Modeling interacting networks of neurons as systems of processes with variable length

Based on joint work with A. Duarte, A. Galves, P. Hodara, G. Ost

I Introduction

- neurons communicate by sending sequences of action potentials (spikes).

Past transmembrane currents of Na^+ / K^+-ions, stimulated by ion pumps

- spikes:  are "stereotyped":

always of same shape

short duration: ≈ 1 ms; followed by refractory period during which the neuron is not able to spike again (≈ 1 ms)

$\Rightarrow$ point process model

I countable set: indices of neurons

$$X_t(i) = \begin{cases} 1 : \text{neuron } i \text{ spikes at time } t \\ 0 : \text{no spike of } i \text{ at } t \end{cases}$$

- $t \in \mathbb{Z}$: index of time window within which we observe the neuron ($\simeq 3$ ms)

Observations: 1 0 1 0 0 1 0 1 1
0 0 0 1 1 0 1 0 0

Dynamics: PCA

$\mathcal{F}_t = \sigma(X_s(i),\ s \le t,\ i \in I)$: past $\le t$

$\forall J \subset I$ finite, $\forall a(j) \in \{0,1\}$

$$P\Big(\bigcap_{i \in J} \{X_{t+1}(i) = a(i)\} \mid \mathcal{F}_t\Big) = \prod_{i \in J} P(X_{t+1}(i) = a(i) \mid \mathcal{F}_t)$$

$$P(X_{t+1}(i) = 1 \mid \mathcal{F}_t) = \varphi_i \Big(\underbrace{\sum_{j \in I} W_{j \to i} \sum_{s=L_t^i+1}^{t} g_i(t+1-s)\, X_s(j)}_{=: U_t(i)} \Big)$$

- the prob. that i spikes at $t+1$ is a fct φ_i of its membrane potential $U_t(i)$

- membrane potential is: sum of spikes of presynaptic neurons of i since its last spike time:

$$L_t^i = \sup\{s \le t : X_s(i) = 1\};$$

i.e. $L_t^i = t \Leftrightarrow i$ spikes at t ;

else $t - L_t^i =$ "age-process" ($\to$ J. Chevallier)
= time elapsed since last spike

Convention: $\sum_{t+1}^{t} \ldots = 0$; i.e. $V_t(i) = 0$ if $X_t(i) = 1$:

at spike times, membrane potential is reset to 0.

$\Rightarrow$ specific memory structure : memory of variable length

- spikes coming from presynaptic neurons are weighted by synaptic weights

$W_{j \to i}$ of neuron j on i

- loss of potential since time s when $X_s(j)$ had spiked and present :

$g_i(t+1-s)$

Conditions 1) Uniform summability of synaptic weights

$$\sup_i \sum_j |W_{j \to i}| < \infty$$

2) $g(t) := \sup_{i \in I} g_i(t) < \infty \quad \forall t, \quad g_i : \mathbb{R}_+ \to \mathbb{R}_+$

3) $\varphi_i \uparrow$, Lipschitz

Example

3	1	1	0	1	1	0	1	1	
2	1	0	1	0	0	0	[1]	0	↓
1	1	0	1	1	0	[1]	0	0	

?

$L_8^3 = 8 \Rightarrow \mathcal{U}_8(3) = 0 \Rightarrow P(X_9(3) = 1 \mid \mathcal{F}_8) = \varphi_3(0)$

$L_8^2 = 7 \Rightarrow \mathcal{U}_8(2) = W_{3\to 2}\, g_2(1)$

$L_8^1 = 6 \Rightarrow \mathcal{U}_8(1) = W_{3\to 1}\,(g_1(1) + g_1(2)) + W_{2\to 1}\, g_1(2)$

Comments

- non-linear Hawkes process in discrete time with a countable number of components and locally a memory structure of variable length
- if $g_i(t) \equiv 1$ (no leakage) or $g_i(t) = e^{-\alpha(t-1)}$, $\alpha > 0 \Rightarrow$ $(\mathcal{U}_t(i), i \in I)$ Markov

Indeed : Let $(W_t(i),\ i\in I,\ t\in\mathbb{Z})$ i.i.d. $\mathcal{U}(0,1)$

$$\Rightarrow U_{t+1}(i) = 1_{\{W_{t+1}^{(i)} > \varphi_i(U_t(i))\}}$$

$$\left[e^{-2} U_t(i) + \sum_{j\to i} W_{j\to i}\, 1_{\{W_{t+1}(j) \le \varphi_j(U_t(j))\}} \right]$$

Exo Show that the above definition of $(U_t(i))$ is equivalent to our PCA-dynamics, if we put :

$$X_t(i) = 1_{\{U_t(i)>0\}}$$

$$= 1_{\{W_t(i) \le \varphi_i(U_{t-1}(i))\}}$$

II Existence of the process

Question : When does a unique stationary version of $(X_t(i),\ i\in I,\ t\in\mathbb{Z})$ exist?

"Stationary" means : $\mathcal{L}(X_t(i),\ i\in I) = \mathcal{L}(X_0(i),\ i\in I)\ \forall t$

Additional assumptions :

→ $\varphi_i(0) \ge p_* > 0$ "minimal spiking rate"

- $\gamma := \sup_i \|\varphi_i\|_{Lip} < \infty$

- $g := \sum_{s=1}^{\infty} g(s) < \infty, \quad g = \sup_i g_i$

In what follows, we work in the Markovian frame (and give some hints concerning the general case at the end)

Fact: $U_t(i) = \sum_j W_{j \to i} \sum_{s=L_t^i+1}^{t} g_i(t+1-s) \, X_s(j)$

satisfies: $|U_t(i)| \le r \cdot g$;

$$r = \sup_i \sum |W_{j \to i}|$$

$\Rightarrow$ compact state space

$\Rightarrow$ $\exists$ invariant measures - But how many?

Wasserstein coupling

Take U, $\tilde{U}$ 2 copies of the above HC, starting from 2 fixed initial conditions at time 0: u, $\tilde{u}$.

Coupling: We use the same innovations $W_t(i)$, $t \ge 0$, $i \in I$, for U and $\tilde{U}$.

Write $D_t(i) = |U_t(i) - \tilde{U}_t(i)|$, $t \ge 0$.

Transitions of $D_t(i)$:

1) If $W_{t+1}(i) \le \varphi_i(U_t(i) \wedge \tilde{U}_t(i))$:

in both processes, i spikes $\Rightarrow$

$U_{t+1}(i) = \tilde{U}_{t+1}(i) = 0$; $D_{t+1}(i) = 0$

2) If $W_{t+1}(i) > \varphi_i(U_t(i) \vee \tilde{U}_t(i))$:

in both processes, i does not spike $\Rightarrow$

$$D_{t+1}(i) \le e^{-\alpha} D_t(i) + \sum_j |W_{j \to i}| \, |X_{t+1}(j) - \tilde{X}_{t+1}(j)|$$

3) If one spikes, but not the other, then $D_{t+1}(i) = \max |U_{t+1}(i)|, |\tilde{U}_{t+1}(i)|$, and we use simply that : $|U_t(i)| \le r_\vartheta$ $\forall t, \forall i$.

Associated probabilities (given $\mathcal{F}_t$)

2) $1 - \varphi_i(U_t(i) \vee \tilde{U}_t(i)) \le 1 - p^*$

3) $|\varphi_i(U_t(i)) - \varphi_i(\tilde{U}_t(i))| \le \gamma \cdot D_t(i)$

$$\Rightarrow E[D_{t+1}(i) \mid \mathcal{F}_t] \le (1-p^*)\left[e^{-\alpha} D_t(i) + \sum_j |W_{j \to i}| \, E[|X_{t+1}(j) - \hat{X}_{t+1}(j)| \mid \mathcal{F}_t]\right] + \gamma D_t(i) r_\vartheta$$

$$= \left[(1-p^*) e^{-\lambda} + \gamma r_\delta \right] D_t(i) + (1-p^*) \cdot \sum_j |W_{j \to i}| \, E\left[|X_{t+1}(j) - \tilde{X}_{t+1}(j)| \mid \mathcal{F}_t \right]$$

<u>But</u> $X_{t+1}(j) = 1_{\{W_{t+1}(j) \le \varphi_j(\mathcal{U}_t(j))\}}$

$$\Rightarrow E\left[|X_{t+1}(j) - \tilde{X}_{t+1}(j)| \mid \mathcal{F}_t \right] \le \gamma \, D_t(j)$$

<u>Take expectations</u> $\delta_t(i) = E\, D_t(i)$

$$\Rightarrow \delta_{t+1}(i) \le \left[(1-p^*) e^{-\lambda} + \gamma r_\delta \right] \delta_t(i) + (1-p^*) \sum_j |W_{j \to i}| \, \gamma \, \delta_t(j)$$

$$H_{ij} = (1-p^*) |W_{j \to i}| \, \gamma, \quad i \ne j$$

$$H_{ii} = (1-p^*) e^{-\lambda} + \gamma r_\delta, \text{ then } \delta_{t+1} \le H \delta_t \le \dots \le H^{t+1} \delta_0$$

<u>Condition</u> $H : \ell_\infty \to \ell_\infty$ (defined on the space of sequences $(x_i, i \in I)$ having bounded elements) satisfies:

$$|||H||| < 1,$$

$$|||H||| = \sup \{ \|Hx\|_\infty : \|x\|_\infty \le 1 \}$$

$$\Leftrightarrow (1-p^*) \left[e^{-\lambda} + r_\delta \right] + \gamma r_\delta < 1$$

OK : if $\gamma r s < 1$ and p^* suff. large.

Prop Under this condition, $\exists !$ invariant measure, and $\delta_t \to 0$ exp. speed.

Rem In the non-Markovian frame, one has to consider $D_t(i) = 1_{\mathcal{L}_t^i \neq \tilde{\mathcal{L}}_t^i}$

One gets the (Bette) criterion :

$$1 - p^* + \gamma r s < 1.$$

III Perfect Simulation by the Clan of Ancestors - Method

$$p_{(i,t)}(1|x) = \varphi_i \Big(\sum_j \sum_{s=\mathcal{L}_{t+1}^i}^{t} w_{j \to i}\, g_i(t+1-s)\, x_s(j) \Big)$$

Write $\mathcal{V}_{\cdot \to i} = \{ j : w_{j \to i} \neq 0 \}$ $\Rightarrow$

- $p_{(i,t)}(1|x)$ depends on $x^t_{\mathcal{L}_{t+1}^i}(\mathcal{V}_{\cdot \to i})$:

  spatio-temporal neighborhood

- it is the transition proba of neuron i at $t+1$

Kalikow-type decomposition

Fix $i \in I$ + growing sequence $V_i(k)$, $k \geq -1$, of subsets of $\mathcal{V}_{\cdot \to i} \cup \{i\}$:

$V_i(-1) = \phi, \quad V_i(0) = \{i\}, \quad V_i(k) \subsetneq V_i(k+1)$

$\cup V_i(k) = \mathcal{V}_{-i} \cup \{i\}$

Continuity modulus

$$\beta(k) = \sup \{ |p_{(i,t)}(1|x) - p_{(i,t)}(1|y)| : x \underset{k}{\equiv} y \}$$

where $x \underset{k}{\equiv} y \Leftrightarrow x_s(j) = y_s(j) \quad \forall t-k \le s \le t$
$\forall j \in V_i(k)$

Rem Under our conditions:

- $\sum |w_{j,i}| < \infty$
- $\sum_{n=1}^{\infty} |g_i(n)| < \infty$
- φ_i Lipschitz

$\Rightarrow \beta(k) \to 0, \ k \to \infty$

Theorem (Kalikow decomposition)

$$p_{(i,t)}(1|x) = \sum_{k=-1}^{\infty} \lambda(k) \, p_i^{[k]}(1|x)$$

- $\lambda(k) \ge 0, \quad \sum \lambda(k) = 1$
- $\forall k, x$: $p_i^{[k]}(\cdot|x)$ proba on $\{0,1\}$,
  depends only on $x_s(j)$: $t-k \le s \le t$
  $j \in V_i(k)$

Explain: $\lambda(-1), \ p^{[-1]}(\cdot)$

- Generalization for Hawkes processes of "random Markov chains" Kalikow 1990, Ferrari, Maass, Martinez + Ney (2000)

Dynamical interpretation

Suppose: Process already constructed up to time t.

1) $\forall i$: Choose Size of context

$$K_{(i,t)} = k \text{ with proba } \lambda(k),\ k \geq -1,$$

independently $\forall i$

2) If $K_{(i,t)} = -1$: Put

$$X_i(t+1) = \begin{cases} 1 & \text{w. proba } p_i^{[-1]}(1) \\ 0 & \text{else} \end{cases}$$

Else: Put $k = K_{(i,t)}$. Inspect the values of the sites belonging to $V_i(k) = [t-k, t] \times V_i(k)$ and choose

$$X_i(t+1) = \begin{cases} 1 & \text{w. proba } p_i^{[k]}(1 | x_{V_i(k)}) \\ 0 & \text{else} \end{cases}$$

$\Rightarrow$ random spatio-temporal interaction range.

Estimates on the reproduction probabilities

$$\lambda_i(0) \le \gamma_i \|g_i\|_1 \sum_j |W_{j\to i}| \le \gamma r s$$

$$\lambda_i(k) \le \gamma_i \Big(\sum_{j \notin V_i(k-1)} |W_{j\to i}| \|g_i\|_1 + \sum_{j \in V_i(k-1)} |W_{j\to i}| \sum_{n \ge k} |g_i(n)| \Big)$$

Perfect simulation from the stationary measure

Intuition: If processes startet from $t=-\infty$, it must be in stationary regime at $t=0$.

Want to sample a finite set of sites from invariant measure - that is, at $t=0$.

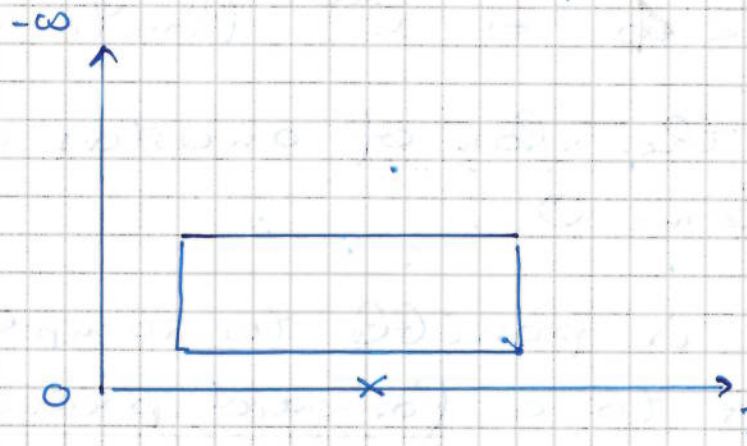
Say: Only site $\{i_0\}$!

Clan of Ancestors: Determine all couples (j,s) that might possibly influence the value of $X_{i_0}(t)$ for $t=0$!

1) Choose $K_{(i_0,0)} = \begin{cases} -1 & \lambda_{i_0}(-1) \\ k & \lambda_{i_0}(k) \end{cases}$

and put $C^1 := \mathcal{V}_{i_0}(h) =$

$[-h-1, -1] \times V_{i_0}(h)$; $C^1 := \phi$ if $h=-1$.

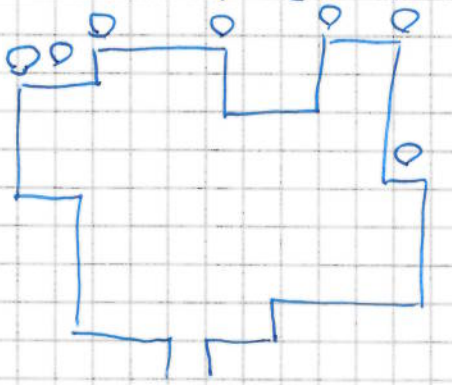


2) $\forall (j,s) \in C^1$: replace (j,s) by $C^1(j,s)$ as above and put

$$C^2 := \left(\bigcup_{(j,s) \in C^1} C^1(j,s) \right) \setminus C^1$$

etc

3) $N^{STOP} = \inf \{ n : C^n = \phi \}$



$N^{STOP} < \infty$ a.s. $\Leftarrow$ associated branching process has offspring mean < 1

$$m < \sum_{k=0}^{\infty} \lambda(k)\,(k+1)\,|V_i(k)|$$

Prop If $m < 1 \Rightarrow C^\infty$ finite a.s.

We call C^∞ the clan of ancestors of neuron i_0 at time 0.

Given C^∞, it is possible to sample from $X_{i_0}(0)$ thanks to a forward procedure:

We assign values 1 or 0 to all sites within C^∞, from up to down, using the probabilities $p_j^{[k]}(\cdot \mid x)$, with the choices of k already fixed in the backward procedure.

IV Estimation of the interaction graph

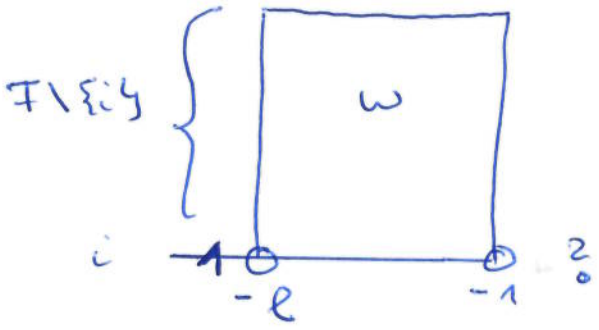
Back to the model in discrete time.

Interaction neighborhood of i:

$$\mathcal{V}_i = \{ j : W_{j-i} \neq 0 \}$$

Want to estimate $\mathcal{V}_i$, given a sample $X_1(T), \dots, X_n(T)$, $T \subset \mathbb{I}$ some finite space window.

Idea: Write $p_i(1|\omega) = \varphi_i \left(\sum_j W_{j-i} \cdot \sum_{k=1}^{\ell} g_i(k)\, \omega_{-k}(j) \right)$, where ω = test block



Suppose: $\mathcal{V}_i \subset T$.

$$\Rightarrow p_i(1|\omega) = P\left(X_{t+1}(i) = 1 \,\middle|\, L_t^i = t-\ell,\ X_{t-\ell+1}^t (T \setminus \{i\}) = \omega \right)$$

$$= P(X_{t+1}(i) = 1 \,|\, \mathcal{F}_t)\, 1_{L_t^i = t-\ell,\ X_{t-\ell+1}^t (T \setminus i) = \omega}$$

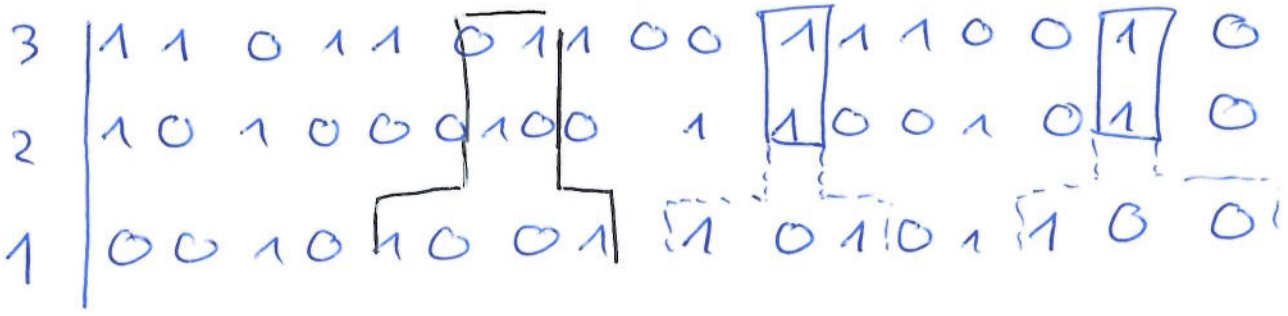
Estimator of $p_i(1|w)$:

$$\hat{P}_{(i,n)}(1|w) = \frac{N_{(i,n)}(w,1)}{N_{(i,n)}(w)}, \text{ where:}$$

$N_{(i,n)}(w,1)$ = number of occurences of w followed by a spike of i, within the sample, when the last spike of i has occurred $\ell+1$ time steps before in the past

$$N_{(i,n)}(w) = N_{(i,n)}(w,0) + N_{(i,n)}(w,1)$$

Ex $\ell = 1$, $w = \begin{matrix} 3 & 1 \\ 2 & 1 \end{matrix}$, $I = \{1,2,3\}$, $i = 1$



$N_{(i,n)}(w,1) = 1$

$N_{(i,n)}(w,0) = 1$

$\Rightarrow \hat{p} = \frac{1}{2}$

Can be done for longer words:

$\ell = 2$, $w = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$ $\Rightarrow$ $N_{(i,n)}(w,1) = 1$, $N_{(i,n)}(w,0) = 0$ $\Rightarrow \hat{p} = 1$

Idea: If $3 \not\to 1$, then a change within the coordinate 3 shouldn't change the spiking behavior of 1.

More precisely: $\forall\, \tilde{\omega}: \tilde{\omega} = \omega$:

$P_1(1|\omega) = P_1(1|\tilde{\omega})$, thus:

$\hat{P}_{(1,n)}(1|\omega) \approx \hat{P}_{(1,n)}(1|\tilde{\omega})$ (for n big)

Rigorous formulation

Admissible test words:

- $T_{(i,n)} = \{\omega \in \bigcup_{\ell=1}^{n-2} \{0,1\}^{\{-\ell,\dots,-1\} \times \mathbb{F} \setminus \{i\}} : N_{(i,n)}(\omega) \geq n^{1/2+\varepsilon}\}$

For some fixed $\varepsilon \in]0, \frac{1}{2}[$.

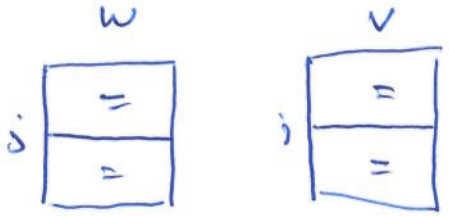
- $L(\omega) = \ell$ if $\omega \in \{0,1\}^{\{-\ell,\dots,-1\} \times \{\mathbb{F} \setminus \{i\}\}}$

- $v, w \in T_{(i,n)}$ with $L(v) = L(w)$: we write

$\boxed{v_{\{j\}^c} = w_{\{j\}^c}}$ if $v_{-k}(m) = w_{-k}(m)$

$\forall\, m \in \mathbb{F},\ m \neq j$

$\forall\, k = 1, \dots, \ell$



Measure of sensibility

$$\Delta_{(i,n)}(j) = \max_{\substack{w \in \mathcal{T}_{(i,n)} \\ v \in \mathcal{T}_{(i,n)} : v_{\{j\}^c} = w_{\{j\}^c}}} \left| \hat{P}_{(i,n)}(1|w) - \hat{P}_{(i,n)}(1|v) \right|$$

Estimator: Fix a threshold $\varepsilon > 0$

$$\hat{\mathcal{V}}^{(\varepsilon)}_{(i,n)} = \{ j \in \mathcal{T} \setminus \{i\} : \Delta_{(i,n)}(j) > \varepsilon \}$$

Rq 1) Spiking probability of each neuron depends on spatio-temporal neighborhoods / portions of the past: $X_t(i)$ depends on all random variables $X_s(j)$: $j \in \mathcal{V}_i$ and

$$L^i_{t-n} + 1 \leq s \leq t-1$$

But since we (suppose to) know the temporal dependencies, L^i_t does not need to be estimated.

2) No spatial pruning procedure, we always take test words within the whole spatial region $\mathcal{T} \setminus \{i\}$.

Conditions

1) We do not need stationarity, only:

$\gamma_T \# < \infty$; $g(t) = \sup_i g_i(t) < \infty \ \forall t$.

2) Transition probabilities are uniformly positive, that is, all φ_i are bounded away from 0 and 1 on the compact where our process lives: $p_* \le \varphi_i(\cdot) \le 1 - p_* \ \forall i$.

Theorem 1 Suppose $v_i \subset \mathcal{F}$.

1) $\forall j \in \mathcal{F} \setminus v_i$:

$$P\left(j \in \hat{v}^{(\varepsilon)}_{(i,n)}\right) \le 4 n^{3/2-\varphi} e^{-\frac{\varepsilon^2 n^{2\varphi}}{2}}$$

2) $\forall j \in v_i$ and ε suff. small:

$$P\left(j \notin \hat{v}^{(\varepsilon)}_{(i,n)}\right) \le 4 \exp\left(-\frac{(m_i - \varepsilon)^2 n^{2\varphi}}{2}\right) + \exp\left(-C n^{1/2+\varphi}\right)$$

Here, $m_i = \inf_{u \in K_i} \varphi_i'(u) \inf_{j \in v_i} |W_{j,i}| \, g_i(1)$

↑ compact, where the membrane potential of neuron i lives under our assumptions.

Idea of proof for the underestimation:

① One shows that $\forall j \in \vartheta_i$: "identifiability"

$$m_{ij} = \max_{w, v \in \{0,1\}^{\{-1\} \times V_i} : w_{j^c} = v_{j^c}} |p_i(1|w) - p_i(1|v)|$$

satisfies: $\inf_{j \in \vartheta_i} m_{ij} \geq m_i > 0$

⇒ the theoretical probabilities "feel" the difference. "Identifiability"

② Take $\varepsilon < m_i$ and $j \in \vartheta_i$ ⇒

$\exists\, w^*, v^*$ with $L(w^*) = L(v^*) = 1$, which coincide, except in j, s.t.

$$|p_i(1|w^*) - p_i(1|v^*)| \geq m_i.$$

Let $C_n = \{ N_{(i,n)}(w^*) \geq n^{\varphi + 1/2},\ N_{(i,n)}(v^*) \geq n^{\varphi + 1/2} \}$

$$\Rightarrow P(j \notin \hat{\vartheta}) \leq P(|\hat{p}_{(i,n)}(1|w^*) - \hat{p}_{(i,n)}(1|v^*)| < \varepsilon, C_n)$$

$$+ P(C_n^c)$$

↑

easy to handle, since all probabilities are strictly case-bounded

1. Term $\leq$

$$P\left(\left|\hat{P}_{(i,n)}(1|\omega^*) - p_i(1|\omega^*)\right| > \frac{m_i - \varepsilon}{2},\ N(\omega^*) \geq n^{1k+\delta}\right)$$

$$+ P\left(\left|\hat{P}_{(i,n)}(1|\nu^*) - p_i(1|\nu^*)\right| > \frac{m_i - \varepsilon}{2},\ N(\nu^*) \geq n^{1k+\delta}\right)$$

Indeed:

$$\varepsilon \geq \left|\hat{P}_{(i,n)}(1|\omega^*) - \hat{P}_{(i,n)}(1|\nu^*)\right| \geq$$

$$\left|\underbrace{p_i(1|\omega^*) - p_i(1|\nu^*)}_{m_i}\right| - \left|\hat{P}_{(i,n)}(1|\omega^*) - p_i(1|\omega^*)\right| - \left|\hat{P}_{(i,n)}(1|\nu^*) - p_i(1|\nu^*)\right|$$

$$\Rightarrow\ m_i - \varepsilon \leq \left|\hat{P}_{(i,n)}(1|\omega^*) - p_i(1|\omega^*)\right| + \left|\hat{P}_{(i,n)}(1|\nu^*) - p_i(1|\nu^*)\right|$$

$\Rightarrow$ <u>need deviation inequality for</u>

$$P\left(\left|\hat{P}_{(i,n)}(1|\omega) - p_i(1|\omega)\right| > \lambda,\ N(\omega) \geq n^{1k+\delta}\right)$$

<u>Conditional Hoeffding inequality</u>

$$H_n^i = N_{(i,n)}(\omega, 1) - \underbrace{p_i(1|\omega)}_{p_i}\ \underbrace{N_{(i,n)}(\omega)}_{N_n}$$

$$\Rightarrow P\left(|H_n| > \lambda\right) \leq 2\exp\left(-\frac{2\lambda^2}{n-\ell+1}\right) P(N_n > 0)$$

where $\ell = \ell(\omega)$

(Proof: Exo)

Once we have proved the conditional Hoeffding inequality, it follows:

$$M_n = \left(\hat{p}_{(i,n)}(1|\omega) - p_i(1|\omega)\right) \cdot N_{(i,n)}(\omega)$$

$$\Rightarrow P\left(|\hat{p} - p| > \lambda,\ N(\omega) \geq n^{1/2+\varepsilon}\right) \leq$$

$$P\left(|M_n| > \lambda n^{1/2+\varepsilon}\right) \leq$$

$$2 \cdot \exp\left(\frac{-2\lambda^2 n^{1+2\varepsilon}}{n-\ell+1}\right) \underbrace{P(N_n > 0)}_{\leq 1 \text{ (not needed here)}}$$

$$\text{Since } \frac{n}{n-\ell+1} \geq 1 \Rightarrow \ldots \leq \underbrace{2\, e^{-2\lambda^2 n^{2\varepsilon}}}_{\leq 1} P(N_n > 0)$$

Idea of proof of overestimation: $j \in \mathcal{T} \setminus V_i$

$$P(j \in \hat{V}) = P\left(\Delta_{(i,n)}(j) > \varepsilon\right)$$

$$= P\left(\exists \text{ 2 admissible } \omega, v : \left|\hat{p}_{(i,n)}(1|\omega) - \hat{p}_{(i,n)}(1|v)\right| > \varepsilon\right)$$

(ω different in j)

Since $j \notin V_i$, even if $\omega = v$ except in j, we have $p_i(1|\omega) = p_i(1|v)$

$$\Rightarrow \left|\hat{p}_{(i,n)}(1|\omega) - \hat{p}_{(i,n)}(1|v)\right| > \varepsilon \text{ implies:}$$

$$\left|\hat{p}_{(i,n)}(1|\omega) - p_i(1|\omega)\right| > \varepsilon/2 \quad \text{or}$$

$$\left|\hat{p}_{(i,n)}(1|v) - p_i(1|v)\right| > \varepsilon/2$$

$$\Rightarrow P(j \in \hat{V}) \leq 2 \cdot$$

$$E\Big[\sum_{w \in J^{(i,n)}} \sum_{\substack{v \in J^{(i,n)} \\ v = w \\ i \neq j}} 1_{|\hat{p}_{(i,n)}(1|w) - p_i(1|w)| > \varepsilon/2} \Big]$$

(1) $|J^{(i,n)}| \leq n^{1/2 - \varphi}$ (since $N_{(i,n)}(w) \geq n^{1/2+\varphi}$ and $\sum N(w) \leq n$)

$$\Rightarrow P(j \in \hat{V}) \leq 2 \cdot n^{1/2-\varphi} \cdot$$

$$\sum_w P\big[|\hat{p}_{(i,n)}(1|w) - p_i(1|w)| > \varepsilon/2,\ N(w) \geq n^{1/2+\varphi} \big)$$

$$\leq \Big(\underbrace{\sum_w P(N_n(w) > 0)}_{E \sum_w 1_{N(w)>0} \leq n} \Big) \cdot 2 \cdot n^{1/2-\varphi} \cdot 2 \cdot \underbrace{e^{-\frac{1}{2}\varepsilon^2 n^{2\varphi}}}_{\text{Exo}}$$

$\uparrow$

$$\boxed{\sum_w N_{(i,n)}(w) \leq n \cdot}$$

$\square$

V Models in continuous time and mean-field limits

- spike counting process in continuous time:
  $\forall i \in I$: $Z^i(]s,t]) = \#$ spikes of neuron i during $]s,t]$
- process characterized by its intensity:
  $P(Z^i \text{ jumps during }]t, t+dt] \mid \mathcal{F}_t) = \lambda_t(i)\,dt$
- Ex: $\lambda_t(i) \equiv \lambda \Rightarrow PP(\lambda)$
- Hawkes intensity

$$\lambda_t(i) = \varphi_i\left(\sum_j W_{j\to i} \int_{[0,t[} g_i(t-s)\, Z_j(ds)\right)$$

→ infinite memory process if g_i not of compact support

→ system of interacting nonlinear Hawkes processes (Brémaud-Massoulié 96, Hawkes 71)

(same conditions on φ_i, $W_{j\to i}$, g_i as before)

Mean field frame

$I = \{1, \dots, N\}$, $\varphi_i \equiv \varphi$, $g_i \equiv g$, $W_{j \to i} = \frac{1}{N}$

$\Rightarrow$ each neuron spikes at same rate

$$\lambda(t) = \varphi\left(\int_{[0,t[} g(t-s)\, d\bar{Z}_N(s) \right)$$

$$\bar{Z}_N = \frac{1}{N} \sum_{j=1}^{N} Z_j(ds)$$

Heuristics $\bar{Z}_N(s) \to m(s)$ $(N \to \infty)$

emp. mean $\to$ theoretical mean

where $m(t) = E(\bar{Z}(t))$,

$\bar{Z}(t)$ = spiking countins process of a typical neuron in the limit

$$\bar{Z}(t) = \int_{[0,t]} \int_{\mathbb{R}_+} 1_{z \le \varphi\left(\int_0^s g(s-u)\, d E(\bar{Z}(u)) \right)} N(ds, dz)$$

where $N(ds, dz)$ PRH on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $ds\, dz$

Rq $\bar{Z}(t)$ has (det) intensity

$$\bar{\lambda}(t) = \varphi\left(\int_0^t g(t-u)\, d \underbrace{E\bar{Z}(u)}_{\bar{\lambda}(u)\, du} \right)$$

$\Rightarrow$ $\bar{Z}(t)$ is an inhomogeneous Poisson process of McKean-Vlasov type

Thm (Delattre, Fournier, Hoffmann, AAP 2016)

1) $\exists!$ solution $\bar{Z}(t)$ of the above equation s.t. $t \mapsto \mathbb{E}\bar{Z}(t)$ locally bounded

2) $\forall K$: $((Z^{N,i}(t))_{t\geq 0}, 1 \leq i \leq K) \xrightarrow{\mathcal{L}} P^{\otimes K}$

where $P = \mathcal{L}((\bar{Z}(t))_{t\geq 0})$

"Propagation of chaos": the neurons get $\perp\!\!\!\perp$ as $N \to \infty$

Proof

1) Let $\bar{Z}(t)$ be any solution. Write $m(t) = \mathbb{E}\bar{Z}(t)$ + take expectations:

$$\bar{m}(t) = \int_0^t \varphi\left(\int_0^s g(s-u)\, d\bar{m}(u)\right) ds$$

Convolution equation: possesses a unique (φ Lipschitz, $g \in L^1_{loc}$) increasing locally bounded solution which is C^1

Take this unique solution and define

$$\bar{Z}(t) = \int_0^t \int_{\mathbb{R}_+} 1_{\{z \le \varphi(\int_0^s g(s-u)\, dm(u))\}} N(ds, dz)$$

We only have to show that $E\bar{Z}(t) = m(t)$ which is evident.

2) Propagation is shown by using the "Sznitman coupling"

→ prepare i.i.d. PRM's $N^k(ds, dz)$

→ construct

$$Z^{N,1}(t) \quad \underline{\text{same noise}} \quad \bar{Z}^1(t)$$
$$\vdots$$
$$Z^{N,N}(t) \quad \underline{\text{same noise}} \quad \bar{Z}^N(t)$$

$\bar{Z}^1(t), \dots, \bar{Z}^N(t)$: i.i.d. copies of limit $\bar{Z}$

"Same noise" means:

$$Z^{N,k}(t) = \int_{[0,t]} \int 1_{z \le \varphi(\int_{[0,s[} g(s-u)\, d\bar{Z}_N(u))} N^k(ds, dz)$$

$$\bar{Z}^k(t) = \int_{[0,t]} \int 1_{z \le \varphi(\int_0^s g(s-u)\, d\bar{m}(u))} N^k(ds, dz)$$

$$\Delta_N^k(t) = \int_0^t |\, d Z^{N,k}(u) - d \bar{Z}^k(u) \,|$$

= total variation norm of the signed measure $d(Z^{N,k}(u) - \bar{Z}^k(u))$ on $[0,t]$

Recall: $\|\mu\|_{TV}(A) = \sup \sum |\mu(E_i)| : (E_i)$ partition of A

Hence: $\Delta_N^k(t) =$ <u>number of distinct jumps of the two processes</u>

By exchangeability: $E\,\Delta_N^k(t) = \delta_N(t)$

<u>Fact:</u> $\delta_N(t) \le \|\varphi\|_{Lip} \frac{1}{\sqrt{N}} \int_0^t \left(\int_0^s g^2(s-u)\, d\bar{m}(u) \right)^{1/2} ds$

$+ \|\varphi\|_{Lip} \int_0^t |g(t-s)|\, \delta_N(s)\, ds$

<u>Proof of fact</u>

$$\Delta_N^1(t) = \int_0^t \int |1_{z \le \varphi(\int_0^s g(s-u)\, d\bar{Z}_N(u))} - 1_{z \le \varphi(\int_0^s g(s-u)\, d\bar{m}(u))}|\, N^1(ds,dz)$$

$$\Rightarrow \delta_N(t) = \int_0^t E\left[\left| \varphi\left(\int_0^s g(s-u)\, d\bar{Z}_N(u)\right) - \varphi\left(\int_0^s g(s-u)\, d\bar{m}(u)\right) \right| \right] ds$$

↑ add + subtract limit process ...

Δ

Hyp $g \in L^2_{loc}$

$$\Rightarrow \int_0^t \left(\int_0^s g^2(s-u)\, d\bar{m}(u) \right)^{1/2} ds \leq C_T \quad \forall t \leq T$$

Gronwall $\Rightarrow \sup_{t \leq T} \delta_N(t) \leq C_T \frac{1}{\sqrt{N}}$,

whence

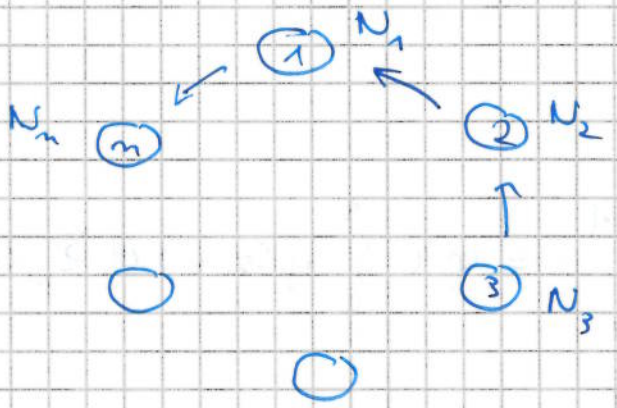
$$E\left[\sup_{t \leq T} \left| Z^{N,k}(t) - \bar{Z}^k(t) \right| \right] \leq C_T \frac{1}{\sqrt{N}}.$$

□

Extension to a multi population frame

(with S. Ditlevsen)

n classes of neurons, n fixed



$N = N_1 + \dots + N_n$, $\forall k$: $\frac{N_k}{N} \to p_k \in]0,1[$

$Z^N_{k,i}(t)$: $1 \leq k \leq n$, $1 \leq i \leq N_k$

All neurons within a given population are exchangeable: they have the same intensity

$$\lambda_k(t) = \varphi_k\left(\int_{[0,t[} h_{k\,k+1}(t-s)\, d\,\overline{Z}^{N}_{k+1}(s) \right)$$

$$\overline{Z}^{N}_{k+1} = \frac{1}{N_{k+1}} \sum_{j=1}^{N_{k+1}} Z^{N}_{k+1,j}(s)$$

Limit System: $(\overline{Z}_1(t), \dots, \overline{Z}_n(t))$ s.t.

$$\forall k \in \{1, \dots, n\}$$

$$\overline{m}_k(t) = \int_0^t \varphi_k\left(\int_0^s h_{k\,k+1}(s-u)\, d\,\overline{m}_{k+1}(u) \right) ds$$

Oscillations of limit System

Suppose $h_{k\,k+1}(t) = c_k\, e^{-\alpha_k t}$

$c_k \neq 0$ factor of impact of pop. $k+1$ on pop. k

$c_k > 0$: excitatory

$c_k < 0$: inhibitory

$\alpha_k > 0$: leakage rate

Markovian frame:

$$x_k(t) := \int_0^t h_{k\,k+1}(t-s)\, d\bar{m}_{k+1}(s)$$

$$\begin{cases} x_k' = -\alpha_k x_k + c_k \bar{m}_{k+1}'(t) \\ \quad\;\; = -\alpha_k x_k + c_k \varphi_{k+1}(x_{k+1}) \end{cases}$$

Monotone cyclic feedback system,

$\delta := \prod_{k=1}^{n} c_k$ total feedback

Prop If $\delta < 0$, then:

1) $\exists!$ equilibrium $x^* \in \mathbb{R}^n$

2) If x^* unstable ($\exists \lambda : \mathrm{Re}(\lambda) > 0$)

$\Rightarrow$ $\exists$ finite number of non-constant periodic orbits

Any ω-limit set is either x^* or one of these orbits.

Mallet-Paret + Smith 1990

Consequence

$\exists$ solutions of the limit process s.t. their law is periodic

(more precisely: the intensity is periodic)

Although no periodicity is present in the microscopic dynamics (at the particle level)

Thm (with S. Ditlevsen)

1) $\exists!$ equilibrium point $x^* \in \mathbb{R}^n$

2) Let us rewrite the system as $\dot{x} = F(x)$:

If $DF(x^*)$ possesses 2 eigenvalues λ s.t. $\mathrm{Re}(\lambda) > 0$

$\Rightarrow \exists$ at least 1 non-constant periodic orbit (and at most a finite number of such orbits), and one of them is asymptotically stable.

Consequence:

$\exists$ solutions of the limit process s.t. their law is periodic!

Although no periodicity is present in the microscopic dynamics (at the particle level)

Oscillations for the finite particle system

Recall: each neuron belonging to population k spikes at rate

$\Phi_k(X_k(t))$ where

$$X_k(t) = \int_{[0,t[} c_k \, e^{-\lambda_k (t-s)} \, d\overline{Z}^N_{k+1}(s)$$

High-intensity - small-jumps - diffusion approximation:

$$d\tilde{X}_k(t) = -\lambda_k \tilde{X}_k(t) + c_k \Phi_{k+1}(\tilde{X}_{k+1}(t))\, dt + c_k \frac{\sqrt{\Phi_{k+1}(\tilde{X}_{k+1}(t))}}{\sqrt{N_{k+1}}} \, d\beta_{k+1}(t)$$

Prop $\| P_t \varphi - \tilde{P}_t \varphi \|_\infty \leq Ct \, \dfrac{\|\varphi\|_{4,\infty}}{N^2}$

Control of the weak error

Thm

1) $\tilde{X}$ is recurrent in the sense of Harris, with invariant probability measure μ^N.

2) $\forall T$ periodic orbit of the limit system:
$\forall x \in \mathbb{R}^n$

$$P_x\left(\tilde{X}_t \in B_\varepsilon(T) \text{ during } [T, T+\delta]\right) > 0$$

$\Rightarrow P_x\left(\tilde{X}_t \text{ makes a complete "tour" around } T \text{ i.o.}\right) = 1$

3) We have even more:

$$\mu^N(D) \sim C\, e^{-N \inf_{x \in D} W(x)}$$

$$W(x) = V(T, x) \wedge \left[V(T, x^*) + V(x^*, x) \right]$$

if there is a single periodic orbit T and x^* the unstable equilibrium.

→ can be extended to higher order memory kernels (Erlang instead of exponential)