Asymptotic stability of the critical Fisher-KPP front using pointwise estimates

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Abstract

We propose a simple alternative proof of a famous result of Gallay regarding the nonlinear asymptotic stability of the critical front of the Fisher-KPP equation which shows that perturbations of the critical front decay algebraically with rate $t^{-3/2}$ in a weighted L^∞ space. Our proof is based on pointwise semigroup methods and the key remark that the faster algebraic decay rate $t^{-3/2}$ is a consequence of the lack of an embedded zero of the Evans function at the origin for the linearized problem around the critical front.

Keywords: Fisher-KPP equation, nonlinear stability, pointwise Green's function

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1 Introduction

We revisit the asymptotic stability analysis of Gallay [7] for the critical Fisher-KPP front of the following scalar parabolic equation

$$u_t = u_{xx} + cu_x + f(u), \quad t > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a $\mathscr{C}^2$ map satisfying $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$ and $f''(u) < 0$ for all $u \in (0, 1)$ and $c > 0$. For such an example, it is well known that for any wavespeed

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$c \geq 2\sqrt{f'(0)} := c_*$, there exist monotone traveling front solutions $q(x)$ connecting $u = 1$ at $-\infty$ and $u = 0$ at $+\infty$ where the front profile q is solution of the second order ODE

$$0 = q_{xx} + cq_x + f(q). \tag{1.2}$$

The stability of traveling fronts for the Fisher-KPP equation has been studied by many authors. For the super-critical family of fronts propagating with speeds $c > c^*$, stability was established by Sattinger using exponential weights to stabilize the essential spectrum and yield exponential in time stability; see [17]. Stability of the critical front was established by [12], with extensions and refinements achieved in [4, 5, 7]. The sharpest of these results for the Fisher-KPP equation is [7], where perturbations of the critical front are shown to converge in an exponentially weighted L^∞ space with algebraic rate $t^{-3/2}$. Of course, we also mention that strong results concerning the convergence of compactly supported initial data to traveling fronts are possible for (1.1) using comparison principle techniques; see for example [2].

The primary challenge presented by the critical front is that it is not possible to stabilize the essential spectrum using exponential weights. This is due to the presence of absolute spectrum at $\lambda = 0$ in the form of a branch point of the dispersion relation of the asymptotic system near $+\infty$. The presence of continuous spectrum near the origin suggests algebraic decay and one might further anticipate heat kernel type decay of perturbations. As we note above, perturbations of the critical front are known to converge slightly faster – in an exponentially weighted L^∞ space with algebraic rate $t^{-3/2}$; see [7].

Our approach is similar to that of [17] where the linear eigenvalue problem is studied in an exponentially weighted space and resolvent estimates are obtained via inverse Laplace transform. For the super-critical fronts studied in [17] the Laplace inversion contours can be placed in the stable half plane thereby simplifying the analysis. No such extension is possible here and we instead approach the problem using pointwise semigroup estimates. Pointwise semigroup methods were introduced by Zumbrun and Howard [19] and have been developed over the past several decades to address stability problems where the essential spectrum can not be separated from the imaginary axis. Applications include stability of viscous shock waves; see [9, 19], stability and instability of spatially periodic patterns; see [10], stability of defects in reaction-diffusion equations; see [3], and more recently stability of stationary reaction-diffusion fronts; see [13], to mention a few.

A rough outline of our approach is as follows.

- Working in an exponentially weighted space with weight $\omega(x)$, we construct bounded solutions $\varphi^\pm(x)$ for the eigenvalue problem $\mathcal{L}p = \lambda p$ on $\mathbb{R}^\pm$ where $\mathcal{L}$ is a linear operator describing the linearized eigenvalue problem transformed to the weighted space.

- Find bounds for the pointwise Green's function,

$$\mathbf{G}_\lambda(x,y) = \begin{cases} \dfrac{\varphi^+(x)\varphi^-(y)}{\mathbb{W}_\lambda(y)}, & x \geq y, \\ \\ \dfrac{\varphi^-(x)\varphi^+(y)}{\mathbb{W}_\lambda(y)}, & x \leq y, \end{cases}$$

  where the Wronskian $\mathbb{W}_\lambda(y) := \varphi^+(y)\varphi^{-'}(y) - \varphi^{+'}(y)\varphi^-(y)$, is often referred to as the Evans function; see [1].

- Apply the inverse Laplace transform, and by a suitable choice of inversion contour show that the Green's function

$$\mathbf{G}(t,x,y) = \frac{1}{2\pi\mathbf{i}}\int_\Gamma e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda, \tag{1.3}$$

  decays pointwise with algebraic rate $t^{-3/2}$.

- Apply L^p estimates to the nonlinear solution expressed using Duhamel's formula,

$$p(t,x) = \int_\mathbb{R}\mathbf{G}(t,x,y)p_0(y)\mathrm{d}y + \int_0^t\int_\mathbb{R}\mathbf{G}(t-\tau,x,y)\omega(y)\mathcal{N}(q_*(y),\omega(y)p(\tau,y))p(\tau,y)\mathrm{d}y\mathrm{d}\tau,$$

  to show that the nonlinear system also exhibits the same algebraic decay rate.

This approach is motivated by the observation in [16] that the faster algebraic decay rate is a consequence of the lack of an embedded zero of the Evans function at $\lambda = 0$ (the analytic extension of the Evans function to the branch point is possible due to the Gap Lemma; see [8, 11]). Indeed, the critical front has weak exponential decay near $x = +\infty$,

$$q_*(x) \underset{+\infty}{\sim} bxe^{-\gamma_* x}, \tag{1.4}$$

where $\gamma_* := c_*/2$ and for some $b > 0$. This weak exponential decay implies that the derivative of the wave also has weak exponential decay and therefore does not lead to a zero of $\mathbb{W}_\lambda(y)$ at $\lambda = 0$. In the outline of our argument, this fact comes into play when we require bounds on the supremum of $\mathbf{G}(t,x,y)$. We find that this quantity is dominated by a region where $\mathbf{G}_\lambda(x,y)$ resembles $Ce^{-\sqrt{\lambda}(x-y)}$. Of course, this is exactly the Laplace transform of the derivative of the heat kernel; from which we naturally expect algebraic decay with rate $t^{-3/2}$.

We now state our main result. Let $\omega(x) > 0$ be a positive, bounded, smooth weight function of the form

$$\omega(x) = \begin{cases} e^{-\gamma_* x} & x \geq 1, \\ e^{\beta x} & x \leq -1, \end{cases}$$

for some $0 < \beta < -\frac{c_*}{2} + \sqrt{\frac{c_*^2}{4} - f'(1)}$.

Theorem 1. *Consider (1.1) with initial data $u(0,x) = q_*(x) + v_0(x)$ satisfying $0 \leq u(0,x) \leq 1$. There exists an $\epsilon > 0$ such that if $v_0(x)$ satisfies,*

$$\left|\left|\frac{1}{(1+|\cdot|)}\frac{v_0(\cdot)}{\omega(\cdot)}\right|\right|_{L^\infty} < \left|\left|(1+|\cdot|)\frac{v_0(\cdot)}{\omega(\cdot)}\right|\right|_{L^1} < \epsilon$$

then the solution $u(t,x)$ is defined for all time and the critical front is nonlinearly stable in the sense that there exists a $C>0$ such that

$$\left|\left|\frac{1}{1+|\cdot|}\frac{v(t,\cdot)}{\omega(\cdot)}\right|\right|_{L^\infty} \leq \frac{C}{(1+t)^{3/2}}, \quad t>0,$$

where $v(t,x) := u(t,x) - q_*(x)$.

Theorem 1 recovers the sharp algebraic in time L^∞ decay rate of perturbations of the critical front that was obtained in [7]. The proof in [7] uses as a weight the derivative of the front profile. In this weighted space, the linearized operator as $x\to\infty$ is equivalent to the radial Laplacian in three dimensions; for which the fundamental solution possesses algebraic decay rate $t^{-3/2}$. The nonlinear argument relies on scaling variables and the application of renormalization group techniques. In comparing Theorem 1 to the main result in [7] we note small differences in the spatial decay rates of the allowable perturbations and note that the result in [7] is stronger than the one presented here in that the author is able to identify an asymptotic profile for the solution in addition to its decay rate.

The main novel contribution of our study is to present an alternative proof based upon pointwise semigroup methods and make rigorous the observation in [16] that the faster algebraic decay rate is a consequence of the lack of an embedded zero of the Evans function at $\lambda=0$. We contend that the proof of Theorem 1 presented here is simpler than that of [7] as it relies on (rather coarse) ODE estimates, contour integration and a standard nonlinear stability argument avoiding the technical PDE estimates and renormalization group theory of [7]. Furthermore, this alternative method paves the way to tackle a broader class of problems. For example, one could consider the extended Fisher-KPP equation

$$u_t = -\gamma u_{xxxx} + u_{xx} + f(u), \quad t>0, \quad x\in\mathbb{R}, \tag{1.5}$$

where $\gamma>0$ is a small parameter and $f(u)$ is as in (1.1). For such an equation, there exists a family of fronts with wavespeed $c\geq c_*(\gamma)$, in the limit $\gamma\to 0$, which were shown to be stable in exponentially weighted spaces [15]. It could be possible to adapt the above ideas to prove that the critical front decays algebraically with rate $t^{-3/2}$ in a weighted L^∞ space. The calculations in that case are more involved (a four-dimensional system of ODEs), but the general key ingredients remain unchanged. Along similar lines, the approach developed in this paper could be used to establish precise stability results for pulled invasion fronts in systems of reaction-diffusion equations. For example, refinements of the stability results in [6, 14] may be achievable. Our aim in the present paper is to illustrate the key ideas in the simple Fisher-KPP setting (1.1), where the analysis is very explicit.

From the perspective of pointwise semigroup methods, the application here is fairly straightforward. To reinforce the discussion in the preceding paragraph, we regard this relative simplicity to be a strength of this paper. One mathematical feature of interest is the presence of the branch point at the origin which prevents the continuation of any contour integrals into the left half of the complex plane. An important reference in this regard is Howard [9] where a marginally stable branch point also arises when considering the stability of degenerate viscous shock waves. The Fisher-KPP equation being studied here is quite different, but considerable similarities remain between the approach taken here and the one in [9].

The rest of the paper is organized as follows. In Section 2, we set up and study the linearized eigenvalue problem. In Section 3, we derive bounds on the pointwise Green's function $\mathbf{G}_\lambda(x,y)$. These estimates are leveraged to obtain bounds on the time Green's function $\mathbf{G}(t,x,y)$ in Section 4. The nonlinear stability argument is then presented in Section 5.

2 Preliminaries and ODE estimates

In this section, we set up the linear stability problem and begin to construct the pointwise Green's function $\mathbf{G}_\lambda(x,y)$.

We work in frame moving to the right with constant velocity c_*, so that equation (1.1) reads

$$u_t = u_{xx} + c_* u_x + f(u), \quad t>0, \quad x \in \mathbb{R}, \tag{2.1}$$

and writing the solutions $u(t,x) = q_*(x) + v(t,x)$, we obtain the following equation for the perturbation $v(t,x)$:

$$v_t = v_{xx} + c_* v_x + f'(q_*)v + f(q_*+v) - f(q_*) - f'(q_*)v, \quad t>0, \quad x\in\mathbb{R}. \tag{2.2}$$

Let $\omega(x) > 0$ be a positive, bounded, smooth weight function of the form

$$\omega(x) = \left\{ \begin{array}{ll} e^{-\gamma_* x} & x \geq 1, \\ e^{\beta x} & x \leq -1, \end{array} \right. \tag{2.3}$$

for some $\beta > 0$ which will be fixed later. Without loss of generality we assume that $\omega(0) = 1$. We perform a change of variable of the form $v = \omega p$, where p now satisfies

$$p_t = p_{xx} + \left(c_* + 2\frac{\omega'}{\omega} \right) p_x + \left(f'(q_*) + c_* \frac{\omega'}{\omega} + \frac{\omega''}{\omega} \right) p + \mathcal{N}(q_*, \omega p) p, \quad t>0, \quad x\in\mathbb{R}, \tag{2.4}$$

where

$$\mathcal{N}(\mu,\nu) := \frac{1}{\nu}\left(f(\mu+\nu) - f(\mu) - f'(\mu)\nu \right).$$

From now on, we will denote by $\mathcal{L}$ the linear operator

$$\mathcal{L}p := p_{xx} + \left(c_* + 2\frac{\omega'}{\omega} \right) p_x + \left(f'(q_*) + c_* \frac{\omega'}{\omega} + \frac{\omega''}{\omega} \right) p, \tag{2.5}$$

with dense domain $H^2(\mathbb{R})$ in $L^2(\mathbb{R})$. Let us note that for $x \geq 1$, the operator $\mathcal{L}$ reduces to

$$\mathcal{L}p = p_{xx} + (f'(q_*) - f'(0))p,$$

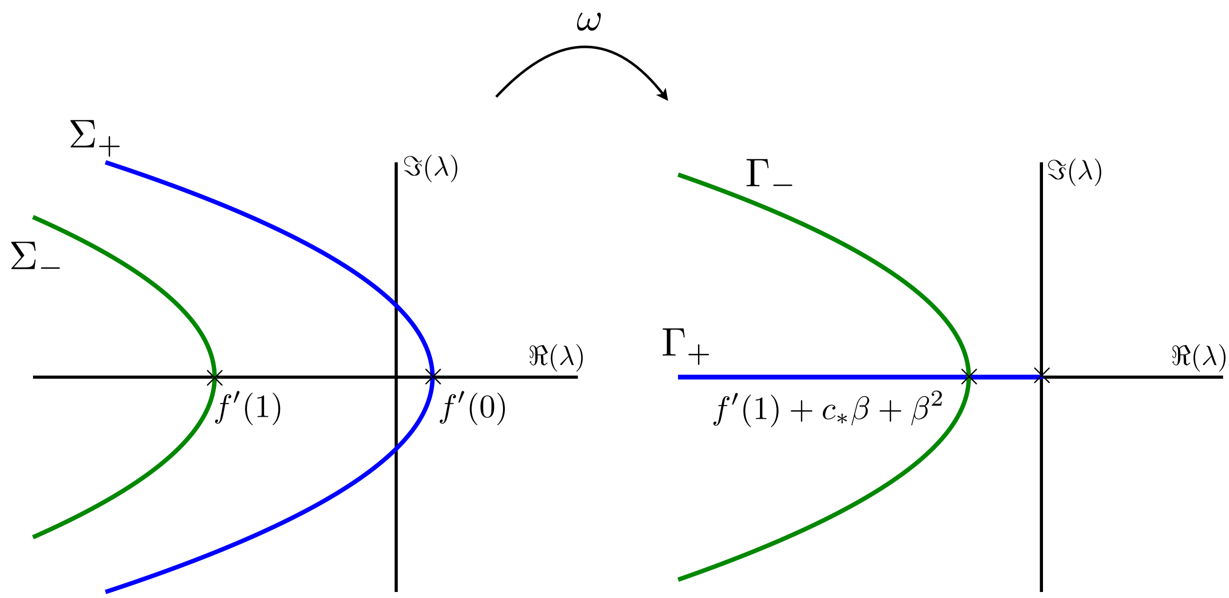

Figure 1: *Illustration of the action of the weight function ω on the boundaries of the essential spectrum of the linearized equation around the critical traveling front solution q_*. Note that Σ_- is mapped to Γ_- while Σ_+ is mapped to the negative real axis Γ_+ inclusive of the point at 0.*

while for $x \leq -1$, it becomes

$$\mathcal{L}p = p_{xx} + (c_* + 2\beta)p_x + \left(f'(q_*) + c_*\beta + \beta^2\right) p,$$

from which we impose that $0 < \beta < -\frac{c_*}{2} + \sqrt{\frac{c_*^2}{4} - f'(1)}$ so that the essential spectrum of

$$\mathcal{L}_\infty^- := \partial_{xx} + (c_* + 2\beta)\partial_x + \left(f'(1) + c_*\beta + \beta^2\right)$$

lies to the left of the imaginary axis. We denote by Γ_- the parabola in the complex plane defined by the boundary of the essential spectrum of $\mathcal{L}_\infty^-$, that is

$$\Gamma_- := \left\{-\ell^2 + (c_* + 2\beta)\mathbf{i}\ell + f'(1) + c_*\beta + \beta^2 \mid \ell \in \mathbb{R}\right\}.$$

We illustrate in Figure 1 the action of the weight function ω on the boundaries of the essential spectrum of the linearized equation around the critical traveling front solution q_*.

For simplicity, we will denote

$$\zeta_1 := c_* + 2\frac{\omega'}{\omega} = \begin{cases} 0 & x \geq 1, \\ (c_* + 2\beta) & x \leq -1, \end{cases} \quad \text{and } \zeta_0 := f'(q_*) + c_*\frac{\omega'}{\omega} + \frac{\omega''}{\omega} = \begin{cases} f'(q_*) - f'(0) & x \geq 1, \\ f'(q_*) + c_*\beta + \beta^2 & x \leq -1, \end{cases}$$

with asymptotics

$$\lim_{x \to -\infty} \zeta_0(x) = f'(1) + c_*\beta + \beta^2 < 0, \qquad \text{and} \qquad \lim_{x \to +\infty} \zeta_0(x) = 0,$$

such that operator $\mathcal{L}$ reads

$$\mathcal{L}p = p_{xx} + \zeta_1 p_x + \zeta_0 p.$$

The pointwise Green's function $\mathbf{G}_\lambda(x,y)$ is a solution of

$$(\mathcal{L}-\lambda)\mathbf{G}_\lambda = -\delta(x-y), \tag{2.6}$$

where $\delta(x)$ is the Dirac distribution. To the right of the essential spectrum solutions can be constructed by identifying exponentially decaying solutions on either half-line and then matching them at $x=y$ enforcing continuity and a jump discontinuity in the derivative,

$$\lim_{x\to y^-} \frac{\partial \mathbf{G}_\lambda}{\partial x} + 1 = \lim_{x\to y^+} \frac{\partial \mathbf{G}_\lambda}{\partial x}.$$

We begin with the construction of the exponentially decaying solutions of

$$\mathcal{L}p = \lambda p. \tag{2.7}$$

Our notations will be to denote by $\varphi^\pm(x)$ the (unique up to multiplication by a constant) exponentially decaying solutions of (2.7) at $\pm\infty$, and by $\psi^\pm$ a choice of exponentially growing solutions at $\pm\infty$. We can already compute the asymptotic decay and growth rates of $\varphi^\pm$ and $\psi^\pm$ from (2.7) at $\pm\infty$. At $+\infty$, we obtain the simpler system

$$p_{xx} = \lambda p,$$

from which we deduce the asymptotic exponential rates $\pm\sqrt{\lambda}$. While at $-\infty$, the system reduces to

$$\mathcal{L}^-_\infty p = \lambda p,$$

with growth rates given by

$$\mu^\pm(\lambda,\beta) := \frac{-(c_*+2\beta)}{2} \pm \frac{1}{2}\sqrt{(c_*+2\beta)^2 - 4(f'(1)+c_*\beta+\beta^2)+4\lambda}.$$

Using the above notations, the Green's function $\mathbf{G}_\lambda(x,y)$ for (2.6) takes the form

$$\mathbf{G}_\lambda(x,y) = \begin{cases} \dfrac{\varphi^+(x)\varphi^-(y)}{\mathbb{W}_\lambda(y)}, & x \geq y, \\ \\ \dfrac{\varphi^-(x)\varphi^+(y)}{\mathbb{W}_\lambda(y)}, & x \leq y, \end{cases}$$

where $\mathbb{W}_\lambda(y)$ is the Wronskian,

$$\mathbb{W}_\lambda(y) := \varphi^+(y){\varphi^-}'(y) - {\varphi^+}'(y)\varphi^-(y),$$

and thus satisfies the equation

$$\partial_y \mathbb{W}_\lambda(y) = -\zeta_1(y)\mathbb{W}_\lambda(y).$$

As a consequence and due to the choice of $\omega(0)=1$ we have that

$$\mathbb{W}_\lambda(y) = \frac{e^{-c_* y}}{\omega^2(y)}\mathbb{W}_\lambda(0), \quad y \in \mathbb{R}. \tag{2.8}$$

From (2.8) and the specific form of the weight function ω, it is then straightforward to check that $\mathbb{W}_\lambda(y)$ simplifies to

$$\mathbb{W}_\lambda(y) = \begin{cases} \mathbb{W}_\lambda(0), & y \geq 1, \\ e^{-(c_*+2\beta)y}\mathbb{W}_\lambda(0), & y \leq -1. \end{cases}$$

Lemma 2.1. *The Wronskian function $\mathbb{W}_\lambda$ satisfies the following properties:*

(i) $\mathbb{W}_0(y) \neq 0$ *for all* $y \in \mathbb{R}$;

(ii) there exists $M_s > 0$ *such that for all* λ *to the right of* Γ_- *and off the negative real axis with* $|\lambda| < M_s$, *we have*

$$\frac{1}{|\mathbb{W}_\lambda(y)|} \leq C$$

for all $y \in \mathbb{R}$ *and some constant* $C > 0$.

Proof. When $\lambda = 0$, we have that $\varphi^- = \omega^{-1}q_*'$. Using the asymptotic behavior of the critical front at $+\infty$, that is that there exist $a > 0$ and $b > 0$ such that

$$q_*(y) = (a + by)e^{-\gamma_* y} + \mathcal{O}(y^2 e^{-2\gamma_* y}),$$

as $y \to +\infty$, we deduce that

$$\varphi^{-'}(y) \underset{+\infty}{\sim} -\gamma_* b,$$

and as a consequence,

$$\mathbb{W}_0(y) \underset{+\infty}{\sim} -\gamma_* b,$$

which in turn implies that $\mathbb{W}_0(0) \neq 0$. We further note that $\mathbb{W}_\lambda(y) \neq 0$ for all λ to the right of the essential spectrum indicating the absence of unstable point spectrum. This is a consequence of Theorem 5.5 in [17]. We therefore obtain (ii) from the definition of $\mathbb{W}_\lambda$ in (2.8). ∎

Lemma 2.2. *Under the assumptions of our main theorem, for the constant* $M_s > 0$ *from Lemma 2.1, and* $0 < \alpha < \gamma_*$, *we have the following estimates on the growth and decay modes* $\varphi^\pm$ *and* $\psi^\pm$ *of* (2.7).

(i) $(0 \leq x)$ *For all* $|\lambda| \leq M_s$, *to the right of* Γ_- *and off the negative real axis,*

$$\begin{aligned} \varphi^+(x) &= e^{-\sqrt{\lambda}x}\left(1 + \theta_1^+(x,\lambda)\right), \\ \varphi^{+'}(x) &= e^{-\sqrt{\lambda}x}\left(-\sqrt{\lambda} + \theta_2^+(x,\lambda)\right), \\ \psi^+(x) &= e^{\sqrt{\lambda}x}\left(1 + \kappa_1^+(x,\lambda)\right), \\ \psi^{+'}(x) &= e^{\sqrt{\lambda}x}\left(\sqrt{\lambda} + \kappa_2^+(x,\lambda)\right), \end{aligned}$$

where

$$\theta_1^+(x,\lambda), \kappa_1^+(x,\lambda) = \mathcal{O}(e^{-\alpha x})$$

while

$$\theta_2^+(x,\lambda), \kappa_2^+(x,\lambda) = \mathcal{O}(\sqrt{\lambda})\mathcal{O}(e^{-\alpha x}).$$

(ii) $(x \leq 0)$ For all $|\lambda| \leq M_s$ and to the right of Γ_-,

$$\begin{aligned}
\varphi^-(x) &= e^{\mu^+(\lambda,\beta)x}\left(1+\theta_1^-(x,\lambda)\right),\\
\varphi^{-\prime}(x) &= e^{\mu^+(\lambda,\beta)x}\left(\mu^+(\lambda,\beta)+\theta_2^-(x,\lambda)\right),\\
\psi^-(x) &= e^{\mu^-(\lambda,\beta)x}\left(1+\kappa_1^-(x,\lambda)\right),\\
\psi^{-\prime}(x) &= e^{\mu^-(\lambda,\beta)x}\left(\mu^-(\lambda,\beta)+\kappa_2^-(x,\lambda)\right),
\end{aligned}$$

where

$$\theta_{1,2}^-(x,\lambda),\kappa_{1,2}^-(x,\lambda)=\mathcal{O}(e^{\alpha x}).$$

Proof. As the analysis is similar for each case, we will develop it only for φ^+ and ψ^+. Following [19], we first write (2.7) as a first order system of differential equations of the form

$$P' = \mathcal{A}(x,\lambda)P, \quad P=(p,p')^{\mathbf{t}}, \tag{2.9}$$

where

$$\mathcal{A}(x,\lambda) := \begin{pmatrix} 0 & 1 \\ \lambda-\zeta_0(x) & -\zeta_1(x)\end{pmatrix},$$

with associated asymptotic matrices at $x=\pm\infty$

$$\mathcal{A}^+(\lambda) := \begin{pmatrix} 0 & 1 \\ \lambda & 0\end{pmatrix}, \quad \text{and} \quad \mathcal{A}^-(\lambda) := \begin{pmatrix} 0 & 1 \\ \lambda-(f'(1)+c_*\beta+\beta^2) & -c_*-2\beta\end{pmatrix}.$$

Case φ^+. Setting $P(x)=e^{-\sqrt{\lambda}x}Z(x)$, we can rewrite $P'=\mathcal{A}(x,\lambda)P$ as

$$Z' = (\mathcal{A}^+(\lambda)+\sqrt{\lambda}I_2)Z + (\underbrace{\mathcal{A}(x,\lambda)-\mathcal{A}^+(\lambda)}_{:=\mathcal{B}(x)})Z, \tag{2.10}$$

where for $x \geq 1$ we have

$$\mathcal{B}(x) = \begin{pmatrix} 0 & 0 \\ -\zeta_0(x) & 0\end{pmatrix},$$

with $|\zeta_0(x)| = |f'(0)-f'(q_*(x))| = \mathcal{O}(e^{-\alpha x})$ as $x\to+\infty$. We remark that the matrix $\mathcal{A}^+(\lambda)+\sqrt{\lambda}I_2$ has two eigenvalues: 0 and $2\sqrt{\lambda}$. As we seek a solution of (2.10) such as $Z(x)\to Z^+(\lambda)=(1,-\sqrt{\lambda})^{\mathbf{t}}$ as $x\to+\infty$ we thus have to look for solutions of the integral equation

$$Z(x) = Z^+ - \int_x^{+\infty} e^{(\mathcal{A}^+(\lambda)+\sqrt{\lambda}I_2)(x-y)}\mathcal{B}(y)Z(y)\mathrm{d}y,$$

for which we have the explicit formula for $e^{(\mathcal{A}^+(\lambda)+\sqrt{\lambda}I_2)z}$ given by

$$e^{(\mathcal{A}^+(\lambda)+\sqrt{\lambda}I_2)z} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}e^{2\sqrt{\lambda}z} & -\frac{1}{2\sqrt{\lambda}}+\frac{1}{2\sqrt{\lambda}}e^{2\sqrt{\lambda}z} \\ -\frac{\sqrt{\lambda}}{2}+\frac{\sqrt{\lambda}}{2}e^{2\sqrt{\lambda}z} & \frac{1}{2}+\frac{1}{2}e^{2\sqrt{\lambda}z}\end{pmatrix}.$$

As a consequence of the specific structure of the matrix $\mathcal{B}(x)$, we obtain a decoupled equation for the first component $z_1(x)$ of $Z(x)$ which we shall obtain as a fixed point of the following map $\mathcal{T}$

$$\mathcal{T}(z_1)(x) := 1 + \int_x^{+\infty} \left(-\frac{1}{2\sqrt{\lambda}} + \frac{1}{2\sqrt{\lambda}} e^{2\sqrt{\lambda}(x-y)} \right) \zeta_0(y) z_1(y) \mathrm{d}y,$$

for $x \geq 1$. The contraction mapping theorem and the remark that $|\zeta_0(x)| = \mathcal{O}(e^{-\alpha x})$ as $x \to +\infty$ implies the existence of a fixed point of $\mathcal{T}$ on $L^\infty([A, \infty))$ for $A > 0$ sufficiently large. Then, we define $\theta_1^+(x, \lambda)$ as

$$\theta_1^+(x, \lambda) := z_1(x) - 1.$$

Using an iterative argument, we readily get that $\theta_1^+(x, \lambda) = \mathcal{O}(e^{-\alpha x})$ as $x \to +\infty$ uniformly in λ. Furthermore the function $\theta_1^+(\cdot, \lambda)$ is analytic in $\sqrt{\lambda}$, as it can directly be inferred from the fact that

$$\int_x^{+\infty} \left(-\frac{1}{2\sqrt{\lambda}} + \frac{1}{2\sqrt{\lambda}} e^{2\sqrt{\lambda}(x-y)} \right) \zeta_0(y) \mathrm{d}y = -\int_x^{+\infty} e^{2\sqrt{\lambda}(x-y)} \left(\int_y^{+\infty} \zeta_0(\tau) \mathrm{d}\tau \right) \mathrm{d}y$$

by integration by parts. From the solution z_1, we directly obtain an expression for the second component z_2 of $Z(x)$ as

$$z_2(x) = -\sqrt{\lambda} + \int_x^{+\infty} \left(\frac{1}{2} + \frac{1}{2} e^{2\sqrt{\lambda}(x-y)} \right) \zeta_0(y) z_1(y) \mathrm{d}y := -\sqrt{\lambda} + \theta_2^+(x, \lambda),$$

with the asymptotics $\theta_2^+(x, \lambda) = \mathcal{O}(\sqrt{\lambda}) \mathcal{O}(e^{-\alpha x})$ as $x \to +\infty$. To obtain the conclusion of the lemma and a solution defined for all $x \geq 0$, it suffices to flow backward the solution of (2.10) from $x = A$ to $x = 0$.

Case ψ^+. For ψ^+, we perform the change of variable $P(x) = e^{\sqrt{\lambda} x} Z(x)$ in (2.9), so that we obtain

$$Z' = (\mathcal{A}^+(\lambda) - \sqrt{\lambda} I_2) Z + \mathcal{B}(x) Z. \tag{2.11}$$

We want to construct ψ^+ as a solution of (2.11) which satisfies the following two conditions:

- $\lim\limits_{x \to +\infty} Z(x) = \tilde{Z}^+(\lambda) = (1, \sqrt{\lambda})^{\mathbf{t}}$,
- strong convergence to $\widetilde{Z}^+(\lambda)$.

To do so, we first write the variation of constants formula for (2.11)

$$Z(x) = e^{(\mathcal{A}^+(\lambda) - \sqrt{\lambda} I_2)(x - x_0)} Z_0 + \int_{x_0}^x e^{(\mathcal{A}^+(\lambda) - \sqrt{\lambda} I_2)(x-y)} \mathcal{B}(y) Z(y) \mathrm{d}y, \tag{2.12}$$

for some $x_0 \geq 1$ and $Z_0 \in \mathbb{R}^2$ to be chosen later. The matrix $\mathcal{A}^+(\lambda) - \sqrt{\lambda} I_2$ has two eigenvalues 0 and $-2\sqrt{\lambda}$ to which we associate a center projection Π_c and a stable projection Π_s. Noticing that we want to impose convergence of $Z(x)$ as $x \to +\infty$ along the center direction $(1, \sqrt{\lambda})^{\mathbf{t}}$, we have that

$$\begin{aligned} Z(x) = {} & e^{(\mathcal{A}^+(\lambda) - \sqrt{\lambda} I_2)(x - x_0)} \Pi_s Z_0 - \int_x^{+\infty} e^{(\mathcal{A}^+(\lambda) - \sqrt{\lambda} I_2)(x-y)} \Pi_c \mathcal{B}(y) Z(y) \mathrm{d}y + \widetilde{Z}^+(\lambda) \\ & + \int_{x_0}^x e^{(\mathcal{A}^+(\lambda) - \sqrt{\lambda} I_2)(x-y)} \Pi_s \mathcal{B}(y) Z(y) \mathrm{d}y. \end{aligned}$$

In order to have strong convergence along the center direction, we further impose

$$e^{-(\mathcal{A}^+(\lambda)-\sqrt{\lambda}I_2)x_0}\Pi_s Z_0 + \int_{x_0}^{+\infty} e^{-(\mathcal{A}^+(\lambda)-\sqrt{\lambda}I_2)y}\Pi_s\mathcal{B}(y)Z(y)\mathrm{d}y = 0,$$

such that we look for bounded solutions of

$$Z(x) = \widetilde{Z}^+(\lambda) - \int_x^{+\infty} e^{(\mathcal{A}^+(\lambda)-\sqrt{\lambda}I_2)(x-y)}\mathcal{B}(y)Z(y)\mathrm{d}y.$$

Once again, the first component decouples and we obtain

$$z_1(x) = 1 + \int_x^{+\infty}\left(\frac{1}{2\sqrt{\lambda}} - \frac{1}{2\sqrt{\lambda}}e^{-2\sqrt{\lambda}(x-y)}\right)\zeta_0(y)z_1(y)\mathrm{d}y.$$

We then look for solutions of the form $z_1(x) = 1 + \kappa_1^+(x,\lambda)$, where $\kappa_1^+(x,\lambda)$ is solution of

$$\kappa(x) = \int_x^{+\infty}\left(\frac{1}{2\sqrt{\lambda}} - \frac{1}{2\sqrt{\lambda}}e^{-2\sqrt{\lambda}(x-y)}\right)\zeta_0(y)\mathrm{d}y + \int_x^{+\infty}\left(\frac{1}{2\sqrt{\lambda}} - \frac{1}{2\sqrt{\lambda}}e^{-2\sqrt{\lambda}(x-y)}\right)\zeta_0(y)\kappa(y)\mathrm{d}y.$$

By integration by parts and as $|\zeta_0(x)| = \mathcal{O}(e^{-\alpha x})$ as $x \to +\infty$, we also have that

$$\int_x^{+\infty}\left(\frac{1}{2\sqrt{\lambda}} - \frac{1}{2\sqrt{\lambda}}e^{-2\sqrt{\lambda}(x-y)}\right)\zeta_0(y)\mathrm{d}y = -\int_x^{+\infty} e^{-2\sqrt{\lambda}(x-y)}\left(\int_y^{+\infty}\zeta_0(\tau)\mathrm{d}\tau\right)\mathrm{d}y,$$

for all λ such that $2\Re(\sqrt{\lambda}) - \alpha < 0$. As a consequence, we have the estimate

$$\left|\int_x^{+\infty}\left(\frac{1}{2\sqrt{\lambda}} - \frac{1}{2\sqrt{\lambda}}e^{-2\sqrt{\lambda}(x-y)}\right)\zeta_0(y)\mathrm{d}y\right| = \mathcal{O}(e^{-\alpha x}),$$

which allows us to apply the contraction mapping theorem and get the existence of κ_1^+ such that $x \mapsto e^{\alpha x}\kappa_1^+(x,\lambda) \in L^\infty([B,\infty))$ for $B > 0$ sufficiently large. ∎

The following Corollary will be essential in the derivation of our bounds for the pointwise Green's function $\mathbf{G}_\lambda(x,y)$. It will turn out that the $\mathcal{O}(e^{-\alpha x})$ bounds for $\kappa_{1,2}^+$ and $\theta_{1,2}^+$ will be insufficient and we will instead require a bound of $\mathcal{O}(\sqrt{\lambda})\mathcal{O}(e^{-\alpha x})$ on their difference. We remark that a similar cancellation occurs in [9] for a different problem where a branch point exists at the origin.

Corollary 2.3. *For $\kappa_{1,2}^+$ and $\theta_{1,2}^+$ defined in Lemma 2.2, we have that*

$$\kappa_{1,2}^+(x,\lambda) - \theta_{1,2}^+(x,\lambda) = \sqrt{\lambda}\Lambda_{1,2}^+(x,\lambda), \tag{2.13}$$

where $\Lambda_{1,2}^+(x,\lambda) = \mathcal{O}(e^{-\alpha x})$ for $x \geq 0$ and $\Lambda_{1,2}^+$ is analytic in $\sqrt{\lambda}$.

Proof. We recall from the previous Lemma that θ_1^+ is solution of

$$\theta(x) = -\int_x^{+\infty} e^{2\sqrt{\lambda}(x-y)}\left(\int_y^{+\infty}\zeta_0(\tau)\mathrm{d}\tau\right)\mathrm{d}y + \int_x^{+\infty}\left(-\frac{1}{2\sqrt{\lambda}} + \frac{1}{2\sqrt{\lambda}}e^{2\sqrt{\lambda}(x-y)}\right)\zeta_0(y)\theta(y)\mathrm{d}y,$$

while κ_1^+ is solution of

$$\kappa(x) = -\int_x^{+\infty} e^{-2\sqrt{\lambda}(x-y)}\left(\int_y^{+\infty}\zeta_0(\tau)\mathrm{d}\tau\right)\mathrm{d}y + \int_x^{+\infty}\left(\frac{1}{2\sqrt{\lambda}} - \frac{1}{2\sqrt{\lambda}}e^{-2\sqrt{\lambda}(x-y)}\right)\zeta_0(y)\kappa(y)\mathrm{d}y.$$

As a consequence, we obtain that

$$\begin{aligned}\kappa_1^+(x,\lambda)-\theta_1^+(x,\lambda) = &-\int_x^{+\infty}\left(e^{-2\sqrt{\lambda}(x-y)}-e^{2\sqrt{\lambda}(x-y)}\right)\left(\int_y^{+\infty}\zeta_0(\tau)\mathrm{d}\tau\right)\mathrm{d}y\\ &+\int_x^{+\infty}\left(\frac{1}{2\sqrt{\lambda}}-\frac{1}{2\sqrt{\lambda}}e^{-2\sqrt{\lambda}(x-y)}\right)\zeta_0(y)\kappa_1^+(y,\lambda)\mathrm{d}y\\ &-\int_x^{+\infty}\left(-\frac{1}{2\sqrt{\lambda}}+\frac{1}{2\sqrt{\lambda}}e^{2\sqrt{\lambda}(x-y)}\right)\zeta_0(y)\theta_1^+(y,\lambda)\mathrm{d}y.\end{aligned}$$

Denoting $\Theta_0(y)=-\int_y^{+\infty}\zeta_0(\tau)\mathrm{d}\tau$, we have

$$\int_x^{+\infty}\left(e^{-2\sqrt{\lambda}(x-y)}-e^{2\sqrt{\lambda}(x-y)}\right)\Theta_0(y)\mathrm{d}y = 2\sqrt{\lambda}\int_x^{+\infty}\left(e^{-2\sqrt{\lambda}(x-y)}+e^{2\sqrt{\lambda}(x-y)}\right)\left(\int_y^{+\infty}\Theta_0(\tau)\mathrm{d}\tau\right)\mathrm{d}y,$$

where

$$\int_x^{+\infty}\left(e^{-2\sqrt{\lambda}(x-y)}+e^{2\sqrt{\lambda}(x-y)}\right)\left(\int_y^{+\infty}\Theta_0(\tau)\mathrm{d}\tau\right)\mathrm{d}y=\mathcal{O}(e^{-\alpha x}),$$

for $x\geq 0$. Iterating the argument in the other integral terms, we finally obtain the desired expression (2.13). The proof is then similar for $\kappa_2^+(x,\lambda)-\theta_2^+(x,\lambda)$. ■

3 Estimates on the Green's function $\mathbf{G}_\lambda(x,y)$

Define the following subset of the complex plane,

$$\Omega_\delta=\{\lambda\in\mathbb{C}\ \mid \Re(\lambda)\geq -\delta_0-\delta_1|\Im(\lambda)|\}, \tag{3.1}$$

where $\delta_{0,1}>0$ are chosen small enough such that $\Gamma_-\cap\Omega_\delta=\emptyset$.

We now derive estimates on $\mathbf{G}_\lambda(x,y)$ in two regimes: sufficiently large λ and the remaining values of λ near the origin.

Lemma 3.1. *There exist some $\eta>0$ and an $M_l>0$ such that if $\lambda\in\Omega_\delta$ and $|\lambda|>M_l$ then*

$$|\mathbf{G}_\lambda(x,y)|\leq\frac{C}{\sqrt{|\lambda|}}e^{-\sqrt{|\lambda|}\eta|x-y|}. \tag{3.2}$$

Proof. This is a standard result, see Proposition 7.3 of [19], but we sketch it here for completeness. To construct $\mathbf{G}_\lambda$, we require solutions of

$$p''+\zeta_1(x)p'+\zeta_0(x)p-\lambda p=0.$$

After scaling the spatial coordinate as $\tilde{x}=\sqrt{|\lambda|}x$, we express this as a first order system,

$$\frac{\mathrm{d}P}{\mathrm{d}\tilde{x}}=\mathcal{A}(\tilde{x},\lambda)P,\quad P=\left(p,q/\sqrt{|\lambda|}\right)^{\mathbf{t}}. \tag{3.3}$$

Let $\tilde{\lambda} = \frac{\lambda}{|\lambda|}$ at which point we observe

$$\mathcal{A}(\tilde{x}, \lambda) := \begin{pmatrix} 0 & 1 \\ \tilde{\lambda} & 0 \end{pmatrix} + \frac{1}{\sqrt{|\lambda|}} \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{|\lambda|}} \zeta_0 \left(\frac{\tilde{x}}{\sqrt{|\lambda|}} \right) & -\zeta_1 \left(\frac{\tilde{x}}{\sqrt{|\lambda|}} \right) \end{pmatrix}.$$

Since $\tilde{\lambda}$ is normalized to lie on the unit circle, there exists an $\eta(\delta) > 0$ such that $\Re\left(\sqrt{\tilde{\lambda}}\right) > \eta$ for all $\lambda \in \Omega_\delta$. Therefore, the leading order system possesses modes that decay at exponential rate $e^{-\sqrt{\tilde{\lambda}}\tilde{x}}$ ($e^{\sqrt{\tilde{\lambda}}\tilde{x}}$) as $\tilde{x} \to \infty$ ($\tilde{x} \to -\infty$). For $\sqrt{|\lambda|}$ sufficiently large, the full system then has solutions that decay with exponential rate $e^{-\eta\tilde{x}}$ ($e^{\eta\tilde{x}}$). Reverting to the original variables and imposing continuity and a jump discontinuity in the derivative at $x = y$ we obtain estimates for $\mathbf{G}_\lambda$ as specified in (3.2).

■

Lemma 3.2. *Under the assumptions of our main theorem and for $|\lambda| \leq M_s$ to the right of Γ_- and off the negative real axis, we have the following estimates.*

(i) $y \leq 0 \leq x$

$$\mathbf{G}_\lambda(x, y) = e^{-\sqrt{\lambda}(x-y)} \mathcal{O}\left(e^{(\mu^+(\lambda,\beta) - \sqrt{\lambda})y} \right);$$

(ii) $x \leq 0 \leq y$

$$\mathbf{G}_\lambda(x, y) = e^{\sqrt{\lambda}(x-y)} \mathcal{O}\left(e^{(\mu^+(\lambda,\beta) - \sqrt{\lambda})x} \right);$$

(iii) $0 \leq y \leq x$

$$\mathbf{G}_\lambda(x, y) = e^{-\sqrt{\lambda}(x-y)} \mathcal{O}\left(e^{(\sqrt{\lambda} - \alpha)y} \right);$$

(iv) $0 \leq x \leq y$

$$\mathbf{G}_\lambda(x, y) = e^{\sqrt{\lambda}(x-y)} \mathcal{O}\left(e^{(\sqrt{\lambda} - \alpha)x} \right);$$

(v) $y \leq x \leq 0$

$$\mathbf{G}_\lambda(x, y) = e^{\mu^-(\lambda,\beta)(x-y)} \mathcal{O}(1);$$

(vi) $x \leq y \leq 0$

$$\mathbf{G}_\lambda(x, y) = e^{\mu^+(\lambda,\beta)(x-y)} \mathcal{O}(1).$$

All terms $\mathcal{O}$ are analytic for the λ considered here.

Proof. Cases (i)-(ii). For $y \leq 0 \leq x$, we have

$$\mathbf{G}_\lambda(x, y) = \frac{\varphi^+(x)\varphi^-(y)}{\mathbb{W}_\lambda(y)},$$

and according to Lemma 2.2, we can write

$$\mathbf{G}_\lambda(x, y) = \frac{1}{\mathbb{W}_\lambda(y)} e^{-\sqrt{\lambda}(x-y)} (1 + \theta_1^+(x, \lambda))(1 + \theta_1^-(y, \lambda)) e^{(\mu^+(\lambda,\beta) - \sqrt{\lambda})y}.$$

Now using Lemma 2.1 and the estimates on $\theta_1^{\pm}(\cdot,\lambda)$, we conclude that

$$\mathbf{G}_\lambda(x,y) = e^{-\sqrt{\lambda}(x-y)}\mathcal{O}\left(e^{(\mu^+(\lambda,\beta)-\sqrt{\lambda})y}\right),$$

where the terms containing $\mathcal{O}$ are analytic to the right of Γ_- and away from the negative real axis. This concludes the case (i), and (ii) can be treated similarly.

Cases (iii)-(iv). Let us fix $0 \leq y \leq x$. When $0 \leq y$ we need to express $\varphi^-(y)$ as a linear combination of $\varphi^+(y)$ and $\psi^+(y)$. Thus, we write

$$\varphi^-(y) = C(y,\lambda)\varphi^+(y) + D(y,\lambda)\psi^+(y).$$

We denote by $\mathbb{J}_\lambda$ and $\mathbb{I}_\lambda$ the following two determinants

$$\mathbb{J}_\lambda(y) := \varphi^+(y){\psi^+}'(y) - {\varphi^+}'(y)\psi^+(y), \quad \mathbb{I}_\lambda(y) := \varphi^-(y){\psi^+}'(y) - {\varphi^-}'(y)\psi^+(y),$$

such that the coefficients C and D are given by

$$C(y,\lambda) = \frac{\mathbb{I}_\lambda(y)}{\mathbb{J}_\lambda(y)}, \quad \text{and} \quad D(y,\lambda) = \frac{\mathbb{W}_\lambda(y)}{\mathbb{J}_\lambda(y)}.$$

Let us first remark that from Lemma 2.2, we have

$$\mathbb{J}_\lambda(y) = (1+\theta_1^+(y,\lambda))(\sqrt{\lambda}+\kappa_2^+(y,\lambda) - (-\sqrt{\lambda}+\theta_2^+(y,\lambda))(1+\kappa_1^+(y,\lambda)) = 2\sqrt{\lambda} + \mathcal{O}(e^{-\alpha y}),$$

as $y \to +\infty$. Finally, as both φ^+ and ψ^+ are solutions of (2.7), we have that $\partial_y \mathbb{J}_\lambda(y) = 0$ for all $y \geq 1$, from which we deduce that

$$\mathbb{J}_\lambda(y) = 2\sqrt{\lambda}, \quad \text{for } y \geq 1.$$

It will be convenient to rewrite $\varphi^-(y)$ as

$$\varphi^-(y) = \left(C(y,\lambda) + \frac{\mathbb{W}_\lambda(y)}{\mathbb{J}_\lambda(y)e^{-2\sqrt{\lambda}y}}\right)\varphi^+(y) + \frac{\mathbb{W}_\lambda(y)}{\mathbb{J}_\lambda(y)}\left(\psi^+(y) - e^{2\sqrt{\lambda}y}\varphi^+(y)\right).$$

Using Lemma 2.2 and the Corollary 2.3, we see that

$$\psi^+(y) - e^{2\sqrt{\lambda}y}\varphi^+(y) = e^{\sqrt{\lambda}y}\left(\kappa_1^+(y,\lambda) - \theta_1^+(y,\lambda)\right) = \sqrt{\lambda}e^{\sqrt{\lambda}y}\Lambda_1^+(y,\lambda),$$

where $\Lambda_1^+(y,\lambda) = \mathcal{O}(e^{-\alpha y})$ for $y \geq 0$. Thus, we get

$$\frac{\mathbb{W}_\lambda(y)}{\mathbb{J}_\lambda(y)}\left(\psi^+(y) - e^{2\sqrt{\lambda}y}\varphi^+(y)\right) = \mathbb{W}_\lambda(y)\frac{\sqrt{\lambda}}{\mathbb{J}_\lambda(y)}e^{\sqrt{\lambda}y}\Lambda_1^+(y,\lambda) = e^{\sqrt{\lambda}y}\mathcal{O}(e^{-\alpha y}),$$

for all $y \geq 1$. On the other hand, we have

$$\begin{aligned}\frac{\mathbb{I}_\lambda(y)}{\mathbb{J}_\lambda(y)} + \frac{\mathbb{W}_\lambda(y)}{\mathbb{J}_\lambda(y)e^{-2\sqrt{\lambda}y}} &= \frac{1}{\mathbb{J}_\lambda(y)}\left\{\varphi^-(y)\left({\psi^+}'(y) - e^{2\sqrt{\lambda}y}{\varphi^+}'\right) - {\varphi^-}'(y)\left(\psi^+(y) - e^{2\sqrt{\lambda}y}\varphi^+\right)\right\}\\ &= \frac{\sqrt{\lambda}}{\mathbb{J}_\lambda(y)}\left\{\varphi^-(y)\Lambda_1^+(y,\lambda) - {\varphi^-}'(y)\Lambda_2^+(y,\lambda)\right\}e^{\sqrt{\lambda}y},\end{aligned}$$

where

$$\varphi^-(y)\Lambda_1^+(y,\lambda) - \varphi^{-'}(y)\Lambda_2^+(y,\lambda) = \mathcal{O}\left(e^{(\sqrt{\lambda}-\alpha)y}\right).$$

Collecting all these results, we obtain that

$$\mathbf{G}_\lambda(x,y) = \frac{\varphi^+(x)\varphi^-(y)}{\mathbb{W}_\lambda(y)} = e^{-\sqrt{\lambda}(x-y)}\mathcal{O}\left(e^{(\sqrt{\lambda}-\alpha)y}\right),$$

for all $0 \leq y \leq x$.

Cases (v)-(vi). Let us fix $y \leq x \leq 0$. In that case, we need to express $\varphi^+(x)$ as a linear combination of $\varphi^-(x)$ and $\psi^-(x)$. Thus we write

$$\varphi^+(x) = A(x,\lambda)\varphi^-(x) + B(x,\lambda)\psi^-(x).$$

We denote by $\mathbb{H}_\lambda$ and $\mathbb{K}_\lambda$ the following two determinants

$$\mathbb{H}_\lambda(x) := \varphi^-(x)\psi^{-'}(x) - \varphi^{-'}(x)\psi^-(x), \quad \mathbb{K}_\lambda(x) := \varphi^+(x)\psi^{-'}(x) - \varphi^{+'}(x)\psi^-(x),$$

such that the coefficients A and B are given by

$$A(x,\lambda) = \frac{\mathbb{K}_\lambda(x)}{\mathbb{H}_\lambda(x)}, \quad \text{and} \quad B(x,\lambda) = -\frac{\mathbb{W}_\lambda(x)}{\mathbb{H}_\lambda(x)}.$$

Once again, using Lemma 2.2, we obtain

$$\mathbb{H}_\lambda(x) = e^{-(c_*+2\beta)x}\left((1+\theta_1^-(x,\lambda))(\mu^-(\lambda,\beta)+\kappa_2^-(x,\lambda)) - (\mu^+(\lambda,\beta)+\theta_2^-(x,\lambda))(1+\kappa_1^-(x,\lambda))\right),$$

from which we deduce that

$$\mathbb{H}_\lambda(x) = e^{-(c_*+2\beta)x}\mathcal{O}(1),$$

for all $x \geq 0$. Recalling that the Wronskian $\mathbb{W}_\lambda$ can be simplified for $x \leq -1$ to

$$\mathbb{W}_\lambda(x) = e^{-(c_*+2\beta)x}\mathbb{W}_\lambda(0)$$

and owing to Lemma 2.1, we have

$$B(x,\lambda) = -\frac{\mathbb{W}_\lambda(x)}{\mathbb{H}_\lambda(x)} = \mathcal{O}(1)$$

for all $x \leq 0$. On the other hand, we have that

$$\partial_x\mathbb{K}_\lambda(x) = -\zeta_1(x)\mathbb{K}_\lambda(x),$$

such that

$$\mathbb{K}_\lambda(x) = e^{-\int_0^x \zeta_1(\tau)\mathrm{d}\tau}\mathbb{K}_\lambda(0).$$

From which, we deduce that for all $x \leq -1$ we further have

$$\mathbb{K}_\lambda(x) = e^{-(c_*+2\beta)x}\mathbb{K}_\lambda(0),$$

such that we deduce

$$A(x,\lambda) = \frac{\mathbb{K}_\lambda(x)}{\mathbb{H}_\lambda(x)} = \mathcal{O}(1),$$

which holds true for all $x \leq 0$. Finally, we get

$$\frac{A(x,\lambda)}{\mathbb{W}_\lambda(y)}\varphi^-(x)\varphi^-(y) = \frac{e^{\mu^+(\lambda,\beta)(x+y)}}{e^{(\mu^+(\lambda,\beta)+\mu^-(\lambda,\beta))y}}\mathcal{O}(1) = e^{\mu^-(\lambda,\beta)(x-y)}\mathcal{O}\left(e^{(\mu^+(\lambda,\beta)-\mu^-(\lambda,\beta))x}\right),$$

and

$$\frac{B(x,\lambda)}{\mathbb{W}_\lambda(y)}\psi^-(x)\varphi^-(y) = \frac{\mu^-(\lambda,\beta)x + e^{\mu^+(\lambda,\beta)y}}{e^{(\mu^+(\lambda,\beta)+\mu^-(\lambda,\beta))y}}\mathcal{O}(1) = e^{\mu^-(\lambda,\beta)(x-y)}\mathcal{O}\left(1\right).$$

And the proof of lemma is thereby complete. ∎

4 Estimates on the Green's function $\mathbf{G}(t,x,y)$

We now use the estimates on the pointwise Green's function $\mathbf{G}_\lambda(x,y)$ to derive bounds on the temporal Green's function $\mathbf{G}(t,x,y)$.

Proposition 4.1. *Under the assumptions of our main theorem, and for some constants $\kappa > 0$, $r > 0$ and M, $T > 1$, the Green's function $\mathbf{G}(t,x,y)$ for $\partial_t p = \mathcal{L}p$ satisfies the following estimates.*

(i) For $|x-y| \geq Kt$ or $t < T$, with K sufficiently large,

$$|\mathbf{G}(t,x,y)| \leq C\frac{1}{t^{-1/2}}e^{-\frac{|x-y|^2}{\kappa t}}.$$

(ii) For $|x-y| \leq Kt$ and $t \geq T$, with K and T as above,

$$|\mathbf{G}(t,x,y)| \leq C\left(\frac{1+|x-y|}{t^{3/2}}\right)e^{-\frac{|x-y|^2}{\kappa t}} + Ce^{-rt}.$$

Proof. Case (i): The proof of this case follows as in [19]. We select a contour of integration consisting of the concatenation of a parabolic contour contained in the region where the estimates of Lemma 3.1 are valid and a ray extending to infinity. Consider the region Ω_δ with $\delta_0 = 0$ for simplicity. We will slightly abuse notation by setting $\delta_1 = \delta > 0$ into the definition (3.1) of Ω_δ. Define $\Gamma_\delta = \{-\delta|\ell| + \mathbf{i}\ell \mid \ell \in \mathbb{R}\}$ as the boundary of this region. We will establish the necessary estimate by appropriate choice of contours on which to apply the inverse Laplace transform. Define a secondary contour Γ_ρ defined for those values of λ for which

$$\sqrt{\lambda} = \rho + \mathbf{i}k\ ,\ \ k \in \mathbb{R},$$

implying

$$\lambda(k) = \rho^2 - k^2 + 2\mathbf{i}\rho k.$$

Suppose that ρ is chosen sufficiently large so that when $\lambda(k) \in \Omega_\delta$ estimate (3.2) applies. Define the contours $\Delta_1 = \Gamma_\rho \cap \Omega_\delta$ and Δ_2 to be the continuation of the contour to ∞ along the contour Γ_δ. Then,

$$|\mathbf{G}(t,x,y)| \leq \frac{1}{2\pi}\left|\int_{\Delta_1} e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda\right| + \frac{1}{2\pi}\left|\int_{\Delta_2} e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda\right|.$$

We apply the estimate from Lemma 3.1 to the integral along the contour Δ_1. Here we have

$$\frac{1}{2\pi}\left|\int_{\Delta_1} e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda\right| \leq C\left|\int_{\Delta_1} e^{\rho^2 t - k^2 t}e^{-\sqrt{\rho^2+k^2}\eta|x-y|}\mathrm{d}k\right|,$$

where we have used that $\frac{|\sqrt{\lambda}|}{\sqrt{|\lambda|}} = 1$. Using the bound, $\sqrt{\rho^2+k^2} \geq \rho$ we then factor

$$\frac{1}{2\pi}\left|\int_{\Delta_1} e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda\right| \leq Ce^{\rho^2 t - \rho\eta|x-y|}\left|\int_{\Delta_1} e^{-k^2 t}\mathrm{d}k\right|.$$

Now, we must select ρ so that

$$\rho^2 t - \rho\eta|x-y| = -\frac{|x-y|^2}{\kappa t},$$

for some $\kappa > 0$. Solving for ρ, we obtain

$$\rho = \frac{|x-y|}{t}\left(\frac{\eta}{2} \pm \frac{1}{2}\sqrt{\eta^2 - \frac{4}{\kappa}}\right).$$

Recall that η is fixed by Lemma 3.1, so we may select κ sufficiently large so that the equation for ρ has real roots. With κ fixed, we may now restrict to $\frac{|x-y|}{t}$ sufficiently large so that the contour Γ_1 lies within the region where the estimate (3.2) applies and we obtain the bound

$$\frac{1}{2\pi}\left|\int_{\Delta_1} e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda\right| \leq \frac{C}{\sqrt{t}}e^{-\frac{|x-y|^2}{\kappa t}}.$$

We now turn our attention to the contribution from the integral along Δ_2, focusing on the portion in the upper half plane; the case in the lower half plane is similar. In this region, the contour can be described as

$$\lambda(\ell) = -\delta\ell + \mathbf{i}\ell.$$

Intersection points of Δ_1 and Δ_2 satisfy

$$\rho^2 - k^2 = -\delta\ell, \quad 2\rho k = \ell,$$

from which we find

$$k^* = \rho\left(\delta + \sqrt{\delta^2+\rho^2}\right), \; \ell_* = 2\rho^2\left(\delta + \sqrt{\delta^2+\rho^2}\right). \tag{4.1}$$

We then have

$$\frac{1}{2\pi}\left|\int_{\Delta_2} e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda\right| \leq C\int_{\ell_*}^{\infty}\frac{1}{\sqrt{\ell}}e^{-\delta\ell t}d\ell \leq C\int_{4\delta\rho^2}^{\infty}\frac{1}{\sqrt{\ell}}e^{-\delta\ell t}d\ell \leq \frac{C}{\sqrt{t}}\int_{2\delta\rho\sqrt{t}}^{\infty}e^{-z^2}dz \leq \frac{C}{\sqrt{t}}e^{-(2\delta\rho\sqrt{t})^2}. \tag{4.2}$$

Substituting for ρ and potentially taking a larger value for κ, we obtain the bound

$$\frac{1}{2\pi}\left|\int_{\Delta_2} e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda\right| \leq \frac{C}{\sqrt{t}}e^{-\frac{|x-y|^2}{\kappa t}},$$

and the result follows in this case.

Finally, parabolic regularity gives the same estimate for short time $t < T$.

Case (ii): In the regime $|x-y| \leq Kt$, the analysis must be carried out in pieces according to the relative positions of x and y. The cases where one, or both, of x and y are positive are similar so we consider the case of $y \leq 0 \leq x$ only.

We first summarize our approach. We will deform the Laplace inversion contour into two pieces – a parabolic segment surrounding the branch point at $\lambda = 0$ and a ray tending to infinity along Γ_δ. Recall the estimate in Lemma 3.2,

$$\mathbf{G}_\lambda(x,y) = e^{-\sqrt{\lambda}(x-y)}\mathcal{O}\left(e^{(\mu^+(\lambda,\beta)-\sqrt{\lambda})y}\right).$$

We select the parabolic contour in such a way that it is contained in the region where

$$\Re\left(\mu^+(\lambda,\beta)-\sqrt{\lambda}\right) > 0. \tag{4.3}$$

The contour integration can then be divided into four integrals depending on the relevant estimates in each region, see Figure 2 for an illustration,

- Γ_1 – a parabolic contour near $\lambda = 0$ for which condition (4.3) holds ,
- Γ_2 – a continuation of the contour along the ray Γ_δ for those λ values where condition (4.3) holds ,
- Γ_3 – a continuation of the contour along the ray Γ_δ for those λ values where condition (4.3) fails, but for which the large λ estimates from Lemma 3.1 do not apply ,
- Γ_4 – a continuation of the contour along the ray Γ_δ for those λ values where the large λ estimates from Lemma 3.1 apply.

Note that at $\lambda = 0$ we have that $\mu^+(\lambda,\beta) > \sqrt{\lambda}$. Then there exists $M_\mu > 0$ such that for all $\lambda \in \Omega_\delta$, off the imaginary axis and with $|\lambda| < M_\mu$ we have that $\mu^+(\lambda,\beta) > \sqrt{\lambda}$.

Define the contour Γ_1 via

$$\sqrt{\lambda} = \rho + \mathbf{i}k,$$

for some $\rho > 0$ and for all k such that $\sqrt{\lambda} \in \Omega_\delta$.

For Γ_2 through Γ_4 we work with λ in the upper half plane without loss of generality – estimates for the lower half plane are analogous. The contours Γ_2 through Γ_4 are all given by

$$\lambda = -\delta_0 + \cos(\theta)\ell + \mathbf{i}\sin(\theta)\ell,$$

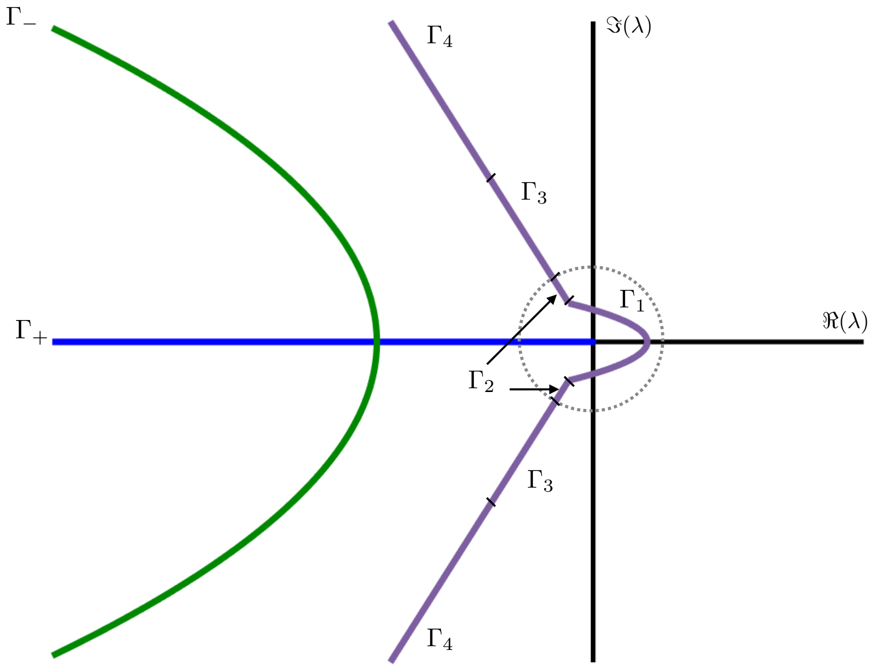

Figure 2: *Visualization of the 4 contours of integration Γ_i, $i = 1, \ldots, 4$, in the regime $|x - y| \leq Kt$. Γ_1 is a parabolic contour near $\lambda = 0$ for which condition (4.3) holds. Γ_2 is a continuation of the contour along the ray Γ_δ for those λ values where condition (4.3) holds. Γ_3 is a continuation of the contour along the ray Γ_δ for those λ values where condition (4.3) fails, but for which the large λ estimates from Lemma 3.1 do not apply where we simply used the uniform boundedness of the Green's function. Γ_4 is a continuation of the contour along the ray Γ_δ for those λ values where the large λ estimates from Lemma 3.1 apply.*

for some fixed $\pi/2 < \theta < \pi$ and varying intervals of ℓ. For any ρ we can compute the intersection points

$$k^*(\rho) = -\rho\cot(\theta) + \sqrt{\rho^2\csc^2(\theta) + \delta_0}, \qquad \ell_1(\rho) = \frac{2\rho k^*(\rho)}{\sin(\theta)}.$$

Recall the estimates in Lemma 3.1 and Lemma 3.2. Write

$$\mathbf{G}_\lambda(x, y) = e^{-\sqrt{\lambda}(x-y)} e^{(\mu^+(\lambda,\beta)-\sqrt{\lambda})x} \mathbf{H}_\lambda(x, y),$$

where $\mathbf{H}_\lambda(x, y)$ is analytic as a function of $\sqrt{\lambda}$ for $\lambda \in \Omega_\delta$ and uniformly bounded in (x, y).

For the parabolic contour, we recall that we are interested in $|x - y| \leq Kt$ so we select

$$\rho = \frac{|x - y|}{Lt}$$

for some L sufficiently large so that the parabolic contour Γ_1 lies within the region for which $\mu^+(\lambda, \beta) > \sqrt{\lambda}$. Let us note that

$$\mathrm{d}\lambda = 2\mathbf{i}\left(\frac{(x-y)}{Lt} + \mathbf{i}k\right)\mathrm{d}k \text{ and } \lambda = \frac{(x-y)^2}{L^2t^2} - k^2 + 2\frac{(x-y)}{Lt}\mathbf{i}k.$$

Then

$$\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_1} e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda = \frac{1}{\pi}e^{\rho^2 t-\rho(x-y)}\int_{-k^*}^{k^*} e^{-k^2 t}e^{2\mathbf{i}\rho k-\mathbf{i}k(x-y)}\mathbf{H}_{\lambda(k)}(x,y)(\rho+\mathbf{i}k)\mathrm{d}k$$
$$= \frac{1}{\pi}e^{\rho^2 t-\rho(x-y)}\int_{-k^*}^{k^*} e^{-k^2 t}\left(H_R(x,y,k)+\mathbf{i}H_I(x,y,k)\right)(\rho+\mathbf{i}k)\mathrm{d}k. \qquad (4.4)$$

Here we have expanded $e^{2\mathbf{i}\rho k-\mathbf{i}k(x-y)}\mathbf{H}_{\lambda(k)}$ into its real and imaginary parts. Since $\mathbf{G}_\lambda$ is holomorphic and the contour is symmetric with respect to the real axis, we have that $H_R(x,y,k)$ is even in k while $H_I(x,y,k)$ is odd. Using this, the integral reduces to

$$\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_1} e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda = \frac{1}{\pi}e^{\rho^2 t-\rho(x-y)}\int_{-k^*}^{k^*} e^{-k^2 t}\left(\rho H_R(x,y,k)-kH_I(x,y,k)\right)\mathrm{d}k.$$

Boundedness of H_R implies that

$$\left|\frac{1}{\pi}e^{\rho^2 t-\rho(x-y)}\int_{-k^*}^{k^*} e^{-k^2 t}\rho H_R(x,y,k)\mathrm{d}k\right| \le C\frac{\rho}{\sqrt{t}}e^{\rho^2 t-\rho(x-y)}.$$

On the other hand, since $H_I(x,y,k)$ is odd in k, it can be expressed as $H_I(x,y,k) = k\tilde{H}_I(x,y,k)$, where $\tilde{H}_I(x,y,k)$ is again bounded. Therefore,

$$\left|\frac{1}{\pi}e^{\rho^2 t-\rho(x-y)}\int_{-k^*}^{k^*} e^{-k^2 t}k^2\tilde{H}_I(x,y,k)\mathrm{d}k\right| \le \frac{C}{t^{3/2}}e^{\rho^2 t-\rho(x-y)}.$$

Using our choice of ρ, we then arrive at the estimate

$$\frac{1}{2\pi}\left|\int_{\Gamma_1} e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda\right| \le C\frac{1+|x-y|}{t^{3/2}}e^{-\frac{|x-y|^2}{\kappa t}},$$

for $\kappa = L^2/(L-1)$.

Next consider the integral along Γ_2. Here we have

$$\frac{1}{2\pi}\left|\int_{\Gamma_2} e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda\right| \le \frac{1}{\pi}e^{-\delta_0 t}\int_{\ell_1}^{\ell_2} e^{\cos(\theta)\ell-(x-y)\mathrm{Re}(\sqrt{-\delta_0+\cos(\theta)\ell+\mathbf{i}\sin(\theta)\ell})}\mathrm{d}\ell.$$

Using that

$$\Re(\sqrt{-\delta_0+\cos(\theta)\ell+\mathbf{i}\sin(\theta)\ell}) < \ell\cos(\theta/2),$$

the estimate simplifies to

$$\frac{1}{2\pi}\left|\int_{\Gamma_2} e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda\right| \le Ce^{-\delta_0 t}\int_{\ell_1}^{\ell_2} e^{\cos(\theta)\ell-\sqrt{\ell}(x-y)\cos(\theta/2)}\mathrm{d}\ell$$
$$\le Ce^{-\delta_0 t}\frac{|x-y|}{t^{3/2}}e^{-\frac{|x-y|^2}{\kappa t}}, \qquad (4.5)$$

for $\kappa = \frac{-4\cos(\theta)}{\cos^2(\theta/2)}$.

Next consider the integral along Γ_3. The key distinction in this case is that (4.3) does not hold. We instead use the fact that $\mathbf{G}_\lambda(x,y)$ is uniformly bounded in this region to obtain

$$\frac{1}{2\pi}\left|\int_{\Gamma_3} e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda\right| \le Ce^{-rt},$$

for some $r > 0$. Finally, we consider the integral along Γ_4. The large λ bounds apply here and the analysis follows as in (4.2). We find

$$\frac{1}{2\pi}\left|\int_{\Gamma_4} e^{\lambda t}\mathbf{G}_\lambda(x,y)\mathrm{d}\lambda\right| \le Ce^{-\delta_0 t}\int_{\ell_3}^{\infty}\frac{1}{\sqrt{\ell}}e^{\cos(\theta)\ell t}\mathrm{d}\ell \le Ce^{-rt},$$

for some $r > 0$.

This concludes the proof of case (ii) with $y \le 0 \le x$. To complete the proof similar estimates are required to the five other orderings as in Lemma 3.2. Since the proofs are analogous to the present case, we omit the details. Note that for cases (v) and (vi) in Lemma 3.2 we could achieve sharper bounds, but these are not required for the proof of Theorem 1 so we do not pursue this here.

■

In the following section, we will require bounds on integrals of the form,

$$\left|\int_{\mathbb{R}}\mathbf{G}(t,x,y)h(y)\mathrm{d}y\right|,$$

for some $h(y)$. Applying Proposition 4.1, we can then estimate

$$\begin{aligned}\left|\int_{\mathbb{R}}\mathbf{G}(t,x,y)h(y)\mathrm{d}y\right| \le &\int_{-\infty}^{x-Kt} C\frac{1}{t^{-1/2}}e^{-\frac{|x-y|^2}{\kappa t}}|h(y)|\mathrm{d}y \\ &+\int_{x-Kt}^{x+Kt} C\left(\left(\frac{1+|x-y|}{t^{3/2}}\right)e^{-\frac{|x-y|^2}{\kappa t}}+e^{-rt}\right)|h(y)|\mathrm{d}y \\ &+\int_{x+Kt}^{\infty} C\frac{1}{t^{-1/2}}e^{-\frac{|x-y|^2}{\kappa t}}|h(y)|\mathrm{d}y. \end{aligned}\tag{4.6}$$

Note that the terms $\frac{1}{t^{-1/2}}e^{-\frac{|x-y|^2}{\kappa t}}$ in the first and last integrals are maximized at the boundary and therefore the contribution from these integrals decay exponentially in time. For the middle integral we take the limits of integration to infinity and use $1+|x-y| \le 1+|x|+|y|+|x||y|$ from which we obtain

$$\left|\int_{\mathbb{R}}\mathbf{G}(t,x,y)h(y)\mathrm{d}y\right| \le Ce^{-rt}\int_{\mathbb{R}}(1+|y|)|h(y)|\mathrm{d}y + C\frac{1+|x|}{(1+t)^{3/2}}\int_{\mathbb{R}}(1+|y|)|h(y)|\mathrm{d}y.$$

Due to the exponential decay of the first integral, we in fact have

$$\left|\int_{\mathbb{R}}\mathbf{G}(t,x,y)h(y)\mathrm{d}y\right| \le C\frac{1+|x|}{(1+t)^{3/2}}\int_{\mathbb{R}}(1+|y|)|h(y)|\mathrm{d}y.\tag{4.7}$$

5 Nonlinear Stability: Proof of Theorem 1

We recall that we look for perturbations $p(t,x)$ which satisfy the evolution equation

$$p_t = p_{xx} + \left(c_* + 2\frac{\omega'}{\omega}\right)p_x + \left(f'(q_*) + c_*\frac{\omega'}{\omega} + \frac{\omega''}{\omega}\right)p + \mathcal{N}(q_*,\omega p)p, \quad t>0, \quad x\in\mathbb{R}. \tag{5.1}$$

The Cauchy problem associated to equation (5.1) with initial condition $p_0 \in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} |y||p_0(y)|\mathrm{d}y < +\infty$ is locally well-posed in $L^\infty(\mathbb{R})$. As a consequence, we let $T_*>0$ be the maximal time of existence of a solution $p\in L^\infty(\mathbb{R})$ with initial condition $p_0 \in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} |y||p_0(y)|\mathrm{d}y < +\infty$ of the associated integral formulation of (5.1) given by

$$p(t,x) = \int_{\mathbb{R}} \mathbf{G}(t,x,y)p_0(y)\mathrm{d}y + \int_0^t\int_{\mathbb{R}} \mathbf{G}(t-\tau,x,y)\mathcal{N}(q_*(y),\omega(y)p(\tau,y))p(\tau,y)\mathrm{d}y\mathrm{d}\tau. \tag{5.2}$$

For $t\in[0,T_*)$, we define

$$\Theta(t) = \sup_{0\leq\tau\leq t}\sup_{x\in\mathbb{R}} (1+\tau)^{3/2}\frac{|p(\tau,x)|}{1+|x|}.$$

Furthermore, by our regularity assumption on f, there exists a positive nondecreasing function $\chi:\mathbb{R}_+\to\mathbb{R}_+$, such that

$$|\mathcal{N}(q_*,\omega p)p| \leq \chi(R)\omega p^2, \quad |\omega p|\leq R.$$

Now, using the fact that ω is exponentially localized, we have that for all $0\leq\tau\leq t$ and all $y\in\mathbb{R}$

$$|\mathcal{N}(q_*(y),\omega(y)p(\tau,y))p(\tau,y)| \leq \chi(\omega_\infty\Theta(t))\omega(y)p^2(\tau,y) \leq \frac{1}{(1+\tau)^3}\chi(\omega_\infty\Theta(t))(1+|y|)^2\omega(y)\Theta(t)^2,$$

where $\omega_\infty := \sup_{x\in\mathbb{R}}((1+|x|)\omega(x)) < \infty$.

We now bound each term of the right hand side of (5.2) by using the estimates derived in Proposition 4.1, see (4.7), to get

$$\left|\int_{\mathbb{R}} \mathbf{G}(t,x,y)p_0(y)\mathrm{d}y\right| \leq \frac{C}{(1+t)^{3/2}}\int_{\mathbb{R}}(1+|x-y|)e^{-\frac{|x-y|^2}{Lt}}|p_0(y)|\mathrm{d}y \leq C\frac{1+|x|}{(1+t)^{3/2}}\int_{\mathbb{R}}(1+|y|)|p_0(y)|\mathrm{d}y,$$

together with

$$|\mathbf{G}(t-\tau,x,y)\mathcal{N}(q_*(y),\omega(y)p(\tau,y))p(\tau,y)| \leq C\frac{\chi(\omega_\infty\Theta(t))\Theta(t)^2}{(1+t-\tau)^{3/2}(1+\tau)^3}e^{-\frac{|x-y|^2}{L(t-\tau)}}(1+|x-y|)(1+|y|)^2\omega(y).$$

Using the fact that [18]

$$\int_0^t \frac{1}{(1+t-\tau)^{3/2}(1+\tau)^3}\mathrm{d}\tau \leq \frac{\tilde{C}}{(1+t)^{3/2}},$$

we obtain

$$\left|\int_0^t\int_{\mathbb{R}} \mathbf{G}(t-\tau,x,y)\mathcal{N}(q_*(y),\omega(y)p(\tau,y))p(\tau,y)\mathrm{d}y\mathrm{d}\tau\right| \leq \widehat{C}\frac{\chi(\omega_\infty\Theta(t))\Theta(t)^2(1+|x|)}{(1+t)^{3/2}}\int_{\mathbb{R}}(1+|y|)^3\omega(y)\mathrm{d}y.$$

As a consequence, for all $t\in[0,T_*)$ we have

$$\Theta(t) \leq C\int_{\mathbb{R}}(1+|y|)|p_0(y)|\mathrm{d}y + \widehat{C}\left(\int_{\mathbb{R}}(1+|y|)^3\omega(y)\mathrm{d}y\right)\chi(\omega_\infty\Theta(t))\Theta(t)^2. \tag{5.3}$$

Let

$$\Gamma = \sup_{x\in\mathbb{R}} \frac{|p_0(x)|}{1+|x|}, \quad \Omega = \int_{\mathbb{R}} (1+|y|)|p_0(y)|\mathrm{d}y.$$

Then (5.3) at $t=0$ implies

$$\Gamma \leq C_1\Omega + C_2\chi(\omega_\infty\Gamma)\Gamma^2.$$

We require that the initial condition $p_0(y)$ satisfy

$$\Omega \leq \min\left\{\frac{1}{2C_1}, \frac{1}{4C_1C_2\chi(\omega_\infty)}\right\}, \quad \Gamma < C_1\Omega.$$

Then it follows that

$$\Theta(t) \leq 1$$

for all $t\in[0,T_*)$. This implies the maximal time of existence is $T_* = +\infty$ and the solution p of (5.1) satisfies

$$\sup_{t\geq 0}\sup_{x\in\mathbb{R}} (1+t)^{3/2}\frac{|p(t,x)|}{1+|x|} < 1.$$

This concludes the proof.

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