Large-scale dynamics for the FitzHugh-Nagumo model

Cristóbal Quiñinao

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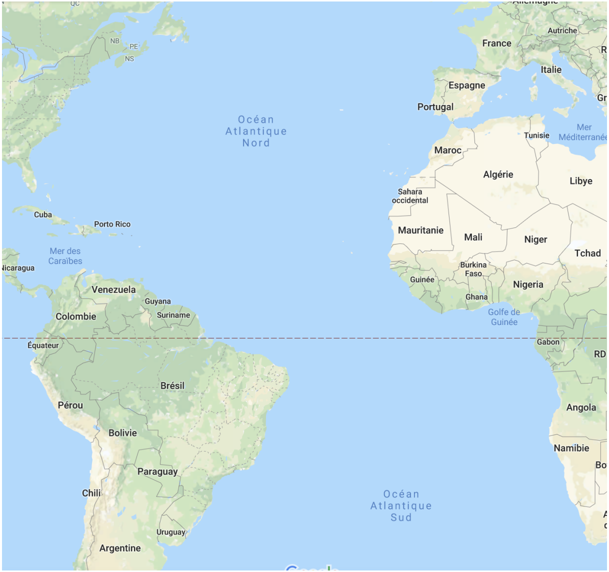
Deterministic and Stochastic Models in Neuroscience

Universidad de O'Higgins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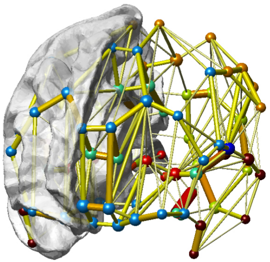
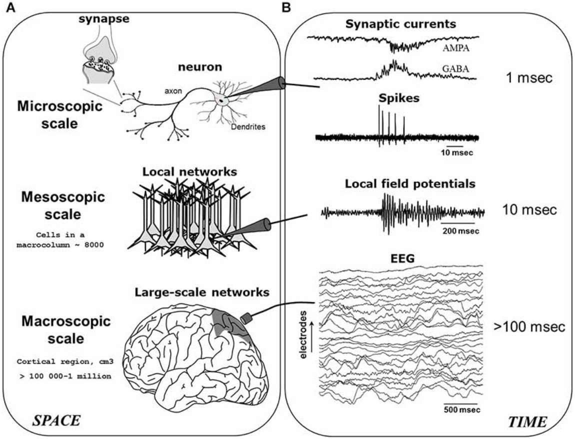


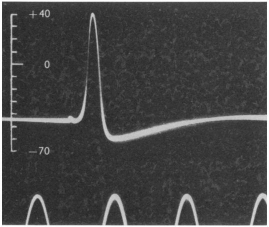
The ability to exploit and transform the environment is remarkable characteristic of humans and it has been well stablished that this ability is due to a very evolved nervous system

Principles of Neural Science, Kandel et al.(2000)

A picture is worth a thousand words



Frontiers in Human Neuroscience, Ros et al.(2014)



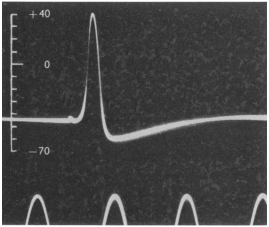
FitzHugh-Nagumo model

Simplification of the HH model conserving the most prominent aspects of it

$$\begin{aligned} dV &= \big(N(V) - w + I_0\big)\, dt, \\ dw &= \frac{1}{\tau}(V + a - bw)\, dt, \end{aligned}$$

where τ, a, b and I_0 are constants, $N(\cdot)$ us a cubic function with negative leading term.

Nature, Lond. Hodgkin & Huxley (1939)



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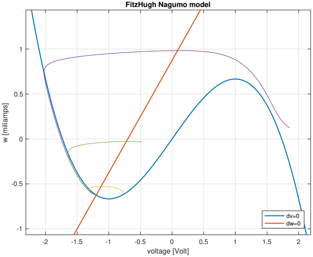
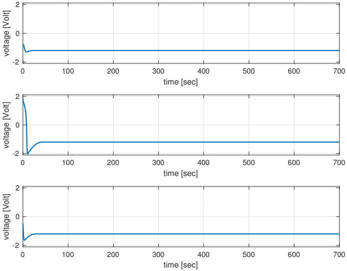
Numerics on the FhN model





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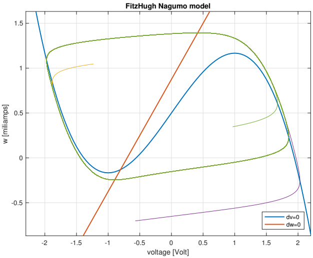

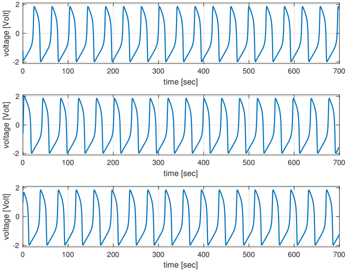

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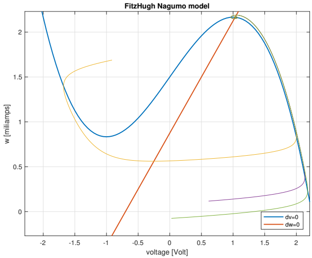
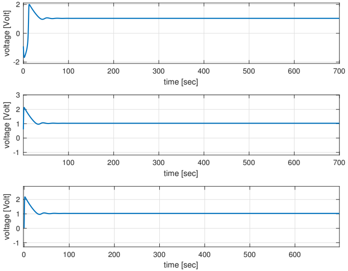
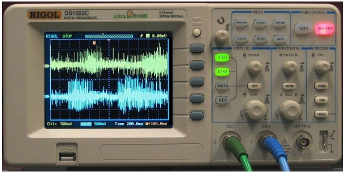




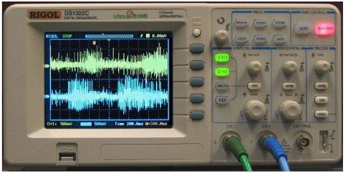


FitzHugh-Nagumo with noisy input

Consider the equation

$$\begin{aligned} dV &= \big(N(V) - w + I_0\big)\,dt + \sigma dW_t, \\ dw &= \frac{1}{\tau}(V + a - bw)\,dt, \end{aligned}$$

where σ is a positive constant, and W_t is a Brownian motion.



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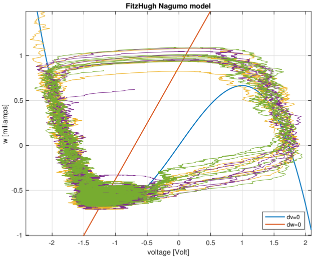
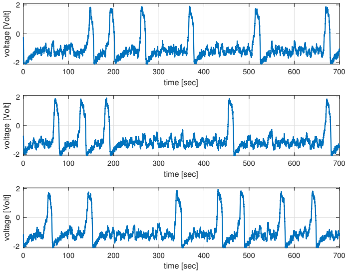
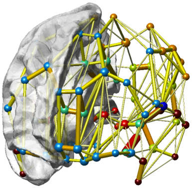
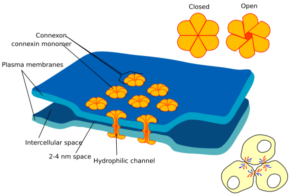
Numerics on the Noisy-FhN model





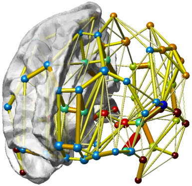


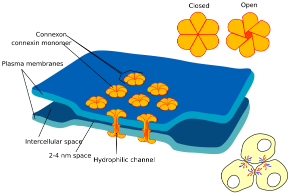


FitzHugh-Nagumo Network

- Consider n FhN neurons $(v_t^i, w_t^i)_{t\geq 0}$, $i = 1, \ldots, n$.
- Neurons interact through the difference of their potential.
- For simplicity, consider a fully connected network with synaptic weights ε/n

$$I_t = -\frac{\varepsilon}{n}\sum_{j=1}^{n}\left(v_t^i - v_t^j\right)$$





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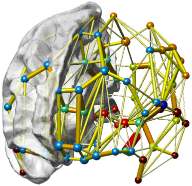

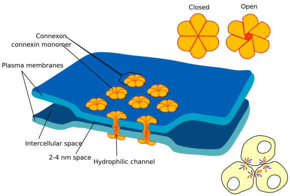

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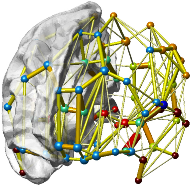

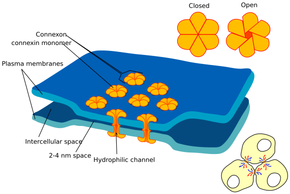

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FitzHugh-Nagumo noisy network

Dynamics of each cell $(v_t^i, w_t^i)_{t\geq 0}$ are then solution to the equations

$$
\begin{aligned}
dv_t^i &= \left(N(v_t^i) - w_t^i + I_0\right) dt - \frac{\varepsilon}{n}\sum_{j=1}^{n}\left(v_t^i - v_t^j\right) dt + \sigma dW_t^i \\
dw_t^i &= \left(v_t^i + a - bw_t^i\right)\frac{dt}{\tau}
\end{aligned}
$$

Nonlinear SDE

Since interaction is linear, the system can be re-written by

$$\begin{cases} dv_t^i &= \big(N(v_t^i) - w_t^i + I_0\big)\,dt - \varepsilon\Big(v_t^i - \frac{1}{n}\sum_{j=1}^n v_t^j\Big)\,dt + \sigma dW_t^i \\ \tau dw_t^i &= (v_t^i + a - bw_t^i)\,dt, \end{cases}$$

and in the case $n \gg 1$, it is natural to consider the mean-field representation

$$\begin{cases} d\bar{v}_t &= \big(N(\bar{v}_t) - \bar{w}_t + I_0\big)\,dt - \varepsilon\big(\bar{v}_t - \mathbb{E}[\bar{v}_t]\big)\,dt + \sigma d\bar{W}_t, \\ \tau d\bar{w}_t &= (\bar{v}_t + a - b\bar{w}_t)dt \end{cases}$$

- The passage micro/macro is due to the propagation of chaos property.
- Proof follows the coupling technique using nice a priori bounds to deal with the cubic nonlinearity (Carrillo, Fournier, etc).

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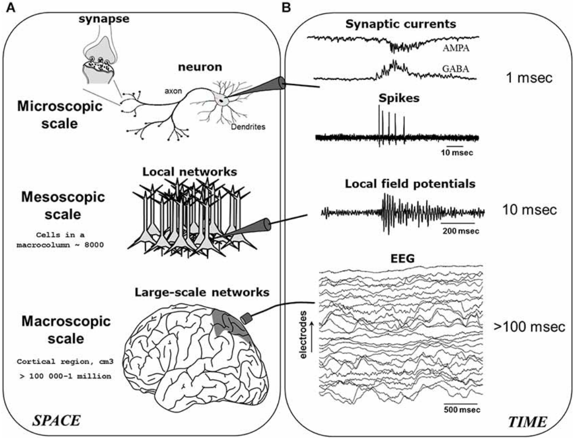
An equation is worth a thousand images



Frontiers in Human Neuroscience, Ros et al.(2014)

FitzHugh-Nagumo mean-field equation

By using the Fokker-Planck equation, we finally find that the law f_t of finding neurones with voltage v and recovery variable w at time t, solves

$$\partial_t f_t = Q_\varepsilon(f_t) f_t = \partial_w (A f_t) + \partial_v (B_\varepsilon[f_t] f_t) + \frac{\sigma^2}{2} \partial^2_{vv} f_t$$

where $A = (bw - a - v)/\tau$,

$$B_\varepsilon[g] = -N(v) + x - I_0 + \varepsilon \left(v - \underbrace{\int_{\mathbb{R}^2} v\, g(w,v)}_{\mathscr{J}(g)} \right)$$

Consequences of the a priori bounds

- Existence of solutions for any coupling value ε. Uniqueness holds true when initial conditions have *finite partial entropy*:

$$\sup_{[0,T]} \int_{\mathbb{R}^2} f \log f + \int_0^t \int_{\mathbb{R}^2} \frac{|\partial_v f|^2}{f} \leq C(T).$$

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Stability results

On the variation $h := f - G_\varepsilon$, the FhN kinetic equation induces the linear integro-differential operator

$$\mathscr{L}_\varepsilon h = Q_\varepsilon[G_\varepsilon]h + \varepsilon \mathscr{J}(h)\partial_v G_\varepsilon$$

which is such that

$$\langle Q_\varepsilon[G_\varepsilon]\, h, h\rangle_{L^2(m)} \quad \leq \quad K_1\|h\|_{L^2(\mathbb{R}^2)} - K_2\|h\|_{L^2(m)}$$

Consequences:

- Existence of stationary solutions, and uniqueness as a function of ε, is a consequence of some semigroup arguments [Mischler & Mouhout].
- Spectral analysis and nonlinear convergence in the weak coupling regime [Krein-Rutman + Duhamel's formula].

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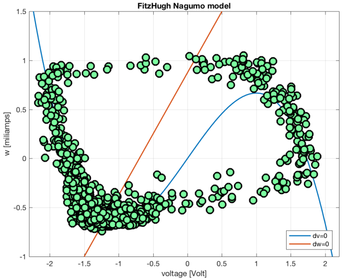
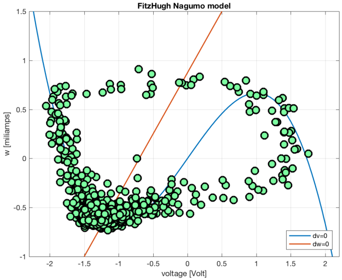

FitzHugh Nagumo model
w [miliamps]
voltage [Volt]
dv=0
dw=0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Beyond the weakly connected case

- The next problem is the characterisation of the system when ε is large.
- To understand this transition we use the Hamilton-Jacobi approach of Roquejoffre, Barles, Perthame et al.
- We present the results for a simplified version of the equation (without the w dependance) but they remain true in the general case.

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The simplified equation

For $\varepsilon > 0$, we are concerned with the behaviour of $g_\varepsilon(t,v)$, solutions to the equation

$$\partial_t g_\varepsilon = \partial_v \Big(\big(-N(v) + \varepsilon^{-1}(v - I_g^\varepsilon(t)) \big) g_\varepsilon + \partial_v g_\varepsilon \Big),$$

coupled with the variable

$$I_g^\varepsilon(t) = \int_{\mathbb{R}} v g_\varepsilon,$$

modelling a *self-induced current*

Formal calculations

In terms of the Hopf-Cole transformation $g_\varepsilon = e^{\frac{\psi_\varepsilon}{\varepsilon}}$, we get

$$\partial_t \psi_\varepsilon = \big(1 - \varepsilon N'(v)\big) + \big(\varepsilon^{-1}(v - I^\varepsilon(t)) - N(v)\big)\,\partial_v \psi_\varepsilon + \varepsilon^{-1}|\partial_v \psi_\varepsilon|^2 + \partial^2_{vv}\psi_\varepsilon$$

At the limit $\varepsilon \to 0$

$$0 = (v - I(t))\partial_v \psi(t,v) + |\partial_v \psi(t,v)|^2,$$

for some $I(t)$ which is unfortunately unknown

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Some remarks

- However, previous equation can be explicitly solved, thus at the limit we expect to have
$$\psi(t,v) = -\frac{1}{2}(v - I(t))^2.$$
- Since $g_\varepsilon = e^{\frac{\psi_\varepsilon}{\varepsilon}}$, we also expect that
$$\psi(t,v) \leq 0.$$
- The points where ψ equals 0 are very important since, once ε is small, the functions $g_\varepsilon(t,\cdot)$ are expected to concentrate around these points

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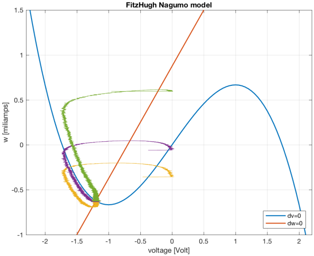
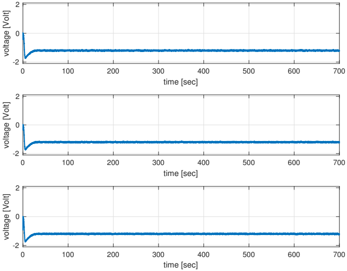
Numerics on the Noisy-FhN model strongly connected





Final remark

In the 2-dimensional case, the limit remains a viscosity solution to

$$0 = (v - I(t))\partial_v \psi(t, v, w) + |\partial_v \psi(t, v, w)|^2.$$

Defining

$$\langle v \rangle_t = I(t) = \int_{\mathbb{R}^2} v f_t \qquad \langle x \rangle_t = \int_{\mathbb{R}^2} x f_t$$

we find that the pair $(\langle v \rangle_t, \langle x \rangle_t)$ is a solution to

$$\begin{aligned} d\langle v \rangle_t &= \big(N(\langle v \rangle_t) - \langle x \rangle_t + I_0\big)\, dt, \\ \tau d\langle x \rangle_t &= (\langle v \rangle_t + a - b\langle x \rangle_t)\, dt, \end{aligned}$$

i.e., to the FhN equation!

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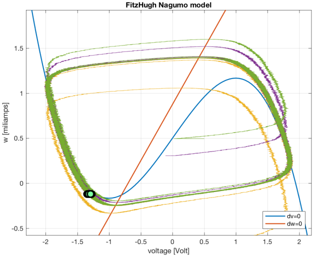
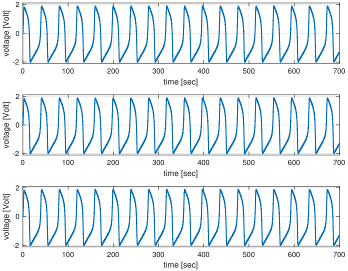
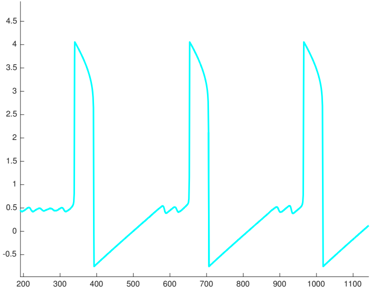
i.e., to the FhN equation!

Numerics on the excited Noisy-FhN model strongly connected





MMO?



FitzHugh-Nagumo synaptic network

And if we consider a more complex model?

$$
\begin{aligned}
dv_t^i &= \left(N(v_t^i) - w_t^i + I_0\right) dt + \sigma dW_t^i \\
&\quad + \frac{\varepsilon^{-1}}{n}\left(g_E(v_t^i)\sum_{j=1}^{n} s_j^E - g_I(v_t^i)\sum_{j=1}^{n} s_j^I\right) dt \\
dw_t^i &= \left(v_t^i + a - bw_t^i\right)\frac{dt}{\tau} \\
ds_t^i &= -s_t^i + \alpha(v_t^i)(1 - s_t^i).
\end{aligned}
$$

Condition:

$$\psi = \frac{1}{2}\left[g_E(v)^2\overline{s}_E - g_I(v)^2\overline{s}_I\right] \leq 0,$$

i.e. the model might converge to a balanced network!

FitzHugh-Nagumo synaptic network

And if we consider a more complex model?

$$\begin{aligned} dv_t^i &= \left(N(v_t^i) - w_t^i + I_0\right) dt + \sigma dW_t^i \\ &\quad + \frac{\varepsilon^{-1}}{n}\left(g_E(v_t^i)\sum_{j=1}^n s_j^E - g_I(v_t^i)\sum_{j=1}^n s_j^I\right) dt \\ dw_t^i &= (v_t^i + a - bw_t^i)\,\frac{dt}{\tau} \\ ds_t^i &= -s_t^i + \alpha(v_t^i)(1 - s_t^i). \end{aligned}$$

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