# Stochastic mean-field theories of cortical networks

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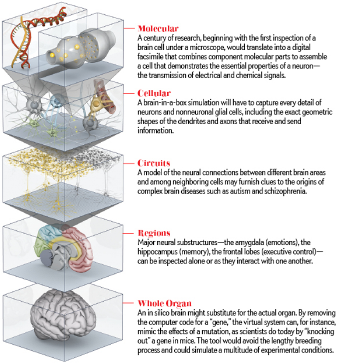
# Mean-field theory for neural networks

**Agenda** mathematical (rigorous) modelling of neural microcircuits




Assumptions

- cell input should be "Poissonian" and "close to" independent
- network should exhibit stable "balanced state", i.e. each cell operates close to threshold

Math challenges

- how to formalize the above assumptions?
- how to characterize "balanced states" in mathematical terms?
- how to derive mean-field limits and fluctuation theory (for finite-size corrections)?

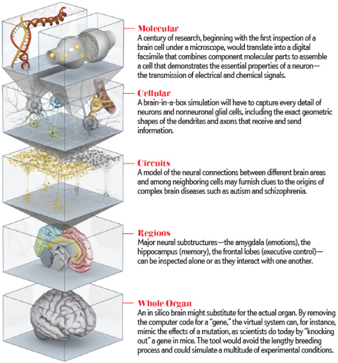
Applications

- statistical inference
- control (of dynamical states)

# Mean-field theory for neural networks

**Agenda** mathematical (rigorous) analysis of neural microcircuits, e.g.:

- How to distinguish network variability from signal noise?
- How to relate various scale parameters, e.g. connectivity vs. population size, precisely?




**References**


F. Farkhooi, W.S.: A complete mean-field theory for dynamics of binary recurrent networks, Phys. Rev. Lett. 119, 2017

(math. background in) W.S.: Stochastic Processes in Neuroscience, Lecture Notes, TU Berlin, Version: August NN, 2017

# Classical model - Binary neural networks

- reduced math. description $n_i(t) = 0 - 1$, $i = 1, \dots, N$
- Markovian dynamics

$$\begin{cases} n_i : 0 \to 1 & \text{with rate } f\left(\gamma \sum_j J_{ij}^{(N)} n_j + I_{ext}\right) \\ n_i : 1 \to 0 & \text{with rate } 1 - f(\dots) \end{cases}$$

for given $0 \leq f \leq 1$, $J_{ij}^{(N)} \in \{0, 1\}$

- sparsity $\sum_j J_{ij}^{(N)} = \mathcal{O}(K)$, $K \ll N$
- operation close to threshold $\gamma = \mathcal{O}\left(\frac{1}{\sqrt{K}}\right)$

**features**

- admits "asynchronous irregular" states
- MFT for the mean rate $\bar{n}(t) = \frac{1}{N} \sum_i n_i(t)$ combines Poissonian and central limit theorem, widely unexplored mathematically
- universality poorly understood
- heterogeneities, in particular impact of motives on dynamical features, poorly understood

# (Stochastic) MFT

class. simplifications for rigorous math analysis:

- **symmetry** $J_{ij} = J_{ji}$
- **all-to-all couplings** $K = \sum_j J_{ij} = \mathcal{O}(N)$

well understood with the help of (equilibrium) statistical mechanics
(since in this case it becomes a gradient type dynamics)
**motivation for asymmetry**

- symmetry lacks neurophysiological plausibility, because synapses operate unidirectional
- the majority of neurons either act excitatorily ($J_{ij} > 0$) or inhibitorily ($J_{ij} < 0$) which also contradicts symmetry
- symmetry creates additional attractors that do not correspond to memorized states (e.g., metastable mixture states, spin-glass attractor)

add. motivation for asymmetric couplings can be found in the survey article:

`Kree, R. and Zippelius, A. (1991). Asymmetrically diluted neural networks, in Models of Neural Networks, ed. van Hemmen, et al., Springer`

# Classical simplifications

For rigorous math analysis:

- **symmetry** $J_{ij} = J_{ji}$
- **all-to-all couplings** $K = \sum_j J_{ij} = \mathcal{O}(N)$

well understood with the help of (equilibrium) statistical mechanics (since in this case it becomes a gradient type dynamics)

**motivation for dilution**

- neural connectivity is high, but far away from all-to-all
- allows for structural/hierarchical models

# An exactly solvable asymmetric neural network model

Derrida, et al., Europhys. Lett., 4, pp. 167-173 (1987)

$$J_{ij} = \frac{1}{K} c_{ij} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu$$

where

- $\xi_i^\mu = \pm 1$ value of neuron $i$ in pattern $\mu$, supposed to be independent random variables with $P\left(\xi_i^\mu = \pm 1\right) = \frac{1}{2}$
- $c_{ij} \in \{0, 1\}$ - random, independent, $P(c_{ij} = 1) = \frac{K}{N}$

**Dynamics**

**parallel** - all neurons updated simultaneously

$$n_i(t+\delta t) = \begin{cases} +1 & \text{with prob.}\ (1 + \exp\left(-2\beta u_i(t)\right))^{-1} \\ -1 & \text{with prob.}\ (1 + \exp\left(+2\beta u_i(t)\right))^{-1} \end{cases} \qquad (1)$$

where

- $\beta = \frac{1}{T}$ is interpreted as inverse temperature
- $u_i(t) = \sum_j J_{ij} n_j(t)$

typical order of $\Delta t = \mathcal{O}(1)$

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- $\xi_i^\mu = \pm 1$ value of neuron $i$ in pattern $\mu$, supposed to be independent random variables with $P\left(\xi_i^\mu = \pm 1\right) = \frac{1}{2}$
- $c_{ij} \in \{0, 1\}$ - random, independent, $P(c_{ij} = 1) = \frac{K}{N}$

**Dynamics**

**random sequential** - choose a neuron $i$ randomly (according to uniform distribution) and update its state according to (1)
typical order of $\Delta t = \mathcal{O}\left(\frac{1}{N}\right)$

**Problem** How to compare both dynamics precisely?

- parallel update is **deterministic**
- sequential update is **random**
- both are not cont.-time MCs

# Main result: dynamical properties as $N \to \infty$

**observable**

$$m(t) = E\left(\frac{1}{N}\sum_{i=1}^{N} \xi_i^\mu n_i(t)\right)$$

overlap with stored pattern $(\xi_1^\mu, \ldots, \xi_N^\mu)$

- parallel dynamics

$$m(t+\Delta t) = f(m(t))$$

- sequential update

$$\frac{d}{dt} m(t) = -m(t) + f(m(t))$$

where

$$f(m) = \sum_{k=0}^{\infty} \frac{K^k e^{-k}}{k!} \sum_{n=0}^{k} \sum_{s=0}^{k(p-1)} \frac{(1+m)^{k-n}(1-m)^n}{2^{kp}} \binom{k}{n} \binom{k(p-1)}{s} \cdot \tanh\left(\beta\left(kp - 2(n-s)\right)\right).$$

# Main result: dynamical properties as $N \to \infty$ - fixed row sum

$$\sum_{j=1}^{N} c_{ij} \equiv K$$

**observable**

$$m(t) = E\left(\frac{1}{N}\sum_{i=1}^{N} \xi_i^{\mu} n_i(t)\right)$$

overlap with stored pattern $(\xi_1^{\mu}, \dots, \xi_N^{\mu})$

- parallel dynamics

$$m(t + \Delta t) = f(m(t))$$

- sequential update

$$\frac{d}{dt} m(t) = -m(t) + f(m(t))$$

where

$$f(m) = \sum_{n=0}^{K} \sum_{s=0}^{K(p-1)} \frac{(1+m)^{K-n}(1-m)^{n}}{2^{Kp}} \binom{K}{n} \binom{K(p-1)}{s} \cdot \tanh\left(\beta\left(Kp - 2(n-s)\right)\right)$$

# General case - Heuristics

recall:

- reduced math. description $n_i(t) = 0 - 1$, $i = 1, \ldots, N$
- Markovian dynamics

$$\begin{cases} n_i : 0 \to 1 & \text{with rate } f\left(\gamma \sum_j J_{ij}^{(N)} n_j - m\right) \\ n_i : 1 \to 0 & \text{with rate } 1 - f(\ldots) \end{cases}$$

for given $0 \leq f \leq 1$, $J_{ij}^{(N)} \in \{0, 1\}$

**Conjecture**

as $N \to \infty$, but $K \ll N$: $n_i(0)$ ind. $\Rightarrow$ $n_i(t)$ asympt. ind.
has been verified for $t \sim \mathcal{O}(1)$ in the case of the parallel update for $K = \mathcal{O}(\log N)$ in
`Derrida, et al., J. Physique 47, 1297-1303, 1986`
suppose also that $n_i(t)$ are identically distributed, then

$$u_i^{(N)}(t) = \gamma \sum_{j=1}^{N} J_{ij}^{(N)} n_j(t) - m \sim \gamma U(t) - m$$

with

- $U(t) \sim \mathrm{Bin}(K, m(t))$
- $m(t) = E(n_i(t)) = E\left(\frac{1}{N} \sum_{j=1}^{N} m_j(t)\right)$

# General case - Heuristics, ctd.

the weak law of large numbers therefore implies

$$\frac{1}{N}\sum_{i=1}^{N} f(u_i^{(N)}(t)) \sim E\left(f\left(\gamma U(t) - m\right)\right)$$

therefore

$$m^{(N)}(t) = \frac{1}{N}\sum_{i=1}^{N} n_i(t) \sim m(t)$$

with

$$\frac{d}{dt} m(t) = -m(t) + E\left(f\left(\gamma U(t) - m\right)\right)$$

# CLT approximation - $K$ large

for increasing $K$

CLT-approximation $\text{Bin}(K, m(t)) \sim N(Km(t), Km(t)(1-m(t)))$ yields

$$\gamma U(t) - m \sim N(\mu_1(t), \mu_2(t))$$

with $\mu_1(t) = \gamma Km(t) - m$, $\mu_2(t) = \gamma^2 Km(t)(1-m(t))$, and thus

$$\frac{d}{dt}m(t) \sim -m(t) + \frac{1}{\sqrt{2\pi\gamma^2 Km(t)(1-m(t))}} \int f(u) e^{-\frac{(u-\gamma Km(t)-m))^2}{2\gamma^2 m(t)(1-m(t))}} \, du$$

obtained in `Van Vreeswijk, et al., Science 1996, Neural Comput. 10, 1998`.

**Our goal** new approach to MFT incl. finite size effects using stochastic analysis

# Elements of MFT: martingale structure of Markov chains

**General setting**

$(X(t))_{t\geq 0}$ - (time-homogeneous) time-continuous Markov chain on finite state space $S$, right-cont. trajectories

$(P(t))_{t\geq 0}$ - family of transition probabilities

$Q$ - generator (rate) matrix, i.e.,

$$Q = \frac{d}{dt}P(t)_{|t=0} \qquad P(t) = e^{tQ}, t \geq 0$$

$\mathcal{F}(t) := \sigma\{X(s) \mid s \leq t\}$, $t \geq 0$, filtration generated by $X(t)$, $t \geq 0$

# Martingale structure, ctd.

## Theorem

*Let $f : S \to \mathbb{R}$ be any bounded function. Then*

$$f(X(t)) = f(X(0)) + M^f(t) + \int_0^t Qf(X(s))\, ds\; t \geq 0\,, \tag{2}$$

*where*

$$M^f(t) := f(X(t)) - f(X(0)) - \int_0^t Qf(X(s))\, ds\,,\; t \geq 0\,,$$

*is a right-cont. martingale w.r.t. $(\mathcal{F}(t))_{t\geq 0}$ with*

$$\begin{aligned} E\left(M^f(t)^2\right) &= E\left(\int_0^t \left(Q\left(f^2\right) - 2fQf\right)(X(s))\, ds\right) \\ &= \int_0^t E\left(\sum_{j\in S} q_{X(s)\,j}\,(f(X(s)) - f(j))^2\right)\, ds \end{aligned} \tag{3}$$

*Moreover,*

$$M^f(t)^2 - \int_0^t \sum_{j\in S} q_{X(s)\,j}\,(f(X(s)) - f(j))^2\, ds\,,\; t \geq 0\,. \tag{4}$$

*is again a right-cont. martingale w.r.t. $(\mathcal{F}(t))_{t\geq 0}$.*

# Remarks on Theorem 1

$$f(X(t)) = f(X(0)) + M^f(t) + \int_0^t Qf(X(s))\,ds\,, t \geq 0\,, \tag{5}$$

**Remarks**

- (5) is called the semimartingale decomposition of the process $f(X(t))$, since it gives a decomposition into a martingale and a process of bounded variation $\int_0^t Qf(X(s))\,ds$.
- (5) is the analogue of the Ito-decomposition of $f(X(t))$ for $f \in C^2$ and $X(t)$ being the solution of a stochastic differential equation
- (5) links two important concepts for stochastic processes: Markov property and martingale property

## Corollary

*Suppose that $P(X(0) = i_0) = 1$ for some initial state $i_0 \in S$. Then*

$$E\left(\left(M^f\right)^2(t)\right) = \int_0^t \sum_{i,j \in S} p_{i_0 j}(s) q_{ij} \left(f(i) - f(j)\right)^2\, ds\,.$$

# Binary neural networks: math. model

- network of $N$ binary neurons $n(t) = (n_1(t), \ldots, n_N(t))$ with $n_i(t) \in \{0, 1\}$
- input $u_i(t)$ to the $i^{th}$ neuron given as

$$u_i(t) = \gamma \sum_{j=1}^{N} J_{ij} n_j(t) - m_i \,, i = 1, \ldots, N \,,$$

  with connectivity matrix $J_{ij} \in \{0, 1\}$ (no further distributional assumptions yet)
- $m_i$ denotes some mean input that will be specified later

**dynamics** time-continuous Markov chain on the state space $I_N = \{0, 1\}^N$ with rate matrix $Q(n, m) = 0$ if $|n - m| \geq 2$ and

$$Q(n, m) = \begin{cases} f(u_i) & \text{if } m - n = e_i \\ 1 - f(u_i) & \text{if } m - n = -e_i \,. \end{cases}$$

Here $e_i$ denotes the $i^{th}$ unit vector.

**Ex** for $f$

Heaviside- function $f(u) = 1_{\{u \geq \theta\}}$ for some given threshold $\theta$

sigmoid-function $f(u) = \frac{1}{1+e^{-\gamma(u-\theta)}}$

# Martingales

given $G : I_N : 0\{0,1\}^N \to \mathbb{R}$ the process

$$M_t = M_t^G = G\left(n(t)\right) - G\left(n(0)\right) - \int_0^t QG\left(n(s)\right)\, ds\,, t \geq 0$$

is a martingale w.r.t. the natural filtration generated by $n(t)$ with

$$E\left(M_t^2\right) = \int_0^t E\Big( \sum_{i:n_i=0}^{N} f(u_i)\left(G\left(n(s)+e_i\right) - G\left(n(s)\right)\right)^2 + \sum_{i:n_i=1}^{N} (1-f(u_i))\left(G\left(n(s)-e_i\right) - G\left(n(s)\right)\right)^2 \Big)\, ds$$

**Ex**

- $G(n) = \pi_i(n) = n_i$, we obtain that

$$M_t^i = n_i(t) - \int_0^t f(u_i(s)) - n_i(s)\, ds$$

- $G(n) = \pi_{ij}(n) = n_i n_j$, $i \neq j$, we obtain that

$$M_t^{ij} = n_i(t)n_j(t) - \int_0^t \left(-2n_i(s)n_j(s) + f(u_i(s))n_j(s) + f(u_j(s))n_i(s)\right)\, ds$$

# Elements of a MFT: LLN

**Laws of large numbers** of the mean activity

$$\bar{n}(t) := \frac{1}{N}\sum_{i=1}^{N} n_i(t)$$

**Scenario** $J_{ij} = J_{ij}^{(N)}$ such that $\sum_{j=1}^{N} J_{ij}^{(N)} \geq K_N$ with $K_N \uparrow \infty$, $m_i \equiv m$, $f$ Lipschitz

In this case:

$$n^{J^{(N)}}(t) := \frac{1}{|J^{(N)}|}\sum_{j\in J^{(N)}} n_j^{(N)}(t) \asymp m^{(N)}(t)$$

with $|J^{(N)}| \geq K_N$, where

$$\dot{m}^{(N)}(t) = -m^{(N)}(t) + f(\gamma_N K_N m^{(N)}(t) - m)\,, m(0) = m_0 \qquad (6)$$

for suitable initial conditions $n_i(0)$, e.g. $n_i(0)$ i.i.d. with $E(n_i(0)) = m_0$.

# LLN

$$d_N(t) := \sup_{\substack{J \subset \{1,\dots,N\} \\ |J| \geq K_N}} E\left(|n^J(t) - m^{(N)}(t)|^2\right)^{\frac{1}{2}}$$

## Theorem

$$d_N(t) \leq d_N(0) + \sqrt{\frac{t}{K_N}} + (\gamma_N K_N \|f\|_{Lip} + 1) \int_0^t d_N(s)\, ds$$

*Gronwall's inequality implies in particular,*

$$d_N(t) \leq \left(d_N(0) + \sqrt{\frac{t}{K_N}}\right) e^{\left(\gamma_N K_N \|f\|_{Lip} + 1\right) t}, \, t \geq 0 \, .$$

*Suppose now that* $K_N \to \infty$, $\sup_{N \geq 1} \gamma_N K_N < \infty$ *and initial conditions* $n(0)$ *are chosen such that* $\lim_{N \to \infty} d_N(0) \to 0$, *e.g.* $n_i(0)$ *i.i.d. with* $E(n_i(0)) = n_0$, *then for every ensemble average* $n^{J^{(N)}}$ *with* $|J^{(N)}| \geq K_N$ *it follows that*

$$\lim_{N \to \infty} E\left(|n^{J^{(N)}}(t) - m^{(N)}(t)|^2\right) = 0 \, .$$

# LLN, ctd.

**Corollary**

*If $\gamma_N K_N \to \gamma_*$, then*

$$\lim_{N\to\infty} E\left(\left|n^{J^{(N)}}(t) - m(t)\right|^2\right) = 0$$

*where $m$ is a solution to the ordinary differential equation*

$$\dot{m}(t) = -m(t) + f\left(\gamma_* m(t) - m\right), m(0) = m_0\,.$$

# main observation

Fix a subset $J \subset \{1, \ldots, N\}$ with $|J| \geq K_N$. $n^J(t)$ admits the following semimartingale decomposition

$$n^J(t) = n^J(0) + \int_0^t Qn^J(s)\, ds + M_t$$

with

- $Qn^J(t) = \frac{1}{|J|} \sum_{i \in J} f(u_i(t)) - n_i(t) \sim f(\gamma_N K_N m^{(N)}(t) - m) - m^{(N)}(t)$
- and

$$\begin{aligned} E\left(M_t^2\right) &= \int_0^t \sum_{i \in J : n_i = 0} E\left( f(u_i(s)) \left(\frac{1}{|J|}\right)^2 \right) ds \\ &\quad + \int_0^t \sum_{i \in J : n_i = 1} E\left( (1 - f(u_i(s))) \left(\frac{1}{|J|}\right)^2 \right) ds \\ &= \frac{1}{|J|^2} \int_0^t E\left( \sum_{i \in J} (1 - n_i(s)) f(u_i(s)) + n_i(s)(1 - f(u_i(s))) \right) ds \sim \frac{t}{|J|} . \end{aligned}$$

# Remarks

- (Universality) no additional distributional assumptions on $(J_{ij}^{(N)})$ required
- noteable implication: $n_i$ become asymptotically uncorrelated: indeed, $f(u_i^{(N)}(t)) \to f(\gamma_* m(t) - m)$ implies:

$$\frac{d}{dt} E(n_i(t))E(n_j(t)) = (f(\gamma_* m(t) - m) - E(n_i(t)))\,(f(\gamma_* m(t) - m) - E(n_j(t)))$$

$$\begin{aligned}\frac{d}{dt} E(n_i(t)n_j(t)) = -2E(n_i(t)n_j(t)) + f(\gamma_* m(t) - m)E(n_i(t)) \\ + f(\gamma_* m(t) - m)E(n_j(t)))\end{aligned}$$

implies

$$\frac{d}{dt}\,(E(n_i(t)n_j(t)) - E(n_i(t))E(n_j(t))) \asymp -2\,(E(n_i(t)n_j(t)) - E(n_i(t))E(n_j(t)))$$

so that $Cov(n_i(t), n_j(t)) \asymp 0$ for $t > 0$ provided the same holds for the initial condition $t = 0$

# Elements of a MFT: CLT - small ensemble size

**The central limit theory** for the mean activity

**Scenario** in addition $\gamma_N K_N \equiv \gamma_*$, hence

$$\bar{n}(t) = \frac{1}{N}\sum_{i=1}^{N} n_i(t) \asymp m(t)$$

where

$$\dot{m}(t) = -m(t) + f(\gamma_* m(t) - m)\,. \tag{7}$$

next define standardized ensemble averages

$$n^{j,*}(t) := \sqrt{|J|}\left(n^J(t) - m(t)\right) = \sqrt{|J|}\left(\frac{1}{|J|}\sum_{i\in J} n_i(t) - m(t)\right)\,.$$

# CLT - small ensemble size

## Theorem

*$J^{(N)} \subset \{1, \ldots, N\}$, $K_N$ and $d_N(0)$ such that*

$$|J^{(N)}| \uparrow \infty \textbf{ but } \sqrt{|J^{(N)}|}\left(d_N(0) + \frac{1}{\sqrt{K_N}}\right) \to 0$$

*Suppose that*

$$P \circ \left(n^{|J^{(N)}|,*}(0)\right)^{-1} \to N(m_0, \sigma_0^2) \text{ in distr./weakly}$$

*(e.g. $n_i(0)$ iid Bernoulli $(m_0)$, hence $\sigma_0^2 = m_0(1 - m_0)$)*

*Then $n^{J^{(N)},*}(t) \to n_\infty(t)$ in distr. (on the Skorokhod space $\mathcal{D}([0, \infty))$), which is a sol. of the sde*

$$dn_\infty(t) = -n_\infty(t)\, dt + \sigma(t)\, dW(t)$$

*where $W(t)$ is 1d-Brownian motion and*

$$\sigma^2(t) := (1 - m(t))f(\gamma_* m(t) - m) + m(t)(1 - f(\gamma_* m(t) - m))$$

# CLT, ctd.

**Rem**

- $f$ no longer enters the drift term, since the argument of $f$ is "faster averaging" than $n^{J^{(N)}}$
- the CLT yields the following "finite size" correction

$$n^{J^{(N)}}(t) \asymp m(t) + \frac{1}{\sqrt{|J^{(N)}|}} n_\infty(t)$$

in the LLN, where

$$\begin{aligned} \dot{m}(t) &= -m(t) + f\left(\gamma_* m(t) - m\right) \\ dn_\infty(t) &= -n_\infty(t)\, dt + \sigma(t)\, dW(t) \end{aligned}$$

with

$$\sigma^2(t) = (1 - m(t)) f(\gamma_* m(t) - m) + m(t)(1 - f(\gamma_* m(t) - m))$$

# Main ingredient

(rescaled) semimartingale decomposition

$$n^{J^{(N)},*}(t) = n^{J^{(N)},*}(0) + \int_0^t \sqrt{|J^{(N)}|}\left(Q^{(N)} n^{J^{(N)}}(s) - \dot{m}(s)\right)\, ds + M^{(N)}(t)$$

where

$$M^{(N)}(t) := \sqrt{|J^{(N)}|}\left(n^{J^{(N)}}(t) - n^{J^{(N)}}(0) - \int_0^t Q^{(N)} n^{J^{(N)}}(s)\, ds\right)$$

with

$$E\left(\left(M^f\right)^2(t)\right) \to \int_0^t \sigma^2(s)\, ds\,,\, N \to \infty$$

and apply the following martingale CLT

# Martingale CLT

## Theorem

*For $n = 1, 2, , \dots ,$ let $(\mathcal{F}_t^n)_{t \geq 0}$ be a filtration and $(M_n(t))_{t \geq 0}$ be an $(\mathcal{F}_t^n)_{t \geq 0}$-martingale with right-continuous sample paths, having left limits at $t > 0$ and starting at 0, i.e. $M_n(0) = 0$, such that*

$$\lim_{n \to \infty} E\left( \sup_{0 \leq s \leq t} |M_n(s) - M_n(s-)| \right) = 0\,.$$

*Assume that there exist nonnegative, nondecreasing, $(\mathcal{F}_t^n)_{t \geq 0}$-adapted processes such that*

$$M_n^2(t) - A_n(t)\,, t \geq 0\,,$$

*is an $(\mathcal{F}_t^n)_{t \geq 0}$-martingale and that*

$$\lim_{n \to \infty} A_n(t) = \int_0^t \sigma^2(s)\,ds \text{ in probability}$$

*for some deterministic function $\sigma : [0, \infty) \to \mathbb{R}$. Then*

$$\lim_{n \to \infty} M_n(t) = \int_0^t \sigma(s)\,dW(s)\,, t \geq 0\,,$$

*weakly on the Skorokhod-space $D[0, \infty)$. Here, $(W(t))_{t \geq 0}$ is a 1d-Brownian motion.*

# CLT - total population average activity

Additional assumption: $f \in C_b^2$, rows of $J_{ij}^{(N)}$ ind.

## Theorem

$$P \circ \left(\bar{n}^{(N),*}(0)\right)^{-1} \to N(m_0, \sigma_0^2) \text{ in distr./weakkly}$$

*(e.g. $n_i(0)$ iid Bernoulli $(m_0)$, hence $\sigma_0^2 = m_0(1-m_0)$).*

*Then*

$$\bar{n}^{(N),*}(t) := \sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N} n_i(t) - m(t)\right) \to n_\infty(t)$$

*in distr./weakly (on the Skorokhod space $\mathcal{D}([0,\infty))$), which is a sol. of the sde*

$$dn_\infty(t) = \left(\gamma_* f'(\gamma_* m(t) - m) n_\infty(t) - n_\infty(t)\right)\, dt + \sigma(t)\, dW(t)$$

*where $W(t)$ is 1d-Brownian motion and*

$$\sigma^2(t) := (1 - m(t)) f(\gamma_* m(t) - m) + m(t)(1 - f(\gamma_* m(t) - m))$$

**Note** drift term now depends on $f'$

# How $f'$ enters the drift term

$$\sqrt{N}\left(Q^{(N)}\bar{n}^{(N)}(t)-\dot{m}(t)\right)$$
$$=\underbrace{\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left(f\left(\gamma_* n^{J_{i\cdot}^{(N)}}(t)-m\right)-f(\gamma_* m(t)-m)\right)}_{=:I}-\underbrace{n^{(N),*}(t)}_{=:II}$$

# How $f'$ enters the drift term, ctd.

Taylor expansion yields for the first term $I$

$$\begin{aligned}
&\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left(f\left(\gamma_* n^{J_{i\cdot}^{(N)}}(t)-m\right)-f(\gamma_* m(t)-m)\right)\\
&\quad=\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left(\gamma_* f'(\gamma_* m(t)-m)\frac{1}{K_N}\left(\sum_{j=1}^{N}J_{ij}^{(N)}n_j(t)-m(t)\right)\right)\\
&\qquad+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\gamma_*^2}{2}f''(\xi_i^{(N)}(t))\frac{1}{K_N^2}\left(\sum_{j=1}^{N}J_{ij}^{(N)}n_j(t)-m(t)\right)^2\right)\\
&\quad=I_a+I_b
\end{aligned}$$

for certain values $\xi_i^{(N)}(t)$ between $u_i^{(N)}(t)-m$ and $\gamma_* m(t)-m$, and now 1st term no longer vanishes

# Extensions - finite $K$, $N \to \infty$

$\sup_N K_N < \infty$ implies that remainder in

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\left(f\left(\gamma_* n^{J_{i\cdot}^{(N)}}(t) - m\right) - f(\gamma_* m(t) - m)\right) \\
&\quad = \frac{1}{N}\sum_{i=1}^{N}\left(\gamma_* f'(\gamma_* m(t) - m)\frac{1}{K_N}\left(\sum_{j=1}^{N} J_{ij}^{(N)} n_j(t) - m(t)\right)\right) \\
&\qquad + \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\gamma_*^2}{2} f''(\xi_i^{(N)}(t))\frac{1}{K_N^2}\left(\sum_{j=1}^{N} J_{ij}^{(N)} n_j(t) - m(t)\right)^2\right) \\
&\quad = I_a + I_b
\end{aligned}
$$

no longer vanishes with increasing $N$

# Extensions - finite $K$, $N \to \infty$

hence look at full Taylor expansion at $\mu_1(t) := \gamma_* m(t) - m$:

$$\frac{1}{N}\sum_{i=1}^{N}\left(f\left(\gamma_* n^{J_{i\cdot}^{(N)}}(t) - m\right) - f(\gamma_* m(t) - m)\right)$$

$$= \sum_{k=0}^{\infty}\frac{f^{(k)}(\mu_1(t))}{k!}\frac{1}{N}\sum_{i=1}^{N}\left(\gamma_* n^{J_{i\cdot}^{(N)}}(t) - m - \mu_1(t)\right)^k$$

$$\asymp \sum_{k=0}^{\infty}\frac{f^{(k)}(\mu_1(t))}{k!}\mu_k(t)$$

with

$$\mu_k(t) = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\left(\gamma_* n^{J_{i\cdot}^{(N)}}(t) - m - \mu_1(t)\right)^k$$

**Remark** expansion of $\mu_k(t)$ w.r.t. $K$ (see `Farkhooi, et. al.`) yields

- $\mu_k(t) = \mathcal{O}\left(K^{1-k}\right)$
- polynomial scaling $\gamma = \gamma_* \frac{1}{\alpha}$, $\alpha \in (0,1]$, leads to
  - $\mu_{2k+1}(t) \sim \mathcal{O}\left(K^{1-(2k+1)\alpha}\right)$
  - $\mu_{2k}(t) \sim \mathcal{O}\left(K^{1-(2k)\alpha}\right) + (2k-1)!!\mu_2(t)^k$

# Extensions - finite $K$, $N \to \infty$

$$\mu_k(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \left( \gamma_* n^{J_{i\cdot}^{(N)}}(t) - m - \mu_1(t) \right)^k$$

**Remark** expansion of $\mu_k(t)$ w.r.t. $K$ (see `Farkhooi, et. al.`)
special case: $\gamma = \gamma_* \frac{1}{\sqrt{K}}$ leads to

- $\mu_{2k+1}(t) \sim \mathcal{O}\left(K^{1-\frac{2k+1}{2}}\right)$
- $\mu_{2k}(t) \sim \mathcal{O}\left(K^{1-\frac{2k}{2}}\right) + (2k-1)!!\mu_2(t)^k$

recovers normal approximation obtained in `van Vreeswijk, et al`
improved convergence of infinite series in terms of corrections terms to the Gaussian approximation leads to

$$\frac{d}{dt} m(t) \asymp -m(t) + \frac{1}{\sqrt{2\pi\gamma^2 K m(t)(1-m(t))}} \int f(u)(1+G_m(u)) e^{-\frac{(u-\gamma K m(t) - m))^2}{2\gamma^2 m(t)(1-m(t))}} du$$

where

- $G_m(u) = \sum_{k=3}^{m} (-1)^k \frac{\mu_k}{k! \mu_2^{\frac{k}{2}}} H_k\left(\frac{u-\mu_1}{\sqrt{\mu_2}}\right)$
- $H_k$ = Hermite polynomial of order $k$