Introduction to mesoscopic models of visual cortical structures

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M2 MVA / M2 Maths-Bio

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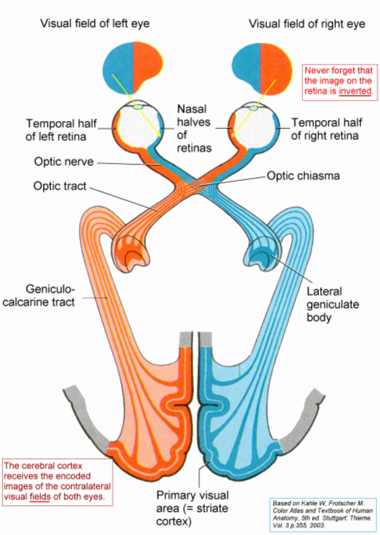
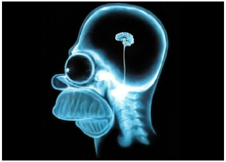
Outline

1. Structure of primary visual cortex (V1)
    - Anatomy
    - Retinotopy
    - Cortical layers organization
2. Functional architecture of V1
3. Neural fields models
4. Applications

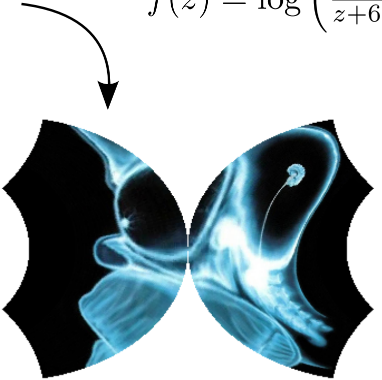
Anatomy of the visual cortex



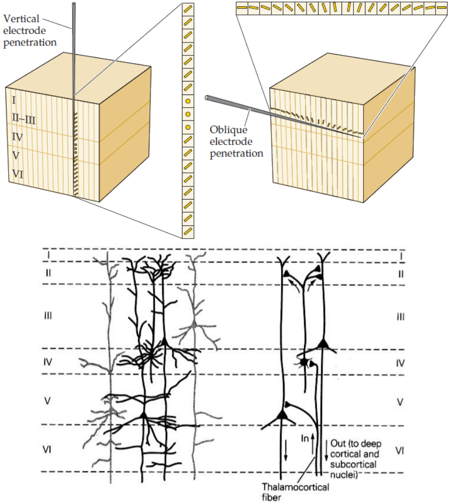
Retinotopy



$$f(z) = \log\left(\frac{z+0.33}{z+6.66}\right)$$



Cortical layers organization of V1 (Purves et al)



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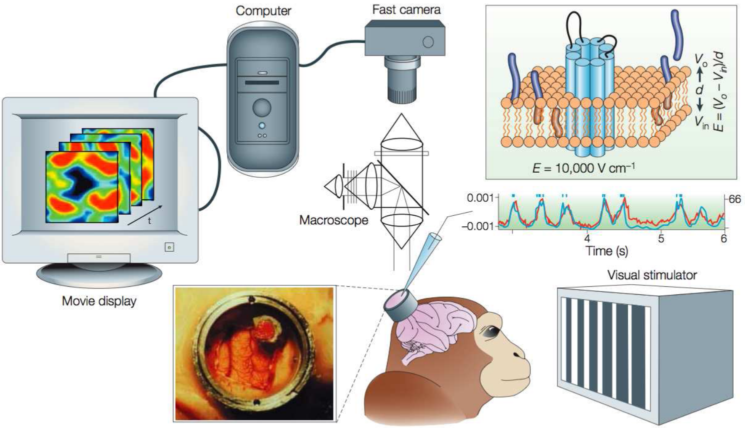
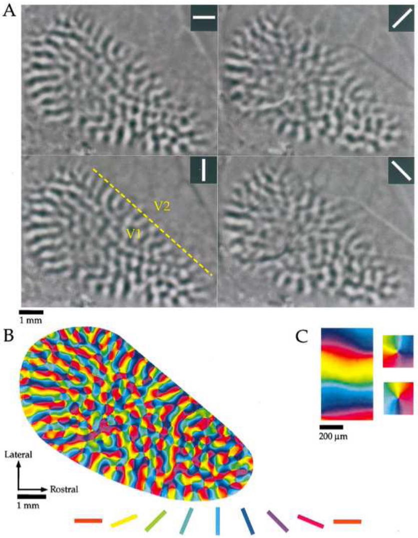
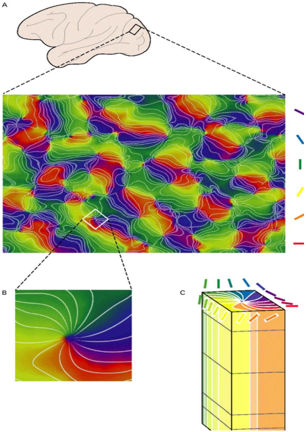
Optical imaging: methods



Results for orientation (Bosking et al 97)



Hypercolumns of orientation in V1



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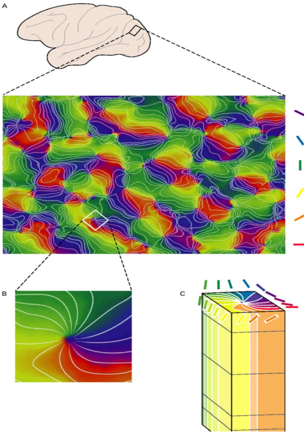
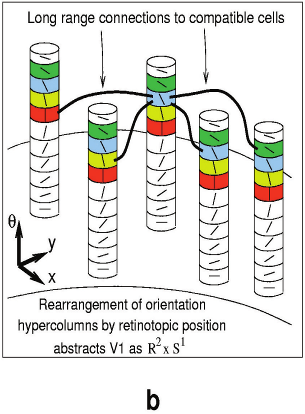
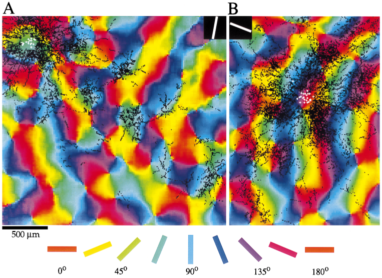




Ben Sahar and Zucker 2004.

Intra-cortical connections in V1: anisotropy?



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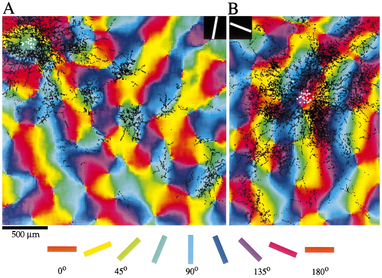

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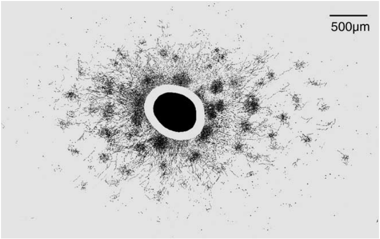
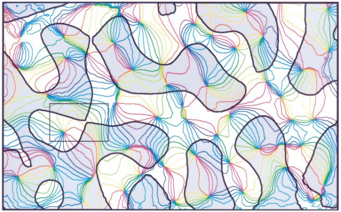


Lund et al 03 (Macaque).

Other cortical maps: ocular dominance, direction of motion etc...



Hubener et al 97 (Cat).

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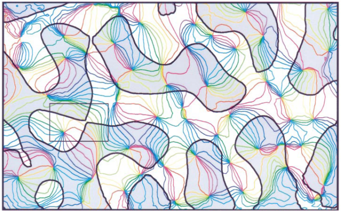

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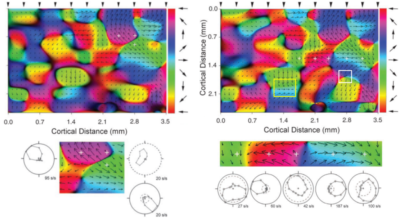

Diogo et al 03 (area MT of Monkey).

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- Existence of particular points: pinwheels (all orientations are represented).
- Local excitatory/inhibitory connections are homogeneous, whereas long-range connections (mainly excitatory neurons) are patchy, modulatory and anisotropic.

Outline

Local models for n interacting neural masses

- each neural population i is described by its average membrane potential $V_i(t)$ or by its average instantaneous firing rate $\nu_i(t)$ with $\nu_i(t) = S_i(V_i(t))$, where S_i is sigmoidal:

$$S_i(x) = \frac{S_{im}}{1 + e^{-\sigma_i(x-\theta_i)}}$$

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- the number of spikes arriving between t and $t + dt$ is $\nu_j(t)dt$, then the average membrane potential of population i is:

$$\begin{aligned} V_i(t) &= \sum_j \int_{t_0}^{t} PSP_{ij}(t-s) S_j(V_j(s)) ds \\ \nu_i(t) &= S_i\left(\sum_j \int_{t_0}^{t} PSP_{ij}(t-s) \nu_j(s) ds\right) \end{aligned}$$

The voltage-based model

- post-synaptic potential has the same shape no matter which presynaptic population caused it, this leads to

$$PSP_{ij}(t) = w_{ij} PSP_i(t)$$

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- if we assume that $PSP_i(t) = e^{-t/\tau_i} H(t)$ or equivalently

$$\tau_i \frac{dPSP_i(t)}{dt} + PSP_i(t) = \delta(t)$$

we end up with a system of ODEs:

$$\tau_i \frac{dV_i(t)}{dt} + V_i(t) = \sum_j w_{ij} S_j(V_j(t)) + I_{ext}^i(t).$$

We rewrite in vector form:

$$\dot{\mathbf{V}}(t) = -\mathbf{L}\mathbf{V}(t) + \mathbf{W}\mathbf{S}(\mathbf{V}(t)) + \mathbf{I}_{ext}(t)$$

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- we also suppose that $PSP_j(t) = e^{-t/\tau_j} H(t)$ and we end up with a system of ODEs:

$$\tau_i \frac{dA_i(t)}{dt} + A_i(t) = S_i \left(\sum_j w_{ij} A_j(t) + I^i_{ext}(t) \right).$$

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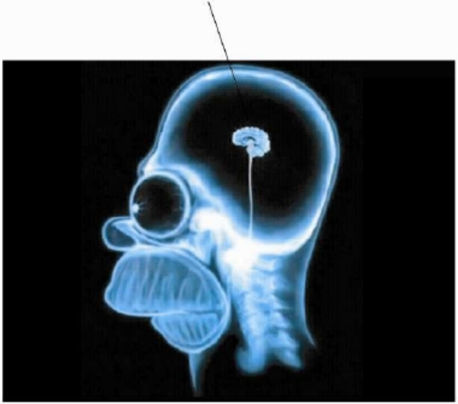
Activity neural fields equation

$$\frac{d\mathbf{A}(\mathbf{r}, t)}{dt} = -\mathbf{L}\mathbf{A}(\mathbf{r}, t) + \mathbf{S}\left(\int_{\Omega} \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{A}(\mathbf{r}', t)d\mathbf{r}' + \mathbf{I}_{ext}(\mathbf{r}, t)\right) \qquad (2)$$

Remarks

$$\tau_s \frac{\partial a(x,t)}{dt} = -a(x,t) + \int W(x|y) f(a(y,t)) dy$$



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- features can be taken into account: $\mathbf{V}(\mathbf{r}, \theta, t)$ in the case of orientation

Cauchy problem

Ω is an open bounded set of $\mathbb{R}^d$. We define $\mathcal{F} = L^2(\Omega, \mathbb{R}^n)$ (Hilbert space). We can rewrite equation (1) in a compact form (function $\mathbf{V}(t)$ is thought of as a mapping $\mathbf{V} : \mathbb{R}^+ \to \mathcal{F}$):

$$\begin{cases} \dfrac{d\mathbf{V}}{dt} & = & -\mathbf{LV} + \mathbf{R}(t, \mathbf{V}) \quad t > 0 \\ \mathbf{V}(0) & = & \mathbf{V}_0 \in \mathcal{F} \end{cases} \tag{3}$$

The nonlinear operator $\mathbf{R}$ is defined by:

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Theorem (Existence and uniqueness of a solution)

If the following two hypotheses are satisfied:

1. $\mathbf{W} \in \mathcal{C}(\mathbb{R}^+, L^\infty(\Omega^2, \mathbb{R}^n))$ *and is uniformly bounded in time,*
2. *the external input* $\mathbf{I}_{ext} \in \mathcal{C}(\mathbb{R}^+, \mathcal{F})$

then for any function $\mathbf{V}_0 \in \mathcal{F}$ *there is a unique solution* $\mathbf{V}$ *defined on* $\mathbb{R}^+$ *and continuously differentiable of the initial value problem* (3).

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3. $\|\mathbf{R}(t, \mathbf{V}_1) - \mathbf{R}(t, \mathbf{V}_2)\|_{\mathcal{F}} \leq DS_m \sup_{t \in \mathbb{R}^+} \|\mathbf{W}(t)\|_{\mathcal{F}} \|\mathbf{V}_1 - \mathbf{V}_2\|_{\mathcal{F}}$ for all $t > 0$ and $\mathbf{V}_1, \mathbf{V}_2 \in \mathcal{F}$, where $DS_m = \sup_{i=1\dots n} \sup_{x \in \mathbb{R}} |S_i'(x)|$ (Lipschitz continuity of $\mathbf{R}$ with respect to its second argument, uniformly with respect to the first)

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4. application of the Cauchy Lipschitz theorem in Banach spaces

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1.
   - if it exists: $D^q\mathbf{R}(t, \mathbf{V}_0)[U_1, \cdots, U_q] = \mathbf{W}(t)\left(\mathbf{S}^{(q)}(\mathbf{V}_0)(U_1 \cdots U_q)\right)$
   - $D^q\mathbf{R}(t, \mathbf{V}_0)$ is well defined because $U_1 \cdots U_q \in L^\infty(\Omega, \mathbb{R}^n)$
   -

$$\|D^q\mathbf{R}(t, \mathbf{V}_0)[U_1, \cdots, U_q]\|_{L^\infty(\Omega, \mathbb{R}^n)} \leq |\Omega| \left\|\mathbf{W}(t)\mathbf{S}^{(q)}(\mathbf{V}_0)\right\|_{L^\infty(\Omega^2, \mathbb{R}^n)} \|U_1 \cdots U_q\|_{L^\infty(\Omega, \mathbb{R}^n)}$$

More properties for the nonlinearity

Lemma

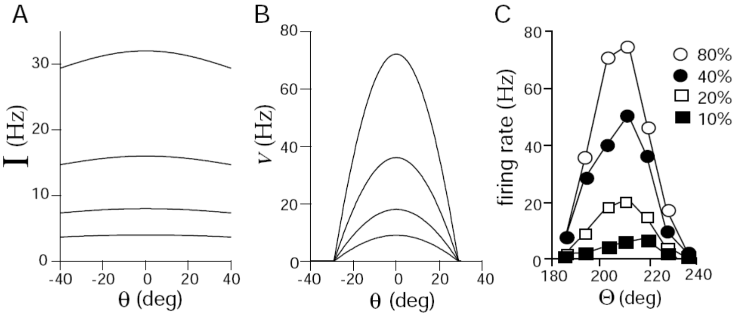
If $\mathbf{W} \in \mathcal{C}(\mathbb{R}^+, L^\infty(\Omega^2, \mathbb{R}^n))$*, then* $\mathbf{R}$ *satisfies the following properties:*

1. $\forall q \in \mathbb{N}$, $\mathbf{R}(t, \cdot) \in \mathcal{C}^q(L^\infty(\Omega, \mathbb{R}^n), L^\infty(\Omega, \mathbb{R}^n))$ *and* $D^q\mathbf{R}(t, \mathbf{V}_0) = \mathbf{W}(t)\mathbf{S}^{(q)}(\mathbf{V}_0)$
2. $\mathbf{R}(t, \cdot)$ *is a compact operator for all* $t > 0$.

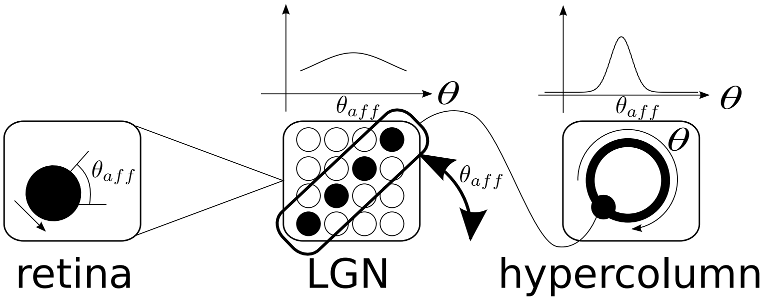
1.
   - if it exists: $D^q\mathbf{R}(t, \mathbf{V}_0)[U_1, \cdots, U_q] = \mathbf{W}(t)\left(\mathbf{S}^{(q)}(\mathbf{V}_0)(U_1 \cdots U_q)\right)$
   - $D^q\mathbf{R}(t, \mathbf{V}_0)$ is well defined because $U_1 \cdots U_q \in L^\infty(\Omega, \mathbb{R}^n)$
   - $$\|D^q\mathbf{R}(t, \mathbf{V}_0)[U_1, \cdots, U_q]\|_{L^\infty(\Omega, \mathbb{R}^n)} \leq |\Omega| \left\|\mathbf{W}(t)\mathbf{S}^{(q)}(\mathbf{V}_0)\right\|_{L^\infty(\Omega^2, \mathbb{R}^n)} \|U_1 \cdots U_q\|_{L^\infty(\Omega, \mathbb{R}^n)}$$
2. direct application of Arzelà-Ascoli theorem

Outline

Ring Model of orientation: facts



Ring Model of orientation: mechanism



Ring Model of orientation: equation

We consider the following equation:

$$\tau \frac{\partial V(\theta, t)}{\partial t} = -V(\theta, t) + \int_{-\pi/2}^{\pi/2} J(\theta - \theta') S(\mu V(\theta')) \frac{d\theta'}{\pi} + \epsilon I(\theta) \qquad (4)$$

where τ is a temporal synaptic contant ($\tau = 1ms$), $J(\theta - \theta')$ is a connectivity function (excitatory/inhibitory) and S is the sigmoidal function:

$$S(x) = \frac{1}{1 + \exp(-x + \kappa)},$$

$I(\theta)$ is an input coming from the LGN given by:

$$I(\theta) = 1 - \beta + \beta \cos(2(\theta - \theta_{aff}))$$

Without loss of generality we take $\theta_{aff} = 0$. Moreover, we take the simplest possible connectivity function:

$$J(\theta) = -1 + J_1 \cos(2\theta), \quad J_1 > 0$$

Ermentrout-Cowan model

We consider the following equation:

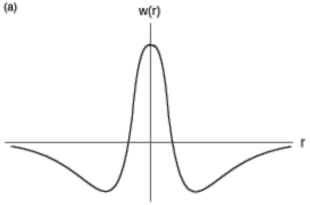
$$\tau \frac{\partial}{\partial t} a(\mathbf{r}, t) = -a(\mathbf{r}, t) + \int_{\mathbf{R}^2} w(\mathbf{r}|\mathbf{r}') S(\mu a(\mathbf{r}', t)) d\mathbf{r}' \quad (5)$$

where τ is a temporal synaptic contant ($\tau = 1ms$), $w(\mathbf{r}|\mathbf{r}') = w(\|\mathbf{r} - \mathbf{r}'\|)$ is a connectivity function (excitatory/inhibitory) and S is the sigmoidal function:

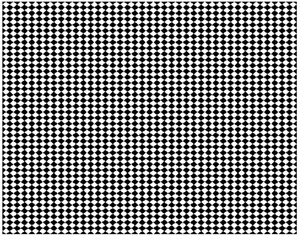
$$S(x) = \frac{1}{1 + \exp(-x + \kappa)} - \frac{1}{1 + \exp(\kappa)},$$

We choose a "Mexican-hat" connectivity function:

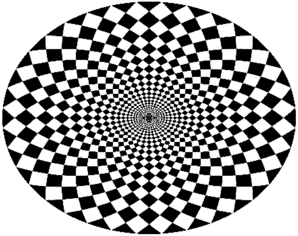
$$w(x) = \frac{A_1}{\sigma_1} e^{-\frac{x^2}{\sigma_1^2}} - \frac{A_2}{\sigma_2} e^{-\frac{x^2}{\sigma_2^2}} \quad (6)$$



Patterns of the Ermentrout-Cowan model



V1



Visual field

Bresslof-Cowan-Golubitsky-Thomas-Wiener model

We consider the following equation:

$$\tau \frac{\partial}{\partial t} a(\mathbf{r}, \theta, t) = -a(\mathbf{r}, \theta, t) + \int_{\mathbf{R}^2} \int_{-\pi/2}^{\pi/2} w(\mathbf{r}, \theta | \mathbf{r}', \theta') S(\mu a(\mathbf{r}', \theta', t)) d\mathbf{r}' \frac{d\theta'}{\pi} \quad (7)$$

with

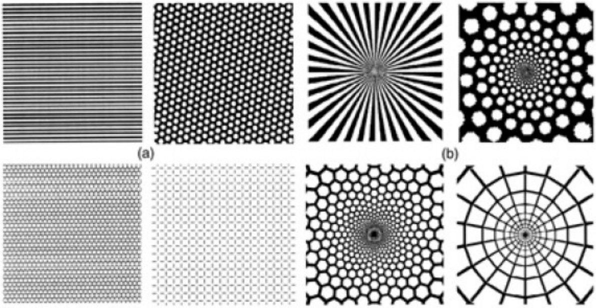
$$S(x) = \frac{1}{1 + \exp(-x + \kappa)} - \frac{1}{1 + \exp(\kappa)},$$

and

$$w(\mathbf{r}, \theta | \mathbf{r}', \theta') = J(\theta - \theta')\delta_{\mathbf{r},\mathbf{r}'} + \beta(1 - \delta_{\mathbf{r},\mathbf{r}'}) w_{lat}(\mathbf{r} - \mathbf{r}', \theta)$$

- for $\beta = 0$, we recover the Ring Model of orientation
- if $a(\mathbf{r}, \theta, t)$ is independent of θ we recover the Ermentrout-Cowan model
- we will try to infer some properties from the case $\beta = 0$ to the case $0 < \beta \ll 1$ and in the same time we will use similar method as for the Ermentrout-Cowan model

Geometric visual hallucinations



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