The threshold phenomena for propagation in fractional Laplacian diffusion equations

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January 27, 2025

Abstract

In this paper, we investigate the threshold phenomenon between extinction and propagation in fractional Laplacian diffusion equations for a class of compactly supported initial data. We provide the first quantitative estimates on the threshold when the reaction nonlinearity is of bistable or ignition type. We mainly use estimates on the fundamental solutions of fractional Laplacian operators together with some accurate upper and lower solutions to show that the solution either propagates or goes extinct.

Keywords: extinction, propagation, threshold phenomena, fractional diffusion

AMS Subject Classification: 35B40, 35K57, 35R11, 45G05, 47G10

1 Introduction

We consider the following reaction-diffusion equation:

$$\partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = f(u(t,x)), \quad t>0, \quad x \in \mathbb{R}, \tag{1.1}$$

supplemented with some compactly supported initial data. The operator $(-\Delta)^\alpha$ denotes the fractional Laplacian of order $\alpha \in (0,1)$ which can be defined by the following singular integral:

$$(-\Delta)^\alpha u(x) = c_\alpha \text{P.V.} \left(\int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+2\alpha}} dy \right),$$

where P.V. denotes the Cauchy principal value of the integral. It is well known that the fractional Laplacian is the generator of a stable Lévy process which models jumps and long-distance interactions [26], and it is commonly used in population dynamics [8, 10, 21]. In the above equation, the nonlinearity f will be always assumed to be of bistable or ignition type with $f(0) = f(1) = 0$. Precise assumptions will be given later on. When specified to $\alpha = 1$, the fractional Laplacian reduces to the standard Laplacian, and equation (1.1) becomes the classical reaction-diffusion equation

$$\partial_t u(t,x) - \Delta u(t,x) = f(u(t,x)), \quad t>0, \quad x \in \mathbb{R}. \tag{1.2}$$

It is well established that such a local equation exhibits a so-called *threshold phenomenon*. Roughly speaking, this means that *small* initial data lead to extinction, whereas *large* initial data lead to propagation. The study of threshold phenomena for equation (1.2) with ignition type nonlinearity can be traced back to Kanel [17]. When considering initial data of the form of the characteristic function of an interval $[-L, L]$ for $L>0$, namely

$$u_0(x) = \mathbb{1}_{[-L,L]}(x), \quad x \in \mathbb{R}, \tag{1.3}$$

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Kanel showed that the long-time behavior of the solution u of the Cauchy problem associated to (1.2) with $u(0,\cdot) = u_0$ depends on L and proved that there are two lengths $L_*^{\rm ext}$ and $L_*^{\rm prop}$ with $0 < L_*^{\rm ext} \leq L_*^{\rm prop} < +\infty$ such that

$$\begin{cases} \lim_{t\to+\infty} u(t,x) = 0 \text{ uniformly in } x \in \mathbb{R} \text{ if } L < L_*^{\rm ext}, \\ \lim_{t\to+\infty} u(t,x) = 1 \text{ uniformly on compacts if } L > L_*^{\rm prop}. \end{cases}$$

Later on, Aronson and Weinberger [4], Fife and McLeod [13], and Flores [14] extended this phenomenon to bistable type nonlinearities. Furthermore, Zlatŏs [29] investigated the *sharp* threshold phenomenon from extinction to propagation in the ignition and bistable problems. Here, we refer to a *sharp* threshold behavior when, for any strictly increasing family of initial data exhibiting extinction for sufficiently small values of the parameter and propagation for sufficiently large values of the parameter, there is exactly one member of the family for which neither extinction nor propagation occurs. More precisely, Zlatŏs considered the asymptotic behavior of (1.2) with ignition or bistable nonlinearity and proved that $L_*^{\rm ext} = L_*^{\rm prop} := L_*$. In the bistable case, at the threshold value L_*, the corresponding solution of (1.2) uniformly converges to the unique positive stationary solution that decays to zero at infinity centered at the origin. Later on, Du and Matano [11] used the zero number argument to extend the result of Zlatŏs to monotone families of compactly supported initial data. They showed that any nonnegative bounded solution with compactly supported initial data converges to a stationary solution as $t \to \infty$. Poláčik [25] studied the threshold solutions and sharp transitions of nonautonomous parabolic equations in higher-dimensional space. Muratov and Zhong investigated the long time behavior of solutions to the Cauchy problem in one-dimensional spaces with bistable, ignition, or monostable nonlinearities, and further extended this result to high-dimensional spaces [23, 24]. More recently, Alfaro, Ducrot, and Faye [1] provided the first quantitative estimates of the threshold phenomena for (1.2). To be more precise, they considered initial data in the form of

$$u_0(x) = (\theta + \varepsilon)\mathbb{1}_{[-L,L]}(x), \quad x \in \mathbb{R}, \tag{1.4}$$

where $\varepsilon \in (0, 1-\theta]$ and $\theta \in (0,1)$ is the threshold of the nonlinearity (see Assumption 2.1 below). In the regime where $\int_0^1 f(u)\mathrm{d}u > 0$, that is the steady state $u = 1$ is more stable than the steady state $u = 0$, following [29, 11], there exist two lengths $L_\varepsilon^{\rm ext}$ and $L_\varepsilon^{\rm prop}$ with $0 < L_\varepsilon^{\rm ext} \leq L_\varepsilon^{\rm prop} < +\infty$ such that the solution u_L^ε of (1.2) satisfies

$$\begin{cases} \lim_{t\to+\infty} u_L^\varepsilon(t,x) = 0 \text{ uniformly in } x \in \mathbb{R} \text{ if } L < L_\varepsilon^{\rm ext}, \\ \lim_{t\to+\infty} u_L^\varepsilon(t,x) = 1 \text{ uniformly on compacts if } L > L_\varepsilon^{\rm prop}. \end{cases}$$

The key outcome of the study in [1] shows that the lengths $L_\varepsilon^{\rm ext}$ and $L_\varepsilon^{\rm prop}$ satisfy, as $\varepsilon \to 0^+$, the following asymptotics

$$0 < \liminf_{\varepsilon\to 0^+} \frac{L_\varepsilon^{\rm ext}}{\ln\left(\frac{1}{\varepsilon}\right)} \leq \limsup_{\varepsilon\to 0^+} \frac{L_\varepsilon^{\rm prop}}{\ln\left(\frac{1}{\varepsilon}\right)} < +\infty,$$

indicating that when the threshold is sharp, that is when $L_\varepsilon^{\rm ext} = L_\varepsilon^{\rm prop} := L_\varepsilon^\star$, then $L_\varepsilon^\star \sim \ln\left(\frac{1}{\varepsilon}\right)$ as $\varepsilon \to 0^+$.

Analogues of the above quantitative result have been very recently derived for nonlocal diffusion equations [2, 6, 7, 20, 27], which are typically used to characterize the movement and interaction of organisms in non-adjacent spatial locations in the modeling of population dynamics. In this setting, nonlocal reaction-diffusion equations write

$$\partial_t u(t,x) = \int_{\mathbb{R}} J(x-y)u(t,y)\mathrm{d}y - u(t,x), \quad t > 0, \quad x \in \mathbb{R}, \tag{1.5}$$

where $J : \mathbb{R} \to \mathbb{R}$ is a nonnegative dispersal kernel with $\int_{\mathbb{R}} J(y)\mathrm{d}y = 1$. Berestycki, Rodríguez [6] and Lim [20] demonstrated that there are two propagation thresholds for the nonlocal diffusion equation satisfying sufficiently high step initial data and exponentially bounded kernel functions, while Zhang, Li, and Yang [27] demonstrated that equation (1.5) with a compactly supported kernel has a sharp threshold between extinction and propagation. Most notably, Alfaro, Ducrot, and Kang [2] investigated the threshold phenomenon for the propagation of system (1.5) for compactly supported initial data of the form (1.4). In the limit $\varepsilon \to 0^+$, they derived various asymptotic limits for the corresponding lengths $L_\varepsilon^{\rm ext}$ and $L_\varepsilon^{\rm prop}$

characterizing the threshold to extinction and propagation respectively for different forms of the kernel functions. In a slightly different but related direction, Besse *et al.* [7] studied the asymptotic behavior of the solution of system (1.5) with bistable type nonlinearity and compactly supported initial conditions. Using the relationship between the nonlinearity, the kernel functions, and the diffusion coefficient, they demonstrated that the solutions can either propagate, go extinct or remain pinned [7], meaning that neither propagation nor extinction occur. We also refer to Andreson *et al.* [3] who recently studied the transition between pinning and unpinning for the solutions of nonlocal systems.

As far as we know, for fractional Laplacian equations such as (1.1), there is no such quantitative estimates of threshold phenomenon in the literature. This paper aims at filling this gap. Specifically, we consider equation (1.1) supplemented with the one-parameter family of initial data (1.4) indexed by the two parameters $L > 0$ and $\varepsilon > 0$, and we shall prove in a first step that for each fixed $\varepsilon > 0$ the solution decays to zero (extinction) when L is small or locally converges to a positive steady state (persistence) when L is large. Next, in the spirit of [1, 2, 7], we use the upper and lower solution methods, the comparison principle, and sharp estimates on the fundamental solutions of the fractional Laplacian diffusion equation to obtain quantitative estimates on the thresholds to extinction and propagation for the solution of system (1.1). More specifically, our main result is that we provide some estimates of the threshold values $L_\varepsilon^{\text{ext}}$ and $L_\varepsilon^{\text{prop}}$ as $\varepsilon \to 0^+$, namely

$$0 < C^- \leq \liminf_{\varepsilon\to 0^+} \frac{L_\varepsilon^{\text{ext}}}{\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha}}}, \quad \text{and} \quad \limsup_{\varepsilon\to 0^+} \frac{L_\varepsilon^{\text{prop}}}{\left(\frac{1}{\varepsilon}\ln\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha}}} \leq C_+ < +\infty, \tag{1.6}$$

for two positive constants $C^\pm$ which depend on $\alpha \in (0,1)$ and the nonlinearity f.

We can see from (1.6) that the "main term" is of magnitude $\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha}}$, but the lower and upper bounds coincide only up to a logarithmic term. This result is consistent with the nonlocal diffusion equation with a heavy-tailed kernel function considered by Alfaro et al. [2]. More specifically, if the Fourier transform of kernel J of equation (1.5) has an expansion $\hat{J}(\xi) = 1 - a|\xi|^\beta + \mathrm{o}\left(|\xi|^\beta\right)$ $(0 < \beta < 2, a > 0)$, as $\xi \to 0^+$, then the quantitative estimates of the threshold is the same as the one for solutions of the evolution given by the $\alpha = \frac{\beta}{2}$ fractional power of the Laplacian. This is somehow not surprising since nonlocal diffusion equations with kernel whose Fourier transform enjoys the above expansion are asymptotically close, in the limit $t \to +\infty$, to the fractional Laplacian equation of order $\alpha = \frac{\beta}{2}$ (see for instance [9]). Let us finally note that it is still an open problem to prove that the transition between propagation and extinction is sharp in the fractional Laplacian case studied here, that is $L_\varepsilon^{\text{ext}} = L_\varepsilon^{\text{prop}}$ for each $\varepsilon \in (0, 1-\theta]$.

2 Assumptions and main results

We state below our assumptions and main results.

Assumption 2.1 (nonlinearity f) *The function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. There is a threshold $\theta \in (0,1)$ such that*

$$f(u) = 0, \forall u \in (-\infty, 0] \cup \{\theta\} \cup [1, \infty),$$

and

$$f(u) > 0, \forall u \in (\theta, 1), \quad \textit{and} \quad \begin{cases} f(u) < 0, \forall u \in (0,\theta), & \textit{(Bistable Case)}, \\ \textit{or} & \\ f(u) = 0, \forall u \in (0,\theta), & \textit{(Ignition Case)}. \end{cases} \tag{2.1}$$

In the bistable case, we further require

$$\int_0^1 f(s)ds > 0.$$

Moreover, in both cases, we require that there are $r^- > 0$ and $\delta \in (\theta, 1)$ such that

$$f(u) \geq r^-(u - \theta), \quad \forall u \in [0, \delta].$$

Notice that, for $r > 0$, the usual cubic bistable nonlinearity

$$f(u) = ru(u-\theta)(1-u)\mathbb{1}_{(0,1)}(u), \quad u \in \mathbb{R},$$

satisfies the Assumption 2.1 as soon as $\theta < \frac{1}{2}$, and so does the ignition nonlinearity

$$f(u) = r(u-\theta)(1-u)\mathbb{1}_{(\theta,1)}(u), \quad u \in \mathbb{R}.$$

Now, for $\varepsilon \in (0, 1-\theta]$ and $L > 0$, we consider the family of initial data ϕ_L^ε given by

$$\phi_L^\varepsilon(x) := (\theta+\varepsilon)\mathbb{1}_{(-L,L)}(x), \quad x \in \mathbb{R}. \tag{2.2}$$

We denote by $u_L^\varepsilon = u_L^\varepsilon(t,x)$ the unique mild solution of (1.1) starting from the initial datum ϕ_L^ε. Then this family of solutions enjoys the so-called threshold property.

Proposition 2.1 (Threshold property) *Let $\alpha \in (0,1)$ and Assumption 2.1 hold. For each $\varepsilon \in (0, 1-\theta]$, there exist two lengths $\widetilde{L}_\varepsilon > 0$ and $\widehat{L}_\varepsilon > 0$ such that*

$$\lim_{t\to+\infty} u_L^\varepsilon(t,x) = \begin{cases} 0 \text{ uniformly in } \mathbb{R}, & \text{if } 0 < L < \widetilde{L}_\varepsilon, \\ 1 \text{ locally uniformly in } \mathbb{R}, & \text{if } L > \widehat{L}_\varepsilon. \end{cases}$$

Now using this and similarly as [1, 2], for each $\varepsilon \in (0, 1-\theta]$, we define the quantities

$$L_\varepsilon^{\text{ext}} := \sup\left\{L > 0 : \lim_{t\to+\infty} u_L^\varepsilon(t,x) = 0 \text{ uniformly on } \mathbb{R}\right\},$$

and

$$L_\varepsilon^{\text{prop}} := \inf\left\{L > 0 : \lim_{t\to+\infty} u_L^\varepsilon(t,x) = 1 \text{ locally uniformly on } \mathbb{R}\right\}.$$

We now state our main result.

Theorem 1 (Quantitative estimates of the thresholds) *Let $\alpha \in (0,1)$ and Assumption 2.1 hold. Then there exist two constants $0 < C^- < C^+$ such that*

$$C^- \leq \liminf_{\varepsilon\to 0^+} \frac{L_\varepsilon^{\text{ext}}}{\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha}}}, \quad \text{and} \quad \limsup_{\varepsilon\to 0^+} \frac{L_\varepsilon^{\text{prop}}}{\left(\frac{1}{\varepsilon}\ln\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha}}} \leq C^+,$$

where the constant C^- can be chosen as

$$C^- := \sqrt{\frac{\theta}{\alpha r^+ C_\alpha}}, \quad \text{with} \quad r^+ := \sup_{u\in(\theta,1]} \frac{f(u)}{u-\theta} > 0,$$

with the constant $C_\alpha > 1$ is defined in (3.6), and $C^+ = C^+(\alpha, \theta, r^-)$.

The organization of this work is as follows. We give some preliminary results in section 3. In section 4, we prove the propagation threshold property from Proposition 2.1. Sections 5 and 6 are dedicated to provide the quantitative estimates on the extinction and propagation threshold respectively.

3 Preliminary

In this section, we introduce some notations and basic results regarding the existence of solutions to the Cauchy problem. We also give a comparison principle for the fractional Laplacian diffusion equation, which is essential to prove the threshold phenomenon. Then, as a direct application, we prove the existence of the extinction threshold.

3.1 Notations and basic results

Let $0 < \alpha < 1$ and consider the Cauchy problem

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = f(u(t,x)), \quad t > 0,\, x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{3.1}$$

with initial datum $u_0 \in L^\infty(\mathbb{R})$. Then, mild solutions $u \in L^\infty((0,\infty)\times\mathbb{R})$ to the problem (3.1) satisfy

$$u(t,x) = \int_{\mathbb{R}} P_t(x-y)u_0(y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y) f(u(s,y))\mathrm{d}y\mathrm{d}s, \text{ a.e. in } (0,\infty)\times\mathbb{R}, \tag{3.2}$$

where $P_t \in C^\infty((0,\infty)\times\mathbb{R})$ is the fractional heat kernel, defined as follows

$$P_t(x) = \frac{1}{t^{\frac{1}{2\alpha}}} P\left(\frac{x}{t^{\frac{1}{2\alpha}}}\right), \tag{3.3}$$

and

$$P(x) := \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{ix\cdot\xi}\mathrm{e}^{-|\xi|^{2\alpha}}\mathrm{d}\xi. \tag{3.4}$$

Furthermore, it follows from [5, 18] that there exists $C_\alpha > 1$ such that

$$\frac{C_\alpha^{-1}}{1+|x|^{1+2\alpha}} \le P(x) \le \frac{C_\alpha}{1+|x|^{1+2\alpha}}, \quad x \in \mathbb{R}. \tag{3.5}$$

Thus, for $t > 0, x \in \mathbb{R}$, one has

$$\frac{C_\alpha^{-1}}{t^{\frac{1}{2\alpha}}(1+|t^{-\frac{1}{2\alpha}}x|^{1+2\alpha})} \le P_t(x) \le \frac{C_\alpha}{t^{\frac{1}{2\alpha}}(1+|t^{-\frac{1}{2\alpha}}x|^{1+2\alpha})}. \tag{3.6}$$

The formula (3.6) of the estimation of fractional heat kernel $P_t(x)$ will play an essential role in deriving some estimates of the threshold for the fractional Laplacian diffusion equation.

From [8] (see also [19, 28]), we know that for each $u_0 \in L^\infty(\mathbb{R})$ there exists a unique mild global in time solution of (3.1) with

$$u \in L^\infty((0,T)\times\mathbb{R}), \quad \forall T > 0\,.$$

Throughout this work, we always assume that the constant $C_\alpha > 1$ is defined in (3.6). Now we state a comparison principle of the fractional Laplace equation, which plays an important role in the study of threshold phenomena, see [8, 12, 28]. For the sake of completeness, we outline its proof.

Proposition 3.1 (Comparison principle) *Let $\alpha \in (0,1)$ and $T > 0$. Assume that $u, w \in L^\infty((0,T)\times\mathbb{R})$ are mild solution of the Cauchy problems*

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = f(u(t,x)), & (t,x) \in (0,T)\times\mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.7}$$

and

$$\begin{cases} \partial_t w(t,x) + (-\Delta)^\alpha w(t,x) = g(w(t,x)), & (t,x) \in (0,T)\times\mathbb{R}, \\ w(0,x) = w_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.8}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous with $f \le g$ on $\mathbb{R}$. If $u_0, w_0 \in L^\infty(\mathbb{R})$ are such that

$$u_0 \le w_0, \ \text{a.e. on } \mathbb{R},$$

then

$$u \le w, \ \text{a.e. on } (0,T)\times\mathbb{R}.$$

Proof. Let $\lambda > \max(\mathrm{Lip}(f), \mathrm{Lip}(g))$ and set

$$U(t,x) = \mathrm{e}^{\lambda t}u(t,x), \quad W(t,x) = \mathrm{e}^{\lambda t}w(t,x). \tag{3.9}$$

Since u and w are mild solutions of (3.7) and (3.8), we can easily get from [8] that U and W are mild solutions of

$$\begin{cases} \partial_t U + (-\Delta)^\alpha U = \tilde{f}(t,U), & \text{in } (0,T)\times\mathbb{R}, \\ U(0,x) = u_0(x), \end{cases} \tag{3.10}$$

and

$$\begin{cases} \partial_t W + (-\Delta)^\alpha W = \tilde{g}(t,W), \ \text{in } (0,T)\times\mathbb{R}, \\ W(0,x) = w_0(x), \end{cases}$$

with $\tilde{f}(t,\zeta) := \lambda\zeta + \mathrm{e}^{\lambda t} f\left(\mathrm{e}^{-\lambda t}\zeta\right)$ and $\tilde{g}(t,\zeta)$ defined similarly. Clearly, $\tilde{f}(t,\zeta) \le \tilde{g}(t,\zeta)$ for all ζ. By the definition of λ, both functions $\tilde{f}$ and $\tilde{g}$ are nondecreasing in ζ. Indeed, for $\zeta_1 \le \zeta_2$ and $t > 0$, one has

$$\tilde{f}(t,\zeta_1) - \tilde{f}(t,\zeta_2) = \lambda(\zeta_1 - \zeta_2) + \mathrm{e}^{\lambda t}\left[f\left(\mathrm{e}^{-\lambda t}\zeta_1\right) - f\left(\mathrm{e}^{-\lambda t}\zeta_2\right)\right] \le (\lambda - \mathrm{Lip}(f))(\zeta_1 - \zeta_2) \le 0.$$

For a given function $u_0 \in L^\infty(\mathbb{R})$ and $t > 0$, we define

$$T_t u_0(x) := (P_t * u_0)(x) = \int_{\mathbb{R}} P_t(x-y)u_0(y)\mathrm{d}y,$$

where $P_t \in C^\infty((0,\infty)\times\mathbb{R})$ is defined in (3.3). Clearly, the family T_t of bounded linear contractions of $L^\infty(\mathbb{R})$ is a semigroup. Note that $U \in L^\infty((0,T)\times\mathbb{R})$ is a mild solution of (3.10) if

$$\begin{aligned} U(t,x) &= \int_{\mathbb{R}} P_t(x-y)u_0(y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)\tilde{f}(s,U(s,y))\mathrm{d}y\mathrm{d}s, \\ &= T_t u_0(x) + \int_0^t T_{t-s}\tilde{f}(s,U(s,y))\mathrm{d}y\mathrm{d}s, \end{aligned}$$

almost everywhere in $(0,T)\times\mathbb{R}$, and similarly for W and $\tilde{g}$.

Consider the iteration scheme:

$$U_{n+1}(t,\cdot) = T_t u_0(x) + \int_0^t T_{t-s}\tilde{f}(s,U_n(s,y))\mathrm{d}y\mathrm{d}s,$$

and similarly for W_n and $\tilde{g}$. The scheme for the U_n converges to the limit U, and the scheme for the W_n converges to the limit W.

We now employ a standard induction argument to show that $U_n \le W_n$ on $(0,T)\times\mathbb{R}$ for all n. Let

$$U_0(t,\cdot) = T_t u_0(\cdot) \text{ and } W_0(t,\cdot) = T_t w_0(\cdot).$$

Since T_t is a positive semigroup on $L^\infty(\mathbb{R})$ and $u_0(\cdot) \le w_0(\cdot)$, then $U_0 \le W_0$ on $(0,T)\times\mathbb{R}$. Suppose $U_n \le W_n$ on $(0,T)\times\mathbb{R}$, then

$$\begin{aligned} U_{n+1}(t,\cdot) &= T_t u_0(x) + \int_0^t T_{t-s}\tilde{f}\left(s,U_n(s,\cdot)\right)\mathrm{d}s \le T_t w_0(x) + \int_0^t T_{t-s}\tilde{f}\left(s,W_n(s,\cdot)\right)\mathrm{d}s \\ &\le T_t w_0(x) + \int_0^t T_{t-s}\tilde{g}\left(s,W_n(s,\cdot)\right)\mathrm{d}s = W_{n+1}(t,\cdot), \end{aligned}$$

which completes the induction step. Taking the limit, this implies that $U \le W$ and therefore also $u \le w$ on $(0,T)\times\mathbb{R}$. This proves the proposition. ∎

We also present a variant of the above comparison principle.

Proposition 3.2 *Let $\alpha \in (0,1)$, $T > 0$ and f be Lipschitz. Assume that $u, v \in L^\infty([0,T)\times\mathbb{R})$ are mild subsolution and supersolution respectively, that is*

$$u(t,x) \le \int_{\mathbb{R}} P_t(x-y)u(0,y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)f(u(s,y))\mathrm{d}y\mathrm{d}s, (0,T)\times\mathbb{R},$$

and

$$v(t,x) \ge \int_{\mathbb{R}} P_t(x-y)v(0,y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)f(v(s,y))\mathrm{d}y\mathrm{d}s, (0,T)\times\mathbb{R},$$

and satisfy $u(0,\cdot) \le v(0,\cdot)$ on $\mathbb{R}$, then $u \le v$ on $[0,T)\times\mathbb{R}$.

3.2 Existence of the extinction threshold $\widetilde{L}_\varepsilon$

In this section, under Assumption 2.1 on the nonlinearity f, we prove the existence of the extinction threshold $\widetilde{L}_\varepsilon$. Let us consider u_L^ε the unique mild solution of (1.1) starting from the initial datum ϕ_L^ε. From the previous comparison principle, since $\varepsilon \in (0, 1-\theta]$, we obtain that

$$0 \le u_L^\varepsilon(t,x) \le 1, \quad t > 0, \quad x \in \mathbb{R}.$$

Upon denoting $M := \sup_{u\in[0,1]} f(u) > 0$, we obtain

$$0 \le u_L^\varepsilon(t,x) \le (\theta+\varepsilon)\int_{-L}^{L} P_t(x-y)\mathrm{d}y + M\int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)\mathrm{d}y\mathrm{d}s, \quad t > 0, \quad x \in \mathbb{R}.$$

Since for each $t > 0$, we have $\int_{\mathbb{R}} P_t(x)\mathrm{d}x = \int_{\mathbb{R}} P(x)\mathrm{d}x = 1$ by definition of P, we further get

$$0 \leq u_L^\varepsilon(t,x) \leq \frac{2LC_\alpha(\theta+\varepsilon)}{t^{\frac{1}{2\alpha}}} + Mt.$$

As a consequence, at $t = t_0 := \frac{\theta}{2M}$, we get

$$0 \leq u_L^\varepsilon(t_0,x) \leq \frac{2LC_\alpha(\theta+\varepsilon)}{t_0^{\frac{1}{2\alpha}}} + \frac{\theta}{2}.$$

Then, we can set $L_\varepsilon := \frac{\theta}{4C_\alpha(\theta+\varepsilon)}\left(\frac{\theta}{2M}\right)^{\frac{1}{2\alpha}} > 0$, and for each $0 < L < L_\varepsilon$, one has

$$0 \leq u_L^\varepsilon(t_0,x) < \theta, \quad x \in \mathbb{R}.$$

The final step of the proof consists in checking that the solution

$$\begin{cases} \partial_t w(t,x) + (-\Delta)^\alpha w(t,x) = f(w(t,x)), & t > 0, x \in \mathbb{R}, \\ w(0,x) = \gamma, \quad x \in \mathbb{R}, \end{cases} \tag{3.11}$$

with $0 < \gamma < \theta$ satisfies

$$w(t,x) \underset{t\to+\infty}{\longrightarrow} 0 \quad \text{uniformly in} \quad x \in \mathbb{R}.$$

Indeed, this follows from the fact that the solution w of the above Cauchy problem (3.11) is constant in space and that the solution of the ODE problem $w' = f(w)$ with $w(0) = \gamma \in (0,\theta)$ converges asymptotically to zero thanks to the bistability property of f. Comparing the solution for $t \geq t_0$ to the problem (3.11) implies that

$$\forall x \in \mathbb{R}, \quad \lim_{t\to+\infty} u_L^\varepsilon(t,x) = 0.$$

4 Propagation threshold

This section is devoted to the proof of the propagation threshold result of Proposition 2.1. We first recall some known properties regarding traveling front solutions associated to (1.1) in the bistable case. By a traveling front solution, we refer to the couple (U,c) with $c \in \mathbb{R}$ and profile U solution of

$$\begin{cases} cU' + (-\Delta)^\alpha U = f(U), & x \in \mathbb{R}, \\ U(-\infty) = 0, \quad U(+\infty) = 1, \end{cases} \tag{4.1}$$

such that $u(t,x) = U(x+ct)$. Gui and Zhao [16] have proved the existence and uniqueness of such traveling fronts for the fractional Laplacian diffusion equation with bistable type, together with asymptotic properties for U_0, as shown in Theorem 2 below (see [22, 15] for related results in the case of an ignition nonlinearity).

Theorem 2 ([16]) *If $\alpha \in (0,1)$, $f \in C^2(\mathbb{R})$ be a bistable nonlinearity, i.e., condition (2.1) holds, then there exists a unique c_0 and profile U_0 (up to translations) solving (4.1). Furthermore, $c_0 > 0$ and U_0 is monotone increasing. Moreover, there exists some constants $B > A > 0$ such that*

$$\frac{A}{|x|^{1+2\alpha}} \leq U_0'(x) \leq \frac{B}{|x|^{1+2\alpha}}, \quad \forall |x| \geq 1.$$

As a consequence, we have $U_0' \in L^p(\mathbb{R})$ for any $1 \leq p \leq \infty$, and

$$\frac{A}{x^{2\alpha}} \leq 1 - U_0(x) \leq \frac{B}{x^{2\alpha}}, \quad \forall x > 1 \text{ and } \frac{A}{|x|^{2\alpha}} \leq U_0(x) \leq \frac{B}{|x|^{2\alpha}}, \quad \forall x < -1.$$

We shall first prove the following result whose proof is largely inspired by the pioneering works of Fife and McLeod [13]. See also [2, 7] for similar results in the case of nonlocal reaction-diffusion equations.

Proposition 4.1 (Propagation threshold in the bistable case) *Let $\alpha \in (0,1)$ and $f \in C^2(\mathbb{R})$ be a bistable nonlinearity satisfying Assumption 2.1. Then, for each $\varepsilon \in (0, 1-\theta]$, there exists $\widehat{L}_\varepsilon > 0$ large enough such that for each $L > \widehat{L}_\varepsilon$ the solution u_L^ε starting from ϕ_L^ε propagates in the sense that*

$$\lim_{t\to+\infty} u_L^\varepsilon(t,x) = 1 \text{ locally uniformly for } x \in \mathbb{R}. \tag{4.2}$$

Proof. We consider (c, U) is an increasing traveling wave solution to (1.1) with $c > 0$ given by Theorem 2, namely, (c, U) is the solution of the following equation:

$$\begin{cases} cU' + (-\Delta)^\alpha U = f(U), & x \in \mathbb{R}, \\ U(-\infty) = 0, \quad U(+\infty) = 1. \end{cases} \tag{4.3}$$

We aim at showing that, for any $\varepsilon \in (0, 1-\theta]$, there exists $L > 0$ large enough such that propagation occurs for u_L^ε the solution to (1.1) starting from (2.2).

Let us consider the function $\underline{u}$ given by

$$\underline{u}(t,x) := U_+(t,x) + U_-(t,x) - 1 - q(t)$$

with $U_\pm(t,x) := U(\xi_\pm(t,x))$, where $\xi_\pm(t,x)$ take the form

$$\xi_+(t,x) = x + ct - \xi(t), \quad \xi_-(t,x) = \xi_+(t,-x) = -x + ct - \xi(t).$$

Here $q = q(t)$ and $\xi = \xi(t)$ are functions to be determined later to ensure that $\underline{u}$ is a sub-solution to (1.1).

From the above and the U-equation of (4.3), we straightforwardly compute, for $t > 0$, $x \in \mathbb{R}$,

$$\begin{aligned} N\underline{u}(t,x) :=& \partial_t \underline{u}(t,x) + (-\Delta)^\alpha \underline{u}(t,x) - f(\underline{u}(t,x)) \\ =& -\xi'(t)[U'(\xi_+(t,x)) + U'(\xi_-(t,x))] + f(U_+(t,x)) + f(U_-(t,x)) \\ & - f(U_+(t,x) + U_-(t,x) - 1 - q(t)) - q'(t). \end{aligned} \tag{4.4}$$

Before going further, let us introduce some notation. Denote $\beta := \theta + \varepsilon \in (\theta, 1]$ the fixed height of the step initial data.

Fix two constants $1 - \beta < \frac{q_0}{2} < \frac{q_1}{2} < 1 - \theta$ such that

$$\theta < 1 - \frac{q_1}{2} < 1 - \frac{q_0}{2} < \beta$$

and define the function Φ, continuous on $\mathbb{R} \times [0, +\infty)$, as

$$\Phi(u,s) = \begin{cases} \dfrac{f(u-s) - f(u)}{s}, & \text{if } s > 0, \\ -f'(u), & \text{if } s = 0. \end{cases}$$

Moreover, for $0 < s \leq \dfrac{q_1}{2}$, we have $\theta < 1 - \dfrac{q_1}{2} \leq 1 - s < 1$, so that $\Phi(1,s) > 0$. Note that $\Phi(1,0) = -f'(1) > 0$. There exists $\mu > 0$ such that $\Phi(1,s) \geq 2\mu$ for $0 \leq s \leq \frac{q_1}{2}$. By continuity, there exists a $\delta > 0$ such that

$$\Phi(u,s) \geq \mu, \text{ for } 1 - \delta \leq u \leq 1 \text{ and } 0 \leq s \leq \frac{q_1}{2}.$$

It then follows that

$$f(u-s) - f(u) \geq \mu s, \quad \forall 1 - \delta \leq u \leq 1, \text{ and } 0 \leq s \leq \frac{q_1}{2}. \tag{4.5}$$

Last, we fix $b > 0$ large enough so that

$$f(u) \leq b(1-u), \quad \forall 0 \leq u \leq 1. \tag{4.6}$$

We select

$$q(t) = \frac{q_0}{2} \frac{1}{(1 + \mu_0 t)^{2\alpha}}, \quad t \geq 0, \tag{4.7}$$

where $\mu_0 > 0$ will be determined later. Next, we divide the proof into three cases according to the value of α.

First case: $\alpha \in (1/2, 1)$. We set

$$\xi(t) = -\xi_0 + \eta(t), \quad \forall t \geq 0, \tag{4.8}$$

where $\xi_0 > 0$ and η is to be selected below with the properties

$$\eta(0) = 0, \quad \eta'(t) > 0, \quad \eta(t) \leq \eta_0 \leq \xi_0, \quad t > 0, \tag{4.9}$$

where the constant $\eta_0 > 0$. By the definition of $\xi(t)$ and (4.9), we get that $\xi(t) = \eta(t) - \xi_0 \leq 0$.

We aim at reaching $N\underline{u}(t,x) \leq 0, \forall x \in \mathbb{R}, t > 0$. Since both $\underline{u}(t,x)$ and $P_t(x)$ are symmetric, it is sufficient to work with $x \geq 0$. Since $U' > 0$, we have, for all $x \geq 0, t > 0$,

$$1 - U_+(t,x) + q(t) = 1 - U(x + ct - \xi(t)) + q(t).$$

According to Theorem 2, there exists $B > 0$ such that

$$1 - U_+(t,x) = 1 - U(x + ct - \xi(t)) \leq \frac{B}{(x + ct - \xi(t))^{2\alpha}}, \quad \alpha \in \left(\frac{1}{2}, 1\right), \forall x > 1. \tag{4.10}$$

It follows from (4.8), (4.9) and (4.10) that

$$\frac{1}{(x + ct - \xi(t))^{2\alpha}} \leq \frac{1}{(-\xi(t))^{2\alpha}} = \frac{1}{(\xi_0 - \eta(t))^{2\alpha}} \leq \frac{1}{(\xi_0 - \eta_0)^{2\alpha}}. \tag{4.11}$$

Recall that $q(t) = \frac{q_0}{2}\frac{1}{(1+\mu_0 t)^{2\alpha}}, t \geq 0$. Furthermore, we choose $s_0 := \xi_0 - \eta_0 > 1$ large enough such that

$$1 - U_+(t,x) \leq \frac{B}{(\xi_0 - \eta_0)^{2\alpha}} = \frac{B}{s_0^{2\alpha}} \leq \frac{q_0}{2}. \tag{4.12}$$

Using (4.7) and (4.12), we have that

$$0 \leq 1 - U_+(t,x) + q(t) \leq \frac{q_0}{2} + q(t) \leq \frac{q_0}{2} + \frac{q_0}{2} = q_0. \tag{4.13}$$

Below we complete the construction of the sub-solution by investigating the sign of $N\underline{u}(t,x)$ for $x \geq 0$, $t > 0$. To do so, recalling that $\delta > 0$ was chosen above for (4.5) to hold, we split our analysis according to the value of $U_-(t,x)$.

For $1 - \delta \leq U_-(t,x) \leq 1$, using (4.5) and (4.13), we have that

$$f(U_-(t,x)) - f(U_-(t,x) - (1 - U_+(t,x) + q(t))) \leq -\mu(1 - U_+(t,x) + q(t)). \tag{4.14}$$

Plugging this into (4.4), using $U' > 0$, $\xi'(t) > 0$, (4.6), (4.10) and (4.11), we have

$$\begin{aligned} N\underline{u}(t,x) \leq& -\mu(1 - U_+(t,x) + q(t)) + b(1 - U_+(t,x)) - q'(t) \\ =&(b - \mu)(1 - U(x + ct - \xi(t))) - \frac{\mu q_0}{2}\frac{1}{(1+\mu_0 t)^{2\alpha}} + \alpha q_0 \mu_0 \frac{1}{(1+\mu_0 t)^{1+2\alpha}} \\ \leq&(b - \mu)\frac{B}{(x + ct - \xi(t))^{2\alpha}} - \frac{\mu q_0}{2}\frac{1}{(1+\mu_0 t)^{2\alpha}} + \alpha q_0 \mu_0 \frac{1}{(1+\mu_0 t)^{1+2\alpha}} \\ \leq& \frac{1}{(1+\mu_0 t)^{2\alpha}}\left(\frac{B(b-\mu)(1+\mu_0 t)^{2\alpha}}{(x + ct - \xi(t))^{2\alpha}} - \frac{\mu q_0}{2} + \alpha q_0 \mu_0\right). \end{aligned} \tag{4.15}$$

Since (4.8), (4.9) and $x \geq 0$, we get that

$$\frac{1}{(x + ct - \xi(t))^{2\alpha}} \leq \frac{1}{(ct - \xi(t))^{2\alpha}} \leq \frac{1}{(ct + \xi_0 - \eta_0)^{2\alpha}} = \frac{1}{(ct + s_0)^{2\alpha}}. \tag{4.16}$$

Thus, combining (4.15) and (4.16), we obtain that

$$N\underline{u}(t,x) \leq \frac{1}{(1+\mu_0 t)^{2\alpha}}\left(B(b-\mu)\left(\frac{1+\mu_0 t}{ct + s_0}\right)^{2\alpha} - \frac{\mu q_0}{2} + \alpha q_0 \mu_0\right).$$

Let

$$g(t) = \frac{1 + \mu_0 t}{ct + s_0},$$

we can see that $g'(t) < 0$ if $\mu_0 < \frac{c}{s_0}$. Thus, we have that

$$g(t) < g(0) = \frac{1}{s_0}, \forall t > 0. \tag{4.17}$$

Using (4.15) and (4.17), we have that

$$N\underline{u}(t,x) \leq \frac{1}{(1+\mu_0 t)^{2\alpha}}\left(\frac{B(b-\mu)}{s_0^{2\alpha}} - \frac{\mu q_0}{2} + \alpha q_0 \mu_0\right) \leq 0,$$

by taking $s_0 > \max\left\{1, \left(\frac{\mu q_0}{2B(b-\mu)}\right)^{2\alpha}\right\}$ and $\mu_0 \leq \min\left\{\frac{c}{s_0}, \frac{\mu}{2\alpha} - \frac{B(b-\mu)}{\alpha q_0 s_0^{2\alpha}}\right\}$.

We consider the case $0 \leq U_-(t,x) \leq \delta$. Let us recall that $f \in C^2(\mathbb{R})$ and $f'(0) < 0$. Therefore, up to modify f on $(-\infty, 0)$ (which is harmless for the problem under consideration since solutions are nonnegative), we may assume that there are $\tilde{\mu}$, and $\tilde{\delta} > 0$ such that

$$f'(u) \leq -\tilde{\mu}, \quad \forall u \in (-\infty, \tilde{\delta}].$$

Also, up to reducing μ and δ appearing in (4.5) if necessary, we may assume $0 < \delta < \tilde{\delta}$ and $0 < \mu \leq \tilde{\mu}$. As a result,

$$f(u) - f(u-s) = \int_{u-s}^{u} f'(\sigma) d\sigma \leq -\mu s, \quad \forall -\infty < u \leq \delta, s \geq 0.$$

From this we deduce (4.14) and conclude as in the first case.

We are thus left with $\delta \leq U_-(t,x) \leq 1-\delta$. If we denote $\kappa > 0$ the Lipschitz constant of f on the interval $[\delta - q_1, 1-\delta]$, we deduce from $\delta \leq U_-(t,x) \leq 1-\delta$ and (4.13) that

$$f(U_-(t,x)) - f(U_-(t,x) - (1 - U_+(t,x) + q(t))) \leq \kappa(1 - U_+(t,x) + q(t)). \tag{4.18}$$

From (4.6), we have

$$f(U_+(t,x)) \leq b(1 - U_+(t,x)). \tag{4.19}$$

Moreover, for $\delta \leq U_-(t,x) \leq 1-\delta$, we have

$$U'(\xi_+(t,x)) + U'(\xi_-(t,x)) \geq U'(\xi_-(t,x)) \geq \min_{U^{-1}(\delta) \leq |z| \leq U^{-1}(1-\delta)} U'(z) := \nu > 0. \tag{4.20}$$

Plugging (4.18), (4.20) and (4.19) into (4.4), we deduce that

$$\begin{aligned} N\underline{u}(t,x) \leq & -\nu \xi'(t) + (\kappa + b)(1 - U_+(t,x)) + \kappa q(t) - q'(t) \\ = & -\nu \eta'(t) + (\kappa+b)(1 - U(x+ct-\xi(t))) + \frac{\kappa q_0}{2}\frac{1}{(1+\mu_0 t)^{2\alpha}} + \alpha q_0 \mu_0 \frac{1}{(1+\mu_0 t)^{1+2\alpha}} \\ \leq & -\nu \eta'(t) + \frac{B(\kappa+b)}{(x+ct-\xi(t))^{2\alpha}} + \frac{\kappa q_0}{2}\frac{1}{(1+\mu_0 t)^{2\alpha}} + \alpha q_0 \mu_0 \frac{1}{(1+\mu_0 t)^{1+2\alpha}}. \end{aligned} \tag{4.21}$$

Using (4.21) and (4.16) again, we obtain that

$$N\underline{u}(t,x) \leq -\nu \eta'(t) + \frac{1}{(1+\mu_0 t)^{2\alpha}}\left(B(\kappa+b)\left(\frac{1+\mu_0 t}{ct+s_0}\right)^{2\alpha} + \frac{\kappa q_0}{2} + \frac{\alpha q_0 \mu_0}{1+\mu_0 t}\right). \tag{4.22}$$

We select $\eta(t)$ solution of

$$\eta'(t) = \frac{1}{\nu(1+\mu_0 t)^{2\alpha}}\left(B(\kappa+b)\left(\frac{1+\mu_0 t}{ct+s_0}\right)^{2\alpha} + \frac{\kappa q_0}{2} + \frac{\alpha q_0 \mu_0}{1+\mu_0 t}\right) > 0.$$

That is we set

$$\eta(t) = \frac{(\kappa+b)B}{\nu}\int_0^t \frac{1}{(cz+s_0)^{2\alpha}} \mathrm{d}z + \frac{\kappa q_0}{2\nu}\int_0^t \frac{1}{(1+\mu_0 z)^{2\alpha}} \mathrm{d}z + \frac{\alpha q_0 \mu_0}{\nu}\int_0^t \frac{1}{(1+\mu_0 z)^{1+2\alpha}} \mathrm{d}z.$$

Thus we have that $\eta(t) \leq \eta(+\infty), \forall t \geq 0$. We estimate $\eta(+\infty)$ as follows:

$$\eta(+\infty) = \frac{(\kappa+b)B}{\nu}\int_0^{+\infty} \frac{1}{(cz+s_0)^{2\alpha}} \mathrm{d}z + \frac{\kappa q_0}{2\nu}\int_0^{+\infty} \frac{1}{(1+\mu_0 z)^{2\alpha}} \mathrm{d}z + \frac{\alpha q_0 \mu_0}{\nu}\int_0^{+\infty} \frac{1}{(1+\mu_0 z)^{1+2\alpha}} \mathrm{d}z. \tag{4.23}$$

Since $\alpha \in \left(\frac{1}{2}, 1\right)$, we can see that (4.23) is integrable. It follows from (4.23) that

$$\eta(+\infty) = \frac{(\kappa+b)B s_0^{1-2\alpha}}{\kappa\nu(2\alpha-1)c} + \frac{\kappa q_0}{2(2\alpha-1)\nu\mu_0} + \frac{\alpha q_0}{2\alpha\nu} := \eta_0. \tag{4.24}$$

With s_0, μ_0 and η_0 chosen as above, we may now set $\xi_0 = \eta_0 + s_0$, such that plugging (4.24) into (4.22), we reach

$$N\underline{u}(t,x) \le 0, \quad x \ge 0.$$

As a consequence, based on the symmetry of the problem, we have obtained that $N\underline{u}(t,x) \le 0$ for all $(t,x) \in (0,+\infty) \times \mathbb{R}$. On the one hand, for $|x| \le L$, we have that

$$\underline{u}(0,x) = U(x+\xi_0) + U(-x+\xi_0) - 1 - \frac{q_0}{2} < 1 - \frac{q_0}{2} < \beta = (\theta+\varepsilon)\mathbb{1}_{(-L,L)}(x).$$

On the other hand, since $U(-\infty) = 0$, for $|x| \ge L$, we obtain that

$$\underline{u}(0,x) = U(x+\xi_0) + U(-x+\xi_0) - 1 - \frac{q_0}{2} < U(-L+\xi_0) - \frac{q_0}{2} < 0,$$

if $L = L(\xi_0) > 0$ is large enough. As a consequence, for such a large $L > 0$,

$$\underline{u}(0,x) \le (\theta+\varepsilon)\mathbb{1}_{(-L,L)}(x), \quad x \in \mathbb{R}.$$

By comparison principle (here $\underline{u}$ is a subsolution in the strong sense and thus a mild subsolution), we obtain that

$$\underline{u}(t,x) \le u_L^\varepsilon(t,x), \quad \forall t > 0, x \in \mathbb{R}.$$

Since $\underline{u}(t,x)$ satisfies (4.2), so does u_L^ε. Thus we have completed the proof in this case.

Second case: $\alpha \in (0,1/2)$. We can choose

$$\eta(t) = \frac{\gamma_0}{\eta_0(1-2\alpha)}\left[(1+\eta_0 t)^{1-2\alpha} - 1\right], \quad \forall t \ge 0,$$

such that one has $\eta(0) = 0$ and

$$\eta'(t) = \frac{\gamma_0}{(1+\eta_0 t)^{2\alpha}} > 0,$$

where $\gamma_0 > 0$ and $\eta_0 > 0$ needs to be fixed appropriately.

We define $\xi(t)$ as

$$\xi(t) = -\xi_0 + \min_{t\ge 0}\left(\frac{ct}{2} - \eta(t)\right) + \eta(t), \quad \forall t \ge 0,$$

with $\xi_0 > 0$ to be fixed later. Let us remark that since $ct/2 - \eta(t) = t\left(c/2 - \frac{\eta(t)}{t}\right) > 0$ for t large enough, the above minimum is well defined and non positive. Next, we remark that

$$ct - \xi(t) = \frac{ct}{2} + \underbrace{\frac{ct}{2} - \eta(t) - \min_{t\ge 0}\left(\frac{ct}{2} - \eta(t)\right)}_{\ge 0} + \xi_0 > \frac{ct}{2} + \frac{\xi_0}{2} > \frac{\xi_0}{2} > 0,$$

for all $t \ge 0$.

With $q(t)$ defined as previously, for $1 - \delta \le U_-(t,x) \le 1$, one gets

$$\begin{aligned}
N\underline{u}(t,x) \le& -\mu(1 - U_+(t,x) + q(t)) + b(1 - U_+(t,x)) - q'(t) \\
\le& (b-\mu)\frac{B}{(ct-\xi(t))^{2\alpha}} - \frac{\mu q_0}{2}\frac{1}{(1+\mu_0 t)^{2\alpha}} + \alpha q_0 \mu_0 \frac{1}{(1+\mu_0 t)^{1+2\alpha}} \\
\le& \frac{1}{(1+\mu_0 t)^{2\alpha}}\left[4^\alpha B(b-\mu)\left(\frac{1+\mu_0 t}{ct+\xi_0}\right)^{2\alpha} - \frac{\mu q_0}{2} + \alpha q_0\mu_0\right] \\
\le& \frac{1}{(1+\mu_0 t)^{2\alpha}}\left[\frac{4^\alpha B(b-\mu)}{\xi_0^{2\alpha}} - \frac{\mu q_0}{2} + \alpha q_0 \mu_0\right],
\end{aligned}$$

where the last inequality holds upon choosing $\mu_0 < c/\xi_0$. As a consequence, if one fixes

$$\xi_0 > \left(\frac{4^{1+\alpha}B(b-\mu)}{\mu q_0}\right)^{1/2\alpha}, \quad 0 < \mu_0 < \min\left(\frac{c}{\xi_0}, \frac{\mu}{4\alpha}\right),$$

then $N\underline{u}(t,x) < 0$. The case $0 \leq U_-(t,x) \leq \delta$ can be handled similarly. We are thus left with $\delta \leq U_-(t,x) \leq 1-\delta$. Still denoting by $\kappa > 0$ the Lipschitz constant of f on $[\delta - q_1, 1-\delta]$, we compute

$$\begin{aligned}
N\underline{u}(t,x) \leq & -\nu\xi'(t) + (\kappa+b)(1-U_+(t,x)) + \kappa q(t) - q'(t) \\
\leq & -\nu\eta'(t) + \frac{B(\kappa+b)}{(ct-\xi(t))^{2\alpha}} + \frac{\kappa q_0}{2}\frac{1}{(1+\mu_0 t)^{2\alpha}} + \alpha q_0\mu_0\frac{1}{(1+\mu_0 t)^{1+2\alpha}} \\
\leq & -\nu\frac{\gamma_0}{(1+\eta_0 t)^{2\alpha}} + \frac{4^\alpha B(\kappa+b)}{(ct+\xi_0)^{2\alpha}} + \frac{\kappa q_0}{2}\frac{1}{(1+\mu_0 t)^{2\alpha}} + \alpha q_0\mu_0\frac{1}{(1+\mu_0 t)^{1+2\alpha}} \\
\leq & \frac{1}{(1+\eta_0 t)^{2\alpha}}\left[-\nu\gamma_0 + 4^\alpha B(\kappa+b)\left(\frac{1+\eta_0 t}{ct+\xi_0}\right)^{2\alpha} + \left(\frac{\kappa q_0}{2} + \alpha q_0\mu_0\right)\left(\frac{1+\eta_0 t}{1+\mu_0 t}\right)^{2\alpha}\right].
\end{aligned}$$

Upon choosing $0 < \eta_0 < \min\left(\frac{c}{\xi_0}, \mu_0\right)$, we get

$$N\underline{u}(t,x) \leq \frac{1}{(1+\eta_0 t)^{2\alpha}}\left[-\nu\gamma_0 + \frac{4^\alpha B(\kappa+b)}{\xi_0^{2\alpha}} + \frac{\kappa q_0}{2} + \alpha q_0\mu_0\right],$$

and thus we can fix $\gamma_0 > 0$ as

$$\gamma_0 > \frac{1}{\nu}\left(\frac{4^\alpha B(\kappa+b)}{\xi_0^{2\alpha}} + \frac{\kappa q_0}{2} + \alpha q_0\mu_0\right).$$

Then, the end of the proof follows similar lines as in the previous case.

Third case: $\alpha = 1/2$. We take

$$\eta(t) = \frac{\gamma_0}{\eta_0}\ln(1+\eta_0 t), \quad \forall t \geq 0,$$

such that $\eta(0) = 0$ and

$$\eta'(t) = \frac{\gamma_0}{1+\eta_0 t} > 0,$$

where $\gamma_0 > 0$ and $\eta_0 > 0$ needs to be fixed appropriately. We define $\xi(t)$ as

$$\xi(t) = -\xi_0 + \eta(t), \quad \forall t \geq 0,$$

with $\xi_0 > 0$ to be fixed appropriately. In fact, since $\frac{\eta(t)}{t} \to 0$ as $t \to +\infty$, similar computations as above shows that $N\underline{u}(t,x) < 0$ holds also in that case, and the proof follows. ■

We can now turn to the proof of Proposition 2.1. In the bistable case, since f is Lipschitz continuous on $[0,1]$, we can use a small C^2 modification from below of the nonlinearity and construct a suitable sub-solution converging to 1 for L large enough by Proposition 4.1 above. Hence, the existence of $\widehat{L}_\varepsilon$ follows from the comparison principle for the bistable case, and thus for the ignition case due to comparison arguments.

5 Quantitative estimate on the extinction threshold

In this section, we provide a quantitative extinction result for the system (1.1) with initial data (2.2). We obtain an asymptotic estimates (as $\varepsilon \to 0^+$) of the size L such that the solution $u_L^\varepsilon(t,x)$ goes extinct at large time. For it, we consider the following problem:

$$\begin{cases} \partial_t w + (-\Delta)^\alpha w = g(w), & t > 0, x \in \mathbb{R}, \\ w(0,x) = (\theta+\varepsilon)\mathbb{1}_{(-L,L)}(x), & x \in \mathbb{R}, \end{cases} \tag{5.1}$$

where $\varepsilon > 0$, $L > 0$ and

$$g(u) := r^+(u-\theta)_+.$$

Here $r^+ > 0$ and $\theta > 0$ are given and fixed parameters, and the subscript $+$ is used to denote the positive part of a real number. We start with a criterion for extinction.

Proposition 5.1 (Extinction criterion) *Let $\alpha \in (0,1)$, $r^+ > 0$ and $\theta > 0$ be fixed. Consider the time*

$$T_\varepsilon := \frac{1}{r^+} \ln \frac{1}{\varepsilon}. \tag{5.2}$$

Then for any $0 < \gamma \le \sqrt{\frac{\theta}{\alpha r^+ C_\alpha}}$, there exists $\varepsilon_0 > 0$ small enough such that, for each $\varepsilon \in (0, \varepsilon_0)$ and for each $0 < L < \gamma \left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha}}$, the solution $w = w(t,x)$ to (5.1) satisfies

$$\sup_{x\in\mathbb{R}} w(T_\varepsilon, x) \le \theta,$$

and is going to extinction at large times.

Proof. Consider $v = v(t,x)$ the solution to the Cauchy problem

$$\begin{cases} \partial_t v + (-\Delta)^\alpha v = 0, & t > 0, x \in \mathbb{R}, \\ v(0,x) = w(0,x), & x \in \mathbb{R}. \end{cases}$$

Then $v(t,\cdot) = (\theta + \varepsilon) P_t(\cdot) * \mathbb{1}_{(-L,L)}(x)$, where the fractional heat kernels $P_t(x)$ defined in (3.3). Thus for all $t > 0$, we have

$$V(t) := \|v(t,\cdot)\|_{L^\infty(\mathbb{R})} = (\theta+\varepsilon) \int_{|x|<L} P_t(x) \mathrm{d}x.$$

Following the strategy developed in [1], we now construct a super solution to (5.1) in the form $W(t,x) := v(t,x)\phi(t)$ with $\phi(0) = 1$ and $\phi(t) > 0$. From $W_t + (-\Delta)^\alpha W = v\phi'$ and the expression of g this requires

$$\phi'(t) \ge r^+ \left(\phi(t) - \frac{\theta}{v(t,x)}\right)_+, \quad \forall (t,x) \in (0,+\infty) \times \mathbb{R},$$

and thus

$$\phi'(t) \ge r^+ \left(\phi(t) - \frac{\theta}{V(t)}\right)_+, \quad \forall t \in (0,+\infty).$$

We choose ϕ as the solution of the Cauchy problem

$$\phi'(t) = r^+ \left(\phi(t) - \frac{\theta}{V(t)}\right), \quad \phi(0) = 1,$$

that is

$$\phi(t) = \mathrm{e}^{r^+ t} \left(1 - \int_0^t r^+ \mathrm{e}^{-r^+ s} \frac{\theta}{V(s)} \mathrm{d}s\right).$$

Observe that $V(0)\phi(0) > \theta$ and denote by $T > 0$ the first time where $V(T)\phi(T) = \theta$ (obviously we let $T = +\infty$ if such a time does not exist). Then

$$\left(\phi(t) - \frac{\theta}{V(t)}\right)_+ = \phi(t) - \frac{\theta}{V(t)}, \quad \forall t \in [0,T).$$

Note that $W(0,x) = v(0,x)\phi(0) = w(0,x)$. Thus, it is clear that $W(t,x) = v(t,x)\phi(t)$ is a super-solution to (5.1) on the time interval $[0,T)$. In particular, if $T < +\infty$, it follows from the comparison principle that $w(T,\cdot) \le W(T,\cdot) \le \theta$, and we are done provided $T \le T_\varepsilon$, a condition we aim at reaching below.

The condition $T < +\infty$ rewrites as: there exists $T > 0$ such that $F_L(T) = 0$ where

$$F_L(t) := \theta\left(1 - \frac{e^{-r^+ t}}{A_L(t)}\right) + \varepsilon - \int_0^t r^+ e^{-r^+ s} \frac{\theta}{A_L(s)} ds = 0,$$

wherein

$$A_L(t) := \int_{|x|<L} P_t(x) \mathrm{d}x = \int_{|x|<L} \frac{1}{t^{\frac{1}{2\alpha}}} P\left(\frac{x}{t^{\frac{1}{2\alpha}}}\right) \mathrm{d}x, \tag{5.3}$$

and $P(x)$ is defined in (3.4). We claim (see the proof of Claim 5.1 below) that $A_L'(t) \le 0$ for all $t > 0$. Since

$$A_L(t) = \int_{|x|<L} \frac{1}{t^{\frac{1}{2\alpha}}} P\left(\frac{x}{t^{\frac{1}{2\alpha}}}\right) \mathrm{d}x = \int_{|x|<\frac{L}{t^{\frac{1}{2\alpha}}}} P(x) \mathrm{d}x,$$

then using (3.4), we have that

$$A_L(0) = \int_{\mathbb{R}} P(x)\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{ix\cdot\xi-|\xi|^{2\alpha}}\mathrm{d}\xi\mathrm{d}x = 1, \tag{5.4}$$

here we used the equation

$$\varphi(0) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{ix\cdot\xi}\varphi(\xi)d\xi\mathrm{d}x.$$

It follows from (5.4) that $F_L(0) = \varepsilon$. Next, we claim that

Claim 5.1 *$A_L'(t) \leq 0$ for all $t > 0$.*

Proof of Claim 5.1. Using (5.3), we have that

$$\begin{aligned} A_L(t) &= \int_{|x|<L} \frac{1}{t^{\frac{1}{2\alpha}}} P\left(\frac{x}{t^{\frac{1}{2\alpha}}}\right)\mathrm{d}x = \int_{|x|<\frac{L}{t^{\frac{1}{2\alpha}}}} P(x)\mathrm{d}x \\ &= \int_{-\frac{L}{t^{\frac{1}{2\alpha}}}}^{0} P(x)\mathrm{d}x + \int_0^{\frac{L}{t^{\frac{1}{2\alpha}}}} P(x)\mathrm{d}x \\ &= 2\int_0^{\frac{L}{t^{\frac{1}{2\alpha}}}} P(x)\mathrm{d}x, \end{aligned} \tag{5.5}$$

this is because $P(x)$ is a even function $\forall x \in \mathbb{R}$. It follows from (5.5) that

$$A_L'(t) = -\frac{L}{\alpha}\frac{1}{t^{\frac{1}{2\alpha}+1}} P\left(\frac{L}{t^{\frac{1}{2\alpha}}}\right). \tag{5.6}$$

By (3.5), we obtain that

$$P\left(\frac{L}{t^{\frac{1}{2\alpha}}}\right) > 0, \quad \forall t > 0, L > 0.$$

This combined with (5.6), we get that

$$A_L'(t) \leq 0, \quad \forall t > 0.$$

We complete the proof this Claim. ■

Thus, using Claim 5.1, we have that $F_L'(t) = \theta e^{-r^+t}\frac{A_L'(t)}{A_L^2(t)} \leq 0$. As a result, since $F_L(0) = \varepsilon$ and $F_L'(t) \leq 0$, the condition $T \leq T_\varepsilon$ is equivalent to requiring $F_L(T_\varepsilon) \leq 0$, that is

$$1 + \frac{\varepsilon}{\theta} \leq r^+ \int_0^{T_\varepsilon} \frac{e^{-r^+s}}{A_L(s)}ds + \frac{e^{-r^+T_\varepsilon}}{A_L(T_\varepsilon)}. \tag{5.7}$$

Note that the right-hand side of the (5.7) is decreasing with respect to L. Using (5.4) and integrating by parts this is equivalent to

$$\frac{\varepsilon}{\theta} \leq -\int_0^{T_\varepsilon} e^{-r^+s}\frac{A_L'(s)}{A_L^2(s)}\mathrm{d}s. \tag{5.8}$$

But, since $A_L'(s) \leq 0, \forall s > 0$ and $0 < A_L(s) \leq 1, \forall s > 0$, we have that

$$\begin{aligned} -\int_0^{T_\varepsilon} e^{-r^+s}\frac{A_L'(s)}{A_L^2(s)}\mathrm{d}s &\geq -\int_0^{T_\varepsilon} e^{-r^+s}A_L'(s)\mathrm{d}s \\ &= \frac{L}{\alpha}\int_0^{T_\varepsilon} e^{-r^+s}\frac{1}{s^{\frac{1}{2\alpha}+1}} P\left(\frac{L}{s^{\frac{1}{2\alpha}}}\right)\mathrm{d}s, \end{aligned} \tag{5.9}$$

which follows from the proof of Claim 5.1 that

$$A_L'(t) = -\frac{L}{\alpha}\frac{1}{t^{\frac{1}{2\alpha}+1}} P\left(\frac{L}{t^{\frac{1}{2\alpha}}}\right), \quad t > 0.$$

Using (3.5) and (5.9), we have that

$$
\begin{aligned}
-\int_0^{T_\varepsilon} e^{-r^+ s}\frac{A_L'(s)}{A_L^2(s)}\mathrm{d}s &\ge \frac{L}{\alpha C_\alpha}\int_0^{T_\varepsilon} e^{-r^+ s}\frac{1}{s^{\frac{1}{2\alpha}+1}}\frac{1}{1+|s^{-\frac{1}{2\alpha}}L|^{1+2\alpha}}\mathrm{d}s \\
&= \frac{L}{\alpha C_\alpha}\int_0^{T_\varepsilon} e^{-r^+ s}\frac{1}{s^{\frac{1}{2\alpha}+1}}\frac{1}{1+(s^{-\frac{1}{2\alpha}}L)^{1+2\alpha}}\mathrm{d}s.
\end{aligned}
$$

We select

$$
L_\varepsilon = \gamma\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha}} \tag{5.10}
$$

for some $\gamma > 0$ to be chosen later. Next, we aim at proving that (5.8) with $L = L_\varepsilon$, is satisfied for $\varepsilon > 0$ small enough. To do so, set

$$
G_\varepsilon := \frac{L_\varepsilon}{\alpha C_\alpha}\int_0^{T_\varepsilon} e^{-r^+ s}\frac{1}{s^{\frac{1}{2\alpha}+1}}\frac{1}{1+(s^{-\frac{1}{2\alpha}}L_\varepsilon)^{1+2\alpha}}\mathrm{d}s.
$$

Hence, we only need to show that

$$
G_\varepsilon \ge \frac{\varepsilon}{\theta}. \tag{5.11}
$$

Let $z = \frac{s}{L_\varepsilon^{2\alpha}}$, then we have that

$$
G_\varepsilon = \frac{1}{\alpha C_\alpha}\int_0^{\frac{T_\varepsilon}{L_\varepsilon^{2\alpha}}} e^{-r^+ z L_\varepsilon^{2\alpha}}\frac{1}{1+z^{1+\frac{1}{2\alpha}}}\mathrm{d}z.
$$

Since function $f(x) = \dfrac{1}{1+x^{1+\frac{1}{2\alpha}}}, \forall x \ge 0$ is monotonically decreasing, we have that

$$
\begin{aligned}
G_\varepsilon &\ge \frac{1}{\alpha C_\alpha}\frac{1}{1+\left(\frac{T_\varepsilon}{L_\varepsilon^{2\alpha}}\right)^{1+\frac{1}{2\alpha}}}\int_0^{\frac{T_\varepsilon}{L_\varepsilon^{2\alpha}}} e^{-r^+ z L_\varepsilon^{2\alpha}}\mathrm{d}z \\
&= \frac{1}{\alpha r^+ C_\alpha}\frac{1}{1+\left(\frac{T_\varepsilon}{L_\varepsilon^{2\alpha}}\right)^{1+\frac{1}{2\alpha}}}\frac{1-\mathrm{e}^{-r^+ T_\varepsilon}}{L_\varepsilon^{2\alpha}}.
\end{aligned} \tag{5.12}
$$

Recalling $T_\varepsilon = \frac{1}{r^+}\ln\frac{1}{\varepsilon}$ and subtituting (5.10) into (5.12), we have that

$$
G_\varepsilon \ge \frac{\varepsilon(1-\varepsilon)}{\alpha r^+ \gamma^2 C_\alpha}\frac{1}{1+\left(\frac{T_\varepsilon}{L_\varepsilon^{2\alpha}}\right)^{1+\frac{1}{2\alpha}}}.
$$

To show (5.11), we only need to prove that

$$
\frac{1-\varepsilon}{\alpha r^+ \gamma^2 C_\alpha}\frac{1}{1+\left(\frac{T_\varepsilon}{L_\varepsilon^{2\alpha}}\right)^{1+\frac{1}{2\alpha}}} \ge \frac{1}{\theta}. \tag{5.13}
$$

Indeed, by the definitions of T_ε and L_ε, we have that

$$
\frac{T_\varepsilon}{L_\varepsilon^{2\alpha}} \to 0 \text{ as } \varepsilon \to 0,
$$

thus (5.13) holds, as $0 < \gamma \le \sqrt{\frac{\theta}{\alpha r^+ C_\alpha}}$ and ε small enough. This completes the proof of Proposition 5.1. ■

We can now conclude the first part of the proof of Theorem 1.

Proof of Theorem 1 – Extinction threshold. By Assumption 2.1, u_L^ε is a sub-solution of system (5.1) with $r^+ > 0$ given by

$$
r^+ := \sup_{u\in(\theta,1]}\frac{f(u)}{u-\theta}.
$$

It follows from Proposition 5.1 and the comparison principle that for each $\varepsilon \in (0, \varepsilon_0)$ and for each $0 < L < \gamma \left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha}}$, one has

$$\forall x \in \mathbb{R}, \quad \lim_{t \to +\infty} u_L^\varepsilon(t,x) = 0,$$

with $0 < \gamma \leq \sqrt{\frac{\theta}{\alpha r^+ C_\alpha}}$. And thus, we deduce that

$$C^- \leq \liminf_{\varepsilon \to 0^+} \frac{L_\varepsilon^{\mathrm{ext}}}{\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha}}},$$

where $C^- = \sqrt{\frac{\theta}{\alpha r^+ C_\alpha}}$. ∎

6 Quantitative estimate on the propagation threshold

We fix $r^- > 0$ and $\theta \in (0,1)$, and define the linear function

$$\tilde{g}(w) := r^-(w - \theta).$$

For $\varepsilon > 0$ and $L > 0$, we consider $w = w(t,x)$ the solution of the Cauchy problem

$$\begin{cases} \partial_t w + (-\Delta)^\alpha w = \tilde{g}(w), & t > 0, x \in \mathbb{R}, \\ w(0,x) = (\theta + \varepsilon) \mathbb{1}_{(-L,L)}(x), & x \in \mathbb{R}. \end{cases} \tag{6.1}$$

We start with a criterion for non-extinction for the solutions of the above Cauchy problem.

Proposition 6.1 (Non-extinction criterion) *Let $\alpha \in (0,1)$, $r^- > 0$ and $\theta \in (0,1)$ be fixed. Let $\theta < \eta' < \eta < 1$ be given. Consider the time*

$$T_\varepsilon = \frac{1}{r^-} \ln \frac{\eta - \theta}{\varepsilon}.$$

Let $\gamma > \left(\frac{C_\alpha(\eta-\theta)}{\alpha(\eta-\eta')}\right)^{\frac{1}{2\alpha}}$ be given. Then for any $0 < k < 1 - \left(\frac{C_\alpha(\eta-\theta)}{\gamma^{2\alpha}\alpha(\eta-\eta')}\right)^{\frac{1}{1+2\alpha}}$, there exists $\varepsilon_0 > 0$ small enough such that, for each $\varepsilon \in (0, \varepsilon_0)$ and for each $L > L_\varepsilon := \gamma \left(\frac{1}{r^- \varepsilon} \ln \frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha}}$. The solution $w = w(t,x)$ of (6.1) satisfies

$$\min_{|x| \leq k L_\varepsilon} w(T_\varepsilon, x) \geq \eta'.$$

Proof. Notice that the solution $w(t,x)$ is explicitly given by

$$w(t,x) = \theta + e^{r^- t} \left(v(t,x) - \theta\right), \tag{6.2}$$

where $v = v(t,x)$ denotes the solution of the linear equation

$$\partial_t v + (-\Delta)^\alpha v = 0, \quad t > 0, \quad x \in \mathbb{R},$$

starting from $v(0,x) = w(0,x) = (\theta + \varepsilon) \mathbb{1}_{(-L,L)}(x)$. From the comparison principle $v(t,x) \leq (\theta + \varepsilon)$ for all $t > 0$ and $x \in \mathbb{R}$, and thus for all $0 < t \leq T_\varepsilon$, we have that

$$w(t,x) \leq \theta + \frac{\eta - \theta}{\varepsilon}(\theta + \varepsilon - \theta) = \eta, \quad \forall x \in \mathbb{R}.$$

Note that $v = v(t,x)$ is given by

$$\begin{aligned} v(t,x) &= (\theta + \varepsilon) P_t(\cdot) * \mathbb{1}_{(-L,L)}(x) \\ &= (\theta + \varepsilon) \int_{|y| < L} \frac{1}{t^{\frac{1}{2\alpha}}} P\left(\frac{x-y}{t^{\frac{1}{2\alpha}}}\right) \mathrm{d}y \\ &= (\theta + \varepsilon) \left[1 - \int_{|y| \geq L} \frac{1}{t^{\frac{1}{2\alpha}}} P\left(\frac{x-y}{t^{\frac{1}{2\alpha}}}\right) \mathrm{d}y\right]. \end{aligned} \tag{6.3}$$

The last equality follows from the property that $\int_{\mathbb{R}} P_t(x)\mathrm{d}x = 1$ for all $t > 0$. Plugging (6.3) into (6.2) and using (3.5), we deduce that

$$
\begin{aligned}
w(t,x) &= \theta + e^{r^- t}\left(\varepsilon - (\theta+\varepsilon)\int_{|y|\ge L} \frac{1}{t^{\frac{1}{2\alpha}}} P\left(\frac{x-y}{t^{\frac{1}{2\alpha}}}\right)\mathrm{d}y\right) \\
&\ge \theta + e^{r^- t}\left(\varepsilon - (\theta+\varepsilon)\int_{|y|\ge L} \frac{C_\alpha}{t^{\frac{1}{2\alpha}}(1+|t^{-\frac{1}{2\alpha}}(x-y)|^{1+2\alpha})}\mathrm{d}y\right).
\end{aligned} \tag{6.4}
$$

We now restrict to x satisfying $|x| \le kL$ for $k \in (0,1)$. Since $|y| \ge L$ ensures that $|x| \le kL \le k|y|$ and $|x-y| \ge (1-k)|y|$, we deduce that

$$
\max_{|x|\le kL} \int_{|y|\ge L} \frac{1}{t^{\frac{1}{2\alpha}}(1+|t^{-\frac{1}{2\alpha}}(x-y)|^{1+2\alpha})}\mathrm{d}y \le \frac{1}{t^{\frac{1}{2\alpha}}} \int_{|y|\ge L} \frac{1}{1+t^{-\frac{1+2\alpha}{2\alpha}}(1-k)^{1+2\alpha}|y|^{1+2\alpha}}\mathrm{d}y.
$$

Noticing that

$$
\frac{1}{t^{\frac{1}{2\alpha}}} \int_{|y|\ge L} \frac{1}{1+t^{-\frac{1+2\alpha}{2\alpha}}(1-k)^{1+2\alpha}|y|^{1+2\alpha}}\mathrm{d}y = \frac{1}{1-k}\int_{|z|\ge \frac{(1-k)L}{t^{\frac{1}{2\alpha}}}} \frac{1}{1+|z|^{1+2\alpha}}\mathrm{d}z,
$$

we obtain for each $|x| \le kL$ that

$$
w(t,x) \ge \theta + e^{r^- t}\left(\varepsilon - (\theta+\varepsilon)\frac{C_\alpha}{1-k}\int_{|z|\ge \frac{(1-k)L}{t^{\frac{1}{2\alpha}}}} \frac{1}{1+|z|^{1+2\alpha}}\mathrm{d}z\right)
$$

At $t = T_\varepsilon = \frac{1}{r^-}\ln\frac{\eta-\theta}{\varepsilon}$ and $\theta+\varepsilon \le 1$, we deduce that

$$
w(T_\varepsilon, x) > \theta + (\eta-\theta)\left(1 - \frac{C_\alpha(\theta+\varepsilon)}{(1-k)\varepsilon}\int_{|z|\ge \frac{(1-k)L}{T_\varepsilon^{\frac{1}{2\alpha}}}} \frac{1}{1+|z|^{1+2\alpha}}\mathrm{d}z\right). \tag{6.5}
$$

We now select

$$
L_\varepsilon := \gamma\left(\frac{1}{r^-\varepsilon}\ln\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha}} > 0,
$$

for some constant $\gamma > 0$ that will be fixed later. In view of (6.5), it is enough to conclude the proof to reach

$$
\theta + (\eta-\theta)\left(1 - \frac{C_\alpha(\theta+\varepsilon)}{(1-k)\varepsilon}\int_{|z|\ge \frac{(1-k)L_\varepsilon}{T_\varepsilon^{\frac{1}{2\alpha}}}} \frac{1}{1+|z|^{1+2\alpha}}\mathrm{d}z\right) \ge \eta'
$$

for $0 < \varepsilon \ll 1$, that is,

$$
I_\varepsilon := \int_{|z|\ge \frac{(1-k)L_\varepsilon}{T_\varepsilon^{\frac{1}{2\alpha}}}} \frac{1}{1+|z|^{1+2\alpha}}\mathrm{d}z \le \frac{(1-k)(\eta-\eta')}{C_\alpha(\eta-\theta)}\varepsilon, \tag{6.6}
$$

since $\theta+\varepsilon \le 1$. Let $X := \dfrac{(1-k)L_\varepsilon}{T_\varepsilon^{\frac{1}{2\alpha}}}$, we have that

$$
I_\varepsilon = 2\int_X^{+\infty} \frac{1}{1+z^{1+2\alpha}}\mathrm{d}z \le 2\int_X^{+\infty} \frac{1}{z^{1+2\alpha}}\mathrm{d}z = \frac{1}{\alpha}X^{-2\alpha}.
$$

In order to prove (6.6), we only need to show that

$$
X^{-2\alpha} \le \frac{\alpha(1-k)(\eta-\eta')}{C_\alpha(\eta-\theta)}\varepsilon, \quad 0 < \varepsilon \ll 1. \tag{6.7}
$$

Recalling $X = \dfrac{(1-k)L_\varepsilon}{T_\varepsilon^{\frac{1}{2\alpha}}}$, $L_\varepsilon := \gamma\left(\dfrac{1}{r^-\varepsilon}\ln\dfrac{1}{\varepsilon}\right)^{\frac{1}{2\alpha}}$ and $T_\varepsilon = \dfrac{1}{r^-}\ln\frac{\eta-\theta}{\varepsilon}$, we have that

$$X^{-2\alpha} = \frac{T_\varepsilon}{(1-k)^{2\alpha}L_\varepsilon^{2\alpha}} = \frac{1}{(1-k)^{2\alpha}\gamma^{2\alpha}}\frac{\ln\dfrac{\eta-\theta}{\varepsilon}}{\ln\dfrac{1}{\varepsilon}}\varepsilon \sim \frac{1}{(1-k)^{2\alpha}\gamma^{2\alpha}}\varepsilon, \text{ as } \varepsilon\to 0.$$

Thus we have (6.7) holds, provided that

$$\frac{C_\alpha(\eta-\theta)}{\gamma^{2\alpha}\alpha(\eta-\eta')} < (1-k)^{1+2\alpha},$$

which holds true since $\gamma > \left(\frac{C_\alpha(\eta-\theta)}{\alpha(\eta-\eta')}\right)^{\frac{1}{2\alpha}}$ and $0<k<1-\left(\frac{C_\alpha(\eta-\theta)}{\gamma^{2\alpha}\alpha(\eta-\eta')}\right)^{\frac{1}{1+2\alpha}}$. This completes the proof of Proposition 6.1. ∎

We can now conclude the second part of the proof of Theorem 1.

Proof of Theorem 1 – Propagation threshold. From Assumption 2.1, we get the existence of $r^->0$ and $\delta\in(\theta,1)$, such that we have

$$f(w)\geq \tilde g(w), \quad \forall w\in(-\infty,\delta),$$

where $\tilde g(w) = r^-(w-\theta)$. Let $0<\beta<\delta-\theta$ be given small enough so that we can define a Lipschitz continuous function $\tilde f:\mathbb{R}\to\mathbb{R}$ such that $\tilde f\leq f$,

$$\tilde f(w) = \begin{cases} f(w), & w\in(-\infty,\theta)\cup[\delta,\infty),\\ r^-(w-\theta), & w\in(\theta,\theta+\beta),\end{cases}$$

and $\tilde f$ satisfies Assumption 2.1. Denote $\tilde w = \tilde w(t,x)$ the solution to

$$\partial_t\tilde w + (-\Delta)^\alpha\tilde w = \tilde f(\tilde w),$$

starting from $\tilde w(0,x) = \phi_L^\varepsilon(x)$, so that $\tilde w(t,x)\leq u_L^\varepsilon(t,x)$ from the comparison principle. Consider the time $T_\varepsilon = \frac{1}{r^-}\ln\frac{\beta}{\varepsilon}$. For $0<\varepsilon\leq\beta$, we know from the proof of Proposition 6.1 (setting $\eta=\theta+\beta$) that $\tilde w\leq\theta+\beta\leq\delta$ on $[0,T_\varepsilon]\times\mathbb{R}$. Since

$$\tilde f(w)\geq\tilde g(w), \quad w\in(\theta,\theta+\beta),$$

it follows from Proposition 6.1 that, for any given $m\in(0,1)$ and $\beta'\in(0,\beta)$, there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ and $L>L_\varepsilon$, we obtain that

$$u_L^\varepsilon(T_\varepsilon,x)\geq\tilde w(T_\varepsilon,x)\geq(\theta+\beta')\mathbb{1}_{(-mL_\varepsilon,mL_\varepsilon)}(x),$$

with $L_\varepsilon = \gamma\left(\frac{1}{r^-\varepsilon}\ln\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha}}$ and $\gamma > \left(\frac{C_\alpha\beta}{\alpha(\beta-\beta')(1-k)^{1+2\alpha}}\right)^{\frac{1}{2\alpha}}$.

From Proposition 2.1 applied to $\varepsilon=\beta'$, we know that $L_{\beta'}^{\text{prop}}<+\infty$ exists, that is for $\ell>L_{\beta'}^{\text{prop}}$ the solution to (1.1) with initial data $(\theta+\beta')\mathbb{1}_{[-\ell,\ell]}(x)$ propagates. As a result, for $\varepsilon>0$ small enough so that $mL_\varepsilon>L_{\beta'}^{\text{prop}}$, we have $u_L^\varepsilon(T_\varepsilon+t,x)\to 1$ as $t\to+\infty$ locally uniformly in $x\in\mathbb{R}$ and therefore $u_L^\varepsilon(t,x)\to 1$ as $t\to+\infty$ locally uniformly in $x\in\mathbb{R}$. We have proved that, for $L_\varepsilon = \gamma\left(\frac{1}{r^-\varepsilon}\ln\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha}}$, one has $L_\varepsilon^{\text{prop}}\leq L_\varepsilon$ for $\varepsilon>0$ small enough. This completes the proof of Theorem 1. ∎

Acknowledgements

This project was initiated during the Labex CIMI research internship of M. Alleysson and L. Davron at the Institut de Mathématiques de Toulouse (June-Jully, 2021). G.F. acknowledges support from the ANR via the project Indyana under grant agreement ANR- 21- CE40-0008, the project ReaCh under grant agreement ANR-23-CE40-0023 and from the Labex CIMI under grant agreement ANR-11-LABX-0040. The research of M.Z. is supported by Natural Science Foundation of Tianjin (No. 23JCQNJC01010).

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