An introduction to bifurcation theory

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October 6, 2011

Abstract

The aim of this chapter is to introduce tools from bifurcation theory which will be necessary in the following sections for the study of neural field equations (NFE) set in the primary visual cortex. In a first step, we deal with elementary bifurcations in low dimensions such as saddle-node, transcritical, pitchfork and Hopf bifurcations. NFEs are dynamical systems defined on Banach spaces and thus are infinite dimensional. Bifurcation analysis for infinite dimensional systems is subtle and can lead to difficult problems. If it is possible, the idea is to locally reduce the problem to a finite dimensional one. This reduction is called the center manifold theory and it will be the main theoretical result of this chapter. The center manifold theory requires some functional analysis tools which will be recalled, especially the notions of linear operator, spectrum, resolvent, projectors etc... We also present some extensions of the center manifold theorem for paramter-dependent and equivarient differential equations. Directly related to the center manifold theory is the normal form theory which is a canonical way to write differential equations. We conclude this chapter with an overview of bifurcations with symmetry and give as a result the Equivariant Branching Lemma. Most of the theorems of this chapter are taken from the excellent book of Haragus-Iooss [4] (center manifolds and normal forms). The last part on the Equivariant Branching Lemma is taken from the very interesting (but difficult) book of Chossat-Lauterbach [2]. One other complementary reference is the book of Golubitsky-Stewart-Schaeffer [3]. For an elementary review on functional analysis the book of Brezis is recommanded [1].

1 Elementary bifurcation

Definition 1.1. *In dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden "qualitative" or topological change in its behaviour. Generally, at a bifurcation, the local stability properties of equilibria, periodic orbits or other invariant sets changes.*

1.1 Bifurcation of dimension 1

In this section, we consider scalar differential equations of the form

$$\frac{du}{dt} = f(u, \mu). \tag{1.1.1}$$

Here the unknown u is a real-valued function of the time t, and the vector field f is real-valued depending, besides u, upon a parameter μ. The parameter μ is the bifurcation parameter. We suppose that equation (1.1.1) is well-defined and satisfies the hypotheses of the Cauchy-Lipschitz theorem, such that for each initial condition there exists a unique solution of equation (1.1.1). Furthermore we assume that the vector field is of class $\mathcal{C}^k$, $k \geq 2$, in a neighborhood of $(0, 0)$ satisfying:

$$f(0,0) = 0, \quad \frac{\partial f}{\partial u}(0,0) = 0. \tag{1.1.2}$$

The first condition shows that $u = 0$ is an equilibrium of equation (1.1.1) at $\mu = 0$. We are interested in local bifurcations that occur in the neighborhood of this equilibrium when we vary the parameter μ. The second condition is a necessary, but not sufficient, condition for the appearance of local bifurcations at $\mu = 0$.

Remark 1.1. *Suppose that the second condition is not satified: $\partial f/\partial u(0,0) \neq 0$. A direct application of the implicit function theorem shows that the equation $f(u,\mu) = 0$ possesses a unique solution $u = u(\mu)$ in a neighborhood of* 0*, for small enough μ. In particular $u = 0$ is the only equilibrium of equation* (1.1.1) *in a neighborhood of* 0 *when $\mu = 0$, and the same property holds for μ small enough. Futhermore, the dynamics of* (1.1.1) *in a neighborhood of* 0 *is qualitatively the same for all sufficiently small values of the parameter μ: no bifurcation occurs for small values of μ.*

1.1.1 Saddle-node bifurcation

Theorem 1.1 (Saddle-node bifurcation)**.** *Assume that the vector field f is of class $\mathcal{C}^k$, $k \geq 2$, in a neighborhood of $(0,0)$ and satisfies:*

$$\frac{\partial f}{\partial \mu}(0,0) =: a \neq 0, \quad \frac{\partial^2 f}{\partial u^2}(0,0) =: 2b \neq 0. \tag{1.1.3}$$

The following properties hold in neighborhood of 0 *in $\mathbb{R}$ for small enough μ:*

(i) if $ab < 0$ (resp. $ab > 0$) the differential equation has no equilibria for $\mu < 0$ (resp. for $\mu > 0$),

(ii) if $ab < 0$ (resp. $ab > 0$) the differential equation possesses two equilibria $u_\pm(\epsilon)$, $\epsilon = \sqrt{|\mu|}$ for $\mu > 0$ (resp. $\mu < 0$), with opposite stabilities. Furthermore, the map $\epsilon \to u_\pm(\epsilon)$ is of class $\mathcal{C}^{k-2}$ in a neighborhood of 0 *in $\mathbb{R}$, and $u_\pm(\epsilon) = O(\epsilon)$.*

Then for equation (1.1.1)*, a saddle-node bifurcation occurs at $\mu = 0$.*

A direct consequence of conditions (1.1.3) is that f has the expansion:

$$f(u,\mu) = a\mu + bu^2 + o(|\mu| + u^2) \text{ as } (u,\mu) \to (0,0)$$

Exercice 1.1. *Consider the truncated equation*

$$\frac{du}{dt} = a\mu + bu^2.$$

Plot bifurcation diagrams in the (u, μ)-plane of this truncated equation for different values of a and b.

Proof. Since $a \neq 0$, we apply the implicit function theorem which implies the existence of unique solution $\mu = g(u)$ for u close to 0 of the equation $f(u, \mu) = 0$, where g is of class $\mathcal{C}^k$, $k \geq 2$ in a neighborhood of the origin with $g(0) = 0$. Its Taylor extansion is given by

$$\mu = -\frac{b}{a}u^2 + o(u^2).$$

Consequently, if $ab\mu > 0$ equation (1.1.1) has no equilibria, one equilibrium $u = 0$ if $\mu = 0$ and a pair of equilibria $u_\pm(\mu) = \pm\sqrt{-a\mu/b} + o(\sqrt{|\mu|})$ if $ab\mu < 0$. Finally, in the case $ab\mu < 0$, we have:

$$\frac{\partial f}{\partial u}(u_\pm(\mu), \mu) = 2bu_\pm(\mu) + o(\sqrt{|\mu|})$$

then the equilibrium $u_-(\mu)$ is attractive, asymptotically stable when $b > 0$ and repelling, unstable when $b < 0$; wheres, the equilibrium $u_+(\mu)$ has opposite stability properties. □

1.1.2 Pitchfork bifurcation

Theorem 1.2 (Pitchfork bifurcation). *Assume that the vector field f is of class $\mathcal{C}^k$, $k \geq 3$, in a neighborhood of $(0, 0)$, that it is satisfies conditions (1.1.2), and that it is odd with repsect to u:*

$$f(-u, \mu) = -f(u, \mu) \tag{1.1.4}$$

Furthermore assume that:

$$\frac{\partial^2 f}{\partial\mu\partial u}(0, 0) =: a \neq 0, \quad \frac{\partial^3 f}{\partial u^3}(0, 0) =: 6b \neq 0. \tag{1.1.5}$$

The following properties hold in neighborhood of 0 in $\mathbb{R}$ for small enough μ:

(i) if $ab < 0$ (resp. $ab > 0$) the differential equation has one trivial equilibrium $u = 0$ for $\mu < 0$ (resp. for $\mu > 0$). This equilibrium is stable when $b < 0$ and unstable when $b > 0$.

(ii) if $ab < 0$ (resp. $ab > 0$) the differential equation possesses the trivial equilibrium $u = 0$ and two nontrivial equilibria $u_\pm(\epsilon)$, $\epsilon = \sqrt{|\mu|}$ for $\mu > 0$ (resp. $\mu < 0$), which are symmetric, $u_+(\epsilon) = -u_-(\epsilon)$. The map $\epsilon \to u_\pm(\epsilon)$ is of class $\mathcal{C}^{k-3}$ in a neighborhood of 0 in $\mathbb{R}$, and $u_\pm(\epsilon) = O(\epsilon)$. The nontrivial equilibria are stable when $b < 0$ and unstable when $b > 0$, whereas the trivial equilibrium has opposite stability.

Then for equation (1.1.1), a pitchfork bifurcation occurs at $\mu = 0$.

A direct consequence of conditions (1.1.2), (1.1.4) and (1.1.5) is that f has the Taylor expansion:

$$f(u,\mu) = uh(u^2,\mu) \quad h(u^2,\mu) = a\mu + bu^2 + o(|\mu| + u^2) \text{ as } (u,\mu) \to (0,0)$$

where h is of class $\mathcal{C}^{(k-1)/2}$ in a neighborhood of $(0,0)$.

Exercice 1.2. *Consider the truncated equation*

$$\frac{du}{dt} = a\mu u + bu^3.$$

- *Plot bifurcation diagrams in the (u,μ)-plane of this truncated equationfor different values of a and b.*
- *Prove the theorem.*

1.1.3 Transcritical bifurcation

Theorem 1.3 (Transcritical bifurcation)**.** *Assume that the vector field f is of class $\mathcal{C}^k$, $k \geq 2$, in a neighborhood of $(0,0)$, that it is satisfies conditions (1.1.2), and also:*

$$\frac{\partial^2 f}{\partial\mu\partial u}(0,0) =: a \neq 0, \quad \frac{\partial^2 f}{\partial u^2}(0,0) =: 2b \neq 0. \tag{1.1.6}$$

The following properties hold in neighborhood of 0 in $\mathbb{R}$ for small enough μ:

(i) the differential equation possesses the trivial equilibrium $u = 0$ and the nontrivial equilibrium $u_0(\mu)$ where the map $\mu \to u_0(\mu)$ is of class $\mathcal{C}^{k-2}$ in a neighborhood of 0 in $\mathbb{R}$, and $u_0(\mu) = O(\mu)$.

(ii) if $a\mu < 0$ (resp. $a\mu > 0$) the trivial equilibrium $u = 0$ is stable (resp. unstable) whereas the nontrivial equilibrium $u_0(\mu)$ is unstbale (resp. stable).

Then for equation (1.1.1), a transcritical bifurcation occurs at $\mu = 0$.

A direct consequence of conditions (1.1.2) and (1.1.6) is that f has the Taylor expansion:

$$f(u,\mu) = a\mu u + bu^2 + o(u|\mu| + u^2) \text{ as } (u,\mu) \to (0,0)$$

Exercice 1.3. *Consider the truncated equation*

$$\frac{du}{dt} = a\mu u + bu^2.$$

- *Plot bifurcation diagrams in the (u,μ)-plane of this truncated equationfor different values of a and b.*
- *Prove the theorem.*

1.2 Bifurcation in dimension 2: Hopf bifurcation

In the remainder of this section we consider differential equations in $\mathbb{R}^2$,

$$\frac{du}{dt} = \mathbf{F}(u, \mu). \tag{1.2.1}$$

Here the unknown u is again a real-valued function that takes values in $\mathbb{R}^2$, and the vector field $\mathbf{F}$ is real-valued depending, besides u, upon a parameter μ. The parameter μ is the bifurcation parameter. We assume that the vector field is of class $\mathcal{C}^k$, $k \geq 3$, in a neighborhood of $(0, 0)$ satisfying:

$$\mathbf{F}(0, 0) = 0 \tag{1.2.2}$$

This condition ensures that $u = 0$ is an equilibrium of equation (1.2.1) at $\mu = 0$. The occurance of a bifurcation is in this case determined by the linearization of the vector field at $(0, 0)$:

$$\mathbf{L} = D_u\mathbf{F}(0, 0)$$

which is a linear operator acting in $\mathbb{R}^2$. When $\mathbf{L}$ has eigenvalues on the imaginary axis, bifurcations may occur at $\mu = 0$. We focus in this section on the case where $\mathbf{L}$ has a pair of complex conjugated purely imaginary eigenvalues. This is called the Hopf bifurcation (or Andronov-Hopf bifurcation).

Hypothesis 1.1. *Assume that the vector field is of class $\mathcal{C}^k$, $k \geq 5$, in a neighborhood of $(0, 0)$, that is satisfies (1.2.2), and the two eigenvalues of the linear operator $\mathbf{L}$ are $\pm i\omega$ for some $\omega > 0$.*

We consider the eigenvector ζ associated to the eigenvalue $i\omega$ of $\mathbf{L}$,

$$\mathbf{L}\zeta = i\omega\zeta$$

If $\mathbf{L}^*$ is the adjoint operator of $\mathbf{L}$ then we define ζ^* as the eigenvector of $\mathbf{L}^*$ satisfying:

$$\mathbf{L}^*\zeta^* = -i\omega\zeta^*, \quad \langle\zeta, \zeta^*\rangle = 1$$

where $\langle\cdot, \cdot\rangle$ denotes the Hermitian scalar product in $\mathbb{C}^2$.

Consider the Taylor extension of the vector field $\mathbf{F}$ in (1.2.1):

$$\mathbf{F}(u, \mu) = \sum_{1 \leq r+q \leq k} \mu^q \mathbf{F}_{rq}(u^{(r)}) + o(|\mu| + \|u\|^k)$$

where $\mathbf{F}_{rq}$ is the r-linear symmetric operator from $(\mathbb{R}^2)^r$ to $\mathbb{R}^2$,

$$\mathbf{F}_{rq} = \frac{1}{r!q!}\frac{\partial^q}{\partial\mu^q} D_u^r \mathbf{F}(0, 0)$$

We define the two coefficients

$$a = \langle \mathbf{F}_{11}\zeta + 2\mathbf{F}_{20}(\zeta, -\mathbf{L}^{-1}\mathbf{F}_{01}), \zeta^* \rangle \tag{1.2.3}$$

$$b = \langle 2\mathbf{F}_{20}\left(\bar{\zeta}, (2i\omega - \mathbf{L})^{-1}\mathbf{F}_{20}(\zeta, \zeta)\right) + 2\mathbf{F}_{20}\left(\zeta, -2\mathbf{L}^{-1}\mathbf{F}_{20}(\zeta, \bar{\zeta})\right) + 3\mathbf{F}_{30}(\zeta, \zeta, \bar{\zeta}), \zeta^* \rangle \tag{1.2.4}$$

Hypothesis 1.2. *We assume that the complex coefficients a and b have nonzero real parts, $a_r \neq 0$ and $b_r \neq 0$. The coefficient $b_r = Re(b)$ is called the first Lyapunov coefficient.*

Definition 1.2. *1. A non-constant solution to the differential equation (1.2.1) is periodic if it exists $T > 0$ such that $u(t) = u(t + T)$. The image of the interval $[0, T]$ under u in the state space $\mathbb{R}^2$ is called the periodic orbit.*

2. A periodic orbit Γ on a plane is called a limit cycle if it is the α-limit set or ω-limit set of some point z not on the periodic orbit, that is, the set of accumulation points of either forward or backward trajectory through z, is exactly Γ. Asymptotically stable and unstable periodic orbits are examples of limit cycles.

Theorem 1.4 (Hopf bifurcation)**.** *Assume that hypotheses 1.1 and 1.2 hold. Then, for the differential equation (1.2.1) a supercritical (resp. subcritical) Hopf bifurcation occurs at $\mu = 0$ when $b_r < 0$ (resp. $b_r > 0$). More precisely, the following properties hold in a neighborhood of 0 in $\mathbb{R}^2$ for small enough μ:*

(i) If $a_r b_r < 0$ (resp. $a_r b_r > 0$) the differential equation has precisely one equilibrium $u(\mu)$ for $\mu < 0$ (resp. $\mu > 0$) with $u(0) = 0$. This equilibrium is stable when $b_r < 0$ and unstable when $b_r > 0$.

(ii) If $a_r b_r < 0$ (resp. $a_r b_r > 0$) the differential equation possesses for $\mu > 0$ (resp. $\mu < 0$) an equilibrium $u(\mu)$ and a unique periodic orbit $u^*(\mu) = O(\sqrt{|u|})$, which surrounds this equilibrium. The periodic orbit is stable when $b_r < 0$ and unstable when $b_r > 0$, whereas the equilibrium has the opposite stability.

Proof. See section on normal forms. □

Remark 1.2. *The number of equilibria of the differential equation stays constant upon varying μ in a neighborhood of 0. The dynamics of the bifurcation change at the bifurcation point $\mu = 0$. Such bifurcations are called dynamic bifurcations, whereas those in which the number of equilibria changes are also called steady bifurcations.*

2 Center manifold

Center manifolds are fundamental for the study of dynamical systems near critical situations and in particular in bifurcation theory. Starting with an infinite-dimensional problem, the center manifold theorem will reduce the study of small solutions, staying sufficiently close to 0, to that of small solutions of a reduced system with finite dimension. The solutions on the center manifold are described by a finite-dimensional system of ordinary differential equations, also called the reduced system.

2.1 Notations and definitions

Consider two (complex or real) Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. We shall use the following notations:

- $\mathcal{C}^k(\mathcal{Y},\mathcal{X})$ is the Banach space of k-times continuously differentiable functions $F : \mathcal{Y} \to \mathcal{X}$ equiped with the norm on all derivatives up to order k,

$$\|F\|_{\mathcal{C}^k} = \max_{j=0,\dots,k}\left(\sup_{y\in\mathcal{Y}}\left(\|D^jF(y)\|_{\mathcal{L}(\mathcal{Y}^j,\mathcal{X})}\right)\right)$$

- $\mathcal{L}(\mathcal{Y},\mathcal{X})$ is the Banach space of linear bounded operators $\mathbf{L} : \mathcal{Y} \to \mathcal{X}$, equiped with operator norm:

$$\|\mathbf{L}\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} = \sup_{\|u\|_{\mathcal{Y}}=1}\left(\|\mathbf{L}u\|_{\mathcal{X}}\right)$$

  if $\mathcal{Y} = \mathcal{X}$, we write $\mathcal{L}(\mathcal{Y}) = \mathcal{L}(\mathcal{Y},\mathcal{X})$.

- For a linear operator $\mathbf{L} : \mathcal{Y} \to \mathcal{X}$, we denote its range by $\mathrm{im}\mathbf{L}$:

$$\mathrm{im}\mathbf{L} = \{\mathbf{L}u \in \mathcal{X} \mid u \in \mathcal{Y}\} \subset \mathcal{X}$$

  and its kernel by $\ker\mathbf{L}$:

$$\ker\mathbf{L} = \{u \in \mathcal{Y} \mid \mathbf{L}u = 0\} \subset \mathcal{Y}$$

- Assume that $\mathcal{Y} \hookrightarrow \mathcal{X}$ with continuous embedding. For a linear operator $\mathbf{L} \in \mathcal{L}(\mathcal{Y},\mathcal{X})$, we denote by $\rho(\mathbf{L})$, or simply ρ, the resolvent set of $\mathbf{L}$:

$$\rho = \{\lambda \in \mathbb{C} \mid \lambda Id - \mathbf{L} : \mathcal{Y} \to \mathcal{X} \text{ is bijective }\}.$$

  The complement of the resolvent set is the spectrum $\sigma(\mathbf{L})$, or simply σ,

$$\sigma = \mathbb{C} \setminus \{\rho\}.$$

Remark 2.1. *When $\mathbf{L}$ is real, both the resolvent set and the spectrum of $\mathbf{L}$ are symmetric with respect to the real axis in the complex plane.*

2.2 Local center manifold

Let $\mathcal{X},\mathcal{Y}$ and $\mathcal{Z}$ be Banach spaces such that:

$$\mathcal{Z} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{X}$$

with continuous embeddings. We consider a differential equation in $\mathcal{X}$ of the form:

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u) \tag{2.2.1}$$

in which we assume that the linear part $\mathbf{L}$ and the nonlinear part $\mathbf{R}$ are such that the following holds.

Hypothesis 2.1 (Regularity)**.** *We assume that $\mathbf{L}$ and $\mathbf{R}$ in* (2.2.1) *have the following properties:*

(i) $\mathbf{L} \in \mathcal{L}(\mathcal{Z},\mathcal{X})$,

(ii) for some $k \geq 2$, there exists a neighborhood $\mathcal{V} \subset \mathcal{Z}$ of 0 such that $\mathbf{R} \in \mathcal{C}^k(\mathcal{V}, \mathcal{Y})$ and

$$\mathbf{R}(0) = 0, \quad D\mathbf{R}(0) = 0.$$

Hypothesis 2.2 (Spectral decomposition)**.** *Consider the spectrum σ of the linear operator $\mathbf{L}$, and write:*

$$\sigma = \sigma_+ \cup \sigma_0 \cup \sigma_-$$

in which

$$\sigma_+ = \{\lambda \in \sigma \mid Re\lambda > 0\}, \quad \sigma_0 = \{\lambda \in \sigma \mid Re\lambda = 0\}, \quad \sigma_- = \{\lambda \in \sigma \mid Re\lambda < 0\}$$

We assume that:

(i) there exists a positive constant γ such that

$$\inf_{\lambda \in \sigma_+} (Re\lambda) > \gamma, \quad \sup_{\lambda \in \sigma_-} (Re\lambda) < -\gamma$$

(ii) the set σ_0 consists of a finite number of eigenvalues with finite algebraic multiplicities.

Hypothesis 2.3 (Resolvent estimates)**.** *Assume that there exist positive constants $\omega_0 > 0$, $c > 0$ and $\alpha \in [0, 1)$ such that for all $\omega \in \mathbb{R}$ with $|\omega| \geq \omega_0$, we have that $i\omega$ belongs to the resolvent set of $\mathbf{L}$ and*

$$\begin{aligned} \|(i\omega Id - \mathbf{L})^{-1}\|_{\mathcal{L}(\mathcal{X})} &\leq \frac{c}{|\omega|} \\ \|(i\omega Id - \mathbf{L})^{-1}\|_{\mathcal{L}(\mathcal{Y},\mathcal{Z})} &\leq \frac{c}{|\omega|^{1-\alpha}} \end{aligned}$$

As a consequence of hypothesis 2.2 (ii), we can define the spectral projection $\mathbf{P}_0 \in \mathcal{L}(\mathcal{X})$, corresponding to σ_0, by the Dunford formula:

$$\mathbf{P}_0 = \frac{1}{2\pi i} \int_\Gamma (\lambda Id - \mathbf{L})^{-1} d\lambda \tag{2.2.2}$$

where Γ is a simple, oriented counterclockwise, Jordan curve surrounding σ_0 and lying entirely in $\{\lambda \in \mathbb{C} \mid |\mathrm{Re}\lambda| < \gamma\}$. Then

$$\mathbf{P}_0^2 = \mathbf{P}_0, \quad \mathbf{P}_0\mathbf{L}u = \mathbf{L}\mathbf{P}_0 u \quad \forall u \in \mathcal{Z},$$

and $\mathrm{im}\mathbf{P}_0$ is finite-dimensional (σ_0 consists of a finite number of eigenvalues with finite algebraic multiplicities). In Particular, it satisfies $\mathrm{im}\mathbf{P}_0 \subset \mathcal{Z}$ and $\mathbf{P}_0 \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$. We define a second projector $\mathbf{P}_h : \mathcal{X} \to \mathcal{X}$ by

$$\mathbf{P}_h = Id - \mathbf{P}_0$$

which also satisfies

$$\mathbf{P}_h^2 = \mathbf{P}_h, \quad \mathbf{P}_h\mathbf{L}u = \mathbf{L}\mathbf{P}_h u \quad \forall u \in \mathcal{Z},$$

and

$$\mathbf{P}_h \in \mathcal{L}(\mathcal{X}) \cap \mathcal{L}(\mathcal{Y}) \cap \mathcal{L}(\mathcal{Z}).$$

We consider the spectral subspaces associated with these two projections:

$$\mathcal{E}_0 = \mathrm{im}\mathbf{P}_0 = \ker \mathbf{P}_h \subset \mathcal{Z}, \quad \mathcal{X}_h = \mathrm{im}\mathbf{P}_h = \ker \mathbf{P}_0 \subset \mathcal{X}$$

which provides the decomposition:

$$\mathcal{X} = \mathcal{X}_h \oplus \mathcal{E}_0.$$

We also denote

$$\mathcal{Z}_h = \mathbf{P}_h \mathcal{Z} \subset \mathcal{Z}, \quad \mathcal{Y}_h = \mathbf{P}_h \mathcal{Y} \subset \mathcal{Y}$$

and denote by $\mathbf{L}_0 \in \mathcal{L}(\mathcal{E}_0)$ and $\mathbf{L}_h \in \mathcal{L}(\mathcal{Z}_h, \mathcal{X}_h)$ the restrictions of $\mathbf{L}$ to $\mathcal{E}_0$ and $\mathcal{Z}_h$. The spectrum of $\mathbf{L}_0$ is σ_0 and the spectrum of $\mathbf{L}_h$ is $\sigma_+ \cup \sigma_-$.

Theorem 2.1 (Center manifold theorem)**.** *Assume that hypotheses 2.1, 2.2 and 2.3 hold. Then there exists a map $\Psi \in \mathcal{C}^k(\mathcal{E}_0, \mathcal{Z}_h)$, with*

$$\Psi(0) = 0, \quad D\Psi(0) = 0,$$

and a neighborhood $\mathcal{O}$ of 0 in $\mathcal{Z}$ such that the manifold:

$$\mathcal{M}_0 = \{u_0 + \Psi(u_0) \mid u_0 \in \mathcal{E}_0\} \subset \mathcal{Z}$$

has the following properties:

(i) $\mathcal{M}_0$ is locally invariant: if u is a solution of equation (2.2.1) satisfying $u(0) \in \mathcal{M}_0 \cap \mathcal{O}$ and $u(t) \in \mathcal{O}$ for all $t \in [0, T]$, then $u(t) \in \mathcal{M}_0$ for all $t \in [0, T]$.

(ii) $\mathcal{M}_0$ contains the set of bounded solutions of (2.2.1) staying in $\mathcal{O}$ for all $t \in \mathbb{R}$.

The manifold $\mathcal{M}_0$ is called a local center manifold of (2.2.1) and the map Ψ is referred to as the reduction function.

Let u be a solution of (2.2.1) which belongs to $\mathcal{M}_0$, then $u = u_0 + \Psi(u_0)$ and u_0 satisfies:

$$\frac{du_0}{dt} = \mathbf{L}_0 u_0 + \mathbf{P}_0 \mathbf{R}(u_0 + \Psi(u_0)) \tag{2.2.3}$$

The reduction function Ψ satisfies:

$$D\Psi(u_0)(\mathbf{L}_0 u_0 + \mathbf{P}_0 \mathbf{R}(u_0 + \Psi(u_0))) = \mathbf{L}_h \Psi(u_0) + \mathbf{P}_h \mathbf{R}(u_0 + \Psi(u_0)) \quad \forall u_0 \in \mathcal{E}_0$$

2.3 Example

Consider

$$\begin{aligned} \frac{dx}{dt} &= xy \\ \frac{dy}{dt} &= -y - x^2 \end{aligned}$$

where $(x, y) \in \mathbb{R}^2$. We have

$$\mathbf{L} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$\mathcal{E}_0$ is equal to the x-axis and $\mathcal{E}_-$ is equal to the y-axis. There exists a center-manifold:

$$\mathcal{M}_0 = \{x + \Psi(x) \mid x \in \mathbb{R}\}$$

where $\Psi : \mathbb{R} \to \mathbb{R}$ satisfies:

$$\Psi'(x)(x\Psi(x)) = -\Psi(x) - x^2, \quad \Psi(0) = \Psi'(0) = 0$$

Setting $\Psi(x) = c_2x^2 + c_3x^3 + o(x^3)$ we get $-c_2x^2 - x^2 = 0$ and $-c_3x^3 = 0$. Consequently:

$$\Psi(x) = -x^2 + o(x^3)$$

The reduced system is

$$\frac{dx}{dt} = -x^3 + o(x^4).$$

2.4 Parameter-dependent center manifold

We consider a parameter-dependent differential equation in $\mathcal{X}$ of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u, \mu) \tag{2.4.1}$$

where $\mathbf{L}$ is a linear operator as in the previous section, and the nonlinear part $\mathbf{R}$ is defined for (u, μ) in a neighborhood of $(0, 0) \in \mathcal{Z} \times \mathbb{R}^m$. Here $\mu \in \mathbb{R}^m$ is a paramter that we assume to be small. More precisely we keep hypotheses 2.2 and 2.3 and replace hypothesis 2.1 by the following:

Hypothesis 2.4 (Regularity). *We assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *in* (2.4.1) *have the following properties:*

(i) $\mathbf{L} \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$,

(ii) for some $k \geq 2$, *there exists a neighborhood* $\mathcal{V}_u \subset \mathcal{Z}$ *and* $\mathcal{V}_\mu \subset \mathbb{R}^m$ *of* 0 *such that* $\mathbf{R} \in \mathcal{C}^k(\mathcal{V}_u \times \mathcal{V}_\mu, \mathcal{Y})$ *and*

$$\mathbf{R}(0, 0) = 0, \quad D_u\mathbf{R}(0, 0) = 0.$$

Theorem 2.2 (Parameter-dependent center manifold theorem). *Assume that hypotheses 2.4, 2.2 and 2.3 hold. Then there exists a map* $\Psi \in \mathcal{C}^k(\mathcal{E}_0 \times \mathbb{R}^m, \mathcal{Z}_h)$, *with*

$$\Psi(0, 0) = 0, \quad D_u\Psi(0, 0) = 0,$$

and a neighborhood $\mathcal{O}_u \times \mathcal{O}_\mu$ *of* 0 *in* $\mathcal{Z} \times \mathbb{R}^m$ *such that for* $\mu \in \mathcal{O}_\mu$ *the manifold:*

$$\mathcal{M}_0(\mu) = \{u_0 + \Psi(u_0, \mu) \mid u_0 \in \mathcal{E}_0\} \subset \mathcal{Z}$$

has the following properties:

(i) $\mathcal{M}_0(\mu)$ *is locally invariant: if* u *is a solution of equation* (2.4.1) *satisfying* $u(0) \in \mathcal{M}_0(\mu) \cap \mathcal{O}_u$ *and* $u(t) \in \mathcal{O}_u$ *for all* $t \in [0, T]$*, then* $u(t) \in \mathcal{M}_0(\mu)$ *for all* $t \in [0, T]$*.*

(ii) $\mathcal{M}_0(\mu)$ *contains the set of bounded solutions of* (2.4.1) *staying in* $\mathcal{O}_u$ *for all* $t \in \mathbb{R}$*.*

Let u *be a solution of* (2.4.1) *which belongs to* $\mathcal{M}_0(\mu)$*, then* $u = u_0 + \Psi(u_0, \mu)$ *and* u_0 *satisfies:*

$$\frac{du_0}{dt} = \mathbf{L}_0 u_0 + \mathbf{P}_0 \mathbf{R}(u_0 + \Psi(u_0, \mu), \mu) \stackrel{def}{=} f(u_0, \mu) \tag{2.4.2}$$

where we observe that $f(0, 0) = 0$ *and* $D_{u_0} f(0, 0) = \mathbf{L}_0$ *has spectrum* σ_0*. The reduction function* Ψ *satiafies:*

$$D_{u_0}\Psi(u_0, \mu) f(u_0, \mu) = \mathbf{L}_h \Psi(u_0, \mu) + \mathbf{P}_h \mathbf{R}(u_0 + \Psi(u_0, \mu), \mu) \quad \forall u_0 \in \mathcal{E}_0$$

2.5 Equivariant systems

Hypothesis 2.5 (Equivariant equation)**.** *We assume that there exists a linear operator* $\mathbf{T} \in \mathcal{L}(\mathcal{X}) \cap \mathcal{L}(\mathcal{Z})$*, which communtes with vector field in equation* (2.2.1)*:*

$$\mathbf{T}\mathbf{L}u = \mathbf{L}\mathbf{T}u, \quad \mathbf{T}\mathbf{R}(u) = \mathbf{R}(\mathbf{T}u)$$

We also assume that the restriction $\mathbf{T}_0$ *of* $\mathbf{T}$ *to* $\mathcal{E}_0$ *is an isometry.*

Theorem 2.3 (Equivariant center manifold)**.** *Under the assumption of theorem 2.1, we further assume that hypothesis 2.5 holds. Then one can find a reduction function* Ψ *which commutes with* $\mathbf{T}$*:*

$$\mathbf{T}\Psi u_0 = \Psi(\mathbf{T}_0 u_0), \quad \forall u_0 \in \mathcal{E}_0$$

and such that the vector field in the reduced equation (2.2.3) *commutes with* $\mathbf{T}_0$*.*

Remark 2.2. *Analogous results hold for the parameter-dependent equation* (2.4.1)*.*

2.6 Empty unstable spectrum

Theorem 2.4 (Center manifold for empty unstable spectrum)**.** *Under the assumptions of theorem 2.1 and assume that* $\sigma_+ = \varnothing$*. Then in addition to propertries of theorem 2.1, the local center manifold* $\mathcal{M}_0$ *is locally attracting: any solution of equation* (2.2.1) *that stays in* $\mathcal{O}$ *for all* $t > 0$ *tends exponentially towards a solution of* (2.2.1) *on* $\mathcal{M}_0$*.*

3 Normal forms

The normal forms theory consists in finding a polynomial change of variable which improves locally a nonlinear system, in order to recognize more easily its dynamics.

3.1 Main theorem

We consider a paramter-dependent differential equations in $\mathbb{R}^n$ of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u, \mu) \tag{3.1.1}$$

in which we assume that $\mathbf{L}$ and $\mathbf{R}$ satisfy the following hypothesis.

Hypothesis 3.1 (Regularity). *Assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *have the following properties:*

(i) $\mathbf{L}$ *is a linear map in* $\mathbb{R}^n$*;*

(ii) for some $k \geq 2$*, there exist neighborhoods* $\mathcal{V}_u \subset \mathbb{R}^n$ *and* $\mathcal{V}_\mu \subset \mathbb{R}^m$ *of* 0 *such that* $\mathbf{R} \in \mathcal{C}^k(\mathcal{V}_u \times \mathcal{V}_\mu, \mathbb{R}^n)$ *and*

$$\mathbf{R}(0, 0) = 0, \quad D_u\mathbf{R}(0, 0) = 0.$$

Theorem 3.1 (Normal form theorem). *Assume that hypothesis 3.1 holds. Then for any positive integer* p*,* $2 \leq p \leq k$*, there exist neighborhoods* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *of* 0 *in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *such that for* $\mu \in \mathcal{V}_2$*, there is a polynomial map* $\Phi_\mu : \mathbb{R}^n \to \mathbb{R}^n$ *of degree* p *with the following properties:*

(i) the coefficients of the monomials of degree q *in* Φ_μ *are functions of* μ *of class* $\mathcal{C}^{k-q}$ *and*

$$\Phi_0(0) = 0, \quad D_u\Phi_0(0) = 0$$

(ii) for $v \in \mathcal{V}_1$*, the polynomial change of variable*

$$u = v + \Phi_\mu(v)$$

transforms equation (3.1.1) *into the normal form:*

$$\frac{dv}{dt} = \mathbf{L}v + \mathbf{N}_\mu(v) + \rho(v, \mu)$$

and the following properties hold:

(a) for any $\mu \in \mathcal{V}_2$*,* $\mathbf{N}_\mu$ *is a polynomial map* $\mathbb{R}^n \to \mathbb{R}^n$ *of degree* p*, with coefficients depending upon* μ*, such that the coefficients of the monomials of degree* q *are of class* $\mathcal{C}^{k-q}$ *and*

$$\mathbf{N}_0(0) = 0, \quad D_v\mathbf{N}_0(0) = 0$$

(b) the equality $\mathbf{N}_\mu(e^{t\mathbf{L}^*}v) = e^{t\mathbf{L}^*}\mathbf{N}_\mu(v)$ *holds for all* $(t, v) \in \mathbb{R} \times \mathbb{R}^n$ *and* $\mu \in \mathcal{V}_2$

(c) the map ρ *belongs to* $\mathcal{C}^k(\mathcal{V}_1 \times \mathcal{V}_2, \mathbb{R}^n)$ *and*

$$\rho(v, \mu) = o(\|v\|^p) \quad \forall \mu \in \mathcal{V}_2$$

3.2 Hopf bifurcation

Consider an equation of the form (3.1.1) with a single parameter $\mu \in \mathbb{R}$ and satisfying the hypotheses in the center manifold theorem 2.2. Assume that the σ_0 of the linear operator $\mathbf{L}$ contains two purely imaginary eigenvalues $\pm i\omega$, which are simple. Under these assumptions, we have $\sigma_0 = \{\pm i\omega\}$ and $\mathcal{E}_0$ is two-dimensional spanned by the eigenvectors $\zeta, \bar{\zeta}$ associated with $i\omega$ and $-i\omega$ respectively. The center manifold theorem 2.2 gives

$$u = u_0 + \Psi(u_0, \mu), \quad u_0 \in \mathcal{E}_0$$

and applying the normal form theorem 3.1 we find

$$u_0 = v_0 + \Phi_\mu(v_0)$$

which gives:

$$u = v_0 + \widetilde{\Psi}(v_0, \mu), \quad u_0 \in \mathcal{E}_0 \tag{3.2.1}$$

For $v_0(t) \in \mathcal{E}_0$, we write

$$v_0(t) = A(t)\zeta + \overline{A(t)\zeta}, \quad A(t) \in \mathbb{C}$$

Lemma 3.1. *The polynomial $\mathbf{N}_\mu$ in theorem 3.1 is of the form:*

$$\mathbf{N}_\mu(A, \overline{A}) = (AQ(|A|^2, \mu), \overline{AQ}(|A|^2, \mu))$$

where Q is a complex-valued polynomial in its argument, satisfying $Q(0,0) = 0$ and of the form:

$$Q(|A|^2, \mu) = a\mu + b|A|^2 + O((|\mu| + |A|^2)^2)$$

We write the Taylor expansion of $\mathbf{R}$ and $\widetilde{\Psi}$:

$$\mathbf{R}(u, \mu) = \sum_{1 \le q+l \le p} \mathbf{R}_{ql}(u^{(q)}, \mu^{(l)}) + o((|\mu| + \|u\|)^p)$$

$$\widetilde{\Psi}(v_0, \mu) = \sum_{1 \le q+l \le p} \widetilde{\Psi}_{ql}(v_0^{(q)}, \mu^{(l)}) + o((|\mu| + \|v_0\|)^p)$$

$$\widetilde{\Psi}_{ql}(v_0^{(q)}, \mu^{(l)}) = \mu^l \sum_{q_1+q_2=q} A^{q_1}\overline{A}^{q_2}\Psi_{q_1 q_2 l}$$

We differentiate equation (3.2.1) and obtain:

$$D_{v_0}\widetilde{\Psi}(v_0, \mu)\mathbf{L}_0 v_0 - \mathbf{L}\widetilde{\Psi}(v_0, \mu) + \mathbf{N}_\mu(v_0) = \mathbf{Q}(v_0, \mu)$$

where

$$\mathbf{Q}(v_0, \mu) = \Pi_p\Big(\mathbf{R}(v_0 + \widetilde{\Psi}(v_0, \mu), \mu) - D_{v_0}\widetilde{\Psi}(v_0, \mu)\mathbf{N}_\mu(v_0)\Big)$$

Here Π_p represents the linear map that associates to map of class $\mathcal{C}^p$ the polynomial of degree p in its Taylor expansion. We then replace the Taylor expansions of $\mathbf{R}$ and $\widetilde{\Psi}$ and by identifying the terms of order $O(\mu), O(A^2)$ and $O(|A|^2)$ we obtain:

$$\begin{aligned} -\mathbf{L}\Psi_{001} &= \mathbf{R}_{01} \\ (2i\omega - \mathbf{L})\Psi_{200} &= \mathbf{R}_{20}(\zeta, \zeta) \\ -\mathbf{L}\Psi_{110} &= 2\mathbf{R}_{20}(\zeta, \bar{\zeta}) \end{aligned}$$

Here the operators $\mathbf{L}$ and $(2i\omega - \mathbf{L})$ are invertible so that Ψ_{001}, Ψ_{200} and Ψ_{110} are uniquely determined. Next we identify the terms of order $O(\mu A)$ and $O(A|A|^2)$

$$\begin{aligned}(i\omega - \mathbf{L})\Psi_{101} &= -a\zeta + \mathbf{R}_{11}(\zeta) + 2\mathbf{R}_{20}(\zeta, \Psi_{001}) \\ (i\omega - \mathbf{L})\Psi_{210} &= -b\zeta + 2\mathbf{R}_{20}(\zeta, \Psi_{110}) + 2\mathbf{R}_{20}(\bar{\zeta}, \Psi_{200}) + 3\mathbf{R}_{30}(\zeta, \zeta, \bar{\zeta})\end{aligned}$$

Since $i\omega$ is a simple isolated eigenvalue of $\mathbf{L}$, the range of $(i\omega - \mathbf{L})$ is of codimension one so we can solve these equations and determine Ψ_{101} and Ψ_{210}, provided the right hand sides satisfy one solvability condition. This solvability condition allows to compute coefficients a and b.

- If $\mathbf{L}$ has an adjoint $\mathbf{L}^*$ acting on the dual space $\mathcal{X}^*$, the solvability condition is that the right hand sides be orthogonal to the kernel of the adjoint $(-i\omega - \mathbf{L}^*)$ of $(i\omega - \mathbf{L})$. The kernel of $(-i\omega - \mathbf{L}^*)$ is just one-dimensional, spanned by $\zeta^* \in \mathcal{X}^*$ with $\langle \zeta, \zeta^* \rangle = 1$. Here $\langle \cdot, \cdot \rangle$ denotes the duality product between $\mathcal{X}$ and $\mathcal{X}^*$. We find:

$$\begin{aligned}a &= \langle \mathbf{R}_{11}(\zeta) + 2\mathbf{R}_{20}(\zeta, \Psi_{001}), \zeta^* \rangle \\ b &= \langle 2\mathbf{R}_{20}\left(\bar{\zeta}, \Psi_{200}\right) + 2\mathbf{R}_{20}\left(\zeta, \Psi_{110}\right) + 3\mathbf{R}_{30}(\zeta, \zeta, \bar{\zeta}), \zeta^* \rangle\end{aligned}$$

- If the adjoint $\mathbf{L}^*$ does not exist, we use a Fredholm alternative since both equations have the form:

$$(i\omega - \mathbf{L})\Psi = \mathbf{R}, \text{ with } \mathbf{R} \in \mathcal{X}$$

  We project with $\mathbf{P}_0$ and $\mathbf{P}_h$ on the subspaces $\mathcal{E}_0$ and $\mathcal{X}_h$ and we obtain

$$\begin{aligned}(i\omega - \mathbf{L}_0)\mathbf{P}_0\Psi &= \mathbf{P}_0\mathbf{R} \\ (i\omega - \mathbf{L}_h)\mathbf{P}_h\Psi &= \mathbf{P}_h\mathbf{R}\end{aligned}$$

  The operator $(i\omega - \mathbf{L}_h)$ is invertible, then the second equation has a unique solution. The first equation is tw-dimensional, there is a solution Ψ_0 provided the solvability condition holds

$$\langle \mathbf{R}_0, \zeta_0^* \rangle = 0$$

  where $\zeta_0^* \in \mathcal{E}_0$ is the eigenvector in the kernel of the adjoint $(-i\omega - \mathbf{L}_0^*)$ in $\mathcal{E}_0$ chosen such that $\langle \zeta, \zeta_0^* \rangle = 1$. If $\mathbf{P}_0^*$ is the adjoint of $\mathbf{P}_0$ and setting $\zeta^* = \mathbf{P}_0^* \zeta_0^*$ the solvability condition becomes $\langle \mathbf{R}, \zeta^* \rangle = 0$ which leads to the same formula for a and b as above.

4 Equivariant bifurcation

4.1 The Euclidean group

In real n-dimensional affine space R_n we chose an origin O and a coordinate frame so that any point P is determined by its coordinates $(x_1, \ldots, x_n)$. The distance between P and Q is given by $d(P, Q) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$. This gives R_n a Euclidean structure. The Euclidean Group $\mathbf{E}(n)$ is the group of all linear or affine linear isometries acting on R_n: all linear transformations which preserve the distances. It can be shown that any such transformation is a composition of an orthogonal transformation $\mathcal{O}$, i.e. an isometry which keeps the origin O fixed, and a translation by a vector $\mathbf{e}$. $\mathbf{e}$ is a vector of $\mathbb{R}^n$. The group of isometries which keeps the origin O fixed is isomorphic to the real orthogonal group $\mathbf{O}(n)$. Given any $g \in \mathbf{E}(n)$ we write $g = (\mathcal{O}, \mathbf{e}) \in \mathbf{O}(n) \times \mathbb{R}^n$. The composition of law is then:

$$g \cdot g' = (\mathcal{O}\mathcal{O}', \mathcal{O}\mathbf{e} + \mathbf{e}')$$

This shows that $\mathbf{E}(n)$ is the semi-product $\mathbf{O}(n) \ltimes \mathbb{R}^n$.

Throughout this section, by "symmetry group" we will mean a closed subgroup of the n-dimensional Euclidean Group $\mathbf{E}(n)$.

4.2 Planar lattice

Name	Holohedry	Basis of $\mathcal{L}$	Basis of $\mathcal{L}^*$
Hexagonal	D_6	$\ell_1 = (\frac{1}{\sqrt{3}}, 1)$, $\ell_2 = (\frac{2}{\sqrt{3}}, 0)$	$\mathbf{k}_1 = (0,1)$, $\mathbf{k}_2 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$
Square	D_4	$\ell_1 = (1,0)$, $\ell_2 = (0,1)$	$\mathbf{k}_1 = (1,0)$, $\mathbf{k}_2 = (0,1)$
Rhombic	D_2	$\ell_1 = (1, -\cot\theta)$, $\ell_2 = (0, \cot\theta)$	$\mathbf{k}_1 = (1,0)$, $\mathbf{k}_2 = (\cos\theta, \sin\theta)$
Rectangular	D_2	$\ell_1 = (1,0)$, $\ell_2 = (0,c)$	$\mathbf{k}_1 = (1,0)$, $\mathbf{k}_2 = (0, \frac{1}{c})$
Oblique	$\mathbb{Z}_2$	$\lvert\ell_1\rvert \neq \lvert\ell_2\rvert$, $\ell_1 \cdot \ell_2 \neq 0$	

Table 1: Lattices in two dimension. $0 < \theta < \frac{\pi}{2}$, $\theta \neq \frac{\pi}{3}$ and $0 < c < 1$.

Let ℓ_1, ℓ_2 be a basis of $\mathbb{R}^2$. The set $\mathcal{L} = \{m_1\ell_1 + m_2\ell_2 \mid (m_1, m_2) \in \mathbb{Z}^2\}$ is a discrete subgroup of $\mathbb{R}^2$. It is called a lattice group because the orbit of a point in R_2, under the action of $\mathcal{L}$ forms a periodic lattice of points in R_2. We set $\widetilde{\mathcal{L}} = \mathcal{L}(O)$. Denote by H the largest subgroup of $\mathbf{O}(n)$ which keeps $\widetilde{\mathcal{L}}$ invariant. Then the symmetry group of $\widetilde{\mathcal{L}}$ is generated by the semi-direct product $H \ltimes \mathcal{L}$. The group H is called the holohedry of the lattice. We define the dual lattice of lattice $\mathcal{L}$ by $\mathcal{L}^* = \{m_1\mathbf{k}_1 + m_2\mathbf{k}_2 \mid (m_1, m_2) \in \mathbb{Z}^2\}$ with $\ell_i \cdot \mathbf{k}_j = \delta_{i,j}$. We summerize in table 1 the different holohedries of the plane.

4.3 Definitions

Definition 4.1. *Given a closed subgroup G of $\mathbf{E}(n)$ and a Banach space $\mathcal{Y}$, a linear action of G on $\mathcal{Y}$ is a continuous homomorphism $\tau : G \to GL(\mathcal{Y})$ from G to the group of invertible linear maps in $\mathcal{Y}$. The map τ is called a representation of G in the space $\mathcal{Y}$.*

If $\ker(\tau) = \{0\}$, the image of G under τ is a group isomorphic to G and we call it the transformation group associated with G. We denote by Γ this group.

Definition 4.2. *If $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ is a smooth operator (of class $\mathcal{C}^k$, $k \geq 2$), then $\mathcal{F}$ is Γ-equivariant if for every $\gamma \in \Gamma$ and every $x \in \mathcal{X}$ we have*

$$\mathcal{F}(\gamma x) = \gamma \mathcal{F}(x)$$

Definition 4.3. *Let Σ be a subgroup of Γ. We denote by $Fix_{\mathcal{Y}}(\Sigma)$ the subspace of $\mathcal{Y}$ consisting of all points which are fixed under Σ:*

$$Fix_{\mathcal{Y}}(\Sigma) = \{y \in \mathcal{Y} \mid \sigma y = y \text{ for all } \sigma \in \Sigma\}$$

Definition 4.4.
- *The largest subgroup of Γ which fixes $x \in \mathcal{X}$ is the isotropy subgroup (or stabilizer) of x, which we denote by $Stab(x)$.*
- *For $x \in \mathcal{X}$, the Γ-orbit of x is the set $\Gamma \cdot x$: the image of x by the action of Γ.*

Definition 4.5. *Let $N(\Sigma)$ be the normalizer of Σ in Γ:*

$$N(\Sigma) = \{\gamma \in \Gamma \mid \gamma\Sigma\gamma^{(-1)} = \Sigma\}$$

4.4 Equivariant Branching Lemma

We consider a paramter-dependent differential equation in $\mathcal{X}$ of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u,\mu) = \mathcal{F}(u,\mu) \tag{4.4.1}$$

where $\mathbf{L}$ is a linear operator as in the previous section, and the nonlinear part $\mathbf{R}$ is defined for (u,μ) in a neighborhood of $(0,0) \in \mathcal{Z} \times \mathbb{R}^m$. Here $\mu \in \mathbb{R}^m$ is a paramter that we assume to be small. We suppose that $\mathcal{F}$ is Γ-equivariant. If we apply the parameter-dependent center manifold 2.3 theorem for equivariant differential equation (4.4.1), the reduced equation on $\mathcal{E}_0$ has the general form:

$$\frac{du_0}{dt} = \mathbf{L}_0 u_0 + \mathbf{P}_0\mathbf{R}(u_0 + \Psi(u_0,\mu),\mu) \stackrel{def}{=} f(u_0,\mu)$$

with

$$\gamma\Psi(u_0,\mu) = \Psi(\gamma u_0,\mu), \quad \forall u_0 \in \mathcal{E}_0 \text{ and } \forall \gamma \in \Gamma$$

Since $\mathcal{E}_0$ is a real space of dimension n, we may regard f as a map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Moreover, Γ acts on $\mathbb{R}^n$ and f is equivariant for this action.

Suppose now that the action of Γ on $\mathbb{R}^n$ possesses an isotropy subgroup Σ with a one-dimensonal fixed point space $\mathrm{Fix}(\Sigma)$. If we look for solutions in $\mathrm{Fix}(\Sigma)$, the reduced equation on the center manifold restricts to a scalar equation.

Hypothesis 4.1. *We suppose that Γ acts absolutely irreducibly on $\mathcal{E}_0$. As a consequence, the linearization of f at the origin is a multiple of the identity and we have $D_u f(0,\mu) = c(\mu)Id$.*

Theorem 4.1 (Steady-state Equivariant Branching Lemma)**.** *We suppose that the assumptions of theorem 2.2 hold. Assume that the compact group Γ acts linearly and that $\mathcal{F}$ is Γ-equivariant. We suppose that Γ acts absolutely irreducibly on $\mathcal{E}_0$. We also suppose that $\mathbf{L}$ has 0 as an isolated eigenvalue with finite multiplicity. If Σ is an isotropy subgroup of Γ with $\dim Fix(\Sigma) = 1$ and if $c'(0) \neq 0$, then it exists a unique branche of solutions with symmetry Σ.*

Furthermore, for each isotropy subgroup Σ of Γ such that $\dim Fix(\Sigma) = 1$ in $\mathcal{E}_0$, either one of the following situations occurs (where $f(u_0,\mu)$ is left hand side of equation (2.2.3) in $Fix(\Sigma)$):

(i) $\Sigma = \Gamma$. *If* $D_\mu f(0,0) \neq 0$, *there exists one branch of solution* $u(\mu)$. *If in addition* $D^2_{uu} f(0,0) \neq 0$, *then* $u^2 = O(\|\mu\|) \Rightarrow$ *saddle-node bifurcation.*

(ii) $\Sigma < \Gamma$ *and the normalizer* $N(\Sigma)$ *acts trivially in Fix*(Σ). *Then* $f(u_0, \mu) = u_0 h(u_0, \mu)$ *and if* $D^2_{u\mu} f(0,0) \neq 0$ *there exists a branch of solution* $u(\mu)$. *If in addition* $D^2_{uu} f(0,0) \neq 0$, *then* $u = O(\|\mu\|) \Rightarrow$ *transcritical bifurcation.*

(iii) $\Sigma < \Gamma$ *and the normalizer* $N(\Sigma)$ *acts as* -1 *in Fix*(Σ). *Then* $f(u_0, \mu) = u_0 h(u_0, \mu)$ *with* h *an even function of* u_0. *If* $D^2_{u\mu} f(0,0) \neq 0$ *there exists a branch of solution* $\pm u(\mu)$ *such that if* $D^3_{uuu} f(0,0) \neq 0$, *then* $u^2 = O(\|\mu\|) \Rightarrow$ *pitchfork bifurcation.*

Remark 4.1.

- *If* $\dim \mathit{Fix}(\Sigma) = 1$, *then* Σ *is a* *maximal* *isotropy subgroup.*
- *When* $\Sigma < \Gamma$, *the bifurcating solutions in Fix*(Σ) *have lower symmetry than the basic solution* $u = 0$. *This effect is called* *spontaneous symmetry breaking.*

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