aligned}
z_{11}' &= -s^* z_{11} - F_u(U_p(\xi)) - z_{11}^2 \\
z_{12}' &= -s^* z_{12} - F_v(U_p(\xi)) - z_{12}(z_{11} + z_{22}) \\
z_{22}' &= -\frac{s^*}{\sigma^*} z_{22} - \frac{1}{\sigma^*} G_v(U_p(\xi)) - z_{22}^2.
\end{aligned} \tag{3.20}$$

Using the expressions for the vectors θ_1 and θ_2, we find corresponding solutions

$$Z_{11}(\xi) = \frac{U_p''(\xi)}{U_p'(\xi)},\ Z_{22}(\xi) = \frac{\phi'(\xi)}{\phi(\xi)},\ Z_{12}(\xi) = \frac{a_2(\xi)}{\phi(\xi)} - \frac{U_p''(\xi)}{U_p'(\xi)}\frac{a_1(\xi)}{\phi(\xi)}.$$

The manifold $W^{ss}(\mathbf{P}_0)$ is then expressed as a graph over p_1 and q_1 coordinates

$$\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} Z_{11}(\xi) & Z_{12}(\xi) \\ 0 & Z_{22}(\xi) \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}.$$

In fact, a calculation reveals that the expression for Z_{12} can be simplified to

$$Z_{12}(\xi) = \frac{1}{\phi(\xi)} \left(\frac{e^{-s^*\xi}}{U_p'(\xi)} \int_\xi^\infty e^{s^*\tau} U_p'(\tau) F_v(U_p(\tau), 0)\phi(\tau) \mathrm{d}\tau \right).$$

It follows from Hypothesis (H4) that as $\xi \to -\infty$,

$$Z_{11}(\xi) \to \nu_u^+(s^*),\ Z_{22}(\xi) \to \nu_v^+(s^*, \sigma^*),\ Z_{12}(\xi) \to \frac{-F_v(\mathbf{p}_1)}{s + \nu_u^+(s^*) + \nu_v^+(s^*, \sigma^*)},$$

which we verify to be fixed points of the system (3.21). These fixed points correspond to the unstable and weak stable eigenvectors for $\mathbf{P}_1$ (see (4.1) below) and we have shown that $\mathrm{span}\{\theta_1, \theta_2\}$ coincides with the weak-unstable eigenspace of $\mathbf{P}_1$ in the limit as $\xi \to -\infty$.

To understand how this heteroclinic perturbs with s and σ, we let

$$\begin{pmatrix} z_{11} \\ z_{12} \\ z_{22} \end{pmatrix} = \begin{pmatrix} Z_{11}(\xi) + \zeta_{11} \\ Z_{12}(\xi) + \zeta_{12} \\ Z_{22}(\xi) + \zeta_{22} \end{pmatrix}.$$

Let $\Xi = (\zeta_{11}, \zeta_{12}, \zeta_{22})^T$, then we obtain

$$\Xi' = A(\xi, s^*, \sigma^*)\Xi + (s - s^*)g(\xi) + (\sigma - \sigma)^* h(\xi) + N(\Xi, s, \sigma), \tag{3.21}$$

where

$$A(\xi, s^*, \sigma^*) = \begin{pmatrix} -s^* - 2Z_{11}(\xi) & 0 & 0 \\ -Z_{12}(\xi) & -s^* - Z_{11}(\xi) + Z_{22}(\xi) & -Z_{12}(\xi) \\ 0 & 0 & -\frac{s^*}{\sigma^*} - 2Z_{22}(\xi) \end{pmatrix}$$

and

$$g(\xi) = \begin{pmatrix} -Z_{11}(\xi) \\ -Z_{12}(\xi) \\ -\frac{1}{\sigma^*} Z_{22}(\xi) \end{pmatrix}, \quad h(\xi) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{(\sigma^*)^2} G_v(U_p(\xi), 0) + \frac{s^*}{(\sigma^*)^2} Z_{22}(\xi) \end{pmatrix}.$$

Since we are only interested in the linear dependence on s and σ, we henceforth ignore the nonlinear terms $N(\Xi, s, \sigma)$. We will also require linearly independent solutions to the associated adjoint equation,

$$\psi' = -A^T(\xi, s^*, \sigma^*)\psi.$$

The adjoint equations form a system

$$\psi_3' = (s^* + 2Z_{11}(\xi))\psi_3 + Z_{12}\psi_4, \tag{3.22}$$

$$\psi_4' = (s^* + Z_{11}(\xi) + Z_{22}(\xi))\psi_4, \tag{3.23}$$

$$\psi_5' = Z_{12}(\xi)\psi_4 + \left(\frac{s^*}{\sigma^*} + 2Z_{22}(\xi)\right)\psi_5, \tag{3.24}$$

We have solutions

$$\psi_3 = \begin{pmatrix} C_3 \\ 0 \\ 0 \end{pmatrix}, \psi_4 = \begin{pmatrix} C_4 \\ D_4 \\ E_4 \end{pmatrix}, \psi_5 = \begin{pmatrix} 0 \\ 0 \\ E_5 \end{pmatrix}.$$

with

$$C_3(\xi) = \left(U_p'(\xi)\right)^2 e^{s^*\xi},$$
$$D_4(\xi) = U_p'(\xi)\phi(\xi)e^{s^*\xi},$$
$$E_5(\xi) = \phi(\xi)^2 e^{\frac{s^*}{\sigma^*}\xi}.$$

Requiring orthogonality of the three vectors at $\xi = \xi_0$ implies that $C_4(\xi_0) = E_4(\xi_0) = 0$. Let $\Phi(\xi, \xi_0)$ be the fundamental matrix solution to $\Xi' = A(\xi, s^*, \sigma^*)\Xi$. Bounded solutions of (3.21) can be expressed in integral form as

$$\Xi(\xi) = \int_{-\infty}^{\xi} \Phi(\xi, \tau)\left((s - s^*)g(\tau) + (\sigma - \sigma)^* h(\tau) + N(\Xi(\tau), s, \sigma)\right) \mathrm{d}\tau.$$

We focus on the leading order dependence on σ. At $\xi = \xi_0$, we write

$$\Xi(\xi_0) = h_3(\sigma)\psi_3(\xi_0) + h_4(\sigma)\psi_4(\xi_0) + h_5(\sigma)\psi_5(\xi_0).$$

Observe that the only contributions come from the projection onto ψ_5

$$\begin{aligned}
h_5(\sigma) = \frac{\langle \psi_5(\xi_0), \Xi(\xi_0)\rangle}{E_5(\xi_0)} &= \frac{(\sigma - \sigma^*)}{E_5(\xi_0)} \int_{-\infty}^{\xi_0} \langle \psi_5(\xi_0), \Phi(\xi_0, \tau)h(\tau)\rangle \mathrm{d}\tau \\
&= \frac{(\sigma - \sigma^*)}{E_5(\xi_0)} \int_{-\infty}^{\xi_0} E_5(\tau)\left(\frac{1}{(\sigma^*)^2} G_v(U_p(\xi), 0) + \frac{s^*}{(\sigma^*)^2} Z_{22}(\xi)\right) \mathrm{d}\tau \\
&= \frac{(\sigma - \sigma^*)}{E_5(\xi_0)} \int_{-\infty}^{\xi_0} e^{\frac{s^*}{\sigma^*}\tau}\left(\frac{1}{(\sigma^*)^2} G_v(U_p(\tau), 0)\phi(\tau)^2 + \frac{s^*}{(\sigma^*)^2}\phi(\tau)\phi'(\tau)\right) \mathrm{d}\tau.
\end{aligned}$$

Returning now to the original change of coordinates, we find

$$\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} Z_{11}(\xi) & Z_{12}(\xi) \\ 0 & Z_{22}(\xi) + E_5(\xi)h_5(\sigma) \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}.$$

This describes a two dimensional subspace of the form,

$$\mathcal{R}(p_1, q_1, \sigma) = \begin{pmatrix} p_1 \\ Z_{11}(\xi_0)p_1 + Z_{12}(\xi_0)q_1 \\ q_1 \\ \left(Z_{22}(\xi_0) + E_5(\xi_0)h_5(\sigma)\right) q_1 \end{pmatrix}. \tag{3.25}$$

We now decompose this subspace into the basis $\{\theta_1, \theta_2, \psi_1, \psi_2\}$. To recover θ_1, we require $\sigma = \sigma^*$, $p_1 = U_p'(\xi_0)$ and $q_1 = 0$. To recover θ_2, we require $\sigma = \sigma^*$, $p_1 = a_1(\xi_0)$ and $q_1 = \phi(\xi_0)$. Projecting onto ψ_2, we find

$$\langle \psi_2, \mathcal{R}(p_1, q_1, \sigma)\rangle = q_1(\sigma - \sigma^*) \int_{-\infty}^{\xi_0} e^{\frac{s^*}{\sigma^*}\tau}\left(\frac{1}{(\sigma^*)^2} G_v(U_p(\tau), 0)\phi(\tau)^2 + \frac{s^*}{(\sigma^*)^2}\phi(\tau)\phi'(\tau)\right) \mathrm{d}\tau,$$

and projecting onto ψ_1, we find

$$\langle \psi_1, \mathcal{R}(p_1, q_1, \sigma)\rangle = q_1(\sigma - \sigma^*) \int_{-\infty}^{\xi_0} b_2(\tau)\left(\frac{1}{(\sigma^*)^2} G_v(U_p(\tau), 0 + \frac{s^*}{(\sigma^*)^2}\frac{\phi'(\tau)}{\phi(\tau)}\right) \mathrm{d}\tau.$$

4 Tracking the unstable manifold $W^u(\mathbf{P}_2)$ forwards

We now derive an expression for $W^u(\mathbf{P}_2)$ in a neighborhood of the fixed point $\mathbf{P}_1$. Hypothesis **(H5)** will be key here. We delay a precise description of this assumption and its consequences until Section 4.2 and instead begin with a required normal form transformation for the traveling wave equation in a neighborhood of $\mathbf{P}_1$.

4.1 A normal form in a neighborhood of $\mathbf{P}_1$

We begin with a local analysis of the dynamics of (1.2) near the fixed point $\mathbf{P}_1 = (u^+, 0, 0, 0)^T$. The Jacobian evaluated at this fixed point is

$$Df(\mathbf{P}_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -F_u(\mathbf{p}_1) & -s & -F_v(\mathbf{p}_1) & 0 \\ 0 & 0 & 0 & \frac{1}{\sigma} \\ 0 & 0 & -\frac{G_v(\mathbf{p}_1)}{\sigma} & -\frac{s}{\sigma} \end{pmatrix},$$

where we note that $G_u(\mathbf{p}_1) = 0$ and hence the linearization is block triangular and the eigenvalues and eigenvectors can be computed explicitly. The characteristic polynomial is $d(\nu) = d_u(\nu)d_v(\nu) = (\nu^2 + s\nu + F_u(\mathbf{p}_1))(\sigma\nu^2 + s\nu + G_v(\mathbf{p}_1))$. The eigenvalues are

$$\begin{aligned} \nu_u^{\pm}(s) &= -\frac{s}{2} \pm \frac{1}{2}\sqrt{s^2 - 4F_u(\mathbf{p}_1)} \\ \nu_v^{\pm}(s,\sigma) &= -\frac{s}{2\sigma} \pm \frac{1}{2\sigma}\sqrt{s^2 - 4\sigma G_v(\mathbf{p}_1)}. \end{aligned}$$

Recall Hypothesis **(H2)** and the assumed ordering $\nu_v^-(s,\sigma) < \nu_u^-(s) < \nu_v^+(s,\sigma) < 0 < \nu_u^+(s)$. The corresponding eigenvectors are

$$e_u^{\pm}(s) = \begin{pmatrix} 1 \\ \nu_u^{\pm}(s) \\ 0 \\ 0 \end{pmatrix}, \quad e_v^{\pm}(s,\sigma) = \begin{pmatrix} -\frac{F_v(\mathbf{p}_1)}{d_u(\nu_v^{\pm}(s,\sigma))} \\ -\frac{F_v(\mathbf{p}_1)\nu_v^{\pm}(s,\sigma)}{d_u(\nu_v^{\pm}(s,\sigma))} \\ 1 \\ \nu_v^{\pm}(s,\sigma) \end{pmatrix}. \tag{4.1}$$

We introduce new coordinates, first by shifting the fixed point $\mathbf{P}_1$ to the origin and then diagonalizing the linearization via

$$\begin{pmatrix} u_1 - u^+ \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = T(s,\sigma) \begin{pmatrix} y^u \\ y^{ss,u} \\ y^{ws} \\ y^{ss,v} \end{pmatrix},$$

where

$$T(s,\sigma) := \begin{pmatrix} 1 & 1 & -\frac{F_v(\mathbf{p}_1)}{d_u(\nu_v^+(s,\sigma))} & -\frac{F_v(\mathbf{p}_1)}{d_u(\nu_v^-(s,\sigma))} \\ \nu_u^+(s) & \nu_u^-(s) & -\frac{F_v(\mathbf{p}_1)\nu_v^+(s,\sigma)}{d_u(\nu_v^+(s,\sigma))} & -\frac{F_v(\mathbf{p}_1)\nu_v^-(s,\sigma)}{d_u(\nu_v^-(s,\sigma))} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \nu_v^+(s,\sigma) & \nu_v^-(s,\sigma) \end{pmatrix}. \tag{4.2}$$

In these new coordinates, the vector field assumes the form,

$$\begin{aligned} \frac{dy^u}{d\xi} &= \nu_u^+(s)y^u + \mathcal{N}_u(y^u, y^{ws}, y^{ss,u}, y^{ss,v}, s, \sigma), \\ \frac{dy^{ws}}{d\xi} &= \nu_v^+(s,\sigma)y^{ws} + \mathcal{N}_{ws}(y^u, y^{ws}, y^{ss,u}, y^{ss,v}, s, \sigma), \\ \frac{dy^{ss,u}}{d\xi} &= \nu_u^-(s)y^{ss,u} + \mathcal{N}_{ss,u}(y^u, y^{ws}, y^{ss,u}, y^{ss,v}, s, \sigma), \\ \frac{dy^{ss,v}}{d\xi} &= \nu_v^-(s,\sigma)y^{ss,v} + \mathcal{N}_{ss,v}(y^u, y^{ws}, y^{ss,u}, y^{ss,v}, s, \sigma). \end{aligned} \tag{4.3}$$

Invariance of the $v_1 = v_2 = 0$ subspace implies that $\mathcal{N}_{ws}(y^u, 0, y^{ws}, 0, s, \sigma) = 0$ and $\mathcal{N}_{ss,v}(y^u, 0, y^{ws}, 0, s, \sigma) = 0$. We expand the nonlinear terms as follows to isolate the quadratic terms,

$$\mathcal{N}_u(y^u, y^{ws}, y^{ss,u}, y^{ss,v}, s, \sigma) = \sum_{i+j+k+l=2} (y^u)^i (y^{ws})^j (y^{ss,u})^k (y^{ss,v})^l \mathbf{N}_u^{(i,j,k,l)}(s,\sigma) + \mathcal{O}(3), \tag{4.4}$$

with the natural analogs for $\mathcal{N}_{ws}$ $\mathcal{N}_{ss,u}$ and $\mathcal{N}_{ss,v}$.

The goal is to perform a Shilnikov type analysis of the origin in (4.3) and obtain asymptotic expansions for solutions that enter a neighborhood of the origin near the weak-stable eigendirection and exit near the unstable manifold. To do this a sequence of near-identity coordinate changes are required to place (4.3) into a suitable normal form. These changes of coordinates are outlined in [12], but we include them in detail here because they will be relevant for deriving the bifurcation equations later.

Straightening of the unstable, stable, and strong stable manifolds The origin is a hyperbolic equilibrium for (4.3) with corresponding stable and unstable manifolds. The following result transforms (4.3) into new coordinates where these stable and unstable manifolds have been straightened.

Lemma 8. *There exists a smooth change of coordinates,*

$$\begin{aligned} z^u &= y^u - \mathcal{H}_s(y^{ws}, y^{ss,u}, y^{ss,v}, s, \sigma) \\ z^{ws} &= y^{ws} \\ z^{ss,u} &= y^{ss,u} - \mathcal{H}_u(y^u, s) \\ z^{ss,v} &= y^{ss,v}, \end{aligned} \tag{4.5}$$

defined on a neighborhood of the origin that transforms (4.3) to the system

$$\begin{aligned} \frac{dz^u}{d\xi} &= \nu_u^+(s)z^u + \mathcal{M}_u(z^u, z^{ws}, Z^s, s, \sigma) \\ \frac{dz^{ws}}{d\xi} &= \nu_v^+(s,\sigma)z^{ws} + \gamma_{11}(z^u, z^{ws}, Z^s, s, \sigma)z^{ws} + \gamma_{12}(z^u, z^{ws}, Z^s, s, \sigma)Z^s \\ \frac{dZ^s}{d\xi} &= \Lambda_{ss}(s,\sigma)Z^s + \gamma_{21}(z^u, z^{ws}, Z^s, s, \sigma)z^{ws} + \gamma_{22}(z^u, z^{ws}, Z^s, s, \sigma)Z^s, \end{aligned} \tag{4.6}$$

where we have let $Z^s = (z^{ss,u}, z^{ss,v})^T$ *and* $\Lambda_{ss}(s,\sigma) = \mathrm{diag}(\nu_u^-(s), \nu_v^-(s,\sigma))$ *and we have that* $\mathcal{M}_u(0, z^{ws}, Z^s, s, \sigma) = 0$*.*

Proof. The origin in (4.3) is hyperbolic with smooth stable and unstable manifolds. The unstable manifold is contained within the invariant sub-space $y^{ws} = y^{ss,v} = 0$ and can be expressed as the graph

$$y^{ss,u} = \mathcal{H}_u(y^u, s),$$

which admits the expansion,

$$\mathcal{H}_u(y^u, s) = \frac{\mathbf{N}_{ss,u}^{(2,0,0,0)}(s,\sigma)}{2\nu_u^+(s) - \nu_u^-(s)}(y^u)^2 + \mathcal{O}(3).$$

Let us remark here that $\mathbf{N}_{ss,u}^{(2,0,0,0)}(s,\sigma)$ does not depend on σ and can be expressed as

$$\mathbf{N}_{ss,u}^{(2,0,0,0)}(s,\sigma) = -\frac{F_{uu}(\mathbf{p}_1)}{2(\nu_u^-(s) - \nu_u^+(s))}.$$

The proof of this statement is left to the Appendix (see Lemma 17). The stable manifold has a similar expansion,

$$\mathcal{H}_s(y^{ws}, y^{ss,u}, y^{ss,v}) = \sum_{j+k+l=2} \mathbf{n}_{(i,j,k)}(s,\sigma)(y^{ws})^j (y^{ss,u})^k (y^{ss,v})^l + \mathcal{O}(3),$$

where

$$\mathbf{n}_{(j,k,l)}(s,\sigma) = \frac{\mathbf{N}_u^{(0,j,k,l)}(s,\sigma)}{j\nu_v^+(s,\sigma) + k\nu_u^-(s) + l\nu_v^-(s,\sigma) - \nu_u^-(s)}.$$

Following these changes of coordinates, we have transformed system (4.3) into (4.6) as required. ∎

Removal of terms $\gamma_{j1}(z^u, 0, 0, s, \sigma)$ We will eventually employ a Shilnikov type analysis where solutions of (4.6) are obtained as solutions of a boundary value problem on the interval $\xi \in [0, T]$ with $T \gg 1$. This boundary value problem imposes conditions on the unstable coordinate at $\xi = T$ and thereby the instability is controlled by evolving that coordinate backwards. One would then hope that the linear behavior would dominate in (4.6). This is not the case due to the presence of the terms $\gamma_{j1}(z^u, 0, 0, s, \sigma)$. To obtain useful asymptotics, we require a further change of coordinates that removes those terms. This is accomplished in the following lemma.

Lemma 9. *There exists functions* $p(z^u, s, \sigma)$ *and* $q(z^u, s, \sigma)$*, with* $p : \mathbb{R}^3 \to \mathbb{R}$*, and* $q : \mathbb{R}^3 \to \mathbb{R}^2$*, valid for* $|z^u| + |s - s^*| + |\sigma - \sigma^*|$ *sufficiently small such that the change of coordinates,*

$$\begin{aligned} \bar{z}^{ws} &= z^{ws}(1 - p(z^u, s, \sigma)) \\ \bar{Z}^s &= Z^s - q(z^u, s, \sigma) z^{ws}, \end{aligned} \tag{4.7}$$

transforms (4.6) to the normal form

$$\begin{aligned} \frac{dz^u}{d\xi} &= \nu_u^+(s) z^u + \overline{\mathcal{M}}_u(z^u, \bar{z}^{ws}, \bar{Z}^s, s, \sigma) \\ \frac{d\bar{z}^{ws}}{d\xi} &= \nu_v^+(s,\sigma)\bar{z}^{ws} + \bar{\gamma}_{11}(z^u, \bar{z}^{ws}, \bar{Z}^s, s, \sigma)\bar{z}^{ws} + \bar{\gamma}_{12}(z^u, \bar{z}^{ws}, \bar{Z}^s, s, \sigma)\bar{Z}^s \\ \frac{d\bar{Z}^s}{d\xi} &= \Lambda_{ss}(s,\sigma)\bar{Z}^s + \bar{\gamma}_{21}(z^u, \bar{z}^{ws}, \bar{Z}^s, s, \sigma)\bar{z}^{ws} + \bar{\gamma}_{22}(z^u, \bar{z}^{ws}, \bar{Z}^s, s, \sigma)\bar{Z}^s, \end{aligned} \tag{4.8}$$

where $\bar{\gamma}_{11}(z^u, 0, 0, s, \sigma) = 0$ *and* $\bar{\gamma}_{21}(z^u, 0, 0, s, \sigma) = 0$*.*

Proof. We use a change of coordinates outlined in [5, 20]. In a first step, we let

$$\begin{aligned}\bar{z}^{ws} &= z^{ws}(1-g^{ws})\\ \bar{Z}^{s} &= Z^{s} - G^{s} z^{ws},\end{aligned}$$

for two smooth functions $g^{ws}:\mathbb{R}\to\mathbb{R}$ and $G^s:\mathbb{R}\to\mathbb{R}^2$. We substitute this change of coordinates into (4.6) and obtain

$$\begin{aligned}\frac{\mathrm{d}\bar{z}^{ws}}{\mathrm{d}\xi} &= \nu_v^+(s,\sigma)\bar{z}^{ws} + \bar{z}^{ws}\left(\gamma_{11}+\gamma_{12}G^s - \frac{1}{1-g^{ws}}\frac{\mathrm{d}g^{ws}}{\mathrm{d}\xi}\right) + \gamma_{12}\bar{Z}^s(1-g^{ws}),\\ \frac{\mathrm{d}\bar{Z}^s}{\mathrm{d}\xi} &= \Lambda_{ss}\bar{Z}^s + \frac{\bar{z}^{ws}}{1-g^{ws}}\left(\Lambda_{ss}G^s - \nu_v^+(s,\sigma)G^s + \gamma_{21} + \gamma_{22}G^s - \frac{\mathrm{d}G^s}{\mathrm{d}\xi} - \gamma_{11}G^s\right)\\ &\quad + \gamma_{22}\bar{Z}^s - G^s\gamma_{12}G^s,\end{aligned} \tag{4.9}$$

where we have suppressed the functional dependence of γ_{ij} for convenience. Recall our original intention – to remove those terms $\gamma_{j1}(z^u,0,0,s,\sigma)$ from (4.6). To accomplish this, we set the terms multiplying $\bar{z}^{ws}$ in (4.9) to zero and find differential equations for g^{ws} and G^s. Since we are interested in these changes of coordinates along the unstable manifold, we augment these equations with the one for z^u and obtain

$$\begin{aligned}\frac{\mathrm{d}g^{ws}}{\mathrm{d}\xi} &= (1-g^{ws})\left(\gamma_{11}(z^u,0,0,s,\sigma)+\gamma_{12}(z^u,0,0,s,\sigma)G^s\right),\\ \frac{\mathrm{d}G^s}{\mathrm{d}\xi} &= \left(\Lambda_{ss}(s,\sigma) - \nu_v^+(s,\sigma)\mathrm{I} + \gamma_{22}(z^u,0,0,s,\sigma) - \gamma_{11}(z^u,0,0,s,\sigma)\right)G^s + \gamma_{21}(z^u,0,0,s,\sigma),\\ \frac{\mathrm{d}z^u}{\mathrm{d}\xi} &= \nu_u^+(s)z^u + \mathcal{M}_u(z^u,0,0,s,\sigma).\end{aligned} \tag{4.10}$$

The origin is a fixed point for (4.10) with one unstable eigenvalue ($\nu_u^+(s)$), one zero eigenvalue and two stable eigenvalues ($\nu_u^-(s)-\nu_v^+(s,\sigma), \nu_v^-(s,\sigma)-\nu_v^+(s,\sigma)$). Thus, there exists a one dimensional unstable manifold given as graphs over the z^u coordinate. These graphs provide the requisite change of variables, namely we have

$$\begin{aligned}g^{ws} &:= p(z^u,s,\sigma),\\ G^s &:= q(z^u,s,\sigma).\end{aligned}$$

We also obtain expansions,

$$p(z^u,s,\sigma) = \frac{\gamma_{11}^{(1)}(s,\sigma)}{\nu_u^+(s)}z^u + \mathcal{O}\left((z^u)^2\right), \tag{4.11a}$$

$$q(z^u,s,\sigma) = -\left(\Lambda_{ss}(s,\sigma) - (\nu_v^+(s,\sigma)+\nu_u^+(s))\mathrm{I}\right)^{-1}\gamma_{21}^{(1)}(s,\sigma)z^u + \mathcal{O}\left((z^u)^2\right), \tag{4.11b}$$

where we have employed the notations

$$\gamma_{i1}(z^u,0,0,s,\sigma) = \gamma_{i1}^{(1)}(s,\sigma)z^u + \gamma_{i1}^{(2)}(s,\sigma)(z^u)^2 + \mathcal{O}\left((z^u)^3\right), \quad i\in\{1,2\}.$$

Quadratic expansions of $p(z^u,s,\sigma)$ and $q(z^u,s,\sigma)$ can be found in Lemma 19 in the Appendix. ∎

The Shilnikov Theorem

Theorem 10. *Consider the boundary value problem consisting of (4.8) with boundary conditions*

$$\bar{z}^{ws}(0) = \kappa,\ \bar{Z}^s(0) = \mathcal{Z}_0,\ z^u(T) = -\kappa,$$

for some $T > 0$. Then there exists a $\delta > 0$ such that for any $|2\kappa + |\mathcal{Z}_0|| < \delta$ and any $T > 1/\delta$ then the boundary value problem has a unique solution and the following asymptotic expansions hold for large T,

$$\begin{aligned} z^u(0) &= -\kappa e^{-\nu_u^+(s)T} + \mathcal{O}(e^{(-\nu_u^+(s)+\omega)T}) \\ \bar{z}^{ws}(T) &= \kappa e^{\nu_v^+(s,\sigma)T} + \mathcal{O}(e^{(\nu_v^+(s,\sigma)-\omega)T}) \\ \bar{Z}^s(T) &= \gamma(s,\sigma)\kappa^2 e^{2\nu_v^+(s,\sigma)T} + \mathcal{O}(e^{(2\nu_v^+(s,\sigma)-\omega)T}), \end{aligned} \tag{4.12}$$

for some $\omega > 0$ where

$$\gamma(s,\sigma) = \frac{\partial\bar{\gamma}_{21}}{\partial\bar{z}^{ws}}(0,0,0,s,\sigma) = \begin{pmatrix} \frac{\mathbf{N}_{ss,u}^{(0,2,0,0)}(s,\sigma)}{2\nu_v^+(s,\sigma)-\nu_u^-(s)} \\ \frac{\mathbf{N}_{ss,v}^{(0,2,0,0)}(s,\sigma)}{2\nu_v^+(s,\sigma)-\nu_v^-(s,\sigma)} \end{pmatrix}.$$

Proof. A full proof of this result is detailed elsewhere and we refer the reader to [19] for example. We sketch the ideas here. Transform the system of differential equations (4.8) into a system of integral equations using variation of constants,

$$\begin{aligned} z^u(\xi) &= e^{\nu_u^+(s)(\xi-T)}z^u(T) - e^{\nu_u^+(s)\xi}\int_\xi^T e^{-\nu_u^+(s)\tau}\overline{\mathcal{M}}_u(z^u(\tau),\bar{z}^{ws}(\tau),\bar{Z}^s(\tau))\mathrm{d}\tau \\ \bar{z}^{ws}(\xi) &= e^{\nu_v^+(s)\xi}\bar{z}^{ws}(0) + e^{\nu_v^+(s,\sigma)\xi}\int_0^\xi e^{-\nu_v^+(s,\sigma)\tau}\big(\bar{\gamma}_{11}(z^u(\tau),\bar{z}^{ws}(\tau),\bar{Z}^s(\tau))\bar{z}^{ws}(\tau) \\ &\quad +\bar{\gamma}_{12}(z^u(\tau),\bar{z}^{ws}(\tau),\bar{Z}^s(\tau))\bar{Z}^s(\tau)\big)\,\mathrm{d}\tau \\ \bar{Z}^s(\xi) &= e^{\Lambda_{ss}(s,\sigma)\xi}\mathcal{Z}_0 + e^{\Lambda_{ss}(s,\sigma)\xi}\int_0^\xi e^{-\Lambda_{ss}(s,\sigma)\tau}\big(\bar{\gamma}_{21}(z^u(\tau),\bar{z}^{ws}(\tau),\bar{Z}^s(\tau))\bar{z}^{ws}(\tau) \\ &\quad +\bar{\gamma}_{22}(z^u(\tau),\bar{z}^{ws}(\tau),\bar{Z}^s(\tau))\bar{Z}^s(\tau)\big)\,\mathrm{d}\tau. \end{aligned}$$

The solution is obtained as a fixed point of the mapping defined by the right hand side of the above equations for any $T > 0$ and $|2\kappa + |\mathcal{Z}_0|| < \delta$ with $\delta > 0$ small enough for the right hand side to be a contraction. The requirement that $T > 1/\delta$ is only to ensure that T is large enough in order to obtain the desired asymptotics.

Recall the non-resonance condition **(H2)**. Under this assumption, the quadratic terms in $\bar{z}^{ws}$ are sufficient to derive an expansion for $\bar{Z}^s(T)$. To do this, we recall that the leading order expansion for $\bar{Z}^s$ can be obtained from the integral equation for $\bar{Z}^s$, where we identify the dominant terms are found in the integral

$$\bar{Z}^s(\xi) = e^{\Lambda_{ss}(s,\sigma)\xi}\int_0^\xi e^{-\Lambda_{ss}(s,\sigma)\tau}\frac{\partial\gamma_{21}}{\partial\bar{z}^{ws}}(-\kappa e^{-\nu_u^+(s)\tau},0,0,s,\sigma)\kappa^2 e^{2\nu_v^+(s,\sigma)\tau}\mathrm{d}\tau.$$

Of these terms, the dominant contribution comes from the quadratic terms that are independent of z^u and we obtain the desired expansion. ∎

4.2 Application of Theorem 10 to the manifold $W^u(\mathbf{P}_2)$

Let $\kappa > 0$ and fix the sections

$$\Sigma^{out} = \{z^u = -\kappa\}, \quad \Sigma^{in} = \{\bar{z}^{ws} = \kappa\}.$$

We suppose that κ is sufficiently small so that these sections intersect the neighborhood on which the changes of variables in Lemma 8 and Lemma 9 are valid and for which the existence of solutions in Theorem 10 holds.

The goal is to derive an expansion for $W^u(\mathbf{P}_2)$ within the section Σ^{out} so as to facilitate a comparison with the manifold $W^{ss}(\mathbf{P}_0)$. Note that for fixed values of σ and s, $W^u(\mathbf{P}_2)$ is a two dimensional manifold, so that its intersection with Σ^{out} is one dimensional. Recall Hypothesis **(H5)**, where we assume that $W^u(\mathbf{P}_2)$ enters a neighborhood of $\mathbf{P}_1$ near the weak-stable eigendirection. In terms of the coordinates of (4.8), this assumption implies that

$$\begin{aligned} z^u(0) &= h_u(\chi, s, \sigma), \\ \bar{z}^{ss,u}(0) &= h_{ss,u}(\chi, s, \sigma), \\ \bar{z}^{ss,v}(0) &= h_{ss,v}(\chi, s, \sigma), \end{aligned} \tag{4.13}$$

where χ parametrizes the intersection and we have that $h_u(0, s, \sigma)$, $h_{ss,u}(0, s, \sigma)$ and $h_{ss,u}(0, s, \sigma)$ are all zero. We first match the terms in the z^u component. We find that to leading order

$$-\kappa e^{-\nu_u^+ T} + \mathcal{O}(e^{(-\nu_u^+ + \omega)T}) = r(s, \sigma)\chi + \mathcal{O}(\chi^2),$$

where $r(s, \sigma) = \frac{\partial h_u}{\partial \chi}(0, s, \sigma)$. We then have the expansion

$$\chi(\rho, s, \sigma) = -\frac{\kappa}{r(s, \sigma)} e^{-\nu_u^+ T} + \tilde{\chi}(T, s, \sigma),$$

see Remark 11. Therefore, for every $T \geq \frac{1}{\delta}$ we can solve for $\chi(\rho, s, \sigma)$ and obtain expressions for $W^u(\mathbf{P}_2)$ within Σ^{out}. These expressions can be given as a graph over the weak-stable direction, namely

$$\begin{aligned} \bar{z}^{ws}(T) &= \rho, \\ \bar{Z}^s(T) &= \rho^2 \left(\gamma(s, \sigma) + \mathcal{Z}_{ss}(\rho, s, \sigma)\right). \end{aligned} \tag{4.14}$$

Remark 11. *It is at this stage that the non-resonance condition in* **(H2)** *comes into play. Were this condition to fail to hold, then the expansions for the strong stable components in (4.12) would depend on the initial character of the manifold $W^u(\mathbf{P}_2)$ within Σ^{in}. Then the particular form of the matching condition $\chi(\rho, s, \sigma)$ would be relevant and it would prove more challenging to match solutions in the following section.*

4.3 Transforming to original coordinates

To compare the description of the manifold $W^u(\mathbf{P}_2)$ in (4.14) to the one for $W^{ss}(\mathbf{P}_0)$ we need to transform back to the original coordinates. To do this, we first transform from $(z^u, \bar{z}^{ws}, \bar{Z}^s)$ coordinates to (z^u, z^{ws}, Z^s) coordinates. This change of coordinates is performed in Lemma 9 and can be inverted explicitly. We obtain

$$\begin{aligned} z^u &= -\kappa \\ z^{ws} &= \frac{\rho}{1 - p(-\kappa, s, \sigma)} \\ Z^s &= \rho \frac{q(-\kappa, s, \sigma)}{1 - p(-\kappa, s, \sigma)} + \rho^2 \left(\gamma(s, \sigma) + \mathcal{Z}_{ss}(\rho, s, \sigma)\right). \end{aligned} \tag{4.15}$$

Next, we need to transform this expression from the coordinates (z^u, z^{ws}, Z^s) to the coordinates $(y^u, y^{ws}, y^{ss,u}, y^{ss,v})$. This involves inverting the change of coordinates given in Lemma 8, i.e. solving the following set of implicit equations,

$$
\begin{aligned}
-\kappa &= y^u - \mathcal{H}_s(y^{ws}, y^{ss,u}, y^{ss,v}) \\
\frac{\rho}{1-p(-\kappa,s,\sigma)} &= y^{ws} \\
\rho\frac{q^{(1)}(-\kappa,s,\sigma)}{1-p(-\kappa,s,\sigma)} + \rho^2\left(\gamma^{(1)}(s,\sigma) + \mathcal{Z}_{ss,u}(\rho,\sigma,s)\right) &= y^{ss,u} - \mathcal{H}_u(y^u) \\
\rho\frac{q^{(2)}(-\kappa,s,\sigma)}{1-p(-\kappa,s,\sigma)} + \rho^2\left(\gamma^{(2)}(s,\sigma) + \mathcal{Z}_{ss,v}(\rho,\sigma,s)\right) &= y^{ss,v}.
\end{aligned}
\tag{4.16}
$$

The change of coordinates can be inverted by first inputting the expressions for $y^{ws}, y^{ss,u}$, and $y^{ss,v}$ into the first equation in (4.16). This yields a scalar equation for y_u,

$$
-\kappa = y^u - \mathcal{H}_s\left(\begin{array}{c} \frac{\rho}{1-p(-\kappa,s,\sigma)} \\ \mathcal{H}_u(y^u) + \rho\frac{q^{(1)}(-\kappa,s,\sigma)}{1-p(-\kappa,s,\sigma)} + \rho^2\left(\gamma^{(1)}(s,\sigma) + \mathcal{Z}_{ss,u}(\rho,\sigma,s)\right) \\ \rho\frac{q^{(2)}(-\kappa,s,\sigma)}{1-p(-\kappa,s,\sigma)} + \rho^2\left(\gamma^{(2)}(s,\sigma) + \mathcal{Z}_{ss,v}(\rho,\sigma,s)\right) \end{array}\right).
$$

Applying the implicit function theorem, we obtain a solution

$$
y^u = \mathcal{Y}_u(\rho,s,\sigma) = \mathcal{Y}_u^0(s) + \rho\mathcal{Y}_u^1(s,\sigma) + \rho^2\mathcal{Y}_u^2(s,\sigma) + \mathcal{O}(\rho^3).
$$

Note that $\mathcal{Y}_u^0(s)$ is a solution of

$$
0 = \kappa + \mathcal{Y}_u^0(s) - \mathcal{H}_s\left(0, \mathcal{H}_u\left(\mathcal{Y}_u^0(s)\right), 0\right)
$$

and we find an expansion in κ of $\mathcal{Y}_u^0(s) = -\kappa + \mathcal{O}(\kappa^4)$. We observe that the independence of the leading order term on σ follows from the fact that the vector field restricted to $y^{ws} = y^{ss,v} = 0$ is indepedent of σ.

We then obtain an explicit representation for $y^{ss,u}$ in terms of $\mathcal{Y}_u$. For convenience we make a similar expansion,

$$
\mathcal{Y}_{uu,s}(\rho,s,\sigma) = \mathcal{H}_u(\mathcal{Y}_u(\rho,s,\sigma)) = \mathcal{Y}_{uu,s}^0(s,\sigma) + \rho\mathcal{Y}_{uu,s}^1(s,\sigma) + \rho^2\mathcal{Y}_{uu,s}^2(s,\sigma) + \mathcal{O}(\rho^3).
$$

These terms have similar expansions in κ, for example

$$
\mathcal{Y}_{uu,s}^0(s,\sigma) = \frac{\mathbf{N}_{ss,u}^{(2,0,0,0)}(s,\sigma)}{2\nu_u^+(s) - \nu_u^-(s)}\kappa^2 + \mathcal{O}(\kappa^3).
$$

To summarize, we have found the expressions

$$
\begin{aligned}
y^u &= \mathcal{Y}_u(\rho,s,\sigma) \\
y^{ws} &= \frac{\rho}{1-p(-\kappa,s,\sigma)} \\
y^{ss,u} &= \rho\frac{q^{(1)}(-\kappa,s,\sigma)}{1-p(-\kappa,s,\sigma)} + \rho^2\left(\gamma^{(1)}(s,\sigma) + \mathcal{Z}_{ss,u}(\rho,\sigma,s)\right) + \mathcal{Y}_{uu,s}(\rho,\sigma,s) \\
y^{ss,v} &= \rho\frac{q^{(2)}(-\kappa,s,\sigma)}{1-p(-\kappa,s,\sigma)} + \rho^2\left(\gamma^{(2)}(s,\sigma) + \mathcal{Z}_{ss,v}(\rho,\sigma,s)\right).
\end{aligned}
\tag{4.17}
$$

Therefore, the manifold $W^u(\mathbf{P}_2) \cap \Sigma^{out}$ in the original variables is

$$\begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u^+ \\ 0 \\ 0 \\ 0 \end{pmatrix} + T(s,\sigma) \begin{pmatrix} \mathcal{Y}_u(\rho,s,\sigma) \\ \rho \frac{q^{(1)}(-\kappa,s,\sigma)}{1-p(-\kappa,s,\sigma)} + \rho^2 \left(\gamma^{(1)}(s,\sigma) + \mathcal{Z}_{ss,u}(\rho,\sigma,s)\right) + \mathcal{Y}_{uu,s}(\rho,\sigma,s) \\ \frac{\rho}{1-p(-\kappa,s,\sigma)} \\ \rho \frac{q^{(2)}(-\kappa,s,\sigma)}{1-p(-\kappa,s,\sigma)} + \rho^2 \left(\gamma^{(2)}(s,\sigma) + \mathcal{Z}_{ss,v}(\rho,\sigma,s)\right) \end{pmatrix}. \quad (4.18)$$

For future reference, we refer to

$$\mathcal{W}(\rho,s,\sigma) := T(s,\sigma) \begin{pmatrix} \mathcal{Y}_u(\rho,s,\sigma) \\ \rho \frac{q^{(1)}(-\kappa,s,\sigma)}{1-p(-\kappa,s,\sigma)} + \rho^2 \left(\gamma^{(1)}(s,\sigma) + \mathcal{Z}_{ss,u}(\rho,\sigma,s)\right) + \mathcal{Y}_{uu,s}(\rho,\sigma,s) \\ \frac{\rho}{1-p(-\kappa,s,\sigma)} \\ \rho \frac{q^{(2)}(-\kappa,s,\sigma)}{1-p(-\kappa,s,\sigma)} + \rho^2 \left(\gamma^{(2)}(s,\sigma) + \mathcal{Z}_{ss,v}(\rho,\sigma,s)\right) \end{pmatrix}. \quad (4.19)$$

4.4 Expansions of relevant quantities

Before proceeding to compare $W^{ss}(\mathbf{P}_0)$ and $W^u(\mathbf{P}_2)$, we first interpret some of the terms in $\mathcal{W}$ and derive alternate expressions that will prove useful later.

Lemma 12. *Recall $\mathcal{W}(\rho,s,\sigma)$ from (4.19). We have that $\mathcal{W}(0,s,\sigma) \subset W^u(\mathbf{P}_1)$. Furthermore, $\mathcal{W}(0,s^*,\sigma)$ is colinear with θ_1 and*

$$\frac{\partial \mathcal{W}}{\partial s}(0,s^*,\sigma) = \theta_s = (\theta_s^1, \theta_s^2, 0, 0)^T, \qquad \frac{\partial \mathcal{W}}{\partial \sigma}(0,s^*,\sigma^*) = 0.$$

Proof. First observe that

$$\mathcal{W}(0,s,\sigma) = T(s,\sigma) \begin{pmatrix} \mathcal{Y}_u^0(s) \\ \mathcal{Y}_{uu,s}^0(s) \\ 0 \\ 0 \end{pmatrix}.$$

Recalling the expression in (4.15) we see that the limit $\rho = 0$ corresponds to a value in the unstable manifold of $\mathbf{P}_1$. When $s = s^*$, the unstable manifold includes the heteroclinic orbit $(U_p(\xi), U_p'(\xi), 0, 0)^T$, with tangent vector θ_1. ∎

Lemma 13. *The vector*

$$\frac{\partial W}{\partial \rho}(0,s^*,\sigma^*) = r_1\theta_1 + r_2\theta_2 = T(s^*,\sigma^*) \begin{pmatrix} \mathcal{Y}_u^1(s^*,\sigma^*) \\ \frac{q^{(1)}(-\kappa,s^*,\sigma^*)}{1-p(-\kappa,s^*,\sigma^*)} + \mathcal{Y}_{uu,s}^1(\sigma^*,s^*) \\ \frac{1}{1-p(-\kappa,s^*,\sigma^*)} \\ \frac{q^{(2)}(-\kappa,s^*,\sigma^*)}{1-p(-\kappa,s^*,\sigma^*)} \end{pmatrix},$$

where it follows that

$$
\begin{aligned}
r_1 &= \frac{1}{\langle\theta_1,\theta_1\rangle}\left(\mathcal{Y}_u^1(s^*,\sigma^*)\langle\theta_1,e_u^+\rangle + \left(\frac{q^{(1)}(-\kappa,s^*,\sigma^*)}{1-p(-\kappa,s^*,\sigma^*)} + \mathcal{Y}_{uu,s}^1(\sigma^*,s^*)\right)\langle\theta_1,e_u^-\rangle\right.\\
&+ \left.\frac{1}{1-p(-\kappa,s^*,\sigma^*)}\langle\theta_1,e_v^+\rangle + \frac{q^{(2)}(-\kappa,s^*,\sigma^*)}{1-p(-\kappa,s^*,\sigma^*)}\langle\theta_1,e_v^-\rangle\right)\\
r_2 &= \frac{1}{\langle\theta_2,\theta_2\rangle}\left(\mathcal{Y}_u^1(s^*,\sigma^*)\langle\theta_2,e_u^+\rangle + \left(\frac{q^{(1)}(-\kappa,s^*,\sigma^*)}{1-p(-\kappa,s^*,\sigma^*)} + \mathcal{Y}_{uu,s}^1(\sigma^*,s^*)\right)\langle\theta_2,e_u^-\rangle\right.\\
&+ \left.\frac{1}{1-p(-\kappa,s^*,\sigma^*)}\langle\theta_2,e_v^+\rangle + \frac{q^{(2)}(-\kappa,s^*,\sigma^*)}{1-p(-\kappa,s^*,\sigma^*)}\langle\theta_2,e_v^-\rangle\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\langle\theta_1,e_u^\pm\rangle &= U_p'(\xi_0) + U_p''(\xi_0)\nu_u^\pm\\
\langle\theta_1,e_v^\pm\rangle &= -\frac{F_u(\mathbf{p}_1)}{d_u(\nu_v^\pm)}\left(U_p'(\xi_0) + U_p''(\xi_0)\nu_v^\pm\right)\\
\langle\theta_2,e_u^\pm\rangle &= a_1(\xi_0) + a_2(\xi_0)\nu_u^\pm\\
\langle\theta_2,e_v^\pm\rangle &= -\frac{F_u(\mathbf{p}_1)}{d_u(\nu_v^\pm)}\left(a_1(\xi_0) + a_2(\xi_0)\nu_v^\pm\right) + \phi(\xi_0) + \nu_v^\pm\phi'(\xi_0).
\end{aligned}
$$

Proof. Recall that those terms that are linear in ρ originate in (4.14) and point and result from following the weak-unstable eigenspace along the unstable manifold of $\mathbf{P}_1$ to the section Σ^{out}. The subspace $\bar{z}^{ss,u} = \bar{z}^{ss,v} = 0$ is invariant in (4.6) and therefore this vector is the weak stable tangent space of $\mathbf{P}_1$ tracked forward along the unstable manifold. In Section 3.1, we calculated that this space coincides with span$\{\theta_1,\theta_2\}$ and the result therefore follows.

∎

Lemma 14. *We have the further expansions of* $\mathcal{W}(\rho,s,\sigma)$

$$
\theta_{\rho\sigma} := \frac{\partial^2\mathcal{W}}{\partial\rho\partial\sigma}(0,s^*,\sigma^*) = r_2\left(\beta_1\psi_1 + \beta_2\psi_2\right),
$$

and

$$
\theta_{\rho^2} := \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial\rho^2}(0,s^*,\sigma^*) = T(s^*,\sigma^*)\begin{pmatrix}\mathcal{Y}_u^2(s^*,\sigma^*)\\ \gamma^{(1)}(s^*,\sigma^*) + \mathcal{Y}_{uu,s}^2(s^*,\sigma^*)\\ 0\\ \gamma^{(2)}(s^*,\sigma^*)\end{pmatrix}.
$$

Proof. The expression for θ_{ρ^2} follows from a calculation.

For $\theta_{\rho\sigma}$, we recall Section 3.1 where the tangent space to the weak- unstable manifold was tracked and its dependence on σ was ascertained; see (3.25) for the expression $\mathcal{R}(p_1,q_1,\sigma)$. Using the parameterization of the subspace in terms of r_1 and r_2, we can write the subspace as

$$
\begin{aligned}
\frac{\partial\mathcal{W}}{\partial\rho}(0,s^*,\sigma) &= r_1\theta_1 + r_2\theta_2 + \frac{\langle\psi_1,\mathcal{R}(r_1U_p'(\xi_0)+r_2a_1(\xi_0),r_2\phi(\xi_0),\sigma)\rangle}{\Omega_1}\psi_1\\
&= \frac{\langle\psi_2,\mathcal{R}((r_1U_p'(\xi_0)+r_2a_1(\xi_0),r_2\phi(\xi_0),\sigma)\rangle}{\Omega_2}\psi_2,
\end{aligned}
$$

where

$$\langle\psi_1,\mathcal{R}\rangle = r_2(\sigma-\sigma^*)\int_{-\infty}^{\xi_0} b_2(\tau)\left(\frac{1}{(\sigma^*)^2}G_v(U_p(\tau),0+\frac{s^*}{(\sigma^*)^2}\frac{\phi'(\tau)}{\phi(\tau)}\right)\mathrm{d}\tau,$$

and

$$\langle\psi_2,\mathcal{R}\rangle = r_2(\sigma-\sigma^*)\int_{-\infty}^{\xi_0} e^{\frac{s^*}{\sigma^*}\tau}\left(\frac{1}{(\sigma^*)^2}G_v(U_p(\tau),0)\phi(\tau)^2+\frac{s^*}{(\sigma^*)^2}\phi(\tau)\phi'(\tau)\right)\mathrm{d}\tau.$$

∎

5 Resolving the bifurcation equation: Proof of Theorem 1

We now establish Theorem 1. Recall the expression (3.15) that describes the manifold $W^{ss}(\mathbf{P}_0)$ near the section Σ^{out}. Similarly, we have expansion (4.18) that describes $W^u(\mathbf{P}_2)$ within the section Σ^{out}. Equating these expressions we obtain an implicit bifurcation equation

$$\begin{aligned}0=\mathcal{F}(\rho,\eta_1,\eta_2,s,\sigma;\xi_0,\kappa) \quad := \quad & \Delta_f(\xi_0)+\eta_1\theta_1+\eta_2\theta_2+h_1(\eta_1,\eta_2,s,\sigma)\psi_1+h_2(\eta_1,\eta_2,s,\sigma)\psi_2\\ & -\mathcal{W}(\rho,s,\sigma).\end{aligned}$$

First, we relate ξ_0 and κ by imposing that $\mathcal{F}(0,0,0,0,s^*,\sigma^*;\xi_0,\kappa)=0$. This is possible since $\Delta_f(\xi_0)$ and $\mathcal{W}(0,s^*,\sigma^*)$ both lie in the heteroclinic orbit $(U_p(\xi),U_p'(\xi),0,0)^T$. We henceforth suppress the dependence of $\mathcal{F}$ on κ.

Using the expansions in Lemma 12 through Lemma 14, we simplify $\mathcal{F}$ to

$$\begin{aligned}\mathcal{F}(\rho,\eta_1,\eta_2,s,\sigma) \quad = \quad & \eta_1\theta_1+\eta_2\theta_2+(s-s^*)\Gamma_0\psi_1+h_1(\eta_1,\eta_2,s,\sigma)\psi_1+h_2(\eta_1,\eta_2,s,\sigma)\psi_2\\ & -(s-s^*)\theta_s-\rho(r_1\theta_1+r_2\theta_2)-\rho^2\theta_{\rho^2}-\rho(s-s^*)\theta_{\rho s}-\rho(\sigma-\sigma^*)\theta_{\rho\sigma}+\mathcal{O}(3).\end{aligned}$$

We wish to employ a Liapunov-Schmidt reduction and so we compute the partials of $\mathcal{F}$,

$$D_{\eta_1,\eta_2,\rho,s}\mathcal{F}=\begin{pmatrix}\theta_1 & \theta_2 & -r_1\theta_1-r_2\theta_2 & \Gamma_0\psi_1-\theta_s\end{pmatrix}.$$

The Jacobian has rank three, so we project onto the range by projecting onto the vectors θ_1,θ_2 and ψ_1. We obtain

$$\begin{aligned}0 \quad = \quad & \eta_1\langle\theta_1,\theta_1\rangle-(s-s^*)\langle\theta_1,\theta_s\rangle-\rho r_1\langle\theta_1,\theta_1\rangle\\ & -\rho^2\langle\theta_1,\theta_{\rho^2}\rangle-\rho(s-\sigma)\langle\theta_1,\theta_{\rho s}\rangle-\rho(\sigma-\sigma^*)\langle\theta_1,\theta_{\rho\sigma}\rangle+\mathcal{O}(3),\end{aligned}$$

$$\begin{aligned}0 \quad = \quad & \eta_2\langle\theta_2,\theta_2\rangle-(s-s^*)\langle\theta_2,\theta_s\rangle-\rho r_2\langle\theta_2,\theta_2\rangle\\ & -\rho^2\langle\theta_2,\theta_{\rho^2}\rangle-\rho(s-\sigma)\langle\theta_2,\theta_{\rho s}\rangle-\rho(\sigma-\sigma^*)\langle\theta_2,\theta_{\rho\sigma}\rangle+\mathcal{O}(3),\end{aligned}$$

$$\begin{aligned}0 \quad = \quad & (s-s^*)\Gamma_0\langle\psi_1,\psi_1\rangle+h_1(\eta_1,\eta_2,s,\sigma)\langle\psi_1,\psi_1\rangle-(s-s^*)\langle\psi_1,\theta_s\rangle\\ & -\rho^2\langle\psi_1,\theta_{\rho^2}\rangle-\rho(s-s^*)\langle\psi_1,\theta_{\rho s}\rangle-\rho(\sigma-\sigma^*)\langle\psi_1,\theta_{\rho\sigma}\rangle+\mathcal{O}(3).\end{aligned}$$

This constitutes an implicit set of equations which we write as $\mathcal{G}(\eta_1, \eta_2, s, \rho, \sigma) = 0$. Now, a simple computation leads to

$$D_{\eta_1,\eta_2,s}\mathcal{G}(0, s^*, \sigma^*) = \begin{pmatrix} \langle\theta_1, \theta_1\rangle & 0 & \langle\theta_1, \theta_s\rangle \\ 0 & \langle\theta_2, \theta_2\rangle & \langle\theta_2, \theta_s\rangle \\ 0 & 0 & \Omega_1\Gamma_0 - \langle\psi_1, \theta_s\rangle \end{pmatrix}.$$

At the same time, we compute

$$D_\rho\mathcal{G}(0, s^*, \sigma^*) = \begin{pmatrix} -r_1\langle\theta_1, \theta_1\rangle \\ -r_2\langle\theta_2, \theta_2\rangle \\ 0 \end{pmatrix}.$$

Therefore, the implicit function theorem ensures a solution $\mathcal{G}(\eta_1(\rho, \sigma), \eta_2(\rho, \sigma), s(\rho, \sigma)) = 0$ with

$$\begin{aligned} \eta_1(\rho, \sigma) &= r_1\rho + \mathrm{g}_1(\rho, \sigma), \\ \eta_2(\rho, \sigma) &= r_2\rho + \mathrm{g}_2(\rho, \sigma), \\ s(\rho, \sigma) - s^* &= \mathrm{G}_s(\rho, \sigma) = \rho^2\frac{1}{\Gamma}\left(\langle\psi_1, \theta_{\rho^2}\rangle - r_1^2\Omega_1\frac{\partial^2 h_1}{\partial\eta_1^2} - r_2^2\Omega_1\frac{\partial^2 h_1}{\partial\eta_2^2}\right) \\ &+ \rho(\sigma - \sigma^*)\frac{1}{\Gamma}\left(\langle\psi_1, \theta_{\rho\sigma}\rangle - r_2\Omega_1\frac{\partial^2 h_1}{\partial\eta_2\partial\sigma}\right) + \mathcal{O}(3), \end{aligned}$$

where the functions g_1, g_2 and G_s are all quadratic order or higher and

$$\Gamma = \Omega_1\Gamma_0 - \langle\psi_1, \theta_s\rangle = \int_{-\infty}^{\infty} e^{s^*\tau}(U_p'(\tau))^2 \mathrm{d}\tau.$$

We then consider the implicit equation

$$\begin{aligned} 0 = \mathcal{H}(\rho, \sigma) &:= \langle\psi_2, \mathcal{F}(\rho, r_1\rho + \mathrm{g}_1(\rho, \sigma), r_2\rho + \mathrm{g}_2(\rho, \sigma), s^* + \mathrm{G}_s(\rho, \sigma), \sigma)\rangle \\ &= h_2(r_1\rho + \mathrm{g}_1(\rho, \sigma), r_2\rho + \mathrm{g}_2(\rho, \sigma), \mathrm{G}_s(\rho, \sigma), \sigma)\langle\psi_2, \psi_2\rangle - \langle\psi_2, \mathcal{W}(\rho, s^* + \mathrm{G}_s(\rho, \sigma), \sigma)\rangle. \end{aligned}$$

Note that $\mathcal{H}(\rho, \sigma) = \rho\tilde{\mathcal{H}}(\rho, \sigma)$. We therefore expand, focusing on quadratic terms in $\mathcal{H}$,

$$\mathcal{H}(\rho, \sigma) = h_2^{(2)}(r_1\rho, r_2\rho, 0, \sigma) - \rho^2\langle\psi_2, \theta_{\rho^2}\rangle - \rho(\sigma - \sigma^*)\langle\psi_2, \theta_{\rho\sigma}\rangle.$$

There are three non-zero terms in $h_2^{(2)}$ that contribute to the quadratic term – namely the terms η_2^2, $\eta_1\eta_2$ and $\eta_2\sigma$. After factoring, we find the solution

$$\rho = M_\rho(\sigma - \sigma^*) + \mathrm{G}_\sigma(\sigma),$$

where

$$M_\rho = \frac{\langle\psi_2, \theta_{\rho\sigma}\rangle - r_2\Omega_2\frac{\partial^2 h_2}{\partial\eta_2\partial\sigma}}{r_2^2\Omega_2\frac{\partial^2 h_2}{\partial\eta_2^2} + r_1r_2\Omega_2\frac{\partial^2 h_2}{\partial\eta_1\partial\eta_2} - \langle\psi_2, \theta_{\rho^2}\rangle},$$

and $\mathrm{G}_\sigma(\sigma)$ collects higher-order terms. We require ρ to be positive to ensure positivity of the solution. Therefore, the sign of M_ρ dictates whether the bifurcation to locked fronts is sub or super critical. With this solution, we can then determine whether the front is sped up or slowed down by inputting this into $\mathrm{G}_s(\rho, \sigma)$.

Simplification of the term M_ρ. We now make several simplifications. First, note that by Lemma 14 the numerator simplifies with

$$\frac{1}{r_2}\langle\psi_2,\theta_{\rho\sigma}\rangle - \Omega_2\frac{\partial^2 h_2}{\partial\eta_2\partial\sigma} = \int_{-\infty}^{\infty} e^{\frac{s^*}{\sigma^*}\xi}\left(\frac{G_v(U_p(\xi),0)}{(\sigma^*)^2}(\phi(\xi))^2 + \frac{s^*}{(\sigma^*)^2}\phi'(\xi)\phi(\xi)\right)\mathrm{d}\xi.$$

We then use the identity $G_v(U_p(\xi),0)\phi(\xi)+s^*\phi'(\xi) = -\sigma^*\phi''(\xi)$ and integrate by parts

$$\begin{aligned}
\frac{1}{r_2}\langle\psi_2,\theta_{\rho\sigma}\rangle - \Omega_2\frac{\partial^2 h_2}{\partial\eta_2\partial\sigma} &= -\frac{1}{\sigma^*}\int_{-\infty}^{\infty} e^{\frac{s^*}{\sigma^*}\xi}\phi(\xi)\phi''(\xi)\mathrm{d}\xi,\\
&= \frac{1}{\sigma^*}\int_{-\infty}^{\infty}\phi'(\xi)\left(\phi'(\xi)e^{\frac{s^*}{\sigma^*}\xi} + \frac{s^*}{(\sigma^*)^2}\phi(\xi)e^{\frac{s^*}{\sigma^*}\xi}\right)\mathrm{d}\xi,\\
&= \frac{1}{\sigma^*}\int_{-\infty}^{\infty}\phi'(\xi)\phi(\xi)e^{\frac{s^*}{\sigma^*}\xi}\left(\frac{\phi'(\xi)}{\phi(\xi)} + \frac{s^*}{(\sigma^*)^2}\right)\mathrm{d}\xi.
\end{aligned}$$

Now, we note that the term inside the parenthesis is positive, since for any ξ we have that $G_v(U_p(\xi),0) > 0$ and therefore

$$Z_{22}(\xi) = \frac{\phi'(\xi)}{\phi(\xi)} > -\frac{s^*}{2\sigma^*} - \frac{s^*}{2\sigma^*}\sqrt{(s^*)^2 - 4G_v(U_p(\xi),0)} > -\frac{s^*}{\sigma^*}.$$

We finally find that

$$\operatorname{sign}\left(\frac{1}{r_2}\langle\psi_2,\theta_{\rho\sigma}\rangle - \Omega_2\frac{\partial^2 h_2}{\partial\eta_2\partial\sigma}\right) = \operatorname{sign}(\phi') = -1,$$

since $\phi' < 0$. And thus $M_\rho \neq 0$ and the sign of M_ρ is determined by the opposite sign of its denominator:

$$\operatorname{sign}M_\rho = -\operatorname{sign}\left(r_2\Omega_2\frac{\partial^2 h_2}{\partial\eta_2^2} + r_1\Omega_2\frac{\partial^2 h_2}{\partial\eta_1\partial\eta_2} - \frac{1}{r_2}\langle\psi_2,\theta_{\rho^2}\rangle\right),$$

where we recall the following expressions for each term

$$\begin{aligned}
\Omega_2\frac{\partial^2 h_2}{\partial\eta_2^2} &= \int_{\xi_0}^{\infty} e^{\frac{s^*}{\sigma^*}\xi}\left(\frac{G_{uv}(U_p(\xi),0)}{\sigma^*}a_1(\xi)\phi(\xi)^2 + \frac{G_{vv}(U_p(\xi),0)}{2\sigma^*}\phi^3(\xi)\right)\mathrm{d}\xi,\\
\Omega_2\frac{\partial^2 h_2}{\partial\eta_1\partial\eta_2} &= \left(\tilde\phi''(\xi_0)\tilde\phi(\xi_0) - (\tilde\phi'(\xi_0))^2\right),\\
\langle\psi_2,\theta_{\rho^2}\rangle &= e^{\frac{s^*}{\sigma^*}\xi_0}\gamma^{(2)}(s^*,\sigma^*)\left(\nu_v^-(s^*,\sigma^*)\phi(\xi_0) - \phi'(\xi_0)\right),
\end{aligned}$$

with from Lemma 18,

$$\gamma^{(2)}(s^*,\sigma^*) = \frac{1}{\sigma(\nu_v^-(s,\sigma)-\nu_v^+(s,\sigma))(2\nu_v^+(s,\sigma)-\nu_v^-(s,\sigma))}\left(\frac{F_v(\mathbf{p}_1)}{d_u(\nu_v^-(s,\sigma))}G_{uv}(\mathbf{p}_1) - \frac{G_{vv}(\mathbf{p}_1)}{2}\right).$$

Expansion of $s - s^*$. With an expansion for ρ as a function of $\sigma - \sigma^*$, we finally obtain an expansion for $s - s^*$ as a function of $\sigma - \sigma^*$. Let

$$s - s^* = M_s(\sigma-\sigma^*)^2 + \mathcal{O}(3),$$

where

$$M_s = \frac{M_\rho}{\Gamma}\left(M_\rho\left(\langle\psi_1,\theta_{\rho^2}\rangle - r_1^2\Omega_1\frac{\partial^2 h_1}{\partial\eta_1^2} - r_2^2\Omega_1\frac{\partial^2 h_1}{\partial\eta_2^2}\right) + r_2\int_{-\infty}^{\infty} b_2(\xi)\left(\frac{G_v(U_p(\xi),0)}{(\sigma^*)^2}\phi(\xi) + \frac{s^*}{(\sigma^*)^2}\phi'(\xi)\right)\mathrm{d}\xi\right).$$

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A Expansions of $h_{1,2}$

We return to derive expressions for those terms in the quadratic expansions of h_1 and h_2 from Lemma 6 that are required for the resolution of the bifurcation equation. To simplify the presentation, we recall some of the notations that were used in Section 3. The maps $h_{1,2}$ are determined by projecting equation (3.16) onto $\psi_{1,2}$ to obtain the expressions

$$h_{1,2}(\eta_1,\eta_2,s,\sigma) = -\frac{1}{\Omega_{1,2}}\int_{\xi_0}^{\infty} \langle \psi_{1,2}(\xi), N(Q^*(\xi,\eta_1,\eta_2,s,\sigma),\xi,s,\sigma)\rangle \,\mathrm{d}\xi, \tag{A.1}$$

where $\Omega_{1,2} = \langle \psi_{1,2}(\xi_0), \psi_{1,2}(\xi_0)\rangle$ and $Q^*(\cdot,\eta_1,\eta_2,s,\sigma)$ is the fixed point solution of the operator T introduced in Lemma 5 (see equation (3.5)). As shown in the proof of Lemma 6, the maps $h_{1,2}$ are at least quadratic or of higher order in all their arguments, and the associated quadratic expansions of $h_{1,2}$ can be obtained by collecting the quadratic expansions of the following quantities:

$$\tilde{h}_{1,2}(\eta_1,\eta_2,s,\sigma) := -\frac{1}{\Omega_{1,2}}\int_{\xi_0}^{\infty} \langle \psi_{1,2}(\xi), N(\eta_1\theta_1(\xi)+\eta_2\theta_2(\xi)+(s-s^*)\theta_s(\xi),\xi,s,\sigma)\rangle \,\mathrm{d}\xi, \tag{A.2}$$

where we approximated $Q^*(\xi,\eta_1,\eta_2,s,\sigma)$ by $Q^0(\xi) = \eta_1\theta_1(\xi)+\eta_2\theta_2(\xi)+(s-s^*)\theta_s(\xi)$. The definition of the nonlinear term $N(z,\xi,s,\sigma)$ is

$$N(z,\xi,s,\sigma) = \begin{pmatrix} 0 \\ N_p(z,\xi,s,\sigma) \\ 0 \\ N_q(z,\xi,s,\sigma) \end{pmatrix}, \quad z = (p_1,p_2,q_1,q_2)^T,$$

with quadratic expansions of $N_{p,q}$ denoted $N_{p,q}^{(2)}$ given by

$$\begin{aligned} N_p^{(2)}(z,\xi,s,\sigma) =& -(s-s^*)p_2 - \frac{F_{uu}(U_p(\xi),0)}{2}p_1^2 - F_{uv}(U_p(\xi),0)p_1q_1 - \frac{F_{vv}(U_p(\xi),0)}{2}q_1^2, \\ N_q^{(2)}(z,\xi,s,\sigma) =& -\frac{1}{\sigma^*}(s-s^*)q_2 + \frac{s^*}{(\sigma^*)^2}(\sigma-\sigma^*)q_2 - \frac{G_{uv}(U_p(\xi),0)}{\sigma^*}p_1q_1 - \frac{G_{vv}(U_p(\xi),0)}{2\sigma^*}q_1^2 \\ &+ \frac{G_v(U_p(\xi),0)}{(\sigma^*)^2}q_1(\sigma-\sigma^*). \end{aligned}$$

To continue, we need expansions for $N_{p,q}^{(2)}$ in terms of η_1, η_2, $(s-s^*)$ and $(\sigma-\sigma^*)$. To accomplish this, we recall that we have

$$\begin{aligned} p_1(\xi) &= \eta_1U_p'(\xi)+\eta_2a_1(\xi)+(s-s^*)\theta_s^1(\xi), \\ p_2(\xi) &= \eta_1U_p''(\xi)+\eta_2a_2(\xi)+(s-s^*)\theta_s^2(\xi), \\ q_1(\xi) &= \eta_2\phi(\xi), \\ q_2(\xi) &= \eta_2\phi'(\xi). \end{aligned}$$

To simplify the presentation, we will use the following notation

$$N_{p,q}^{(2)}(z,\xi,s,\sigma) = \sum_{i+j+k+l=2} \eta_1^i \eta_2^j (s-s^*)^k (\sigma-\sigma^*)^l \mathbf{N}_{p,q}^{(i,j,k,l)}(\xi).$$

We then obtain the expressions:

$$\begin{aligned}
\mathcal{O}(\eta_1^2): \quad \mathbf{N}_p^{(2,0,0,0)}(\xi) &= -\tfrac{F_{uu}(U_p(\xi),0)}{2}\left(U_p'(\xi)\right)^2,\\
\mathcal{O}(\eta_1\eta_2): \quad \mathbf{N}_p^{(1,1,0,0)}(\xi) &= -\left(F_{uu}(U_p(\xi),0)U_p'(\xi)a_1(\xi) + F_{uv}(U_p(\xi),0)U_p'(\xi)\phi(\xi)\right),\\
\mathcal{O}(\eta_2^2): \quad \mathbf{N}_p^{(0,2,0,0)}(\xi) &= -\left(\tfrac{F_{uu}(U_p(\xi),0)}{2}a_1^2(\xi) + F_{uv}(U_p(\xi),0)a_1(\xi)\phi(\xi) + \tfrac{F_{vv}(U_p(\xi),0)}{2}\phi^2(\xi)\right)\\
\mathcal{O}(\eta_1|s-s^*|): \quad \mathbf{N}_p^{(1,0,1,0)}(\xi) &= -\left(U_p''(\xi) + F_{uu}(U_p(\xi),0)U_p'(\xi)\theta_s^1(\xi)\right),\\
\mathcal{O}(\eta_2|s-s^*|): \quad \mathbf{N}_p^{(0,1,1,0)}(\xi) &= -\left(a_2(\xi) + F_{uv}(U_p(\xi),0)\phi(\xi)\theta_s^1(\xi) + F_{uu}(U_p(\xi),0)a_1(\xi)\theta_s^1(\xi)\right),\\
\mathcal{O}(|s-s^*|^2): \quad \mathbf{N}_p^{(0,0,2,0)}(\xi) &= -\left(\theta_s^2(\xi) + \tfrac{F_{uu}(U_p(\xi),0)}{2}(\theta_s^1(\xi))^2\right),
\end{aligned}$$

all other quadratic terms in the expansion being equal to zero. Regarding $N_q^{(2)}$, we get

$$\begin{aligned}
\mathcal{O}(\eta_1\eta_2): \quad \mathbf{N}_q^{(1,1,0,0)}(\xi) &= -\tfrac{G_{uv}(U_p(\xi),0)}{\sigma^*}U_p'(\xi)\phi(\xi),\\
\mathcal{O}(\eta_2^2): \quad \mathbf{N}_q^{(0,2,0,0)}(\xi) &= -\left(\tfrac{G_{uv}(U_p(\xi),0)}{\sigma^*}a_1(\xi)\phi(\xi) + \tfrac{G_{vv}(U_p(\xi),0)}{2\sigma^*}\phi^2(\xi)\right),\\
\mathcal{O}(\eta_2|s-s^*|): \quad \mathbf{N}_q^{(0,1,1,0)}(\xi) &= -\left(\tfrac{1}{\sigma^*}\phi'(\xi) + \tfrac{G_{uv}(U_p(\xi),0)}{\sigma^*}\theta_s^1(\xi)\phi(\xi)\right),\\
\mathcal{O}(\eta_2|\sigma-\sigma^*|): \quad \mathbf{N}_q^{(0,1,0,1)}(\xi) &= \tfrac{G_v(U_p(\xi),0)}{(\sigma^*)^2}\phi(\xi) + \tfrac{s^*}{(\sigma^*)^2}\phi'(\xi),
\end{aligned}$$

all other quadratic terms in the expansion being equal to zero.

As a consequence, we can now collect all quadratic terms in the expansions of the maps $h_{1,2}$ by identification. Namely, if one sets

$$h_{1,2}(\eta_1,\eta_2,s,\sigma) := \sum_{\substack{i,j,k,l\geq 0\\ i+j+k+l\geq 2}} \eta_1^i\eta_2^j(s-s^*)^k(\sigma-\sigma^*)^l \mathbf{h}_{1,2}^{(i,j,k,l)},$$

then using equation (A.2), we get the following relations for the quadratic terms. For all $i,j,k,l\geq 0$ with $i+j+k+l=2$ we have

$$\begin{aligned}
\mathbf{h}_1^{(i,j,k,l)} &= -\frac{1}{\Omega_1}\int_{\xi_0}^{\infty}\left(e^{s^*\xi}U_p'(\xi)\mathbf{N}_p^{(i,j,k,l)}(\xi) + b_2(\xi)\mathbf{N}_q^{(i,j,k,l)}(\xi)\right)\mathrm{d}\xi,\\
\mathbf{h}_2^{(i,j,k,l)} &= -\frac{1}{\Omega_2}\int_{\xi_0}^{\infty} e^{\frac{s^*}{\sigma^*}\xi}\phi(\xi)\mathbf{N}_q^{(i,j,k,l)}(\xi)\mathrm{d}\xi.
\end{aligned}$$

We have the following Lemma which summarizes the previous computations.

Lemma 15. *The nonlinear maps $h_1(\eta_1,\eta_2,s,\sigma)$ and $h_2(\eta_1,\eta_2,s,\sigma)$ from Lemma 6 admit the following* *quadratic expansions. For all $i, j, k, l \geq 0$ with $i + j + k + l = 2$ we have for $h_1(\eta_1, \eta_2, s, \sigma)$:*

$$
\begin{aligned}
\mathcal{O}(\eta_1^2): \ \mathbf{h}_1^{(2,0,0,0)} &= \frac{1}{\Omega_1} \int_{\xi_0}^{\infty} e^{s^*\xi} \frac{F_{uu}(U_p(\xi),0)}{2} \left(U_p'(\xi)\right)^3 \mathrm{d}\xi, \\
\mathcal{O}(\eta_1\eta_2): \ \mathbf{h}_1^{(1,1,0,0)} &= \frac{1}{\Omega_1} \int_{\xi_0}^{\infty} e^{s^*\xi} U_p'(\xi) \left(F_{uu}(U_p(\xi),0)U_p'(\xi)a_1(\xi) + F_{uv}(U_p(\xi),0)U_p'(\xi)\phi(\xi)\right) \mathrm{d}\xi \\
&\quad + \frac{1}{\Omega_1} \int_{\xi_0}^{\infty} b_2(\xi) \frac{G_{uv}(U_p(\xi),0)}{\sigma^*} U_p'(\xi)\phi(\xi) \mathrm{d}\xi, \\
\mathcal{O}(\eta_2^2): \ \mathbf{h}_1^{(0,2,0,0)} &= \frac{1}{\Omega_1} \int_{\xi_0}^{\infty} e^{s^*\xi} U_p'(\xi) \left(\frac{F_{uu}(U_p(\xi),0)}{2} a_1^2(\xi) + F_{uv}(U_p(\xi),0)a_1(\xi)\phi(\xi)\right) \mathrm{d}\xi \\
&\quad + \frac{1}{\Omega_1} \int_{\xi_0}^{\infty} b_2(\xi) \left(\frac{G_{uv}(U_p(\xi),0)}{\sigma^*} a_1(\xi)\phi(\xi) + \frac{G_{vv}(U_p(\xi),0)}{2\sigma^*}\phi^2(\xi)\right) \mathrm{d}\xi \\
&\quad + \frac{1}{\Omega_1} \int_{\xi_0}^{\infty} e^{s^*\xi} U_p'(\xi) \frac{F_{vv}(U_p(\xi),0)}{2} \phi^2(\xi) \mathrm{d}\xi, \\
\mathcal{O}(\eta_1|s-s^*|): \ \mathbf{h}_1^{(1,0,1,0)} &= \frac{1}{\Omega_1} \int_{\xi_0}^{\infty} e^{s^*\xi} U_p'(\xi) \left(U_p''(\xi) + F_{uu}(U_p(\xi),0)U_p'(\xi)\theta_s^1(\xi)\right) \mathrm{d}\xi, \\
\mathcal{O}(\eta_2|s-s^*|): \ \mathbf{h}_1^{(0,1,1,0)} &= \frac{1}{\Omega_1} \int_{\xi_0}^{\infty} e^{s^*\xi} U_p'(\xi) \left(a_2(\xi) + F_{uv}(U_p'(\xi),0)\phi(\xi)\theta_s^1(\xi) + F_{uu}(U_p(\xi),0)a_1(\xi)\theta_s^1(\xi)\right) \mathrm{d}\xi \\
&\quad + \frac{1}{\Omega_1} \int_{\xi_0}^{\infty} b_2(\xi) \left(\frac{1}{\sigma^*}\phi'(\xi) + \frac{G_{uv}(U_p(\xi),0)}{\sigma^*}\theta_s^1(\xi)\phi(\xi)\right) \mathrm{d}\xi, \\
\mathcal{O}(\eta_2|\sigma-\sigma^*|): \ \mathbf{h}_1^{(0,1,0,1)} &= -\frac{1}{\Omega_1} \int_{\xi_0}^{\infty} b_2(\xi) \left(\frac{G_v(U_p(\xi),0)}{(\sigma^*)^2}\phi(\xi) + \frac{s^*}{(\sigma^*)^2}\phi'(\xi)\right) \mathrm{d}\xi, \\
\mathcal{O}(|s-s^*|^2): \ \mathbf{h}_1^{(0,0,2,0)} &= \frac{1}{\Omega_1} \int_{\xi_0}^{\infty} e^{s^*\xi} U_p'(\xi) \left(\theta_s^2(\xi) + \frac{F_{uu}(U_p(\xi),0)}{2}(\theta_s^1(\xi))^2\right) \mathrm{d}\xi,
\end{aligned}
$$

and for $h_2(\eta_1, \eta_2, s, \sigma)$:

$$
\begin{aligned}
\mathcal{O}(\eta_1\eta_2): \ \mathbf{h}_2^{(1,1,0,0)} &= \frac{1}{\Omega_2} \int_{\xi_0}^{\infty} e^{\frac{s^*}{\sigma^*}\xi} \frac{G_{uv}(U_p(\xi),0)}{\sigma^*} U_p'(\xi) \left(\phi(\xi)\right)^2 \mathrm{d}\xi, \\
\mathcal{O}(\eta_2^2): \ \mathbf{h}_2^{(0,2,0,0)} &= \frac{1}{\Omega_2} \int_{\xi_0}^{\infty} e^{\frac{s^*}{\sigma^*}\xi} \left(\frac{G_{uv}(U_p(\xi),0)}{\sigma^*} a_1(\xi)\phi(\xi)^2 + \frac{G_{vv}(U_p(\xi),0)}{2\sigma^*}\phi^3(\xi)\right) \mathrm{d}\xi, \\
\mathcal{O}(\eta_2|s-s^*|): \ \mathbf{h}_2^{(0,1,1,0)} &= \frac{1}{\Omega_2} \int_{\xi_0}^{\infty} e^{\frac{s^*}{\sigma^*}\xi} \left(\frac{1}{\sigma^*}\phi'(\xi)\phi(\xi) + \frac{G_{uv}(U_p(\xi),0)}{\sigma^*}\theta_s^1(\xi)(\phi(\xi))^2\right) \mathrm{d}\xi, \\
\mathcal{O}(\eta_2|\sigma-\sigma^*|): \ \mathbf{h}_2^{(0,1,0,1)} &= -\frac{1}{\Omega_2} \int_{\xi_0}^{\infty} e^{\frac{s^*}{\sigma^*}\xi} \left(\frac{G_v(U_p(\xi),0)}{(\sigma^*)^2}(\phi(\xi))^2 + \frac{s^*}{(\sigma^*)^2}\phi'(\xi)\phi(\xi)\right) \mathrm{d}\xi.
\end{aligned}
$$

All stated integrals converge in the limit $\xi_0 \to -\infty$.

Proof. Asymptotic exponential decay rates for the relevant quantities are collected in Table 1. We focus on the convergence of the integrands as $\xi_0 \to -\infty$. Recall Hypothesis (H2) and the assumed ordering of the eigenvalues

$$\nu_v^-(s^*, \sigma^*) < \nu_u^-(s^*) < \nu_v^+(s^*, \sigma^*) < 0 < \nu_u^+(s^*),$$

as well as the non-resonance condition $\nu_u^-(s^*) < 2\nu_v^+(s^*, \sigma^*)$.

Term	Exponential rate as $\xi \to -\infty$
$\phi(\xi)$, $a_1(\xi)$, $a_2(\xi)$	$\nu_v^+(s^*,\sigma^*) = -\frac{s^*}{2\sigma^*} + \frac{1}{2\sigma^*}\sqrt{(s^*)^2 - 4\sigma^* g(\mathbf{p}_1)}$
$U_p'(\xi)$, $U_p''(\xi)$,$\theta_s^1(\xi)$, $\theta_s^2(\xi)$	$\nu_u^+(s^*) = -\frac{s^*}{2} + \frac{1}{2}\sqrt{(s^*)^2 - 4F_u(\mathbf{p}_1)}$
$b_2(\xi)$	$-\nu_u^-(s^*) = \frac{s^*}{2} + \frac{1}{2}\sqrt{(s^*)^2 - 4F_u(\mathbf{p}_1)}$

Table 1: *Asymptotic exponential decay rates of the terms arising in Lemma 15. These expressions are derived from (3.9) for a_1 and a_2, (3.13) for b_2 and (3.19) for θ_s^1 and θ_s^2.*

We now proceed through the terms in the quadratic expansions of $h_{1,2}$ and show that each of the integrands converge exponentially as $\xi \to -\infty$. The non-resonance condition is key for the convergence of the integrals listed – in particular those that are quadratic in $\eta_{1,2}$.

- For $\mathbf{h}_1^{(2,0,0,0)}$, the asymptotic exponential rate of the integrand is $s^* + 3\nu_u^+ > 0$ and the integral converges as $\xi \to -\infty$.
- For $\mathbf{h}_1^{(1,1,0,0)}$, the asymptotic exponential rate of the first term in the expansions is

$$s^* + 2\nu_u^+(s^*) + \nu_v^+(s^*,\sigma^*) > s^* + 2\nu_u^+(s^*) + \frac{\nu_u^-(s^*)}{2} = \frac{1}{2}s^* + \frac{3}{4}\nu_u^+(s^*) > 0.$$

  The second term has exponential rate

$$-\nu_u^-(s^*) + \nu_u^+(s^*) + \nu_v^+(s^*,\sigma^*) > \nu_u^+(s^*) - \nu_v^+(s^*,\sigma^*) > 0.$$

- For $\mathbf{h}_1^{(0,2,0,0)}$, the asymptotic exponential rate of the first term in the expansions is $s^* + \nu_u^+(s^*) + 2\nu_v^+(s^*,\sigma^*) > s^* + \nu_u^+(s^*) + \nu_u^-(s^*) = 0$ and those terms converge. For the second integral, the rate is $-\nu_u^-(s^*) + 2\nu_v^+(s^*,\sigma^*) > 0$ and the final integral has exponential rate $s^* + \nu_u^+(s^*) + 2\nu_v^+(s^*,\sigma^*) = -\nu_u^-(s^*) + 2\nu_v^+(s^*,\sigma^*) > 0$.
- For $\mathbf{h}_1^{(1,0,1,0)}$ all exponential rates are positive and the integral therefore converges as $\xi \to -\infty$.
- For the first integral in $\mathbf{h}_1^{(0,1,1,0)}$, the term $e^{s^*\xi}U_p'(\xi)a_2(\xi)$ as asymptotic exponential rate $s^* + \nu_u^+(s^*) + \nu_v^+(s^*,\sigma^*) > 0$ and therefore converges. All other terms in the first integral possess stronger decay rates and therefore also converge. The exponential rate of the first term in the second integral is $-\nu_u^-(s^*) + \nu_v^+(s^*,\sigma^*) > -\nu_v^+(s^*,\sigma^*) > 0$. The second term has stronger decay and therefore the second integral also converges.
- For $\mathbf{h}_1^{(0,1,0,1)}$, the asymptotic exponential rate is again $-\nu_u^-(s^*) + \nu_v^+(s^*,\sigma^*) > 0$ and the integral converges.
- For $\mathbf{h}_1^{(0,0,2,0)}$, all exponential rates are positive and the integral converges.
- For $\mathbf{h}_2^{(1,1,0,0)}$, the asymptotic exponential rate is

$$\frac{s^*}{\sigma^*} + \nu_u^+(s^*) + 2\nu_v^+(s^*,\sigma^*) > \frac{s^*}{\sigma^*} + \nu_u^+(s^*) + \nu_v^+(s^*,\sigma^*) + \nu_v^-(s^*,\sigma^*) = \nu_u^+(s^*) > 0,$$

  and the integral converges.

- For $\mathbf{h}_2^{(0,2,0,0)}$, the asymptotic exponential rate is

$$\frac{s^*}{\sigma^*} + 3\nu_v^+(s^*, \sigma^*) > \frac{s^*}{\sigma^*} + \nu_v^+(s^*, \sigma^*) + \nu_v^-(s^*, \sigma^*) = 0,$$

and the integral converges.

- For $\mathbf{h}_2^{(0,1,1,0)}$, the asymptotic rate of the first term in the integral is

$$\frac{s^*}{\sigma^*} + 2\nu_v^+(s^*, \sigma^*) > \frac{s^*}{\sigma^*} + \nu_v^+(s^*, \sigma^*) + \nu_v^-(s^*, \sigma^*) = 0,$$

while the term gives

$$\frac{s^*}{\sigma^*} + \nu_u^+(s^*) + 2\nu_v^+(s^*, \sigma^*) > \frac{s^*}{\sigma^*} + \nu_u^+(s^*) + \nu_v^+(s^*, \sigma^*) + \nu_v^-(s^*, \sigma^*) = \nu_u^+(s^*) > 0,$$

and the integral converge. A similar argument implies the convergence of $\mathbf{h}_2^{(0,1,0,1)}$.

∎

Lemma 16. *We have that*

$$\mathbf{h}_2^{(1,1,0,0)} = \frac{1}{\Omega_2}\left(\tilde{\phi}''(\xi_0)\tilde{\phi}(\xi_0) - (\tilde{\phi}'(\xi_0))^2\right),$$

and

$$\mathbf{h}_2^{(1,1,0,0)} \sim \gamma_{11} e^{(\nu_v^+(s^*,\sigma^*)-\nu_v^-(s^*,\sigma^*)+\nu_u^+(s^*))\xi_0},$$

as $\xi_0 \to -\infty$ *where*

$$\operatorname{sign}(\gamma_{11}) = \operatorname{sign}\left(g_u(\mathbf{p}_1)\right).$$

Proof. Recall from Lemma 15 that

$$\mathbf{h}_2^{(1,1,0,0)} = \frac{1}{\Omega_2}\int_{\xi_0}^{\infty} e^{\frac{s^*}{\sigma^*}\xi}\frac{G_{uv}(U_p(\xi),0)}{\sigma^*}U_p'(\xi)\left(\phi(\xi)\right)^2 \mathrm{d}\xi.$$

Observe that

$$G_{uv}(U_p(\xi),0)U_p'(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi}g(U_p(\xi),0).$$

After also recalling that $\phi(\xi) = e^{-\frac{s^*}{2\sigma^*}\xi}\tilde{\phi}(\xi)$ we are able to transform the integral as follows and obtain the desired result

$$\begin{aligned}
\mathbf{h}_2^{(1,1,0,0)} &= \frac{1}{\sigma^*\Omega_2}\int_{\xi_0}^{\infty}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}g(U_p(\xi),0)\right)\tilde{\phi}^2(\xi)\mathrm{d}\xi \\
&= \frac{1}{\sigma^*\Omega_2}\left[g(U_p(\xi),0)\tilde{\phi}^2(\xi)\right]_{\xi=\xi_0}^{\infty} - \frac{2}{\sigma^*\Omega_2}\int_{\xi_0}^{\infty} g(U_p(\xi),0)\tilde{\phi}(\xi)\tilde{\phi}'(\xi)\mathrm{d}\xi \\
&= -\frac{1}{\sigma^*\Omega_2}g(U_p(\xi_0),0)\tilde{\phi}^2(\xi_0) - \frac{2}{\sigma^*\Omega_2}\int_{\xi_0}^{\infty}\left(-\sigma^*\tilde{\phi}''(\xi) + \frac{(s^*)^2}{4\sigma^*}\phi(\xi)\right)\tilde{\phi}'(\xi)\mathrm{d}\xi \\
&= -\frac{1}{\sigma^*\Omega_2}\left(g(U_p(\xi_0),0)\tilde{\phi}^2(\xi_0) + \sigma^*\left(\tilde{\phi}'(\xi_0)\right)^2 - \frac{(s^*)^2}{4\sigma^*}\left(\tilde{\phi}(\xi_0)\right)^2\right) \\
&= \frac{1}{\Omega_2}\left(\tilde{\phi}''(\xi_0)\tilde{\phi}(\xi_0) - (\tilde{\phi}'(\xi_0))^2\right).
\end{aligned}$$

To determine the asymptotics of the final form, we expand the second order system defining $\tilde{\phi}(\xi)$ into a system,

$$\begin{aligned}\tilde{\phi}' &= \tilde{\psi}\\ \tilde{\psi}' &= \frac{(s^*)^2}{4(\sigma^*)^2}\tilde{\phi} - \frac{g(U_p(\xi),0)}{\sigma^*}\tilde{\phi}\end{aligned}$$

We then diagonalize and expand $g(U_p,0) = g(\mathbf{p}_1) + g_u(\mathbf{p}_1)U_p + \mathcal{O}(2)$, arriving at the following system that is relevant for the determination of the asymptotic decay rates,

$$\begin{aligned}\tilde{\phi}'_{ws} &= \tilde{\nu}_v^+(s^*,\sigma^*)\tilde{\phi}_{ws} + \frac{g_u(\mathbf{p}_1)}{\sigma^*(\tilde{\nu}_v^-(s^*,\sigma^*) - \tilde{\nu}_v^+(s^*,\sigma^*))}\left(U_p - u^+\right)\tilde{\phi}_{ws} + \mathcal{O}(2),\\ U_p' &= \nu_u^+(s^*)(U_p - u^+) + \mathcal{O}(2),\end{aligned}$$

where $\tilde{\nu}_v^\pm = \frac{1}{2\sigma^*}\sqrt{(s^*)^2 - 4\sigma^* g(\mathbf{p}_1)}$. Then

$$U_p(\xi) \sim u^+ - c_u e^{\nu_u^+(s^*)\xi},$$

from which we determine that

$$\left(\tilde{\phi}''(\xi_0)\tilde{\phi}(\xi_0) - (\tilde{\phi}'(\xi_0))^2\right) = -C^2\frac{g_u(\mathbf{p}_1)\nu_u^+(s^*)}{\sigma^*(\tilde{\nu}_v^-(s^*,\sigma^*) - \tilde{\nu}_v^+(s^*,\sigma^*))}e^{\left(2\tilde{\nu}_v^+(s^*,\sigma^*)+\nu_u^+(s^*)\right)\xi}.$$

Therefore γ_{11} is the constant multiplying the exponential and the exponential decay rate is obtained by noting that $2\tilde{\nu}_v^+ = \nu_v^+ - \nu_v^-$. ∎

B Expressions for $\mathbf{N}_u^{(0,0,2,0)}(s,\sigma)$ and $\mathbf{N}_{ss,u}^{(2,0,0,0)}(s,\sigma)$

Lemma 17. *The coefficients $\mathbf{N}_u^{(0,0,2,0)}(s,\sigma)$ and $\mathbf{N}_{ss,u}^{(2,0,0,0)}(s,\sigma)$ appearing in the expansions of $\mathcal{N}_u$ and $\mathcal{N}_{ss,u}$ defined in equation (4.4) depend only on s and have the expressions:*

$$\mathbf{N}_u^{(0,0,2,0)}(s,\sigma) = \frac{F_{uu}(\mathbf{p}_1)}{2(\nu_u^-(s) - \nu_u^+(s))}, \tag{B.1a}$$

$$\mathbf{N}_{ss,u}^{(2,0,0,0)}(s,\sigma) = -\frac{F_{uu}(\mathbf{p}_1)}{2(\nu_u^-(s) - \nu_u^+(s))}. \tag{B.1b}$$

Proof. Let us recall that we have the change of variables

$$\begin{pmatrix} u_1 - u^+ \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = T(s,\sigma)\begin{pmatrix} y^u \\ y^{ss,u} \\ y^{ws} \\ y^{ss,v} \end{pmatrix},$$

where $T(s,\sigma)$ is defined in (4.2). We set $U := (u_1 - u^+, u_2, v_1, v_2)^T$ and $Y := (y^u, y^{ss,u}, y^{ws}, y^{ss,v})^T$. Let us also remark that in the original coordinates, the quadratic terms in $(u_1 - u^+, u_2, v_1, v_2)^T$ of the nonlinear part are given by

$$N_2(U) := \begin{pmatrix} 0 \\ -\frac{F_{uu}(\mathbf{p}_1)}{2}(u_1-u^+)^2 - F_{uv}(\mathbf{p}_1)(u_1-u^+)v_1 - \frac{F_{vv}(\mathbf{p}_1)}{2}v_1^2 \\ 0 \\ -\frac{G_{uv}(\mathbf{p}_1)}{\sigma}(u_1-u^+)v_1 - \frac{G_{vv}(\mathbf{p}_1)}{2\sigma}v_1^2 \end{pmatrix},$$

where we have used the fact that $G_{uu}(\mathbf{p}_1) = 0$. Then, we note that with our change of variables both v_1 and v_2 in the new coordinates do not depend in y^u and $y^{ss,u}$. As consequence, if one keeps only the quadratic terms in y^u and $y^{ss,u}$ in the expression of N_2, expressed in the new coordinates, we get

$$N_2(T(s,\sigma)Y) = \begin{pmatrix} 0 \\ -\frac{F_{uu}(\mathbf{p}_1)}{2}(y^u)^2 - \frac{F_{uu}(\mathbf{p}_1)}{2}(y^{ss,u})^2 - F_{uu}(\mathbf{p}_1)y^u y^{ss,u} \\ 0 \\ 0 \end{pmatrix} + \mathcal{O}(2).$$

To conclude, it is enough to remark that

$$\begin{pmatrix} \mathcal{N}_u(Y,s,\sigma) \\ \mathcal{N}_{ss,u}(Y,s,\sigma) \\ \mathcal{N}_{ws}(Y,s,\sigma) \\ \mathcal{N}_{ss,v}(Y,s,\sigma) \end{pmatrix} = T(s,\sigma)^{-1} N_2(T(s,\sigma)Y) + \mathcal{O}(3),$$

and that the matrix $T(s,\sigma)$ is block triangular so that

$$T(s,\sigma) = \begin{pmatrix} T_{11}(s) & T_{12}(s,\sigma) \\ 0 & T_{22}(s,\sigma) \end{pmatrix} \text{ and } T(s,\sigma)^{-1} = \begin{pmatrix} T_{11}^{-1}(s) & -T_{11}^{-1}(s)T_{1,2}(s,\sigma)T_{22}^{-1}(s,\sigma) \\ 0 & T_{22}^{-1}(s,\sigma) \end{pmatrix}.$$

Finally, a direct computation shows that

$$T_{11}^{-1}(s) = \frac{1}{\nu_u^-(s) - \nu_u^+(s)} \begin{pmatrix} \nu_u^-(s) & -1 \\ -\nu_u^+(s) & 1 \end{pmatrix},$$

which in turns implies that the quadratic terms in y^u and $y^{ss,u}$ in the expression of $\mathcal{N}_u(Y,s,\sigma)$ are

$$\mathcal{N}_u(Y,s,\sigma) = \frac{1}{\nu_u^-(s) - \nu_u^+(s)} \left(\frac{F_{uu}(\mathbf{p}_1)}{2}(y^u)^2 + \frac{F_{uu}(\mathbf{p}_1)}{2}(y^{ss,u})^2 + F_{uu}(\mathbf{p}_1)y^u y^{ss,u} \right) + \mathcal{O}(2),$$

and similarly for $\mathcal{N}_{ss,u}(Y,s,\sigma)$

$$\mathcal{N}_{ss,u}(Y,s,\sigma) = -\frac{1}{\nu_u^-(s) - \nu_u^+(s)} \left(\frac{F_{uu}(\mathbf{p}_1)}{2}(y^u)^2 + \frac{F_{uu}(\mathbf{p}_1)}{2}(y^{ss,u})^2 + F_{uu}(\mathbf{p}_1)y^u y^{ss,u} \right) + \mathcal{O}(2),$$

which concludes the proof. ∎

C Expression for $\mathbf{N}_{ss,v}^{(0,2,0,0)}(s,\sigma)$

Lemma 18. *The coefficient* $\mathbf{N}_{ss,v}^{(0,2,0,0)}(s,\sigma)$ *appearing in the expansion of* $\mathcal{N}_{ss,v}$ *defined in equation* (4.4) *has the following expression:*

$$\mathbf{N}_{ss,v}^{(0,2,0,0)}(s,\sigma) = \frac{1}{\sigma(\nu_v^-(s,\sigma)-\nu_v^+(s,\sigma))}\left(\frac{F_v(\mathbf{p}_1)}{d_u(\nu_v^-(s,\sigma))}G_{uv}(\mathbf{p}_1) - \frac{G_{vv}(\mathbf{p}_1)}{2}\right). \tag{C.1}$$

Proof. The proof is similar to the proof of Lemma 17. One only needs to keep track of the quadratic terms y^{ws} in the nonlinear part of the system and notice that

$$N_2(T(s,\sigma)Y) = \begin{pmatrix} 0 \\ -\frac{F_{uu}(\mathbf{p}_1)}{2}\left(\frac{F_v(\mathbf{p}_1)}{d_u(\nu_v^-(s,\sigma))}\right)^2 (y^{ws})^2 + F_{uv}(\mathbf{p}_1)\frac{F_v(\mathbf{p}_1)}{d_u(\nu_v^-(s,\sigma))}(y^{ws})^2 - \frac{F_{vv}(\mathbf{p}_1)}{2}(y^{ws})^2 \\ 0 \\ \frac{F_v(\mathbf{p}_1)}{d_u(\nu_v^-(s,\sigma))}\frac{G_{uv}(\mathbf{p}_1)}{\sigma}(y^{ws})^2 - \frac{G_{vv}(\mathbf{p}_1)}{2\sigma}(y^{ws})^2 \end{pmatrix} + \mathcal{O}(2).$$

This implies that

$$\mathcal{N}_{ss,v}(Y,s,\sigma) = \frac{1}{\sigma(\nu_v^-(s,\sigma)-\nu_v^+(s,\sigma))}\left(\frac{F_v(\mathbf{p}_1)}{d_u(\nu_v^-(s,\sigma))}G_{uv}(\mathbf{p}_1) - \frac{G_{vv}(\mathbf{p}_1)}{2}\right)(y^{ws})^2 + \mathcal{O}(2),$$

where we have used the explicit form of the inverse of $T_{22}^{-1}(s,\sigma)$. ■

D Quadratic expansions of $p(z^u,s,\sigma)$ and $q(z^u,s,\sigma)$

In the following two Lemmas we will use the notations

$$\gamma_{ij}(z^u,0,0,s,\sigma) = \gamma_{ij}^{(1)}(s,\sigma)z^u + \gamma_{ij}^{(2)}(s,\sigma)(z^u)^2 + \mathcal{O}\left((z^u)^3\right), \quad i,j\in\{1,2\}$$

together with

$$\mathcal{M}_u(z^u,0,0,s,\sigma) = \mathcal{M}_u^{(2)}(s,\sigma)(z^u)^2 + \mathcal{O}\left((z^u)^3\right),$$

where γ_{ij} and $\mathcal{M}_u$ are defined in equation (4.6), Lemma 8.

Lemma 19. *The quadratic expansions for the maps* $p(z^u,s,\sigma)$ *and* $q(z^u,s,\sigma)$ *defined in equations* (4.7) *from Lemma 9 are:*

$$\begin{aligned} p(z^u,s,\sigma) &= \mathcal{P}_1(s,\sigma)z^u + \mathcal{P}_2(s,\sigma)(z^u)^2 + \mathcal{O}\left((z^u)^3\right), \\ q(z^u,s,\sigma) &= \mathcal{Q}_1(s,\sigma)z^u + \mathcal{Q}_2(s,\sigma)(z^u)^2 + \mathcal{O}\left((z^u)^3\right), \end{aligned}$$

with

$$\begin{aligned} \mathcal{P}_1(s,\sigma) &= \frac{\gamma_{11}^{(1)}(s,\sigma)}{\nu_u^+(s)}, \\ \mathcal{Q}_1(s,\sigma) &= -\left(\Lambda_{ss}(s,\sigma) - (\nu_v^+(s,\sigma)+\nu_u^+(s))\mathrm{I}\right)^{-1}\gamma_{21}^{(1)}(s,\sigma), \end{aligned}$$

and

$$\begin{aligned} \mathcal{P}_2(s,\sigma) &= \frac{1}{2\nu_u^+(s)}\left[-\mathcal{P}_1(s,\sigma)\left(\gamma_{11}^{(1)}(s,\sigma)+\mathcal{M}_u^{(2)}(s,\sigma)\right) + \gamma_{11}^{(2)}(s,\sigma) + \gamma_{12}^{(1)}(s,\sigma)\mathcal{Q}_1(s,\sigma)\right], \\ \mathcal{Q}_2(s,\sigma) &= \left(\Lambda_{ss}(s,\sigma) - (\nu_v^+(s,\sigma)+2\nu_u^+(s))\mathrm{I}\right)^{-1}\left(-\gamma_{21}^{(2)}(s,\sigma) + (\mathcal{M}_u^{(2)}(s,\sigma) - \gamma_{22}^{(1)}(s,\sigma) + \gamma_{11}^{(1)}(s,\sigma))\mathcal{Q}_1(s,\sigma)\right). \end{aligned}$$

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