Existence of strong solutions to a fluid–structure system with a structure given by a finite number of parameters

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Abstract. We study the existence of strong solutions to a 2d fluid–structure system. The fluid is modelled by the incompressible Navier–Stokes equations. The structure represents a steering gear and is described by a finite number of parameters and its equations are derived from a virtual work principle. The global domain represents a wind tunnel and imposes mixed boundary conditions to the fluid velocity. Our method reposes on the analysis of the linearized system. Under compatibility conditions on the initial conditions, we can guarantee local existence in time of strong solutions to the fluid–structure problem.

MSC numbers: 74F10, 74H20, 74H25, 74H30, 76D03.

1 Introduction

The goal of this study is to prove the existence of a solution to a fluid–structure problem. The fluid is modelled by the incompressible Navier–Stokes equations and the structure, immersed in the fluid, is governed by a finite number of parameters.

For the sake of simplicity, only two parameters θ_1 and θ_2 are considered. However, all results remain valid for any finite number of parameters.

1.1 Modelling of the problem

The considered structure lies inside an open bounded domain $\Omega \subset \mathbb{R}^2$ and deforms itself over time. The couple of parameters (θ_1, θ_2) lies in an admissible domain $\mathbb{D}_\Theta$ which is an open connected subset of $\mathbb{R}^2$. Let S_{ref}, a smooth closed connected subset of Ω, be the reference configuration for the structure (for instance S_{ref} is the volume occupied by the structure for $\theta_1 = \theta_2 = 0$). We consider a function $\mathbf{X}$ defined on $\mathbb{D}_\Theta \times S_{\text{ref}}$ that computes the position in the structure according to the reference position in S_{ref} and to the value of the parameters $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$.

The volume occupied by the structure for parameters $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$ is a closed bounded connected subset of Ω denoted $S(\theta_1, \theta_2) = \mathbf{X}(\theta_1, \theta_2, S_{\text{ref}})$. We further assume that for every $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$, $S(\theta_1, \theta_2) \subset \Omega$, i.e. there is no contact between the structure $S(\theta_1, \theta_2)$ and the boundary of the domain $\partial\Omega$.

1.1.1 Motivations

Structures depending only on a finite number of parameters arise in the field of aeronautics. For instance, let us consider a steering gear structure. In a first approach, we can model this structure by two rigid solids. Solid S_1 is tied to the fixed frame by a pivoting link O and solid S_2 is tied to solid S_1 by a pivoting link P. The whole model is represented in Fig. 1a. Note that S_1 can be thought of as the aerofoil of a wing and S_2 as a steering gear such as an aileron. For a given S_{ref}, the function $\mathbf{X}^a$ representing the motion of this structure with respect to (θ_1, θ_2) is given below

$$\mathbf{X}^a(\theta_1, \theta_2, \mathbf{y}) = \chi_{S_1}(\mathbf{y}) R_{\theta_1} \mathbf{y} + \chi_{S_2}(\mathbf{y})(R_{\theta_1} \mathbf{y}_P^{\text{ref}} + R_{\theta_1+\theta_2}(\mathbf{y} - \mathbf{y}_P^{\text{ref}})), \quad \forall \mathbf{y} \in S_{\text{ref}}, \quad \forall (\theta_1, \theta_2) \in \mathbb{D}_\Theta,$$

where $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the rotation matrix of angle θ, $\mathbf{y}_P^{\text{ref}} = (y_{P,1}, y_{P,2})^T$ is the coordinate of point P in the reference configuration S_{ref} and χ_E is the characteristic function over a set $E \subset \mathbb{R}^2$ given below

$$\forall \mathbf{y} \in \Omega, \quad \chi_E(\mathbf{y}) = \begin{cases} 1 & \text{if } \mathbf{y} \in E, \\ 0 & \text{else.} \end{cases} \tag{1.1}$$

In the previous example, the domain of definition $\mathbb{D}_\Theta$ of (θ_1, θ_2) is chosen such that no overlaps of the structure occur.

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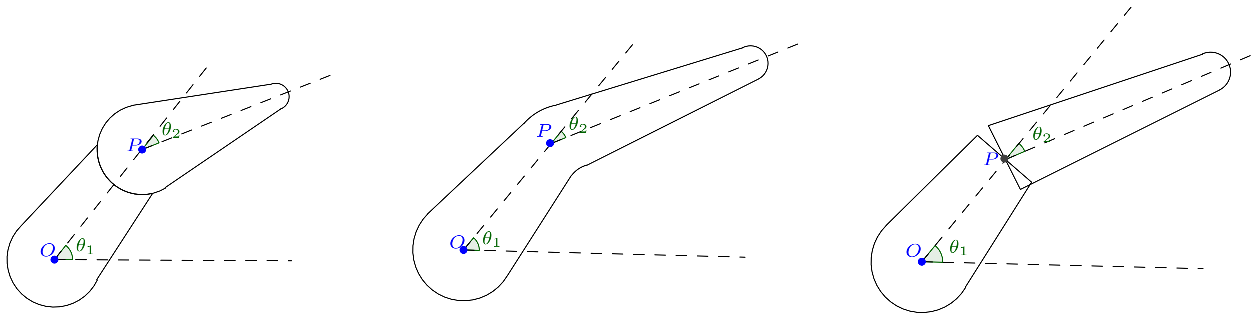

(a) A pivoting link P

(b) A structure continuously deforming

(c) A limit configuration

Figure 1: Three different kinds of structure deformation.

Note that for $\theta_2 \neq 0$, the function $\mathbf{X}^a(\theta_1, \theta_2, .)$ is not a diffeomorphism as it is discontinuous through the interface $\partial S_1 \cap \partial S_2$ between the two solids. In the same way, for $\dot{\theta}_2 \neq 0$, the velocity field is discontinuous inside the structure (we denote $\dot{\theta}_2$ the time derivative of θ_2). In other words, if we keep S_1 at rest and rotate S_2 around P, a discontinuity of the velocity appears through the interface between the two solids. This discontinuity can reduce the regularity expected for the fluid velocity. Indeed, if we assume no–slip boundary conditions between the fluid and the structure and if at time t the trace of the velocity is discontinuous on $\partial S(\theta_1(t), \theta_2(t))$, then the best regularity in space for the velocity of the fluid is the Sobolev space $L^2(0, T; \mathbf{H}^1(\Omega \backslash S(\theta_1(t), \theta_2(t))))$ [1], while for strong solutions we usually expect the velocity in the Sobolev space $L^2(0, T; \mathbf{H}^2(\Omega \backslash S(\theta_1(t), \theta_2(t))))$ [1].

This loss of regularity would harm the estimates of the nonlinear terms (see Appendix A). That is why we consider a smooth approximation $\mathbf{X}^b$ of the deformation $\mathbf{X}^a$.

In the sequel, $\mathbf{y} = (y_1, y_2)$ is the Lagrangian coordinate and $\mathbf{y}^\perp = (-y_2, y_1)$ is normal to $\mathbf{y}$. The behaviour of the smooth structure is represented in Fig. 1b, we give $\mathbf{X}^b$ below

$$\mathbf{X}^b(\theta_1, \theta_2, \mathbf{y}) = g_1(y_1)\mathbf{e}_{r1} + g_2(y_1)\mathbf{e}_{r2} + y_2 \frac{\mathbf{N}(y_1)}{\|\mathbf{N}(y_1)\|}, \quad \mathbf{y} \in S_{\text{ref}}, \quad (\theta_1, \theta_2) \in \mathbb{D}_\Theta, \tag{1.2}$$

where g_1 and g_2 are real–valued functions. We use the notations: $\mathbf{e}_{r1} = (\cos\theta_1, \sin\theta_1)^T$, $\mathbf{e}_{r2} = (\cos(\theta_1 + \theta_2), \sin(\theta_1 + \theta_2))^T$, $\mathbf{N}(y_1) = g_1'(y_1)\mathbf{e}_{\theta 1} + g_2'(y_1)\mathbf{e}_{\theta 2}$, where $\mathbf{e}_{\theta 1} = \mathbf{e}_{r1}^\perp$ and $\mathbf{e}_{\theta 2} = \mathbf{e}_{r2}^\perp$. Moreover, we have $\|\mathbf{N}(y_1)\| = ((N_1(y_1))^2 + (N_2(y_1))^2)^{1/2}$, where N_i is the i^{th} coordinate of $\mathbf{N}$.

The function $y_1 \mapsto g_1(y_1)\mathbf{e}_{r1} + g_2(y_1)\mathbf{e}_{r2}$ gives for $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$ the position of a reference curve. Every fibre of matter that is normal to this curve in the reference configuration stays normal when (θ_1, θ_2) changes. The normal direction to the curve for abscissa y_1 is given by $\mathbf{N}(y_1)$. This model is inspired from the fish–like model described in [17, Section 7].

To enforce smoothness of $\mathbf{X}^b$, g_1 and g_2 are taken as $\mathscr{C}^\infty$ functions which are smooth approximations of respectively $y_{P,1} + (y_1 - y_{P,1})\chi_{[0, y_{P,1}]}(y_1)$ and $(y_1 - y_{P,1})\chi_{[y_{P,1}, y_{\max}]}(y_1)$, where χ_I is defined in a similar way as (1.1) for $I \subset \mathbb{R}$. For instance, let $\varepsilon > 0$ and consider μ_ε a $\mathscr{C}^\infty$ cut–off function such that

$$\begin{cases} \mu_\varepsilon(y_1) = 1, & \text{for } y_1 < y_{P,1}, \\ \mu_\varepsilon(y_1) \in [0, 1], & \text{for } y_{P,1} \leq y_1 \leq y_{P,1} + \varepsilon, \\ \mu_\varepsilon(y_1) = 0, & \text{for } y_{P,1} + \varepsilon < y_1. \end{cases}$$

Then, we can use

$$\begin{cases} g_1(y_1) = y_{P,1} + \mu_\varepsilon(y_1)(y_1 - y_{P,1}), \\ g_2(y_1) = (1 - \mu_\varepsilon(y_1))(y_1 - y_{P,1}), \end{cases}$$

in (1.2) to get a smooth deformation as in Fig. 1b. The velocity field of the structure is not any more discontinuous, we can thus expect the fluid to have the usual regularity of strong solutions.

Remark 1.1. When ε tends to 0, these functions become

$$\begin{cases} g_1(y_1) = \chi_{[a,b[}(y_1)y_1 + \chi_{[b,c]}(y_1)y_{P,1}, \\ g_2(y_1) = \chi_{[b,c]}(y_1)(y_1 - y_{P,2}). \end{cases} \tag{1.3}$$

In this case, we recover the behaviour of a pivoting structure with two rigid solids (see Fig. 1c), corresponding to a transformation denoted $\mathbf{X}^c$. However the two solids overlap with this definition, so that we will not use it either in the sequel. Also let us remark that the limit $\mathbf{X}^c$ of our smooth approximation $\mathbf{X}^b$ is not the original model $\mathbf{X}^a$.

[1]These spaces are given here in an informal manner. They will be defined more precisely later.

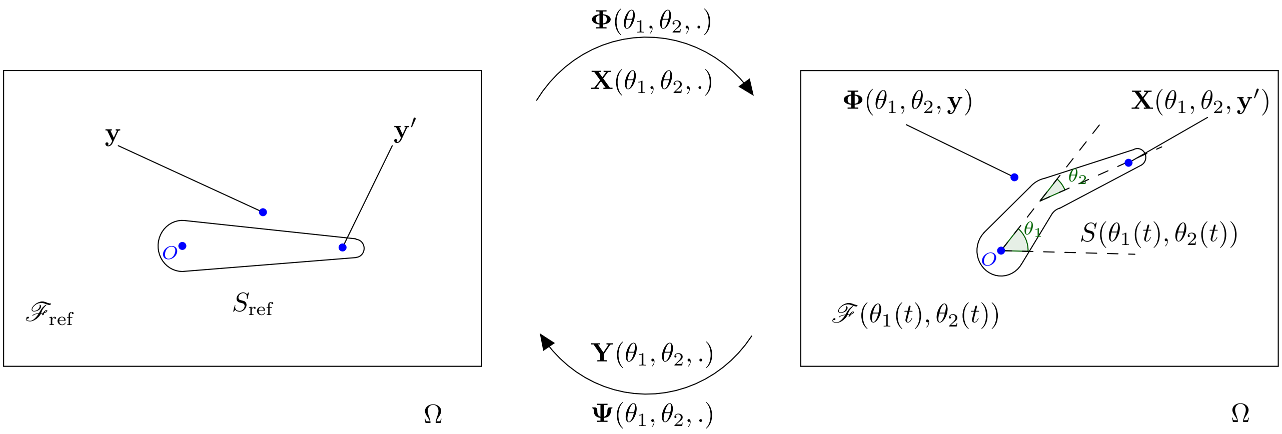

Figure 2: Correspondance between real and reference structure configurations.

Now, let us show that we can choose $\mathbb{D}_\Theta$ such that for every $(\theta_1,\theta_2)\in\mathbb{D}_\Theta, \mathbf{X}^b(\theta_1,\theta_2,.)$ is a $\mathscr{C}^\infty$ diffeomorphism. We can compute the jacobian $\mathcal{J}_{\mathbf{X}^b}(\theta_1,\theta_2,.)$ of $\mathbf{X}^b(\theta_1,\theta_2,.)$, it fulfils $\det(\mathcal{J}_{\mathbf{X}^b}(\theta_1,\theta_2,.)) = \|\mathbf{N}\| + \dfrac{y_2}{\|\mathbf{N}\|^2}\sin(\theta_2)(g_1''g_2' - g_2''g_1')$. This shows that for a given reference configuration and for θ_2 small enough, $\det(\mathcal{J}_{\mathbf{X}^b}(\theta_1,\theta_2,.)) > 0$ everywhere. Hence this proves that $\mathbf{X}^b(\theta_1,\theta_2,.)$ is a $\mathscr{C}^\infty$ diffeomorphism for θ_2 small enough.

We shall therefore keep in mind only the example of $\mathbf{X}^b$ (see Fig. 1b) though our original motivation was to deal with $\mathbf{X}^a$ (see Fig. 1a). More generally, our approach will be applicable to many more choices of deformations $\mathbf{X}$. Let us list below the assumptions used in the sequel.

Modelling Assumptions.

- $S_{\rm ref}$ is a smooth simply connected closed subset of Ω. (1.4)
- For every $(\theta_1,\theta_2)\in\mathbb{D}_\Theta$, we have $\mathbf{X}(\theta_1,\theta_2,S_{\rm ref})\subset\Omega$. (1.5)
- For every $(\theta_1,\theta_2)\in\mathbb{D}_\Theta$, $\mathbf{X}(\theta_1,\theta_2,.)$ is a smooth diffeomorphism from $S_{\rm ref}$ to its image. (1.6)
- The function $\mathbf{X}$ is $\mathscr{C}^\infty$ on $\mathbb{D}_\Theta\times S_{\rm ref}$. (1.7)
- The modes $\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,.)$ and $\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,.)$ form a free family in $\mathbf{L}^2(\partial S_{\rm ref})$ for every (θ_1,θ_2) in $\mathbb{D}_\Theta$. (1.8)

Assumption (1.6) enables us to use a change of variables. This is a crucial step in our approach, as we shall see in Section 3.1. Assumption (1.7) has been chosen to ensure continuity of the velocity field inside the structure and on its boundary. This assumption could be weakened, as $\mathscr{C}^n$ would be sufficient for n large enough, but we keep $\mathscr{C}^\infty$ for simplicity. In our approach, Assumption (1.8) is natural and mandatory to determine the equations of the structure, as we shall see below in Section 1.1.2.

The inverse diffeomorphism of $\mathbf{X}(\theta_1,\theta_2,.)$ whose existence is guaranteed by (1.6) is denoted $\mathbf{Y}(\theta_1,\theta_2,.)$, we have

$$\forall(\theta_1,\theta_2)\in\mathbb{D}_\Theta,\quad \forall\mathbf{y}\in S_{\rm ref},\quad \mathbf{Y}(\theta_1,\theta_2,\mathbf{X}(\theta_1,\theta_2,\mathbf{y})) = \mathbf{y}. \tag{1.9}$$

The diffeomorphisms $\mathbf{X}(\theta_1,\theta_2,.)$ and $\mathbf{Y}(\theta_1,\theta_2,.)$ are illustrated in Fig. 2.

1.1.2 Dynamics of the structure

In order to simplify the equations of the structure, we consider the following assumption for the dynamics of the structure.

Modelling Assumption.

- No friction and no elastic energy are considered in the structure. (1.10)

The equations satisfied by the structure are obtained by a virtual work principle [2, p. 14–17]. We know that the admissible parameters of the structure are $(\theta_1,\theta_2)\in\mathbb{D}_\Theta$, and that the admissible velocities satisfy

$$\mathbf{v}_s \in \mathrm{Vect}(\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,.), \partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,.)).$$

Thus, the virtual work principle can be formulated at every time $t \in [0,T]$ as

$$\left\{\begin{array}{l} \text{Find } (\theta_1(t),\theta_2(t)) \in \mathbb{D}_\Theta, \text{ such that for every } \mathbf{w} \in \mathrm{Vect}(\partial_{\theta_1}\mathbf{X}(\theta_1(t),\theta_2(t),.), \partial_{\theta_2}\mathbf{X}(\theta_1(t),\theta_2(t),.)), \\ \displaystyle\int_{S_{\mathrm{ref}}} \rho\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{y})) - \mathbf{f}_{body}(t,\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{y}))\right)\cdot \mathbf{w}(\mathbf{y})\mathrm{d}\mathbf{y} \\ \qquad\qquad \displaystyle - \int_{\partial S(\theta_1(t),\theta_2(t))} \mathbf{f}_{\mathcal{F}\to S}(\gamma_x)\cdot \mathbf{w}(\mathbf{Y}(\theta_1(t),\theta_2(t),\gamma_x))\mathrm{d}\gamma_x = 0, \end{array}\right. \tag{1.11}$$

where $\mathbf{f}_{body}$ is a distributed source term in the body (modelling for instance the gravity), ρ is a positive constant that represents the mass per unit volume of the structure in the reference configuration S_{ref} and $\mathbf{f}_{\mathcal{F}\to S}$ is the force exerted by the fluid on the structure along $\partial S(\theta_1(t),\theta_2(t))$.

Note that the presence of $\mathbf{f}_{body}$ is compatible with Assumption (1.10), as this term represents external forces. It does not depend on θ_1, θ_2 and their derivatives.

Remark 1.2. Assumption (1.10) has been used in (1.11) as no interior works have been considered.

Let us denote respectively $\dot{\theta}$ and $\ddot{\theta}$ the first and second time derivatives of the function θ. Then, the velocity of the structure can be written as

$$\mathbf{v}_s(t,\mathbf{y}) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{y}) = \sum_{j=1}^{2}\dot{\theta}_j(t)\partial_{\theta_j}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{y}), \quad \forall t\in[0,T], \quad \forall \mathbf{y}\in S_{\mathrm{ref}}, \tag{1.12}$$

and its acceleration as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_s(t,\mathbf{y}) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{y})) = \sum_{j=1}^{2}\ddot{\theta}_j(t)\partial_{\theta_j}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{y}) + \sum_{j,k=1}^{2}\dot{\theta}_j(t)\dot{\theta}_k(t)\partial_{\theta_j\theta_k}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{y}).$$

Now, problem (1.11) can be rewritten as follows

$$\left\{\begin{array}{l} \text{Find } (\theta_1,\theta_2)\in\mathbb{D}_\Theta, \text{ such that for } i\in\{1,2\}, \text{ we have,} \\ \displaystyle\int_{S_{\mathrm{ref}}}\rho\sum_j\ddot{\theta}_j\partial_{\theta_j}\mathbf{X}(\theta_1,\theta_2,\mathbf{y})\cdot\partial_{\theta_i}\mathbf{X}(\theta_1,\theta_2,\mathbf{y})\mathrm{d}\mathbf{y} = -\int_{S_{\mathrm{ref}}}\rho\sum_{j,k}\dot{\theta}_j\dot{\theta}_k\partial_{\theta_j\theta_k}\mathbf{X}(\theta_1,\theta_2,\mathbf{y})\cdot\partial_{\theta_i}\mathbf{X}(\theta_1,\theta_2,\mathbf{y})\mathrm{d}\mathbf{y} \\ \qquad\qquad\displaystyle + \int_{S_{\mathrm{ref}}}\mathbf{f}_{body}(t,\mathbf{X}(\theta_1,\theta_2,\mathbf{y}))\cdot\partial_{\theta_i}\mathbf{X}(\theta_1,\theta_2,\mathbf{y})\mathrm{d}\mathbf{y} \\ \qquad\qquad\displaystyle + \int_{\partial S(\theta_1,\theta_2)}\mathbf{f}_{\mathcal{F}\to S}(\gamma_x)\cdot\partial_{\theta_i}\mathbf{X}(\theta_1,\theta_2,\mathbf{Y}(\theta_1,\theta_2,\gamma_x))\mathrm{d}\gamma_x. \end{array}\right.$$

Let us denote the structure body source term

$$(f_s)_i = \int_{S_{\mathrm{ref}}}\mathbf{f}_{body}(t,\mathbf{X}(\theta_1,\theta_2,\mathbf{y}))\cdot\partial_{\theta_i}\mathbf{X}(\theta_1,\theta_2,\mathbf{y})\mathrm{d}\mathbf{y}. \tag{1.13}$$

On a matrix form, the equations of the structure read

$$\mathcal{M}_{\theta_1,\theta_2}\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} = \mathbf{M_I}(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2) + \mathbf{M_A}(\theta_1,\theta_2,\mathbf{f}_{\mathcal{F}\to S}) + \mathbf{f_s} \quad \text{on } (0,T), \tag{1.14}$$

where $\mathbf{f_s} = ((f_s)_1,(f_s)_2)^T$ and

$$\mathcal{M}_{\theta_1,\theta_2} = \begin{pmatrix}(\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2),\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2))_S & (\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2),\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2))_S\\ (\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2),\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2))_S & (\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2),\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2))_S\end{pmatrix}\in\mathbb{R}^{2\times2}, \tag{1.15}$$

$$\mathbf{M_I}(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2) = \begin{pmatrix}-(\dot{\theta}_1^2\partial_{\theta_1\theta_1}\mathbf{X}(\theta_1,\theta_2)+2\dot{\theta}_1\dot{\theta}_2\partial_{\theta_1\theta_2}\mathbf{X}(\theta_1,\theta_2)+\dot{\theta}_2^2\partial_{\theta_2\theta_2}\mathbf{X}(\theta_1,\theta_2),\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2))_S\\ -(\dot{\theta}_1^2\partial_{\theta_1\theta_1}\mathbf{X}(\theta_1,\theta_2)+2\dot{\theta}_1\dot{\theta}_2\partial_{\theta_1\theta_2}\mathbf{X}(\theta_1,\theta_2)+\dot{\theta}_2^2\partial_{\theta_2\theta_2}\mathbf{X}(\theta_1,\theta_2),\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2))_S\end{pmatrix}\in\mathbb{R}^2, \tag{1.16}$$

$$\mathbf{M_A}(\theta_1,\theta_2,\mathbf{f}_{\mathcal{F}\to S}) = \begin{pmatrix}\displaystyle\int_{\partial S(\theta_1,\theta_2)}\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,\mathbf{Y}(\theta_1,\theta_2,\gamma_x))\cdot\mathbf{f}_{\mathcal{F}\to S}(\gamma_x)\mathrm{d}\gamma_x\\ \displaystyle\int_{\partial S(\theta_1,\theta_2)}\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,\mathbf{Y}(\theta_1,\theta_2,\gamma_x))\cdot\mathbf{f}_{\mathcal{F}\to S}(\gamma_x)\mathrm{d}\gamma_x\end{pmatrix}\in\mathbb{R}^2, \tag{1.17}$$

where $(.,.)_S$ is the scalar product

$$(\mathbf{\Phi},\mathbf{\Psi})_S = \int_{S_{\mathrm{ref}}}\rho\mathbf{\Phi}(\mathbf{y})\cdot\mathbf{\Psi}(\mathbf{y})\mathrm{d}\mathbf{y}. \tag{1.18}$$

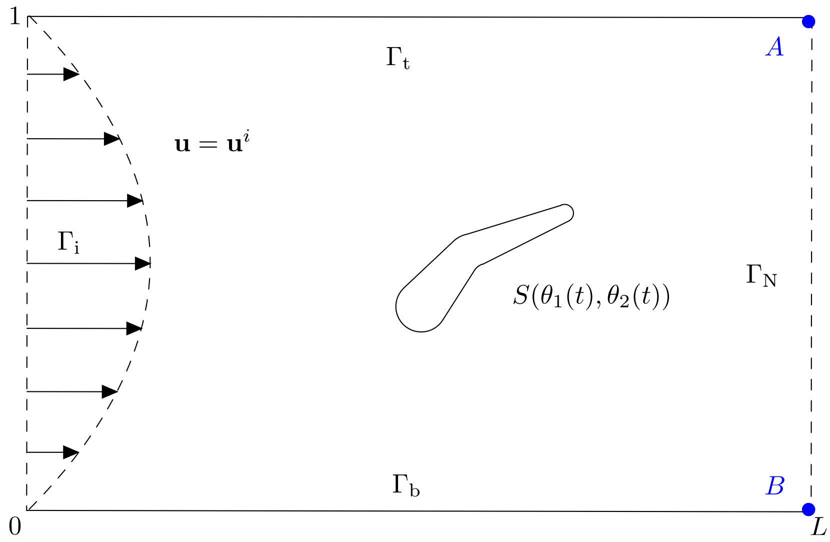

Figure 3: The geometrical configuration.

The matrix $\mathcal{M}_{\theta_1,\theta_2}$ in (1.15) is the Gram matrix of the family $(\partial_{\theta_1}\mathbf{X}(\theta_1, \theta_2), \partial_{\theta_2}\mathbf{X}(\theta_1, \theta_2))$ with respect to the scalar product $(.,.)_S$. It is invertible due to Assumption (1.8) (if two $\mathscr{C}^\infty$ functions are colinear in $\mathbf{L}^2(S_{\mathrm{ref}})$ then they are colinear in $\mathbf{L}^2(\partial S_{\mathrm{ref}})$).

We also consider the following initial displacement and velocity for the structure

$$\begin{cases} \theta_1(0) = \theta_{1,0}, & \theta_2(0) = \theta_{2,0}, \\ \dot{\theta}_1(0) = \omega_{1,0}, & \dot{\theta}_2(0) = \omega_{2,0}. \end{cases} \tag{1.19}$$

1.1.3 Equations of the fluid

In our study, the global domain $\Omega = (0, L) \times (0, 1)$ represents a wind tunnel of length $L > 0$, see Fig. 3. Hence its boundary is composed of four regions: an inflow region $\Gamma_{\mathrm{i}} = \{0\} \times (0, 1)$, a bottom region $\Gamma_{\mathrm{b}} = (0, L) \times \{0\}$, a top region $\Gamma_{\mathrm{t}} = (0, L) \times \{1\}$ and an outflow region $\Gamma_{\mathrm{N}} = \{L\} \times (0, 1)$. We denote $\Gamma_{\mathrm{w}} = \Gamma_{\mathrm{t}} \cup \Gamma_{\mathrm{b}}$ the part of the boundary corresponding to walls and $\Gamma_{\mathrm{D}} = \Gamma_{\mathrm{i}} \cup \Gamma_{\mathrm{w}}$ the part of the boundary where Dirichlet conditions are applied.

At time t, the structure occupies the volume $S(\theta_1(t), \theta_2(t))$, therefore the fluid fills the domain $\mathscr{F}(\theta_1(t), \theta_2(t)) = \Omega \backslash S(\theta_1(t), \theta_2(t))$.

The velocity of the fluid is modelled by the incompressible Navier–Stokes equations

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t}(t, \mathbf{x}) + (\mathbf{u} \cdot \nabla)\mathbf{u}(t, \mathbf{x}) - \operatorname{div} \sigma_F(\mathbf{u}(t, \mathbf{x}), p(t, \mathbf{x})) = \mathbf{f}_{\mathscr{F}}(t, \mathbf{x}), & t \in (0, T), \quad \mathbf{x} \in \mathscr{F}(\theta_1(t), \theta_2(t)), \\ \operatorname{div} \mathbf{u}(t, \mathbf{x}) = 0, & t \in (0, T), \quad \mathbf{x} \in \mathscr{F}(\theta_1(t), \theta_2(t)), \\ \mathbf{u}(t, \mathbf{x}) = \mathbf{u}^i(t, \mathbf{x}), & t \in (0, T), \quad \mathbf{x} \in \Gamma_{\mathrm{i}}, \\ \mathbf{u}(t, \mathbf{x}) = 0, & t \in (0, T), \quad \mathbf{x} \in \Gamma_{\mathrm{w}}, \\ \sigma_F(\mathbf{u}(t, \mathbf{x}), p(t, \mathbf{x}))\mathbf{n}(\mathbf{x}) = 0, & t \in (0, T), \quad \mathbf{x} \in \Gamma_{\mathrm{N}}, \\ \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), & \qquad\qquad\quad \mathbf{x} \in \mathscr{F}(\theta_{1,0}, \theta_{2,0}), \end{cases} \tag{1.20}$$

where $\mathbf{u}(t, \mathbf{x})$ and $p(t, \mathbf{x})$ are velocity and pressure of the fluid at point $\mathbf{x}$ and time t, and

$$\sigma_F(\mathbf{u}, p) = \nu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) - pI,$$

is the stress tensor of the fluid, where $\nu > 0$ is the kinematic viscosity of the fluid. The term $\mathbf{f}_{\mathscr{F}}(t, \mathbf{x})$ in $(1.20)_1$ is a force per unit mass exerted on the fluid. Moreover, a nonhomogeneous Dirichlet boundary condition $\mathbf{u}^i$ is imposed on the inflow region Γ_{i} and we consider an initial condition $\mathbf{u}_0$ for the fluid velocity. Of course, these equations should be completed with suitable boundary conditions on $\partial S(\theta_1(t), \theta_2(t))$ that will be made precise in Section 1.1.4.

1.1.4 Interface between the fluid and the structure

The velocity $\mathbf{u}$ of the fluid fulfils an adherence condition with the boundary of the structure whose velocity is given in (1.12),

$$\mathbf{u}(t, \mathbf{X}(\theta_1(t), \theta_2(t), \mathbf{y})) = \sum_{j=1}^{2} \dot{\theta}_j(t) \partial_{\theta_j} \mathbf{X}(\theta_1(t), \theta_2(t), \mathbf{y}), \quad t \in (0, T), \quad \mathbf{y} \in \partial S_{\mathrm{ref}}.$$

Note that this no–slip boundary condition corresponds to the continuity of the velocity through the interface between the fluid and the structure and can also be rewritten as

$$\mathbf{u}(t,\mathbf{x}) = \sum_{j=1}^{2} \dot\theta_j(t)\partial_{\theta_j}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{Y}(\theta_1(t),\theta_2(t),\mathbf{x})), \quad t\in(0,T), \quad \mathbf{x}\in\partial S(\theta_1(t),\theta_2(t)). \tag{1.21}$$

The forces exerted by the fluid on the structure are given by the stress tensor of the fluid

$$\mathbf{f}_{\mathcal{F}\to S} = -\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}, \quad t\in(0,T), \quad \mathbf{x}\in\partial S(\theta_1(t),\theta_2(t)), \tag{1.22}$$

where $\mathbf{n}_{\theta_1,\theta_2}(\mathbf{x})$ is the outward unitary normal to the fluid domain $\mathscr{F}(\theta_1(t),\theta_2(t))$.

1.1.5 The complete set of equations

The full set of equations is given by (1.14), (1.19), (1.20), (1.21) and (1.22). Note that the coupling between fluid and structure appears in equations (1.20) (as the fluid domain depends on θ_1 and θ_2), (1.21) and (1.22).

The final system considered is given by the following set of equations

$$\left\{\begin{array}{lll}
\dfrac{\partial \mathbf{u}}{\partial t}(t,\mathbf{x}) + (\mathbf{u}(t,\mathbf{x})\cdot\nabla)\mathbf{u}(t,\mathbf{x}) - \operatorname{div}\sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x})) = \mathbf{f}_{\mathscr{F}}(t,\mathbf{x}), & t\in(0,T), & \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)),\\
\operatorname{div}\mathbf{u}(t,\mathbf{x}) = 0, & t\in(0,T), & \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)),\\
\mathbf{u}(t,\mathbf{x}) = \displaystyle\sum_{j=1}^{2}\dot\theta_j(t)\partial_{\theta_j}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{Y}(\theta_1(t),\theta_2(t),\mathbf{x})), & t\in(0,T), & \mathbf{x}\in\partial S(\theta_1(t),\theta_2(t)),\\
\mathbf{u}(t,\mathbf{x}) = \mathbf{u}^i(t,\mathbf{x}), & t\in(0,T), & \mathbf{x}\in\Gamma_{\mathrm{i}},\\
\mathbf{u}(t,\mathbf{x}) = 0, & t\in(0,T), & \mathbf{x}\in\Gamma_{\mathrm{w}},\\
\sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x}))\mathbf{n}(\mathbf{x}) = 0, & t\in(0,T), & \mathbf{x}\in\Gamma_{\mathrm{N}},\\
\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), & & \mathbf{x}\in\mathscr{F}(\theta_{1,0},\theta_{2,0}),\\
\mathcal{M}_{\theta_1,\theta_2}\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix} = \mathbf{M_I}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2) + \mathbf{M_A}(\theta_1,\theta_2,-\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}) + \mathbf{f_s}, & t\in(0,T), & \\
\theta_1(0) = \theta_{1,0}, \quad \theta_2(0) = \theta_{2,0}, & & \\
\dot\theta_1(0) = \omega_{1,0}, \quad \dot\theta_2(0) = \omega_{2,0}. & &
\end{array}\right. \tag{1.23}$$

1.2 Statement of the main result

In this section, after setting up the functional framework, we present our existence result for solutions to problem (1.23). In the sequel, $\mathscr{F}_0 = \mathscr{F}(\theta_{1,0},\theta_{2,0})$ denotes the initial fluid domain and $S_0 = S(\theta_{1,0},\theta_{2,0})$ the initial configuration for the structure. For the sake of simplicity, the initial displacement of the structure is taken equal to zero,

$$\theta_{1,0} = \theta_{2,0} = 0.$$

This can be done without loss of generality by the change of variables

$$(\theta_1,\theta_2) \mapsto (\theta_1-\theta_{1,0},\theta_2-\theta_{2,0}).$$

Moreover, the reference configuration for the structure S_{ref} and for the fluid $\mathscr{F}_{\mathrm{ref}}$ are taken as the initial configuration,

$$S_{\mathrm{ref}} = S_0 = S(0,0),\ \mathscr{F}_{\mathrm{ref}} = \mathscr{F}_0 = \mathscr{F}(0,0).$$

1.2.1 Functional spaces

Sobolev spaces. In the sequel, $H^s(\mathscr{F}_0)$ is the usual Sobolev space of order $s\geq 0$. We identify $L^2(\mathscr{F}_0)$ with $H^0(\mathscr{F}_0)$. We will denote $\mathbf{L}^2(\mathscr{F}_0) = (L^2(\mathscr{F}_0))^2$, $\mathbf{H}^s(\mathscr{F}_0) = (H^s(\mathscr{F}_0))^2$ and so on.

Corners issues. The domain considered for the fluid has four corners of angle $\pi/2$. The ones that are located between Dirichlet and Neumann boundary conditions induce singularities, we denote them $A=(L,1)$ and $B=(L,0)$ (see Fig. 3). We also denote $\mathscr{J}_{d,n} = \{A,B\}$ the set of these corners and we define the distance of a point $\mathbf{x}$ from these corners

$$\text{for } j\in\mathscr{J}_{d,n}, \quad \text{for } \mathbf{x}\in\Omega, \quad r_j(\mathbf{x}) = \mathrm{d}(\mathbf{x},j). \tag{1.24}$$

Note that corners between two Dirichlet boundary conditions do not induce singularities as soon as suitable compatibility conditions are satisfied. We report to [11] for more details.

Weighted Sobolev spaces. The strong solution to the Stokes problem in the domain with corners A and B and with a source term in $\mathbf{L}^2(\mathscr{F}_0)$ belongs to a classical Sobolev space of lower order than what we usually have in smooth domains. In order to get the usual gain of regularity between solutions and source terms, we have to study the solution in adapted Sobolev spaces. As the loss of regularity is located around corners A and B, we can recover the usual regularity if we consider norms that are suitably weighted near these corners. The weighted Sobolev spaces are then defined for $\beta > 0$ as

$$\mathbf{H}^2_\beta(\mathscr{F}_0) = \{\mathbf{u} : \|\mathbf{u}\|_{\mathbf{H}^2_\beta(\mathscr{F}_0)} < +\infty\},$$
$$H^1_\beta(\mathscr{F}_0) = \{p : \|p\|_{H^1_\beta(\mathscr{F}_0)} < +\infty\},$$

where the norms $\|.\|_{\mathbf{H}^2_\beta(\mathscr{F}_0)}$ and $\|.\|_{H^1_\beta(\mathscr{F}_0)}$ are given by

$$\|\mathbf{u}\|^2_{\mathbf{H}^2_\beta(\mathscr{F}_0)} = \sum_{|\alpha|=0}^{2}\sum_{i=1}^{2}\int_{\mathscr{F}_0}\left(\prod_{j\in\mathscr{J}_{d,n}} r_j^{2\beta}(\mathbf{y})\right)|\partial^\alpha u_i(\mathbf{y})|^2 \mathrm{d}\mathbf{y}, \tag{1.25}$$

and

$$\|p\|^2_{H^1_\beta(\mathscr{F}_0)} = \sum_{|\alpha|=0}^{1}\int_{\mathscr{F}_0}\left(\prod_{j\in\mathscr{J}_{d,n}} r_j^{2\beta}(\mathbf{y})\right)|\partial^\alpha p(\mathbf{y})|^2 \mathrm{d}\mathbf{y}. \tag{1.26}$$

Here the sum is on all multi–index α of length $|\alpha| \leq 2$ for (1.25) and $|\alpha| \leq 1$ for (1.26) and r_j is defined in (1.24).

Steady Stokes problem with corners. Let us denote $\mathbf{n}_0$ the outward unitary normal to $\mathscr{F}_0$. The following lemma from [13] explains how and why the spaces $\mathbf{H}^2_\beta$ and H^1_β appear in the context of corners. It gives the result expected for the steady Stokes problem in $\mathscr{F}_0$ with weigthed Sobolev spaces and the regularity obtained in the classical Sobolev spaces.

Lemma 1.3. *[13, Theorem 2.5.] Let us assume that $\mathbf{f}_\mathscr{F} \in \mathbf{L}^2(\mathscr{F}_0)$. The unique solution $(\mathbf{u}, p)$ to the Stokes problem*

$$\begin{cases} -\mathrm{div}\ \sigma_F(\mathbf{u}, p) = \mathbf{f}_\mathscr{F} & \text{in } \mathscr{F}_0,\\ \mathrm{div}\ \mathbf{u} = 0 & \text{in } \mathscr{F}_0,\\ \mathbf{u} = 0 & \text{on } \Gamma_\mathrm{D} \cup \partial S_0,\\ \sigma_F(\mathbf{u}, p)\mathbf{n}_0 = 0 & \text{on } \Gamma_\mathrm{N}, \end{cases} \tag{1.27}$$

belongs to $\mathbf{H}^2_\beta(\mathscr{F}_0) \times H^1_\beta(\mathscr{F}_0)$ for $\beta = 0.42$ and to $\mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_0) \times H^{1/2+\varepsilon_0}(\mathscr{F}_0)$ for $\varepsilon_0 = 0.08$. Moreover, we have the following estimate

$$\|\mathbf{u}\|_{\mathbf{H}^2_\beta(\mathscr{F}_0)\cap\mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_0)} + \|p\|_{H^1_\beta(\mathscr{F}_0)\cap H^{1/2+\varepsilon_0}(\mathscr{F}_0)} \leq C\|\mathbf{f}_\mathscr{F}\|_{\mathbf{L}^2(\mathscr{F}_0)}. \tag{1.28}$$

Remark 1.4. In the sequel, we only use the fact that β and ε_0 belong to $(0, 1/2)$.

Note that the regularity proven in Lemma 1.3 gives a meaning to all integrations by parts as $p_{|\partial\mathscr{F}_0}$ and $\partial_{\mathbf{n}_0}\mathbf{u}_{|\partial\mathscr{F}_0}$ are well defined traces for $(\mathbf{u}, p) \in \mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_0) \times H^{1/2+\varepsilon_0}(\mathscr{F}_0)$.

Also note that according to [8, Theorem 1.4.3.1], there exists a continuous extension operator from $\mathbf{H}^s(\mathscr{F}_0)$ to $\mathbf{H}^s(\mathbb{R}^2)$ for every $s > 0$. This implies that all the classical Sobolev injections and interpolations are valid despite the presence of corners as they can be led in $\mathbb{R}^2$.

1.2.2 Local existence of a strong solution to the problem

The diffeomorphism $\boldsymbol{\Phi}$. A classical difficulty in fluid–structure problems is that the fluid domain changes over time. The classical way to get rid of this difficulty is to use a change of variables on $\mathbf{u}$ and p in order to bring the study back into a fixed domain. This procedure uses a diffeomorphism that we have to define properly. When the state of the structure depends only on a finite number of parameters, it is convenient to construct this diffeomorphism as an extension of the deformation of the structure into the fluid domain.

The diffeomorphism used is defined as an extension of the diffeomorphism $\mathbf{X}$ given for the structure. Hence, we need the extension operator defined below.

Lemma 1.5. *There exists a linear extension operator $\mathcal{E} : \mathbf{W}^{3,\infty}(S_\mathrm{ref}) \to \mathbf{W}^{3,\infty}(\Omega) \cap \mathbf{H}^1_0(\Omega)$ such that for every $\mathbf{w} \in \mathbf{W}^{3,\infty}(S_\mathrm{ref})$*

(*i*) $\mathcal{E}(\mathbf{w}) = \mathbf{w}$ almost everywhere in S_ref,
(*ii*) $\mathcal{E}(\mathbf{w})$ has support within $\Omega_\varepsilon = \{\mathbf{x} \in \Omega \mid \mathrm{d}(x, \partial\Omega) > \varepsilon\}$ for some $\varepsilon > 0$ such that $\mathrm{d}(S(\theta_1, \theta_2), \partial\Omega) > 2\varepsilon$ for all $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$,
(*iii*) $\|\mathbf{w}\|_{\mathbf{W}^{3,\infty}(\Omega)} \leq C\|\mathbf{w}\|_{\mathbf{W}^{3,\infty}(S_\mathrm{ref})}$.

Proof. Extension results are classical, we can for instance find an extension result for smooth domains in [10, Lemma 12.2]. We can get the result by multiplying the extension function of [10, Lemma 12.2] by a cut–off function in $\mathcal{D}(\Omega_\varepsilon)$. □

Then we define the following function

$$\mathbf{\Phi}(\theta_1,\theta_2,\mathbf{y}) = \mathbf{y} + \mathcal{E}\Big(\mathbf{X}(\theta_1,\theta_2,.) - \mathrm{Id}\Big)(\mathbf{y}), \quad \forall(\theta_1,\theta_2) \in \mathbb{D}_\Theta, \quad \forall \mathbf{y} \in \Omega, \tag{1.29}$$

where Id denotes the identity function.

We have $\nabla\mathbf{\Phi}(0,0,\mathbf{y}) = I$ for every $\mathbf{y} \in \Omega$, hence $\det(\nabla\mathbf{\Phi}(0,0,\mathbf{y})) = 1$. Then for every $(\theta_1,\theta_2) \in \mathbb{D}_\Theta$ small enough, the function $\mathbf{\Phi}(\theta_1,\theta_2,.)$ is a diffeomorphism close to the identity function. We denote $\mathbf{\Psi}(\theta_1,\theta_2,.)$ the inverse diffeomorphism of $\mathbf{\Phi}(\theta_1,\theta_2,.)$

$$\forall(\theta_1,\theta_2) \in \mathbb{D}_\Theta, \quad \forall\mathbf{y} \in \Omega, \quad \mathbf{\Psi}(\theta_1,\theta_2,\mathbf{\Phi}(\theta_1,\theta_2,\mathbf{y})) = \mathbf{y}. \tag{1.30}$$

We can prove that $\mathbf{\Phi}$ and $\mathbf{\Psi}$ belong to $\mathscr{C}^\infty(\mathbb{D}_\Theta, \mathbf{W}^{3,\infty}(\Omega))$. These diffeomorphisms are represented in Fig. 2.

The properties of $\mathcal{E}$ imply the following important properties

$$\text{for every } (\theta_1,\theta_2) \in \mathbb{D}_\Theta, \quad \mathbf{\Phi}(\theta_1,\theta_2,S_{\mathrm{ref}}) = S(\theta_1,\theta_2) \quad \text{and} \quad \forall\mathbf{y} \in \Omega\backslash\Omega_\varepsilon, \quad \mathbf{\Phi}(\theta_1,\theta_2,\mathbf{y}) = \mathbf{y}. \tag{1.31}$$

The inflow boundary conditions. We use the following space to define the admissible boundary conditions on the inflow part of the boundary Γ_i,

$$\mathbf{U}^i = \left\{ \begin{array}{lll} \mathbf{u}^i \in \mathbf{H}^{3/2}(\Gamma_\mathrm{i}) & | \quad \mathbf{u}^i_{|\partial\Gamma_\mathrm{i}} = 0, & \displaystyle\int_0^{1/4} \frac{|\partial_{y_2} u_2^i(y_2)|^2}{y_2} \mathrm{d}y_2 < +\infty, \\ & & \displaystyle\int_{3/4}^1 \frac{|\partial_{y_2} u_2^i(y_2)|^2}{1-y_2} \mathrm{d}y_2 < +\infty \end{array} \right\}. \tag{1.32}$$

The conditions with integrals in the definition of $\mathbf{U}^i$ are chosen to match the homogeneous boundary conditions on Γ_w. We now state the following existence theorem.

Theorem 1.6 (Local existence in time of a solution)**.** *Let $T_0 > 0$, let $\mathbf{u}^i \in H^1(0,T_0;\mathbf{U}^i)$, $\mathbf{u}_0 \in \mathbf{H}^1(\mathscr{F}_0)$ and $(\omega_{1,0},\omega_{2,0}) \in \mathbb{R}^2$ satisfying the compatibility conditions*

$$\begin{cases} \operatorname{div} \mathbf{u}_0 = 0 & \text{in } \mathscr{F}_0, \\ \mathbf{u}_0(.) = \displaystyle\sum_{j=1}^2 \omega_{j,0}\partial_{\theta_j}\mathbf{X}(0,0,.) & \text{on } \partial S_0, \\ \mathbf{u}_0 = \mathbf{u}^i(0,.) & \text{on } \Gamma_\mathrm{i}, \\ \mathbf{u}_0 = 0 & \text{on } \Gamma_\mathrm{w}. \end{cases} \tag{1.33}$$

Let $\mathbf{f}_\mathscr{F} \in L^2(0,T_0;\mathbf{W}^{1,\infty}(\Omega))$ and $\mathbf{f_s} \in L^2(0,T_0;\mathbb{R}^2)$. Then there exists a time $T \in (0,T_0)$ such that problem (1.23) *admits a unique solution $(\mathbf{u},p,\theta_1,\theta_2)$ with the following regularity*

$$\begin{aligned} &(\theta_1,\theta_2) \in H^2(0,T;\mathbb{D}_\Theta), \\ &\mathbf{u}(t,\mathbf{\Phi}(\theta_1(t),\theta_2(t),\mathbf{y})) \in L^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0)) \cap \mathscr{C}^0([0,T];\mathbf{H}^1(\mathscr{F}_0)) \cap H^1(0,T;\mathbf{L}^2(\mathscr{F}_0)), \\ &p(t,\mathbf{\Phi}(\theta_1(t),\theta_2(t),\mathbf{y})) \in L^2(0,T;H^1_\beta(\mathscr{F}_0)). \end{aligned}$$

Moreover, we have the estimate

$$\begin{aligned} &\|\mathbf{u}(t,\mathbf{\Phi}(\theta_1(t),\theta_2(t),\mathbf{y}))\|_{L^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0))\cap\mathscr{C}^0([0,T];\mathbf{H}^1(\mathscr{F}_0))\cap H^1(0,T;\mathbf{L}^2(\mathscr{F}_0))} \\ &\quad + \|p(t,\mathbf{\Phi}(\theta_1(t),\theta_2(t),\mathbf{y}))\|_{L^2(0,T;H^1_\beta(\mathscr{F}_0))} + \|(\theta_1,\theta_2)\|_{H^2(0,T;\mathbb{D}_\Theta)} \\ &\qquad \leq C(\|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)} + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}_\mathscr{F}\|_{L^2(0,T_0;\mathbf{L}^2(\mathscr{F}_0))} + \|\mathbf{u}^i\|_{H^1(0,T_0;\mathbf{H}^{3/2}(\Gamma_\mathrm{i}))} + \|\mathbf{f_s}\|_{L^2(0,T_0;\mathbb{R}^2)}). \end{aligned}$$

The proof of Theorem 1.6 mainly follows the one in [4] and is presented in Section 3.3.

1.3 Scientific context

Existence of strong solutions to fluid–structure problems is already available for several cases. For instance the problems of a fluid coupled with rigid bodies [7, 18, 19], a plate [16] or a beam [9, 12] have already been investigated.

Existence of a weak solution has also been proven for a fluid coupled with a plate [5].

In the current study, we focus on a deformable structure depending on a finite number of parameters. A close situation has already been investigated for a finite dimensional approximation of a plate [4]. This makes the modes $\partial_{\theta_j}\mathbf{X}$ fulfil a relation mandatory to ensure the null divergence of the fluid.

In contrast to [4], the case considered in the current paper deals with an intrinsically finite dimensional structure. Hence, the modes $\partial_{\theta_j}\mathbf{X}$ do not fulfil such a relation and some parts of the proof in [4] have then to be modified.

Additional difficulties are induced by the corners on $\partial\Omega$, more information can be found in [11, 13].

1.4 Outline of the paper

In Section 2, we study the linearized problem in the fixed domain $\mathscr{F}_0$. We prove existence of strong solutions to this linearized problem. Then, in Section 3, we prove local existence of solutions to the nonlinear system. We extend the previous result to the nonlinear system with a fixed point argument.

The proof of the estimates of the nonlinear terms can be found in Appendix A.

2 Existence of solution to the linearized problem

In this section we study the linearization of problem (1.23), first with only source terms $\mathbf{f}$ and $\mathbf{s}$ and then with all source terms. These equations are written in the fixed domain $\mathscr{F}_0$ using a change of variables explained in Section 3.1. In the sequel, $(\tilde{\mathbf{u}}, \tilde{p})$ denotes velocity and pressure of the fluid in the fixed domain $\mathscr{F}_0$. We denote $T > 0$ the considered final time.

2.1 Linearized problem with nonhomogeneous source terms

Let us study the following problem

$$\begin{cases} \dfrac{\partial \tilde{\mathbf{u}}}{\partial t} - \nu \Delta \tilde{\mathbf{u}} + \nabla \tilde{p} = \mathbf{f} & \text{in } (0,T) \times \mathscr{F}_0, \\ \operatorname{div} \tilde{\mathbf{u}} = 0 & \text{in } (0,T) \times \mathscr{F}_0, \\ \tilde{\mathbf{u}} = \dot{\theta}_1 \partial_{\theta_1} \mathbf{\Phi}(0,0,.) + \dot{\theta}_2 \partial_{\theta_2} \mathbf{\Phi}(0,0,.) & \text{on } (0,T) \times \partial S_0, \\ \tilde{\mathbf{u}} = 0 & \text{on } (0,T) \times \Gamma_{\mathrm{i}}, \\ \tilde{\mathbf{u}} = 0 & \text{on } (0,T) \times \Gamma_{\mathrm{w}}, \\ \sigma_F(\tilde{\mathbf{u}}, \tilde{p}) \mathbf{n}_0 = 0 & \text{on } (0,T) \times \Gamma_{\mathrm{N}}, \\ \tilde{\mathbf{u}}(0,\mathbf{y}) = \mathbf{u}_0(\mathbf{y}) & \text{in } \mathscr{F}_0, \\ \mathcal{M}_{0,0} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\partial S_0} [\tilde{p} I - \nu(\nabla \tilde{\mathbf{u}} + (\nabla \tilde{\mathbf{u}})^T)] \mathbf{n}_0 \cdot \partial_{\theta_1} \mathbf{\Phi}(0,0,\gamma_y) \mathrm{d}\gamma_y \\ \displaystyle\int_{\partial S_0} [\tilde{p} I - \nu(\nabla \tilde{\mathbf{u}} + (\nabla \tilde{\mathbf{u}})^T)] \mathbf{n}_0 \cdot \partial_{\theta_2} \mathbf{\Phi}(0,0,\gamma_y) \mathrm{d}\gamma_y \end{pmatrix} + \mathbf{s} & \text{on } (0,T), \\ \dot{\theta}_1(0) = \omega_{1,0}, \quad \dot{\theta}_2(0) = \omega_{2,0}, \\ \theta_1(0) = 0, \quad \theta_2(0) = 0, \end{cases} \tag{2.1}$$

where the unkwnows are $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)$ and the source terms are $(\mathbf{f}, \mathbf{s}) \in L^2(0,T; \mathbf{L}^2(\mathscr{F}_0)) \times L^2(0,T; \mathbb{R}^2)$. We will show later that this system corresponds to the linearization of the nonlinear problem (1.23) transported in the fixed initial configuration.

Remark 2.1. The state $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)$ of problem (2.1) can be reduced to $(\tilde{\mathbf{u}}, \tilde{p}, \dot{\theta}_1, \dot{\theta}_2)$. Considering the velocity of the structure instead of its position is sufficient to solve (2.1). However, we prefered to consider the full state $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)$, as it is anyway used to deal with the nonlinear case.

In the sequel, the following spaces will be used

$$\mathbb{U}_T = L^2(0,T; \mathbf{H}^2_\beta(\mathscr{F}_0)) \cap \mathscr{C}^0([0,T]; \mathbf{H}^1(\mathscr{F}_0)) \cap H^1(0,T; \mathbf{L}^2(\mathscr{F}_0)), \tag{2.2}$$

$$\mathbb{P}_T = L^2(0,T; H^1_\beta(\mathscr{F}_0)), \tag{2.3}$$

$$\Theta_T = H^2(0,T; \mathbb{R}^2), \tag{2.4}$$

$$\mathbb{F}_T = L^2(0,T; \mathbf{L}^2(\mathscr{F}_0)), \tag{2.5}$$

$$\mathbb{G}_T = H^1(0,T; \mathbf{H}^{3/2}(\partial S_0)), \tag{2.6}$$

$$\mathbb{S}_T = L^2(0,T; \mathbb{R}^2). \tag{2.7}$$

We endow Θ_T with the following norm

$$\|(\theta_1, \theta_2)\|_{\Theta_T} = \|(\theta_1, \theta_2)\|_{H^2(0,T)} + \|(\theta_1, \theta_2)\|_{L^\infty(0,T)} + \|(\dot{\theta}_1, \dot{\theta}_2)\|_{L^\infty(0,T)},$$

the other spaces are endowed with their usual norms. The norm $\|.\|_{\Theta_T}$ has been chosen so that we have the estimate $\|(\theta_1, \theta_2)\|_{L^\infty(0,T)} + \|(\dot{\theta}_1, \dot{\theta}_2)\|_{L^\infty(0,T)} \leq C \|(\theta_1, \theta_2)\|_{\Theta_T}$ where C does not depend on T. Note that with the natural norm of Θ_T, C would depend on T.

Let us fix an arbitrary time $T_0 > 0$, e.g. $T_0 = 1$. We want to prove the following result.

Proposition 2.2. *There exists a constant $C > 0$ such that for all $T \in (0, T_0)$, C does not depend on T, for all $\mathbf{u}_0 \in \mathbf{H}^1(\mathscr{F}_0)$ and $(\omega_{1,0}, \omega_{2,0}) \in \mathbb{R}^2$ satisfying the compatibility conditions* (1.33) *(with $\mathbf{u}^i = 0$) and every $(\mathbf{f}, \mathbf{s}) \in \mathbb{F}_T \times \mathbb{S}_T$, problem* (2.1) *admits a unique solution*

$$(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2) \in \mathbb{U}_T \times \mathbb{P}_T \times \Theta_T.$$

Moreover, the following estimate holds

$$\|\tilde{\mathbf{u}}\|_{\mathbb{U}_T} + \|\tilde{p}\|_{\mathbb{P}_T} + \|(\theta_1, \theta_2)\|_{\Theta_T} \leq C(\|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)} + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}\|_{\mathbb{F}_T} + \|\mathbf{s}\|_{\mathbb{S}_T}). \tag{2.8}$$

In order to prove Proposition 2.2, we will study the problem (2.1) under its semigroup formulation. Let us define the space

$$\mathbb{H} = \left\{ \begin{array}{ll} (\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2)^T \in \mathbf{L}^2(\mathscr{F}_0) \times \mathbb{R}^4, & \operatorname{div} \tilde{\mathbf{u}} = 0 \text{ in } \mathscr{F}_0, \quad \tilde{\mathbf{u}} \cdot \mathbf{n}_0 = 0 \text{ on } \Gamma_{\mathrm{D}}, \\ & \tilde{\mathbf{u}} \cdot \mathbf{n}_0 = \sum_j \omega_j \partial_{\theta_j} \mathbf{\Phi}(0,0,.) \cdot \mathbf{n}_0 \text{ on } \partial S_0 \end{array} \right\}, \tag{2.9}$$

where $\mathbf{n}_0$ is the unitary outward normal to the fluid domain $\mathscr{F}_0$. This space $\mathbb{H}$ is endowed with the scalar product

$$\begin{aligned}\big((\mathbf{v}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a)^T, (\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b)^T\big)_{\mathbb{H}} = & \int_{\mathscr{F}_0} \mathbf{v}^a \cdot \mathbf{v}^b \mathrm{d}\mathbf{y} + \sum_j \theta_j^a \theta_j^b \\ & + \sum_{j,k} \omega_j^a \omega_k^b (\partial_{\theta_j} \mathbf{X}(0,0,.), \partial_{\theta_k} \mathbf{X}(0,0,.))_S,\end{aligned}$$

where $(.,.)_S$ is defined in (1.18). We also define

$$\mathbb{V} = \left\{ \begin{array}{ll} (\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2)^T \in \mathbf{H}^1(\mathscr{F}_0) \times \mathbb{R}^4, & \operatorname{div} \tilde{\mathbf{u}} = 0 \text{ in } \mathscr{F}_0, \quad \tilde{\mathbf{u}} = 0 \text{ on } \Gamma_{\mathrm{D}}, \\ & \tilde{\mathbf{u}} = \sum_j \omega_j \partial_{\theta_j} \mathbf{\Phi}(0,0,.) \text{ on } \partial S_0 \end{array} \right\}, \tag{2.10}$$

endowed with the scalar product

$$\begin{aligned}\big((\mathbf{v}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a)^T, (\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b)^T\big)_{\mathbb{V}} = & \int_{\mathscr{F}_0} (\mathbf{v}^a \cdot \mathbf{v}^b + \nabla \mathbf{v}^a : \nabla \mathbf{v}^b) \mathrm{d}\mathbf{y} + \sum_j \theta_j^a \theta_j^b \\ & + \sum_{j,k} \omega_j^a \omega_k^b (\partial_{\theta_j} \mathbf{X}(0,0,.), \partial_{\theta_k} \mathbf{X}(0,0,.))_S.\end{aligned}$$

Remark 2.3. Note that $(.,.)_{\mathbb{H}}$ and $(.,.)_{\mathbb{V}}$ are also respectively scalar products on $\mathbf{L}^2(\mathscr{F}_0) \times \mathbb{R}^4$ and $\mathbf{H}^1(\mathscr{F}_0) \times \mathbb{R}^4$.

Lemma 2.4. *The orthogonal space to $\mathbb{H}$ in $\mathbf{L}^2(\mathscr{F}_0) \times \mathbb{R}^4$ with respect to the scalar product $(.,.)_{\mathbb{H}}$ is*

$$(\mathbb{H})^\perp = \left\{ \left(\nabla p, 0, 0, -\mathcal{M}_{0,0}^{-1} \left(\int_{\partial S_0} p \mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi}(0,0,\gamma_y) \mathrm{d}\gamma_y \right)_{j=1,2} \right)^T \;\middle|\; p \in H^1(\mathscr{F}_0)\,,\ p = 0 \text{ on } \Gamma_{\mathrm{N}} \right\},$$

where $\mathcal{M}_{0,0}$ is defined in (1.15).

Proof of Lemma 2.4. Let $(\mathbf{v}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a)^T \in \mathbf{L}^2(\mathscr{F}_0) \times \mathbb{R}^4$ such that for every $(\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b)^T \in \mathbb{H}$,

$$\big((\mathbf{v}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a)^T, (\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b)^T\big)_{\mathbb{H}} = 0.$$

By taking $\mathbf{v}^b = 0$ and $\omega_1^b = \omega_2^b = 0$, we easily obtain $\theta_1^a = \theta_2^a = 0$. With $\omega_1^b = \omega_2^b = 0$, we also get

$$\int_{\mathscr{F}_0} \mathbf{v}^a \cdot \mathbf{v}^b \mathrm{d}\mathbf{y} = 0, \quad \forall \mathbf{v}^b \in \mathbf{L}^2(\mathscr{F}_0) \text{ such that } \operatorname{div} \mathbf{v}^b = 0 \text{ in } \mathscr{F}_0 \text{ and } \mathbf{v}^b \cdot \mathbf{n}_0 = 0 \text{ on } \Gamma_{\mathrm{D}} \cup \partial S_0,$$

which implies, according to [13, Lemma 2.2], $\mathbf{v}^a = \nabla p$, where $p \in H^1(\mathscr{F}_0)$ and $p = 0$ on Γ_{N}. Now,

$$\int_{\mathscr{F}_0} \nabla p \cdot \mathbf{v}^b \mathrm{d}\mathbf{y} + \sum_{j,k} \omega_j^a \omega_k^b (\partial_{\theta_j} \mathbf{X}(0,0,.), \partial_{\theta_k} \mathbf{X}(0,0,.))_S = 0,$$

becomes with the divergence formula and the compatibility condition in (2.9)

$$\sum_j \omega_j^b \int_{\partial S_0} p \mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi}(0,0,\gamma_y) \mathrm{d}\gamma_y + \sum_{j,k} \omega_j^a \omega_k^b (\partial_{\theta_j} \mathbf{X}(0,0,.), \partial_{\theta_k} \mathbf{X}(0,0,.))_S = 0,$$

then

$$\int_{\partial S_0} p \mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi}(0,0,\gamma_y) \mathrm{d}\gamma_y + \sum_k \omega_k^a (\partial_{\theta_j} \mathbf{X}(0,0,.), \partial_{\theta_k} \mathbf{X}(0,0,.))_S = 0,$$

which yields a first inclusion. The converse inclusion is obtained via an integration by parts. $\square$

We define the operator $(A, D(A))$ on $\mathbb{H}$ as

$$D(A) = \left\{ \begin{array}{r} (\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2)^T \in \mathbb{V},\ \tilde{\mathbf{u}} \in \mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_0), \exists \tilde{p} \in H^{1/2+\varepsilon_0}(\mathscr{F}_0) \text{ such that} \\ \operatorname{div} \sigma_F(\tilde{\mathbf{u}}, \tilde{p}) \in \mathbf{L}^2(\mathscr{F}_0) \text{ and } \sigma_F(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n}_0 = 0 \text{ on } \Gamma_{\mathrm{N}} \end{array} \right\}, \tag{2.11}$$

where $\varepsilon_0 = 0.08$ is introduced in Lemma 1.3, and

$$A \begin{pmatrix} \tilde{\mathbf{u}} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \Pi_{\mathbb{H}} \begin{pmatrix} \operatorname{div} \sigma_F(\tilde{\mathbf{u}}, \tilde{p}) \\ \omega_1 \\ \omega_2 \\ \mathcal{M}_{0,0}^{-1} \left[\int_{\partial S_0} -\sigma_F(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi}(0,0,\gamma_y) \mathrm{d}\gamma_y \right]_{j=1,2} \end{pmatrix}, \tag{2.12}$$

where $\Pi_{\mathbb{H}}$ is the orthogonal projector from $\mathbf{L}^2(\mathscr{F}_0) \times \mathbb{R}^4$ onto $\mathbb{H}$ with respect to $\big(.,.\big)_{\mathbb{H}}$.

Lemma 2.5. *The operator A is uniquely defined.*

Proof of Lemma 2.5. Let $(\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2)^T \in D(A)$ and consider two functions p, $q \in H^{1/2+\varepsilon_0}(\mathscr{F}_0)$ satisfying the conditions appearing into the definition of $D(A)$. Then, $\operatorname{div} \sigma_F(0, p-q) = -\nabla(p-q) \in \mathbf{L}^2(\mathscr{F}_0)$ implies $p - q \in H^1(\mathscr{F}_0)$, and $\sigma_F(0, p-q)\mathbf{n}_0 = 0$ on Γ_{N} implies $p - q = 0$ on Γ_{N}.

Now,

$$\begin{pmatrix} \operatorname{div} \sigma_F(\tilde{\mathbf{u}}, p) \\ \omega_1 \\ \omega_2 \\ \mathcal{M}_{0,0}^{-1} \left[\int_{\partial S_0} -\sigma_F(\tilde{\mathbf{u}}, p)\mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi}(0,0,\gamma_y) \mathrm{d}\gamma_y \right]_{j=1,2} \end{pmatrix} - \begin{pmatrix} \operatorname{div} \sigma_F(\tilde{\mathbf{u}}, q) \\ \omega_1 \\ \omega_2 \\ \mathcal{M}_{0,0}^{-1} \left[\int_{\partial S_0} -\sigma_F(\tilde{\mathbf{u}}, q)\mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi}(0,0,\gamma_y) \mathrm{d}\gamma_y \right]_{j=1,2} \end{pmatrix}$$
$$= \begin{pmatrix} \nabla(p-q) \\ 0 \\ 0 \\ -\mathcal{M}_{0,0}^{-1} \left[\int_{\partial S_0} (p-q)\mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi}(0,0,\gamma_y) \mathrm{d}\gamma_y \right]_{j=1,2} \end{pmatrix},$$

which belongs to $\mathbb{H}^{\perp}$ according to Lemma 2.4. Therefore A is uniquely defined. ☐

The key point of this section is the following lemma.

Lemma 2.6. *The operator A generates an analytic semigroup on $\mathbb{H}$. Moreover, for $\lambda \in \mathbb{R}$ large enough, $\lambda I - A$ is positive and $D((\lambda I - A)^{1/2}) = \mathbb{V}$.*

Proof of Lemma 2.6. We define the operator $(A_1, D(A_1))$ on $\mathbb{H}$ with

$$D(A_1) = D(A),$$

and

$$A_1 \begin{pmatrix} \mathbf{v} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \Pi_{\mathbb{H}} \begin{pmatrix} \operatorname{div} \sigma_F(\mathbf{v}, q) \\ 0 \\ 0 \\ \mathcal{M}_{0,0}^{-1} \left[\int_{\partial S_0} -\sigma_F(\mathbf{v}, q)\mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi}(0,0,\gamma_y) \mathrm{d}\gamma_y \right]_{j=1,2} \end{pmatrix}.$$

A direct adaptation of Lemma 2.5 proves that A_1 is uniquely defined. Now, we first prove the properties of Lemma 2.6 on the self–adjoint operator A_1 and then we extend it to A with a perturbation argument. For $(\mathbf{v}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a) \in D(A_1)$ and $(\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b) \in \mathbb{V}$,

$$\begin{aligned} &\big(A_1(\mathbf{v}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a)^T, (\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b)^T\big)_{\mathbb{H}} \\ &= \int_{\partial S_0} (\sigma_F(\mathbf{v}^a, q^a)\mathbf{n}_0) \cdot \mathbf{v}^b \mathrm{d}\gamma_y - \int_{\mathscr{F}_0} \sigma_F(\mathbf{v}^a, q^a) : \nabla \mathbf{v}^b \mathrm{d}\mathbf{y} + \sum_j \omega_j^b \int_{\partial S_0} -\sigma_F(\mathbf{v}^a, q^a)\mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi}(0,0,\gamma_y) \mathrm{d}\gamma_y \\ &= -\nu \int_{\mathscr{F}_0} (\nabla \mathbf{v}^a + (\nabla \mathbf{v}^a)^T) : \nabla \mathbf{v}^b \mathrm{d}\mathbf{y} \\ &= -\mathrm{a}_1((\mathbf{v}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a)^T, (\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b)^T), \end{aligned}$$

where

$$\mathrm{a}_1((\mathbf{v}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a)^T, (\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b)^T) = \frac{\nu}{2} \int_{\mathscr{F}_0} (\nabla \mathbf{v}^a + (\nabla \mathbf{v}^a)^T) : (\nabla \mathbf{v}^b + (\nabla \mathbf{v}^b)^T) \mathrm{d}\mathbf{y},$$

is a continuous bilinear form on $\mathbb{V}$.

For the first equality we use $(\partial_{\theta_j}\mathbf{X}(0,0,.),\partial_{\theta_k}\mathbf{X}(0,0,.))_S = (\mathcal{M}_{0,0})_{jk}$ and for the second $q^a I : \nabla \mathbf{v}^b = q^a \text{div } \mathbf{v}^b = 0$. The final equality shows that $-A_1$ is non-negative and self-adjoint, so we can easily conclude that $D((-A_1)^{1/2}) = \mathbb{V}$.

Moreover, according to Korn's inequality [6, p. 110], there exists $c > 0$ such that

$$\forall (\mathbf{v},\theta_1,\theta_2,\omega_1,\omega_2)^T \in \mathbb{V},\ \mathrm{a}_1((\mathbf{v},\theta_1,\theta_2,\omega_1,\omega_2)^T,(\mathbf{v},\theta_1,\theta_2,\omega_1,\omega_2)^T) + \frac{\nu}{2}\|\mathbf{v}\|^2_{\mathbf{L}^2(\mathscr{F}_0)} \geq c\|\mathbf{v}\|^2_{\mathbf{H}^1(\mathscr{F}_0)},$$

so that,

$$\begin{aligned}&\forall (\mathbf{v},\theta_1,\theta_2,\omega_1,\omega_2)^T \in \mathbb{V},\\ &\mathrm{a}_1((\mathbf{v}^a,\theta_1^a,\theta_2^a,\omega_1^a,\omega_2^a)^T,(\mathbf{v}^b,\theta_1^b,\theta_2^b,\omega_1^b,\omega_2^b)^T) + \max\left(\frac{\nu}{2},c\right)\|(\mathbf{v},\theta_1,\theta_2,\omega_1,\omega_2)\|^2_{\mathbb{H}} \geq c\|(\mathbf{v},\theta_1,\theta_2,\omega_1,\omega_2)\|^2_{\mathbb{V}}.\end{aligned}$$

Hence, according to [3, Theorem 2.12, p. 115], A_1 generates an analytic semigroup on $\mathbb{H}$.

Now, we use the fact that $A - A_1 \in \mathcal{L}(\mathbb{H})$, then according to [14, Corollary 2.2.], A generates an analytic semigroup on $\mathbb{H}$.

A consequence of the previous result is that there exists $\lambda > 0$ such that $\lambda I - A$ is positive. Moreover, $D(\lambda I - A) = D(A_1)$, then by interpolation, $D((\lambda I - A)^{1/2}) = D((-A_1)^{1/2}) = \mathbb{V}$. □

We are now in position to prove Proposition 2.2.

Proof of Proposition 2.2. Let us denote $\mathbf{F} = \Pi_{\mathbb{H}}(\mathbf{f},0,0,\mathcal{M}_{0,0}^{-1}\mathbf{s})^T$ and $\mathbf{z}_0 = (\mathbf{u}_0,0,0,\omega_{1,0},\omega_{2,0})^T$. We have $\mathbf{F} \in L^2(0,T;\mathbb{H})$ and $\mathbf{z}_0 \in D(A^{1/2})$.

According to [3, Theorem 3.1, p. 143] and Lemma 2.6, the problem

$$\left\{\begin{array}{l}\mathbf{z}'(t) = A\mathbf{z}(t) + \mathbf{F}(t),\ t \geq 0,\\ \mathbf{z}(0) = \mathbf{z}_0,\end{array}\right. \tag{2.13}$$

admits a unique solution $\mathbf{z} \in L^2(0,T;D(A)) \cap H^1(0,T;\mathbb{H})$ and there exists $C > 0$ such that

$$\|\mathbf{z}\|_{L^2(0,T;D(A))\cap H^1(0,T;\mathbb{H})} \leq C(\|\mathbf{F}\|_{L^2(0,T;\mathbb{H})} + \|\mathbf{z}_0\|_{\mathbb{V}}). \tag{2.14}$$

With the Sobolev embedding

$$L^2(0,T;D(A)) \cap H^1(0,T;\mathbb{H}) \hookrightarrow \mathscr{C}^0([0,T];\mathbb{V}),$$

we have

$$\|\mathbf{z}\|_{L^2(0,T;D(A))\cap\mathscr{C}^0([0,T];\mathbb{V})\cap H^1(0,T;\mathbb{H})} \leq C(\|\mathbf{F}\|_{L^2(0,T;\mathbb{H})} + \|\mathbf{z}_0\|_{\mathbb{V}}). \tag{2.15}$$

Moreover, C is independent from $T \in (0,T_0)$. To prove this last statement, we consider

$$\forall t \in [0,T_0], \quad \tilde{\mathbf{F}}(t) = \left\{\begin{array}{ll}\mathbf{F}(t) & \text{if } t \in [0,T],\\ 0 & \text{if } t \in]T,T_0].\end{array}\right.$$

If $\tilde{\mathbf{z}}$ is the solution on $[0,T_0]$ of

$$\left\{\begin{array}{l}\tilde{\mathbf{z}}' = A\tilde{\mathbf{z}} + \tilde{\mathbf{F}},\\ \tilde{\mathbf{z}}(0) = \mathbf{z}_0,\end{array}\right.$$

then for $t \leq T$, $\tilde{\mathbf{z}}(t) = \mathbf{z}(t)$. And we have the inequality

$$\|\tilde{\mathbf{z}}\|_{L^2(0,T_0;D(A))\cap\mathscr{C}^0([0,T_0];\mathbb{V})\cap H^1(0,T_0;\mathbb{H})} \leq C(\|\tilde{\mathbf{F}}\|_{L^2(0,T_0;\mathbb{H})} + \|\mathbf{z}_0\|_{\mathbb{V}}),$$

where C does not depend on T, while

$$\|\mathbf{z}\|_{L^2(0,T;D(A))\cap\mathscr{C}^0([0,T];\mathbb{V})\cap H^1(0,T;\mathbb{H})} \leq \|\tilde{\mathbf{z}}\|_{L^2(0,T_0;D(A))\cap\mathscr{C}^0([0,T_0];\mathbb{V})\cap H^1(0,T_0;\mathbb{H})},$$

and

$$\|\tilde{\mathbf{F}}\|_{L^2(0,T_0;\mathbb{H})} = \|\mathbf{F}\|_{L^2(0,T;\mathbb{H})}.$$

We get (2.15) with C independent from T.

Now, if we write $\mathbf{z} = (\tilde{\mathbf{u}},\theta_1,\theta_2,\omega_1,\omega_2)^T$, problem (2.13) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\tilde{\mathbf{u}}\\ \theta_1\\ \theta_2\\ \omega_1\\ \omega_2\end{pmatrix} = \Pi_{\mathbb{H}}\begin{pmatrix}\text{div } \sigma_F(\tilde{\mathbf{u}},p) + \mathbf{f}\\ \omega_1\\ \omega_2\\ \mathcal{M}_{0,0}^{-1}\left(\mathbf{s} + \left(\displaystyle\int_{\partial S_0} -\sigma_F(\tilde{\mathbf{u}},p)\mathbf{n}_0\cdot\partial_{\theta_j}\boldsymbol{\Phi}(0,0,.)\right)_{j=1..2}\right)\end{pmatrix},$$

where $p \in L^2(0,T;H^{1/2+\varepsilon_0}(\mathscr{F}_0))$. Then, Lemma 2.4 implies that there exists $q \in L^2(0,T;H^1(\mathscr{F}_0))$ such that $(\tilde{\mathbf{u}}, p+q, \theta_1, \theta_2)$ satisfies the linear problem (2.1). Moreover, according to (2.15), we have $(\theta_1, \theta_2) \in H^2(0,T;\mathbb{R}^2)$, $\tilde{\mathbf{u}} \in H^1(0,T;\mathbf{L}^2(\mathscr{F}_0)) \cap \mathscr{C}^0([0,T];\mathbf{H}^1(\mathscr{F}_0)) \cap L^2(0,T;\mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_0))$, $\tilde{p} = p+q \in L^2(0,T;H^{1/2+\varepsilon_0}(\mathscr{F}_0))$ and

$$\begin{aligned}\|\tilde{\mathbf{u}}\|_{L^2(0,T;\mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_0))\cap\mathscr{C}^0([0,T];\mathbf{H}^1(\mathscr{F}_0))\cap H^1(0,T;\mathbf{L}^2(\mathscr{F}_0))} + \|\tilde{p}\|_{L^2(0,T;\mathbf{H}^{1/2+\varepsilon_0}(\mathscr{F}_0))} + \|(\theta_1,\theta_2)\|_{\Theta_T} \\ \leq C(\|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)} + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\mathscr{F}_0))} + \|\mathbf{s}\|_{L^2(0,T;\mathbb{R}^2)}).\end{aligned}$$

We still have to show $\tilde{\mathbf{u}} \in L^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0))$ and $\tilde{p} \in L^2(0,T;H^1_\beta(\mathscr{F}_0))$. According to [13, Theorem 2.16], there exists $\mathbf{v} \in H^1(0,T;\mathbf{H}^2(\mathscr{F}_0))$ satisfying

$$\begin{cases} \mathrm{div}\ \mathbf{v} = 0 & \text{in } (0,T)\times\mathscr{F}_0, \\ \mathbf{v} = \sum_j \dot{\theta}_j \partial_{\theta_j}\mathbf{\Phi}(0,0,.) & \text{on } (0,T)\times\partial S_0, \\ \mathbf{v} = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{D}}, \\ (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\mathbf{n}_0 = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{N}}, \end{cases}$$

with

$$\|\mathbf{v}\|_{H^1(0,T;\mathbf{H}^2(\mathscr{F}_0))} \leq C\|(\theta_1,\theta_2)\|_{\Theta_T}.$$

The velocity $\tilde{\mathbf{u}} - \mathbf{v}$ and the pressure $\tilde{p}$ satisfy for almost every t in $(0,T)$

$$\begin{cases} -\nu\Delta(\tilde{\mathbf{u}}-\mathbf{v}) + \nabla\tilde{p} = \mathbf{f} - \partial_t\tilde{\mathbf{u}} + \nu\Delta\mathbf{v} & \text{in } (0,T)\times\mathscr{F}_0, \\ \mathrm{div}\ (\tilde{\mathbf{u}}-\mathbf{v}) = 0 & \text{in } (0,T)\times\mathscr{F}_0, \\ \tilde{\mathbf{u}}-\mathbf{v} = 0 & \text{on } (0,T)\times(\Gamma_{\mathrm{D}}\cup\partial S_0), \\ \sigma_F(\tilde{\mathbf{u}}-\mathbf{v},\tilde{p})\mathbf{n}_0 = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{N}}, \end{cases}$$

then, according to Lemma 1.3, $\tilde{\mathbf{u}} - \mathbf{v} \in L^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0))$ and $\tilde{p} \in L^2(0,T;H^1_\beta(\mathscr{F}_0))$, where β is introduced in Lemma 1.3. Moreover, (1.28) yields

$$\|\tilde{\mathbf{u}}-\mathbf{v}\|_{L^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0))} + \|\tilde{p}\|_{L^2(0,T;H^1_\beta(\mathscr{F}_0))} \leq \|\mathbf{f} - \partial_t\tilde{\mathbf{u}} + \nu\Delta\mathbf{v}\|_{L^2(0,T;\mathbf{L}^2(\mathscr{F}_0))}.$$

With the estimate $\|\tilde{\mathbf{u}}\|_{L^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0))} \leq \|\tilde{\mathbf{u}}-\mathbf{v}\|_{L^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0))} + \|\mathbf{v}\|_{L^2(0,T;\mathbf{H}^2(\mathscr{F}_0))}$, we get

$$\begin{aligned}\|\tilde{\mathbf{u}}\|_{L^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0))} + \|\tilde{p}\|_{L^2(0,T;H^1_\beta(\mathscr{F}_0))} &\leq C(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\mathscr{F}_0))} + \|\tilde{\mathbf{u}}\|_{H^1(0,T;\mathbf{L}^2(\mathscr{F}_0))} + \|\mathbf{v}\|_{H^1(0,T;\mathbf{H}^2(\mathscr{F}_0))}) \\ &\leq C(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\mathscr{F}_0))} + \|\tilde{\mathbf{u}}\|_{H^1(0,T;\mathbf{L}^2(\mathscr{F}_0))} + \|\theta\|_{H^2(0,T;\mathbb{R}^2)}).\end{aligned}$$

This concludes the proof of Proposition 2.2. □

2.2 Linearized problem with nonhomogeneous boundary data

Let us now consider two more source terms: one source term $\mathbf{g}$ on the boundary of the structure ∂S_0 and one source term $\mathbf{u}^i$ on the inflow boundary region Γ_{i}. Let $T_0 > 0$, we study

$$\begin{cases} \dfrac{\partial\tilde{\mathbf{u}}}{\partial t} - \nu\Delta\tilde{\mathbf{u}} + \nabla\tilde{p} = \mathbf{f} & \text{in } (0,T)\times\mathscr{F}_0, \\ \mathrm{div}\ \tilde{\mathbf{u}} = 0 & \text{in } (0,T)\times\mathscr{F}_0, \\ \tilde{\mathbf{u}} = \dot{\theta}_1\partial_{\theta_1}\mathbf{\Phi}(0,0,.) + \dot{\theta}_2\partial_{\theta_2}\mathbf{\Phi}(0,0,.) + \mathbf{g} & \text{on } (0,T)\times\partial S_0, \\ \tilde{\mathbf{u}} = \mathbf{u}^i & \text{on } (0,T)\times\Gamma_{\mathrm{i}}, \\ \tilde{\mathbf{u}} = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{w}}, \\ \sigma_F(\tilde{\mathbf{u}},\tilde{p})\mathbf{n}_0 = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{N}}, \\ \tilde{\mathbf{u}}(0,\mathbf{y}) = \mathbf{u}_0(\mathbf{y}) & \text{in } \mathscr{F}_0, \\ \mathcal{M}_{0,0}\begin{pmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\partial S_0} [\tilde{p}I - \nu(\nabla\tilde{\mathbf{u}} + (\nabla\tilde{\mathbf{u}})^T)]\mathbf{n}_0\cdot\partial_{\theta_1}\mathbf{\Phi}(0,0,\gamma_y)\mathrm{d}\gamma_y \\ \displaystyle\int_{\partial S_0} [\tilde{p}I - \nu(\nabla\tilde{\mathbf{u}} + (\nabla\tilde{\mathbf{u}})^T)]\mathbf{n}_0\cdot\partial_{\theta_2}\mathbf{\Phi}(0,0,\gamma_y)\mathrm{d}\gamma_y \end{pmatrix} + \mathbf{s} & \text{on } (0,T), \\ \dot{\theta}_1(0) = \omega_{1,0}, \quad \dot{\theta}_2(0) = \omega_{2,0}, \\ \theta_1(0) = 0, \quad \theta_2(0) = 0, \end{cases} \tag{2.16}$$

where the source terms are $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\mathscr{F}_0))$, $\mathbf{g} \in H^1(0,T;\mathbf{H}^{3/2}(\partial S_0))$, $\mathbf{u}^i \in H^1(0,T_0;\mathbf{U}^i)$ and $\mathbf{s} \in L^2(0,T;\mathbb{R}^2)$.

We have the following result:

Proposition 2.7. *There exists a constant $C > 0$ such that for all $T \in (0, T_0)$, for all $\mathbf{u}^i \in H^1(0, T_0; \mathbf{U}^i)$, $\mathbf{u}_0 \in \mathbf{H}^1(\mathscr{F}_0)$ and $(\omega_{1,0}, \omega_{2,0}) \in \mathbb{R}^2$ satisfying the compatibility conditions* (1.33) *and every $(\mathbf{f}, \mathbf{g}, \mathbf{s}) \in \mathbb{F}_T \times \mathbb{G}_T \times \mathbb{S}_T$ with $\mathbf{g}(0) = 0$, problem* (2.16) *admits a unique solution*

$$(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2) \in \mathbb{U}_T \times \mathbb{P}_T \times \Theta_T,$$

with

$$\|\tilde{\mathbf{u}}\|_{\mathbb{U}_T} + \|\tilde{p}\|_{\mathbb{P}_T} + \|(\theta_1, \theta_2)\|_{\Theta_T} \leq C(\|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)} + |\omega_{1,0}| + |\omega_{2,0}| \\ + \|\mathbf{f}\|_{\mathbb{F}_T} + \|\mathbf{g}\|_{\mathbb{G}_T} + \|\mathbf{s}\|_{\mathbb{S}_T} + \|\mathbf{u}^i\|_{H^1(0,T_0;\mathbf{U}^i)}). \tag{2.17}$$

Proposition 2.7 is proven at the end of the section. The proof uses the following lifting result for the new terms $\mathbf{g}$ and $\mathbf{u}^i$:

Lemma 2.8. *For every $\mathbf{g} \in \mathbf{H}^{3/2}(\partial S_0)$ and every $\mathbf{u}^i \in \mathbf{U}^i$, there exists $\overline{\mathbf{u}} \in \mathbf{H}^2(\mathscr{F}_0)$ satisfying*

$$\begin{cases} \operatorname{div} \overline{\mathbf{u}} = 0 & \text{in } \mathscr{F}_0, \\ \overline{\mathbf{u}} = \mathbf{g} & \text{on } \partial S_0, \\ \overline{\mathbf{u}} = \mathbf{u}^i & \text{on } \Gamma_{\mathrm{i}}, \\ \overline{\mathbf{u}} = 0 & \text{on } \Gamma_{\mathrm{w}}, \\ (\nabla \overline{\mathbf{u}} + (\nabla \overline{\mathbf{u}})^T)\mathbf{n}_0 = 0 & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{2.18}$$

with

$$\|\overline{\mathbf{u}}\|_{\mathbf{H}^2(\mathscr{F}_0)} \leq C(\|\mathbf{u}^i\|_{\mathbf{U}^i} + \|\mathbf{g}\|_{\mathbf{H}^{3/2}(\partial S_0)}). \tag{2.19}$$

Note that despite the presence of corners, we recover the expected regularity of the lifting for smooth domains.

Remark 2.9. For the sake of readability, from this point onwards all terms $\mathrm{d}\mathbf{y}$ and $\mathrm{d}\gamma_y$ are omitted in the integrals.

Proof of Lemma 2.8. The lifting result has been established for the condition $\mathbf{u}^i = 0$ on the inflow region in [13, Theorem 2.16]. We first lift the input boundary condition $\mathbf{u}^i \neq 0$ in Ω and then we use the aforementioned result.

Lifting of the inflow boundary condition. Let us look for a function $\mathbf{v}$ defined on the entire domain Ω and satisfying

$$\begin{cases} \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{u}^i & \text{on } \Gamma_{\mathrm{i}}, \\ \mathbf{v} = 0 & \text{on } \Gamma_{\mathrm{w}}, \\ (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)\mathbf{n}_0 = 0 & \text{on } \Gamma_{\mathrm{N}}. \end{cases} \tag{2.20}$$

As $\mathbf{v}$ is divergence-free and Ω is simply connected, we look for it under the form $\mathbf{v} = \nabla^{\perp}\psi$, where ψ is a scalar-valued function. In the geometry considered, Γ_{i}, Γ_{t}, Γ_{b} and Γ_{N} are straight lines, hence $\partial_{\mathbf{n}_0}$ is written as $\pm\partial_{y_1}$ or $\pm\partial_{y_2}$ according on which part of the boundary we consider it.

We can prove that ψ has to satisfy the conditions

$$\begin{array}{lll} \partial_{y_2}\psi = -u_1^i & \text{and } \partial_{y_1}\psi = u_2^i & \text{on } \Gamma_{\mathrm{i}}, \\ \partial_{y_1}\psi = 0 & \text{and } \partial_{y_2}\psi = 0 & \text{on } \Gamma_{\mathrm{b}}, \\ \partial_{y_1}\psi = 0 & \text{and } \partial_{y_2}\psi = 0 & \text{on } \Gamma_{\mathrm{t}}, \\ \partial_{y_1}\partial_{y_2}\psi = 0 & \text{and } \partial_{y_1}^2\psi - \partial_{y_2}^2\psi = 0 & \text{on } \Gamma_{\mathrm{N}}. \end{array}$$

We chose to meet these conditions in the following way:

$$\begin{array}{llll} & \psi(y_2) = -\displaystyle\int_0^{y_2} u_1^i & \text{and } \partial_{y_1}\psi = u_2^i & \text{on } \Gamma_{\mathrm{i}}, \\ & \psi = 0 & \text{and } \partial_{y_2}\psi = 0 & \text{on } \Gamma_{\mathrm{b}}, \\ & \psi = -\displaystyle\int_{\Gamma_{\mathrm{i}}} u_1^i & \text{and } \partial_{y_2}\psi = 0 & \text{on } \Gamma_{\mathrm{t}}. \\ \psi(y_2) = -\eta(y_2)\displaystyle\int_{\Gamma_{\mathrm{i}}} u_1^i, & \partial_{y_1}\psi = 0 & \text{and } \partial_{y_1}^2\psi = -\mathrm{d}_{y_2}^2\eta(y_2)\displaystyle\int_{\Gamma_{\mathrm{i}}} u_1^i & \text{on } \Gamma_{\mathrm{N}}, \end{array} \tag{2.21}$$

where η is a $\mathcal{C}^\infty$ function on $[0,1]$ satisfying

$$\forall\, y_2 \in [0,1], \quad \eta(y_2) = \begin{cases} 0 & \text{if } \; y_2 \in [0, 1/4], \\ \in [0,1] & \text{if } \; y_2 \in]1/4, 3/4[, \\ 1 & \text{if } \; y_2 \in [3/4, 1]. \end{cases} \tag{2.22}$$

The theorem [8, Theorem 1.6.1.5, p.69] with $m = 3$ and $d = 2$ gives the existence of $\psi \in H^3(\Omega)$ fulfilling (2.21) under the compatibility conditions:

$$\text{there exist } \alpha_1 \text{ and } \alpha_2 > 0 \text{ such that } \left\{ \begin{array}{l} \displaystyle\int_0^{\alpha_1} \frac{|\partial_{y_2} u_2^i|^2}{y_2} < +\infty, \\ \displaystyle\int_{1-\alpha_2}^1 \frac{|\partial_{y_2} u_2^i|^2}{1-y_2} < +\infty. \end{array} \right. \tag{2.23}$$

These conditions are the ones in the definition of $\mathbf{U}^i$ in (1.32) with $\alpha_1 = \alpha_2 = 1/4$. Moreover we have the estimate

$$\|\mathbf{v}\|_{\mathbf{H}^2(\Omega)} \leq c\|\psi\|_{H^3(\Omega)} \leq C\|\mathbf{u}^i\|_{\mathbf{H}^{3/2}(\Gamma_\mathrm{i})}. \tag{2.24}$$

The divergence-free field $\mathbf{v} = \nabla^\perp \psi \in \mathbf{H}^2(\Omega)$ satisfies (2.20).

Lifting of the structure velocity. Now, $\tilde{\mathbf{v}} = \overline{\mathbf{u}} - \mathbf{v}_{|\mathscr{F}_0}$ has to satisfy

$$\left\{ \begin{array}{ll} \operatorname{div} \tilde{\mathbf{v}} = 0 & \text{in } \mathscr{F}_0, \\ \tilde{\mathbf{v}} = \mathbf{g} - \mathbf{v} & \text{on } \partial S_0, \\ \tilde{\mathbf{v}} = 0 & \text{on } \Gamma_\mathrm{i}, \\ \tilde{\mathbf{v}} = 0 & \text{on } \Gamma_\mathrm{w}, \\ (\nabla\tilde{\mathbf{v}} + (\nabla\tilde{\mathbf{v}})^T)\mathbf{n}_0 = 0 & \text{on } \Gamma_\mathrm{N}. \end{array} \right.$$

According to [13, Theorem 2.16], such $\tilde{\mathbf{v}}$ exists in $\mathbf{H}^2(\mathscr{F}_0)$ as soon as $\mathbf{g} - \mathbf{v} \in \mathbf{H}^{3/2}(\partial S_0)$. Moreover, we have the estimate

$$\|\tilde{\mathbf{v}}\|_{\mathbf{H}^2(\mathscr{F}_0)} \leq C\|\mathbf{g} - \mathbf{v}\|_{\mathbf{H}^{3/2}(\partial S_0)} \leq C(\|\mathbf{g}\|_{\mathbf{H}^{3/2}(\partial S_0)} + \|\mathbf{v}\|_{\mathbf{H}^2(\Omega)}). \tag{2.25}$$

This yields the expected result since $\overline{\mathbf{u}} = \tilde{\mathbf{v}} + \mathbf{v}_{|\mathscr{F}_0}$, the estimate (2.19) comes from (2.24) and (2.25). □

We can now prove Proposition 2.7 in the following way.

Proof of Proposition 2.7. Let $\mathbf{u}^i \in H^1(0,T_0;\mathbf{U}^i)$, $\mathbf{u}_0 \in \mathbf{H}^1(\mathscr{F}_0)$ and $(\omega_{1,0}, \omega_{2,0}) \in \mathbb{R}^2$ satisfying the compatibility conditions (1.33). Let $(\mathbf{f}, \mathbf{g}, \mathbf{s}) \in \mathbb{F}_T \times \mathbb{G}_T \times \mathbb{S}_T$ with $\mathbf{g}(0) = 0$.

Let $\overline{\mathbf{u}} \in H^1(0,T;\mathbf{H}^2(\mathscr{F}_0))$ be the solution to (2.18), it fulfils

$$\|\overline{\mathbf{u}}\|_{H^1(0,T;\mathbf{H}^2(\mathscr{F}_0))} \leq C(\|\mathbf{u}^i\|_{H^1(0,T_0;\mathbf{U}^i)} + \|\mathbf{g}\|_{H^1(0,T;\mathbf{H}^{3/2}(\partial S_0))}). \tag{2.26}$$

The lifting $\overline{\mathbf{u}}$ also belongs to $\mathscr{C}^0([0,T];\mathbf{H}^2(\mathscr{F}_0))$, and as $\mathbf{g}(0) = 0$, we have

$$\|\overline{\mathbf{u}}\|_{\mathscr{C}^0([0,T];\mathbf{H}^2(\mathscr{F}_0))} \leq C(\|\mathbf{u}^i\|_{H^1(0,T_0;\mathbf{U}^i)} + \|\mathbf{g}\|_{H^1(0,T;\mathbf{H}^{3/2}(\partial S_0))}), \tag{2.27}$$

where C does not depend on T.

Let $(\hat{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)$ be the solution to

$$\left\{ \begin{array}{ll} \dfrac{\partial\hat{\mathbf{u}}}{\partial t} - \nu\Delta\hat{\mathbf{u}} + \nabla\tilde{p} = \mathbf{f} - \dfrac{\partial\overline{\mathbf{u}}}{\partial t} + \nu\Delta\overline{\mathbf{u}} & \text{in } (0,T) \times \mathscr{F}_0, \\ \operatorname{div} \hat{\mathbf{u}} = 0 & \text{in } (0,T) \times \mathscr{F}_0, \\ \hat{\mathbf{u}} = \dot{\theta}_1 \partial_{\theta_1}\mathbf{\Phi}(0,0,.) + \dot{\theta}_2 \partial_{\theta_2}\mathbf{\Phi}(0,0,.) & \text{on } (0,T) \times \partial S_0, \\ \hat{\mathbf{u}} = 0 & \text{on } (0,T) \times \Gamma_\mathrm{i}, \\ \hat{\mathbf{u}} = 0 & \text{on } (0,T) \times \Gamma_\mathrm{w}, \\ \sigma_F(\hat{\mathbf{u}}, \tilde{p})\mathbf{n}_0 = 0 & \text{on } (0,T) \times \Gamma_\mathrm{N}, \\ \hat{\mathbf{u}}(0,.) = \mathbf{u}_0(.) - \overline{\mathbf{u}}(0,.) & \text{in } \mathscr{F}_0, \\ \mathcal{M}_{0,0} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\partial S_0} [\tilde{p}I - \nu(\nabla(\hat{\mathbf{u}} + \overline{\mathbf{u}}) + (\nabla(\hat{\mathbf{u}} + \overline{\mathbf{u}}))^T)]\mathbf{n}_0 \cdot \partial_{\theta_1}\mathbf{\Phi}(0,0,\gamma_y) \\ \displaystyle\int_{\partial S_0} [\tilde{p}I - \nu(\nabla(\hat{\mathbf{u}} + \overline{\mathbf{u}}) + (\nabla(\hat{\mathbf{u}} + \overline{\mathbf{u}}))^T)]\mathbf{n}_0 \cdot \partial_{\theta_2}\mathbf{\Phi}(0,0,\gamma_y) \end{pmatrix} + \mathbf{s} & \text{on } (0,T), \\ \dot{\theta}_1(0) = \omega_{1,0}, \quad \dot{\theta}_2(0) = \omega_{2,0}, & \\ \theta_1(0) = 0, \quad \theta_2(0) = 0. & \end{array} \right.$$

We have

$$\begin{array}{l} \mathbf{f} - \dfrac{\partial\overline{\mathbf{u}}}{\partial t} + \nu\Delta\overline{\mathbf{u}} \in L^2(0,T;\mathbf{L}^2(\mathscr{F}_0)), \\ \mathbf{u}_0(.) - \overline{\mathbf{u}}(0,.) = 0 \text{ on } \Gamma_\mathrm{i}, \\ s_j + \displaystyle\int_{\partial S_0} -\nu(\nabla\hat{\mathbf{u}} + (\nabla\hat{\mathbf{u}})^T)\mathbf{n}_0 \cdot \partial_{\theta_j}\mathbf{\Phi}(0,0,\gamma_y) \in L^2(0,T). \end{array}$$

Then, according to Proposition 2.2, $(\hat{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2) \in \mathbb{U}_T \times \mathbb{P}_T \times \Theta_T$ and we have (2.8) with $\hat{\mathbf{u}} = \tilde{\mathbf{u}}$.

Now, we consider $\tilde{\mathbf{u}} = \hat{\mathbf{u}} + \overline{\mathbf{u}}$, then $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2) \in \mathbb{U}_T \times \mathbb{P}_T \times \Theta_T$ and (2.17) is a consequence of (2.26)–(2.27) and (2.8). □

Note that a larger space than $H^1(0,T_0;\mathbf{U}^i)$ could be considered for $\mathbf{u}^i$. Indeed, we use a lifting in space only, inducing the requirement $\mathbf{u}^i \in H^1(0,T_0;\mathbf{U}^i)$. Using a space-time lifting would be slightly more complicated (see [15]), but would allow a larger space for the inflow boundary datum $\mathbf{u}^i$.

3 Local existence of solution to the full problem

In this section, we study the nonlinear problem. We recall that $\theta_{1,0} = \theta_{2,0} = 0$. At first, we rewrite the equations (1.23) in the fixed domain $\mathscr{F}_0$, then, we prove existence of a solution to this problem.

3.1 The equations in a fixed domain

Our goal is to write the equations (1.23) in the fixed domain $\mathscr{F}_0$. To do so, we use the diffeomorphism defined in (1.29). We denote $\mathcal{J}_{\mathbf{\Phi}}$ its Jacobian matrix and $\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}})$ the cofactor matrix of $\mathcal{J}_{\mathbf{\Phi}}$. We use the change of variables $\tilde{\mathbf{u}}(t,\mathbf{y}) = \mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1(t),\theta_2(t),\mathbf{y}))^T \mathbf{u}(t,\mathbf{\Phi}(\theta_1(t),\theta_2(t),\mathbf{y}))$ and $\tilde{p}(t,\mathbf{y}) = p(t,\mathbf{\Phi}(\theta_1(t),\theta_2(t),\mathbf{y}))$ for every $t \in [0,T]$ and $\mathbf{y} \in \mathscr{F}_0$.

This choice is motivated by the fact that, according to [4, Lemma 3.1], we get $\mathrm{div}\,\tilde{\mathbf{u}} = 0$.

In the sequel, v_i denotes the i^{th} component of the vector $\mathbf{v}$. To compute the equations satisfied by $(\tilde{\mathbf{u}},\tilde{p},\theta_1,\theta_2)$, we use the following explicit formula:

$$\mathbf{u}(t,\mathbf{x}) = \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1(t),\theta_2(t),\mathbf{x}))^T \tilde{\mathbf{u}}(t,\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x})),$$

we have

$$\begin{aligned}\partial_t \mathbf{u}(t,\mathbf{x}) = {} & \mathrm{cof}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_{\mathbf{\Psi}}(\theta_1(t),\theta_2(t),\mathbf{x})\right)^T \tilde{\mathbf{u}}(t,\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x})) \\ & + \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1(t),\theta_2(t),\mathbf{x}))^T \partial_t \tilde{\mathbf{u}}(t,\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x})) \\ & + \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1(t),\theta_2(t),\mathbf{x}))^T \nabla_{\mathbf{y}} \tilde{\mathbf{u}}(t,\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x})) \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x}),\end{aligned}$$

$$\begin{aligned}\partial_{x_j} \mathbf{u}(t,\mathbf{x}) = {} & \mathrm{cof}(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1(t),\theta_2(t),\mathbf{x}))^T \tilde{\mathbf{u}}(t,\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x})) \\ & + \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1(t),\theta_2(t),\mathbf{x}))^T \nabla_{\mathbf{y}} \tilde{\mathbf{u}}(t,\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x})) \partial_{x_j}\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x}),\end{aligned}$$

and

$$\begin{aligned}\partial^2_{x_j} \mathbf{u}(t,\mathbf{x}) = {} & \mathrm{cof}(\partial^2_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1(t),\theta_2(t),\mathbf{x}))^T \tilde{\mathbf{u}}(t,\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x})) \\ & + 2\mathrm{cof}(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1(t),\theta_2(t),\mathbf{x}))^T \nabla_{\mathbf{y}} \tilde{\mathbf{u}}(t,\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x})) \partial_{x_j}\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x}) \\ & + \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1(t),\theta_2(t),\mathbf{x}))^T \sum_k \partial_{y_k} \nabla_{\mathbf{y}} \tilde{\mathbf{u}}(t,\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x})) \partial_{x_j}\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x}) \partial_{x_j}\Psi_k(\theta_1(t),\theta_2(t),\mathbf{x}) \\ & + \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1(t),\theta_2(t),\mathbf{x}))^T \nabla_{\mathbf{y}} \tilde{\mathbf{u}}(t,\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x})) \partial^2_{x_j}\mathbf{\Psi}(\theta_1(t),\theta_2(t),\mathbf{x}).\end{aligned}$$

Problem (1.23) in the fixed domain reads (2.16) where $\mathbf{f},\mathbf{g},\mathbf{s}$ are defined by

$$\begin{cases} \mathbf{f} = \mathbf{F}(\theta_1,\theta_2,\tilde{\mathbf{u}},\tilde{p}) + \mathbf{f}_{\mathscr{F}}(t,\mathbf{\Phi}(\theta_1(t),\theta_2(t),\mathbf{y})), \\ \mathbf{g} = \mathbf{G}(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2), \\ \mathbf{s} = \mathbf{S}(\theta_1,\theta_2,\tilde{\mathbf{u}},\tilde{p}) + \mathbf{f}_{\mathbf{s}}, \end{cases} \tag{3.1}$$

and we can decompose $\mathbf{F}(\theta_1,\theta_2,\tilde{\mathbf{u}},\tilde{p}) = \mathbf{F}^1 + \mathbf{F}^2 + \mathbf{F}^3 + \mathbf{F}^4 + \mathbf{F}^5$, where $\mathbf{F}^i$, $\mathbf{G}$ and $\mathbf{S}$ are given below in (3.2).

We write $\mathbf{\Phi}(\theta_1, \theta_2, .)$ under the simpler notation $\mathbf{\Phi}$. The nonlinear terms are given as follows:

$$
\begin{aligned}
\mathbf{F}^1(\theta_1,\theta_2,\tilde{\mathbf{u}}) &= (I-\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))^T)\frac{\partial \tilde{\mathbf{u}}}{\partial t},\\
\mathbf{F}^2(\theta_1,\theta_2,\tilde{\mathbf{u}}) &= -\mathrm{cof}(\dot{\theta}_1\nabla_{\mathbf{x}}\partial_{\theta_1}\mathbf{\Psi}(\theta_1,\theta_2,\mathbf{\Phi})+\dot{\theta}_2\nabla_{\mathbf{x}}\partial_{\theta_2}\mathbf{\Psi}(\theta_1,\theta_2,\mathbf{\Phi}))^T\tilde{\mathbf{u}}(t,\mathbf{y})\\
&\quad -\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))^T(\nabla_{\mathbf{y}}\tilde{\mathbf{u}})\Big(\dot{\theta}_1\partial_{\theta_1}\mathbf{\Psi}(\theta_1,\theta_2,\mathbf{\Phi})+\dot{\theta}_2\partial_{\theta_2}\mathbf{\Psi}(\theta_1,\theta_2,\mathbf{\Phi})\Big),\\
\mathbf{F}^3(\theta_1,\theta_2,\tilde{\mathbf{u}})_i &= \nu\sum_{j,k,l,m}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))_{ki}\frac{\partial^2\tilde{u}_k}{\partial y_l\partial y_m}\frac{\partial\Psi_l}{\partial x_j}(\theta_1,\theta_2,\mathbf{\Phi})\frac{\partial\Psi_m}{\partial x_j}(\theta_1,\theta_2,\mathbf{\Phi})\\
&\quad +2\nu\sum_{j,k,l}\frac{\partial}{\partial x_j}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))_{ki}\frac{\partial\tilde{u}_k}{\partial y_l}\frac{\partial\Psi_l}{\partial x_j}(\theta_1,\theta_2,\mathbf{\Phi})\\
&\quad +\nu\sum_{j,k,l}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))_{ki}\frac{\partial\tilde{u}_k}{\partial y_l}\frac{\partial^2\Psi_l}{\partial x_j^2}(\theta_1,\theta_2,\mathbf{\Phi})\\
&\quad +\nu\sum_{j,k}\frac{\partial^2}{\partial x_j^2}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))_{ki}\tilde{u}_k-\nu\Delta_{\mathbf{y}}\tilde{u}_i(t,\mathbf{y}),\\
\mathbf{F}^4(\theta_1,\theta_2,\tilde{\mathbf{u}})_i &= -\sum_{j,k,r}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))_{kj}\frac{\partial}{\partial x_j}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2))_{ri}\tilde{u}_k\tilde{u}_r\\
&\quad -\sum_{k,r}\det(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))^2\frac{\partial\Phi_i}{\partial y_r}\tilde{u}_k\frac{\partial\tilde{u}_r}{\partial y_k},\\
\mathbf{F}^5(\theta_1,\theta_2,\tilde{p}) &= (I-\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))^T\nabla_{\mathbf{y}}\tilde{p},\\
\mathbf{G}(\theta_1,\theta_2,\omega_1,\omega_2) &= \sum_{j=1}^2\omega_j\Big(\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1,\theta_2))^T\partial_{\theta_j}\mathbf{\Phi}(\theta_1,\theta_2,\mathbf{y})-\partial_{\theta_j}\mathbf{\Phi}(0,0,\mathbf{y})\Big),\\
\mathbf{S}(\theta_1,\theta_2,\tilde{\mathbf{u}},\tilde{p}) &= -(\mathcal{M}_{\theta_1,\theta_2}-\mathcal{M}_{0,0})\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix}+\mathbf{M_I}(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2)\\
&\quad +\begin{pmatrix}\displaystyle\int_{\partial S_0}|\mathcal{J}_{\mathbf{\Phi}}\mathbf{t}_0|\,[\tilde{p}I-\nu(\mathcal{G}(\theta_1,\theta_2,\tilde{\mathbf{u}})+\mathcal{G}(\theta_1,\theta_2,\tilde{\mathbf{u}})^T)]\mathbf{n}_{\theta_1,\theta_2}(\mathbf{\Phi})\cdot\partial_{\theta_1}\mathbf{\Phi}(\theta_1,\theta_2,\gamma_y)\\ \displaystyle\int_{\partial S_0}|\mathcal{J}_{\mathbf{\Phi}}\mathbf{t}_0|\,[\tilde{p}I-\nu(\mathcal{G}(\theta_1,\theta_2,\tilde{\mathbf{u}})+\mathcal{G}(\theta_1,\theta_2,\tilde{\mathbf{u}})^T)]\mathbf{n}_{\theta_1,\theta_2}(\mathbf{\Phi})\cdot\partial_{\theta_2}\mathbf{\Phi}(\theta_1,\theta_2,\gamma_y)\end{pmatrix}\\
&\quad -\begin{pmatrix}\displaystyle\int_{\partial S_0}[\tilde{p}I-\nu(\nabla\tilde{\mathbf{u}}+(\nabla\tilde{\mathbf{u}})^T)]\mathbf{n}_0\cdot\partial_{\theta_1}\mathbf{\Phi}(0,0,\gamma_y)\\ \displaystyle\int_{\partial S_0}[\tilde{p}I-\nu(\nabla\tilde{\mathbf{u}}+(\nabla\tilde{\mathbf{u}})^T)]\mathbf{n}_0\cdot\partial_{\theta_2}\mathbf{\Phi}(0,0,\gamma_y)\end{pmatrix},
\end{aligned}
\tag{3.2}
$$

where $\mathbf{t}_0$ is a unitary tangent vector to ∂S_0, $\mathbf{M_I}$ and $\mathcal{M}_{\theta_1,\theta_2}$ are defined in (1.15), (1.16) and

$$
\mathcal{G}(\theta_1,\theta_2,\tilde{\mathbf{u}})_{ij}=\sum_k\mathrm{cof}\left[\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,.))\circ\mathbf{\Phi}\right]_{ki}\tilde{u}_k+\sum_{k,l}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))_{ki}\frac{\partial\tilde{u}_k}{\partial y_l}\frac{\partial\Psi_l}{\partial x_j}(\theta_1,\theta_2,\mathbf{\Phi}). \tag{3.3}
$$

We can state the following theorem.

Theorem 3.1. *Let $T_0 > 0$. Let $\mathbf{u}^i \in H^1(0,T_0;\mathbf{U}^i)$. For every $(\mathbf{u}_0,\omega_{1,0},\omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}_0)\times\mathbb{R}^2$ satisfying the compatibility conditions* (1.33)*, there exists $T \in (0,T_0)$ such that for every $(\mathbf{f}_{\mathscr{F}},\mathbf{f_s}) \in L^2(0,T;\mathbf{W}^{1,\infty}(\Omega)) \times L^2(0,T;\mathbb{R}^2)$ problem* (2.16) *where the source terms are given by* (3.1) *admits a unique solution $(\tilde{\mathbf{u}},\tilde{p},\theta_1,\theta_2) \in \mathbb{U}_T\times\mathbb{P}_T\times(\Theta_T\cap L^2(0,T;\mathbb{D}_\Theta))$ satisfying the following estimate*

$$
\|\tilde{\mathbf{u}}\|_{\mathbb{U}_T}+\|\tilde{p}\|_{\mathbb{P}_T}+\|(\theta_1,\theta_2)\|_{\Theta_T}\leq C(\|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)}+|\omega_{1,0}|+|\omega_{2,0}|+\|\mathbf{f}_{\mathscr{F}}\|_{L^2(0,T_0;\mathbf{L}^2(\mathscr{F}_0))}+\|\mathbf{u}^i\|_{H^1(0,T_0;\mathbf{U}^i)}+\|\mathbf{f_s}\|_{L^2(0,T_0)}),
$$

where C does not depend on T, $\mathbf{f}_{\mathscr{F}}$, $\mathbf{f_s}$ and $\mathbf{u}^i$.

This theorem is the rewriting of Theorem 1.6 in the fixed domain $\mathscr{F}_0$. To prove Theorem 3.1, we use the results of Section 2 and a fixed point argument.

3.2 Proof of Theorem 3.1

Proof. We work in the fixed fluid domain $\mathscr{F}_0$. Let $T_0 > 0$.
Let $\mathbf{u}^i \in H^1(0,T_0;\mathbf{U}^i)$ and $(\mathbf{u}_0,\omega_{1,0},\omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}_0)\times\mathbb{R}^2$ satisfying the compatibility conditions (1.33).

We define the space

$$
\mathbb{N}_T=\mathbb{U}_T\times\mathbb{P}_T\times(\Theta_T\cap L^2(0,T;\mathbb{D}_\Theta)),
$$

endowed with the norm

$$
\|(\tilde{\mathbf{u}},\tilde{p},\theta_1,\theta_2)\|_{\mathbb{N}_T}=\|\tilde{\mathbf{u}}\|_{\mathbb{U}_T}+\|\tilde{p}\|_{\mathbb{P}_T}+\|(\theta_1,\theta_2)\|_{\Theta_T}. \tag{3.4}
$$

We also define an application Λ^T on $\mathbb{N}_T$ such that for every $(\overline{\mathbf{u}}, \overline{p}, \overline{\theta}_1, \overline{\theta}_2) \in \mathbb{N}_T$, $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2) = \Lambda^T(\overline{\mathbf{u}}, \overline{p}, \overline{\theta}_1, \overline{\theta}_2) \in \mathbb{U}_T \times \mathbb{P}_T \times \Theta_T$ is the solution to problem (2.16), where the nonhomogeneous terms are given by

$$\begin{aligned}
&\mathbf{f} = \mathbf{F}(\overline{\theta}_1, \overline{\theta}_2, \overline{\mathbf{u}}, \overline{p}) + \mathbf{f}_{\mathscr{F}}(t, \boldsymbol{\Phi}(\overline{\theta}_1, \overline{\theta}_2, \mathbf{y})),\\
&\mathbf{g} = \mathbf{G}(\overline{\theta}_1, \overline{\theta}_2, \dot{\overline{\theta}}_1, \dot{\overline{\theta}}_2),\\
&\mathbf{s} = \mathbf{S}(\overline{\theta}_1, \overline{\theta}_2, \overline{\mathbf{u}}, \overline{p}) + \mathbf{f}_{\mathbf{s}},
\end{aligned}$$

where $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{S}$ are given by (3.2). Note that Λ^T depends on the initial data $(\mathbf{u}_0, \omega_{1,0}, \omega_{2,0})$ and on the source term $\mathbf{u}^i$.

We take

$$R = 2C(\|\mathbf{u}^i\|_{H^1(0,T_0;\mathbf{U}^i)} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)} + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}_{\mathscr{F}}\|_{L^2(0,T_0;\mathbf{L}^2(\mathscr{F}_0))} + \|\mathbf{f}_{\mathbf{s}}\|_{L^2(0,T_0)}),$$

where C is the constant of Proposition 2.7, so that by Proposition 2.7 , we have

$$\|\Lambda^T(0,0,0,0)\|_{\mathbb{N}_T} \leq C(\|\mathbf{u}^i\|_{H^1(0,T_0;\mathbf{U}^i)} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)} + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}_{\mathscr{F}}\|_{L^2(0,T_0;\mathbf{L}^2(\mathscr{F}_0))} + \|\mathbf{f}_{\mathbf{s}}\|_{L^2(0,T_0)}) = R/2. \quad (3.5)$$

The strategy adopted is based on the existence of $T > 0$ such that Λ^T is a contraction on

$$\mathbb{B}_R(T) = \left\{ \ (\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2) \in \mathbb{N}_T \quad | \quad \|(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)\|_{\mathbb{N}_T} \leq R, \ (\theta_1, \theta_2)(0) = (0,0) \ \right\}. \quad (3.6)$$

Remark 3.2. The domain $\mathbb{D}_\Theta$ is an open subset of $\mathbb{R}^2$ and $(0,0) \in \mathbb{D}_\Theta$, then there exists $r > 0$ such that $B((0,0),r) \subset \mathbb{D}_\Theta$. Then for $T < r/R$, if $\|\dot{\theta}_j\|_{L^\infty(0,T)} \leq R$ and $\theta_j(0) = 0$, we have

$$\|\theta_j\|_{L^\infty(0,T)} \leq T\|\dot{\theta}_j\|_{L^\infty(0,T)} \leq RT \leq r,$$

and we have for all $t \in (0,T)$, $(\theta_1(t), \theta_2(t)) \in \mathbb{D}_\Theta$. In the sequel we choose $T_0 > 0$ such that $T_0 < r/R$.

The solution to the nonlinear problem will be obtained as a fixed point of the application Λ^T. We use the estimates of the following lemma.

Lemma 3.3. *For every $R' > 0$, there exists a constant $C' = C'(R') > 0$, such that for every $T \in (0, T_0)$, and every $(\tilde{\mathbf{u}}^j, \tilde{p}^j, \theta_1^j, \theta_2^j) \in \mathbb{B}_{R'}(T)$, we have*

$$\begin{aligned}
\|\mathbf{F}(\theta_1^a, \theta_2^a, \tilde{\mathbf{u}}^a, \tilde{p}^a) - \mathbf{F}(\theta_1^b, \theta_2^b, \tilde{\mathbf{u}}^b, \tilde{p}^b)\|_{\mathbb{F}_T} &\leq C'T^{1/4}(\|\tilde{\mathbf{u}}^a - \tilde{\mathbf{u}}^b\|_{\mathbb{U}_T} + \|\tilde{p}^a - \tilde{p}^b\|_{\mathbb{P}_T} + \|\theta^a - \theta^b\|_{\Theta_T}), & (3.7)\\
\|\mathbf{G}(\theta_1^a, \theta_2^a, \dot{\theta}_1^a, \dot{\theta}_2^a) - \mathbf{G}(\theta_1^b, \theta_2^b, \dot{\theta}_1^b, \dot{\theta}_2^b)\|_{\mathbb{G}_T} &\leq C'T\|\theta^a - \theta^b\|_{\Theta_T}, & (3.8)\\
\|\mathbf{S}(\theta_1^a, \theta_2^a, \tilde{\mathbf{u}}^a, \tilde{p}^a) - \mathbf{S}(\theta_1^b, \theta_2^b, \tilde{\mathbf{u}}^b, \tilde{p}^b)\|_{\mathbb{S}_T} &\leq C'T^{1/2}(\|\tilde{\mathbf{u}}^a - \tilde{\mathbf{u}}^b\|_{\mathbb{U}_T} + \|\tilde{p}^a - \tilde{p}^b\|_{\mathbb{P}_T} + \|\theta^a - \theta^b\|_{\Theta_T}), & (3.9)\\
\|\mathbf{f}_{\mathscr{F}}(t, \boldsymbol{\Phi}(\theta_1^a, \theta_2^a, \mathbf{y})) - \mathbf{f}_{\mathscr{F}}(t, \boldsymbol{\Phi}(\theta_1^b, \theta_2^b, \mathbf{y}))\|_{\mathbb{F}_T} &\leq C'T\|\theta^a - \theta^b\|_{\Theta_T}. & (3.10)
\end{aligned}$$

These estimates are proven in Appendix A.

For $(\tilde{\mathbf{u}}^j, \tilde{p}^j, \theta_1^j, \theta_2^j) \in \mathbb{B}_R(T)$, we have $\mathbf{G}(\theta_1^j, \theta_2^j, \dot{\theta}_1^j, \dot{\theta}_2^j)(0) = 0$, hence Proposition 2.7 yields the estimate

$$\begin{aligned}
&\|\Lambda^T(\tilde{\mathbf{u}}^a, \tilde{p}^a, \theta_1^a, \theta_2^a) - \Lambda^T(\tilde{\mathbf{u}}^b, \tilde{p}^b, \theta_1^b, \theta_2^b)\|_{\mathbb{U}_T \times \mathbb{P}_T \times \Theta_T}\\
&\quad \leq C(\|\mathbf{F}(\theta_1^a, \theta_2^a, \tilde{\mathbf{u}}^a, \tilde{p}^a) - \mathbf{F}(\theta_1^b, \theta_2^b, \tilde{\mathbf{u}}^b, \tilde{p}^b)\|_{\mathbb{F}_T} + \|\mathbf{G}(\theta_1^a, \theta_2^a, \dot{\theta}_1^a, \dot{\theta}_2^a) - \mathbf{G}(\theta_1^b, \theta_2^b, \dot{\theta}_1^b, \dot{\theta}_2^b)\|_{\mathbb{G}_T} \quad (3.11)\\
&\qquad + \|\mathbf{S}(\theta_1^a, \theta_2^a, \tilde{\mathbf{u}}^a, \tilde{p}^a) - \mathbf{S}(\theta_1^b, \theta_2^b, \tilde{\mathbf{u}}^b, \tilde{p}^b)\|_{\mathbb{S}_T} + \|\mathbf{f}_{\mathscr{F}}(t, \boldsymbol{\Phi}(\theta_1^a, \theta_2^a, \mathbf{y})) - \mathbf{f}_{\mathscr{F}}(t, \boldsymbol{\Phi}(\theta_1^b, \theta_2^b, \mathbf{y}))\|_{\mathbb{F}_T}),
\end{aligned}$$

and with Lemma 3.3, we have

$$\|\Lambda^T(\tilde{\mathbf{u}}^a, \tilde{p}^a, \theta_1^a, \theta_2^a) - \Lambda^T(\tilde{\mathbf{u}}^b, \tilde{p}^b, \theta_1^b, \theta_2^b)\|_{\mathbb{U}_T \times \mathbb{P}_T \times \Theta_T} \leq KT^{1/4}(\|\tilde{\mathbf{u}}^a - \tilde{\mathbf{u}}^b\|_{\mathbb{U}_T} + \|\tilde{p}^a - \tilde{p}^b\|_{\mathbb{P}_T} + \|\theta^a - \theta^b\|_{\Theta_T}), \quad (3.12)$$

where $K = 4CC'(R)$ depends on R but not on T. Then, for $T \in (0, T_0)$ such that

$$KT^{1/4} \leq 1/2,$$

the application Λ^T is a contraction on $\mathbb{B}_R(T)$. Moreover, (3.12) and (3.5) yield

$$\|\Lambda^T(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)\|_{\mathbb{U}_T \times \mathbb{P}_T \times \Theta_T} \leq \|\Lambda^T(0,0,0,0)\|_{\mathbb{N}_T} + KT^{1/4}(\|\tilde{\mathbf{u}}\|_{\mathbb{U}_T} + \|\tilde{p}\|_{\mathbb{P}_T} + \|\theta\|_{\Theta_T}) \leq R/2 + KRT^{1/4} \leq R.$$

According to Remark 3.2, we have proven that $\Lambda^T : \mathbb{B}_R(T) \to \mathbb{B}_R(T)$ is a contraction. Then, according to the Picard fixed point theorem, there exists a unique fixed point to Λ^T in $\mathbb{B}_R(T)$. This fixed point is the solution to problem (2.16) where the source terms are given by (3.1). This proves Theorem 3.1. ☐

Remark 3.4. Fixing the initial condition for (θ_1, θ_2) in (3.6) is necessary to get the term T in the estimates of Lemma 3.3. We could also have fixed $\dot{\theta}_1(0)$ and $\dot{\theta}_2(0)$ in (3.6). As it is not needed in our proof of Lemma 3.3, we have chosen to let it free.

3.3 Proof of the result in the moving domain, Theorem 1.6

We consider $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)$ in $\mathbb{U}_T \times \mathbb{P}_T \times \Theta_T$ the solution to problem (2.16) with (3.1) given by Theorem 3.1. Let $\mathbf{u}(t,\mathbf{x}) = \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1(t), \theta_2(t), \mathbf{x}))^T \tilde{\mathbf{u}}(t, \mathbf{\Psi}(\theta_1(t), \theta_2(t), \mathbf{x}))$ and $p(t,\mathbf{x}) = \tilde{p}(t, \mathbf{\Psi}(\theta_1(t), \theta_2(t), \mathbf{x}))$. Then the quadruplet $(\mathbf{u}, p, \theta_1, \theta_2)$ is solution to the problem in the moving domain. This proves Theorem 1.6.

A Proof of Lemma 3.3

This section is devoted to the proof of Lemma 3.3. We start with some intermediate lemmas that will be used to decompose the intricate terms of Lemma 3.3 in smaller pieces.

A.1 Technical Lemmas

The following lemma contains Lipschitz estimates on several terms.

Lemma A.1. *For $R > 0$, there exists a constant $C = C(R) > 0$ such that for every $T \in (0, T_0)$ and every $(.,.,\theta_1^j, \theta_2^j) \in \mathbb{B}_R(T)$, the following estimates hold*

$$\|\mathbf{\Phi}(\theta_1^a, \theta_2^a) - \mathbf{\Phi}(\theta_1^b, \theta_2^b)\|_{L^\infty(0,T;\mathbf{L}^\infty(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.1}$$
$$\|\mathcal{J}_{\mathbf{\Phi}}(\theta_1^a, \theta_2^a) - \mathcal{J}_{\mathbf{\Phi}}(\theta_1^b, \theta_2^b)\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.2}$$
$$\|\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a, \theta_2^a) - \mathcal{J}_{\mathbf{\Psi}}(\theta_1^b, \theta_2^b)\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.3}$$
$$\|(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a, \theta_2^a)) \circ \mathbf{\Phi}(\theta_1^a, \theta_2^a) - (\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b, \theta_2^b)) \circ \mathbf{\Phi}(\theta_1^b, \theta_2^b)\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.4}$$
$$\|(\partial^2_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a, \theta_2^a)) \circ \mathbf{\Phi}(\theta_1^a, \theta_2^a) - (\partial^2_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b, \theta_2^b)) \circ \mathbf{\Phi}(\theta_1^b, \theta_2^b)\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.5}$$
$$\|\mathcal{M}_{\theta_1^a,\theta_2^a} - \mathcal{M}_{\theta_1^b,\theta_2^b}\|_{L^\infty(0,T)} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.6}$$
$$\|\mathbf{n}_{\theta_1^a,\theta_2^a}(\mathbf{\Phi}(\theta_1^a, \theta_2^a)) - \mathbf{n}_{\theta_1^b,\theta_2^b}(\mathbf{\Phi}(\theta_1^b, \theta_2^b))\|_{L^\infty(0,T;\mathbf{L}^\infty(\partial S_0))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.7}$$
$$\|\det(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a, \theta_2^a)) - \det(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b, \theta_2^b))\|_{L^\infty(0,T;\mathbf{L}^\infty(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.8}$$
$$\|\partial_{\theta_j}\mathbf{\Phi}(\theta_1^a, \theta_2^a, .) - \partial_{\theta_j}\mathbf{\Phi}(\theta_1^b, \theta_2^b, .)\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.9}$$
$$\|\partial_{\theta_k\theta_j}\mathbf{\Phi}(\theta_1^a, \theta_2^a, .) - \partial_{\theta_k\theta_j}\mathbf{\Phi}(\theta_1^b, \theta_2^b, .)\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.10}$$
$$\|\, |\mathcal{J}_{\mathbf{\Phi}}(\theta_1^a, \theta_2^a)\mathbf{t}_0| - |\mathcal{J}_{\mathbf{\Phi}}(\theta_1^b, \theta_2^b)\mathbf{t}_0| \,\|_{L^\infty(0,T;L^\infty(\partial S_0))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.11}$$

and

$$\|\partial_t\mathcal{J}_{\mathbf{\Phi}}(\theta_1^a, \theta_2^a) - \partial_t\mathcal{J}_{\mathbf{\Phi}}(\theta_1^b, \theta_2^b)\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} \leq C\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.12}$$
$$\|\partial_t(\mathbf{\Psi}(\theta_1^a, \theta_2^a)) \circ \mathbf{\Phi}(\theta_1^a, \theta_2^a) - \partial_t(\mathbf{\Psi}(\theta_1^b, \theta_2^b)) \circ \mathbf{\Phi}(\theta_1^b, \theta_2^b)\|_{L^\infty(0,T;\mathbf{L}^\infty(\Omega))} \leq C\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.13}$$
$$\|\partial_t(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a, \theta_2^a)) \circ \mathbf{\Phi}(\theta_1^a, \theta_2^a) - \partial_t(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b, \theta_2^b)) \circ \mathbf{\Phi}(\theta_1^b, \theta_2^b)\|_{L^\infty(0,T;\mathbf{L}^\infty(\Omega))} \leq C\|\theta^a - \theta^b\|_{\Theta_T}. \tag{A.14}$$

Moreover, for every $(\tilde{\mathbf{u}}^j, ., \theta_1^j, \theta_2^j) \in \mathbb{B}_R(T)$, the following estimates hold on $\mathcal{G}$ defined in (3.3)

$$\|\mathcal{G}(\theta_1^a, \theta_2^a, \tilde{\mathbf{u}}^a) - \mathcal{G}(\theta_1^b, \theta_2^b, \tilde{\mathbf{u}}^b)\|_{L^2(0,T;\mathbf{L}^2(\partial S_0))} \leq C(\|\theta^a - \theta^b\|_{\Theta_T} + \|\tilde{\mathbf{u}}^a - \tilde{\mathbf{u}}^b\|_{\mathbb{U}_T}), \tag{A.15}$$
$$\|\nabla\tilde{\mathbf{u}}^a - \mathcal{G}(\theta_1^a, \theta_2^a, \tilde{\mathbf{u}}^a) - \nabla\tilde{\mathbf{u}}^b + \mathcal{G}(\theta_1^b, \theta_2^b, \tilde{\mathbf{u}}^b)\|_{L^2(0,T;\mathbf{L}^2(\partial S_0))} \leq CT(\|\theta^a - \theta^b\|_{\Theta_T} + \|\tilde{\mathbf{u}}^a - \tilde{\mathbf{u}}^b\|_{\mathbb{U}_T}). \tag{A.16}$$

In particular, as a direct application of Lemma A.1, using that $(0,0,0,0) \in \mathbb{B}_R(T)$, we obtain the following lemma.

Lemma A.2. *For $R > 0$, there exists a constant $C = C(R) > 0$, such that for every $T \in (0, T_0)$ and every $(.,.,\theta_1, \theta_2) \in \mathbb{B}_R(T)$, the following estimates hold*

$$\|\mathcal{J}_{\mathbf{\Phi}}(\theta_1, \theta_2) - I\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} \leq CT, \tag{A.17}$$
$$\|\mathcal{J}_{\mathbf{\Psi}}(\theta_1, \theta_2, \mathbf{\Phi}(\theta_1, \theta_2)) - I\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} \leq CT, \tag{A.18}$$
$$\|\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1, \theta_2) \circ \mathbf{\Phi}(\theta_1, \theta_2)\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} \leq CT, \tag{A.19}$$
$$\|\partial^2_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1, \theta_2) \circ \mathbf{\Phi}(\theta_1, \theta_2)\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq CT, \tag{A.20}$$
$$\|\mathcal{M}_{\theta_1,\theta_2} - \mathcal{M}_{0,0}\|_{L^\infty(0,T)} \leq CT, \tag{A.21}$$
$$\|\mathbf{n}_{\theta_1,\theta_2}(\mathbf{\Phi}(\theta_1, \theta_2)) - \mathbf{n}_0\|_{L^\infty(0,T;\mathbf{L}^\infty(\partial S_0))} \leq CT, \tag{A.22}$$
$$\|\, |\mathcal{J}_{\mathbf{\Phi}}\mathbf{t}_0| - 1\|_{L^\infty(0,T;L^\infty(\partial S_0))} \leq CT, \tag{A.23}$$

and

$$\|\partial_t(\mathcal{J}_{\mathbf{\Phi}}(\theta_1,\theta_2))\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} \leq C, \tag{A.24}$$

$$\left\|\frac{\partial}{\partial t}(\mathbf{\Psi}(\theta_1,\theta_2))\circ\mathbf{\Phi}(\theta_1,\theta_2)\right\|_{L^\infty(0,T;\mathbf{L}^\infty(\Omega))} \leq C, \tag{A.25}$$

$$\left\|\frac{\partial}{\partial t}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2))\circ\mathbf{\Phi}(\theta_1,\theta_2)\right\|_{L^\infty(0,T;\mathbf{L}^\infty(\Omega))} \leq C. \tag{A.26}$$

Moreover, for every $(\tilde{\mathbf{u}},.,\theta_1,\theta_2)\in\mathbb{B}_R(T)$*, we have the following estimate on* $\mathcal{G}$

$$\|\nabla\tilde{\mathbf{u}}-\mathcal{G}(\theta_1,\theta_2,\tilde{\mathbf{u}})\|_{L^2(0,T;\mathbf{L}^2(\partial S_0))} \leq CT. \tag{A.27}$$

Proof of Lemma A.1. Three kinds of estimates have to be proven. First estimates (A.1)–(A.10) are of the type

$$\|\alpha(\theta_1^a,\theta_2^a)-\alpha(\theta_1^b,\theta_2^b)\|_{L^\infty(0,T;\mathbb{X})} \leq CT\|(\theta_1^a,\theta_2^a)-(\theta_1^b,\theta_2^b)\|_{\Theta_T},$$

where α is a differentiable function defined on $\mathbb{D}_\Theta$ and valued in $\mathbb{X}$. We thus use Taylor series and get

$$\|\alpha(\theta_1^a,\theta_2^a)-\alpha(\theta_1^b,\theta_2^b)\|_{L^\infty(0,T;\mathbb{X})} \leq \sup_{(\theta_1,\theta_2)\in\mathbb{D}_\Theta}\|\nabla_\theta\alpha(\theta_1,\theta_2)\|_{L^\infty(0,T;\mathbb{X})}\|\theta^a-\theta^b\|_{L^\infty(0,T)}.$$

According to the definition of $\mathbb{B}_R(T)$ in (3.6), $\theta^a(0)=\theta^b(0)=(0,0)$, we finish with

$$\|\theta^a-\theta^b\|_{L^\infty(0,T)} \leq T\|\theta^a-\theta^b\|_{\Theta_T}.$$

The second type of estimates (A.12)–(A.14) is of the form

$$\|\alpha(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a)-\alpha(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)\|_{L^\infty(0,T;\mathbb{X})} \leq C\|(\theta_1^a,\theta_2^a)-(\theta_1^b,\theta_2^b)\|_{\Theta_T},$$

where α is now a function defined on $\mathbb{D}_\Theta\times\mathbb{R}^2$ with values in $\mathbb{X}$. We use the same strategy and get

$$\begin{aligned}
&\|\alpha(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a)-\alpha(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)\|_{L^\infty(0,T;\mathbb{X})}\\
&\qquad\leq \sup_{\substack{(\theta_1,\theta_2)\in\mathbb{D}_\Theta\\ |\omega_1|+|\omega_2|\leq R}}\|\nabla_{\theta,\omega}\alpha(\theta_1,\theta_2,\omega_1,\omega_2)\|_{L^\infty(0,T;\mathbb{X})}(\|\theta^a-\theta^b\|_{L^\infty(0,T)}+\|\dot\theta^a-\dot\theta^b\|_{L^\infty(0,T)}).
\end{aligned}$$

Note that contrary to the first type of estimates, we do not have the decay in T because we did not enforce $\dot\theta^a(0)=\dot\theta^b(0)$.

Estimate (A.15) is a direct consequence of (A.16). The last estimate to prove is (A.16). We do it via the computation

$$\begin{aligned}
(\mathcal{G}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a)&-\mathcal{G}(\theta_1^b,\theta_2^b,\tilde{\mathbf{u}}^b)-\nabla\tilde{\mathbf{u}}^a+\nabla\tilde{\mathbf{u}}^b)_{ij}\\
&=\sum_k\left(\mathrm{cof}(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a,.)\circ\mathbf{\Phi}^a)_{ki}-\mathrm{cof}(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b,.)\circ\mathbf{\Phi}^b)_{ki}\right)\tilde u_k^a\\
&\quad+\sum_k\mathrm{cof}(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b,.)\circ\mathbf{\Phi}^b)_{ki}(\tilde u_k^a-\tilde u_k^b)\\
&\quad+\sum_{k,l}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a,\mathbf{\Phi}^a)-\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b,\mathbf{\Phi}^b))_{ki}\frac{\partial\tilde u_k^a}{\partial y_l}\frac{\partial\Psi_l}{\partial x_j}(\theta_1^a,\theta_2^a,\mathbf{\Phi}^a)\\
&\quad+\sum_{k,l}\left(\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b,\mathbf{\Phi}^b))_{ki}\frac{\partial\Psi_l}{\partial x_j}(\theta_1^a,\theta_2^a,\mathbf{\Phi}^a)-\delta_{ki}\delta_{lj}\right)\left(\frac{\partial\tilde u_k^a}{\partial y_l}-\frac{\partial\tilde u_k^b}{\partial y_l}\right)\\
&\quad+\sum_{k,l}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b,\mathbf{\Phi}^b))_{ki}\frac{\partial\tilde u_k^b}{\partial y_l}\left(\frac{\partial\Psi_l}{\partial x_j}(\theta_1^a,\theta_2^a,\mathbf{\Phi}^a)-\frac{\partial\Psi_l}{\partial x_j}(\theta_1^b,\theta_2^b,\mathbf{\Phi}^b)\right),
\end{aligned}$$

and with the use of estimates (A.3), (A.4), (A.18) and (A.19) we get estimate (A.16). □

A.2 Detailed proof of Lemma 3.3

Proof. In all the estimates we use Lemmas A.1 and A.2.

• **Estimate** (3.7)**:** using (A.18) and (A.3), we get

$$\begin{aligned}
\|\mathbf{F}^1(\theta_1^a,\theta_2^a,\mathbf{v}^a)&-\mathbf{F}^1(\theta_1^b,\theta_2^b,\mathbf{v}^b)\|_{\mathbb{F}_T}\\
&\leq\left\|I-\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a))^T\right\|_{L^\infty(\mathbf{L}^\infty)}\left\|\frac{\partial\mathbf{v}^a}{\partial t}-\frac{\partial\mathbf{v}^b}{\partial t}\right\|_{L^2(\mathbf{L}^2)}\\
&\quad+\left\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b))^T-\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a))^T\right\|_{L^\infty(\mathbf{L}^\infty)}\left\|\frac{\partial\mathbf{v}^b}{\partial t}\right\|_{L^2(\mathbf{L}^2)}\\
&\leq KT(\|\theta^a-\theta^b\|_{\Theta_T}+\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathbb{U}_T}).
\end{aligned}$$

Now, using (A.26), (A.25), (A.14), (A.13), (A.3) and the estimate $\|\mathbf{v}\|_{L^2(0,T;\mathbf{H}^1(\mathscr{F}_0))} \leq T^{1/2}\|\mathbf{v}\|_{L^\infty(0,T;\mathbf{H}^1(\mathscr{F}_0))}$, we obtain

$$
\begin{aligned}
\|\mathbf{F}^2(\theta_1^a,\theta_2^a,\mathbf{v}^a) &-\mathbf{F}^2(\theta_1^b,\theta_2^b,\mathbf{v}^b)\|_{\mathbb{F}_T} \\
&\leq \|\mathrm{cof}(\partial_t \mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b)\circ \mathbf{\Phi}(\theta_1^b,\theta_2^b) - \partial_t \mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a)\circ \mathbf{\Phi}(\theta_1^a,\theta_2^a))^T\|_{L^\infty(\mathbf{L}^\infty)}\|\mathbf{v}^a\|_{L^2(\mathbf{L}^2)} \\
&\quad +\|\mathrm{cof}(\partial_t \mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b)\circ \mathbf{\Phi}(\theta_1^b,\theta_2^b))^T\|_{L^\infty(\mathbf{L}^\infty)}\|\mathbf{v}^b-\mathbf{v}^a\|_{L^2(\mathbf{L}^2)} \\
&\quad +\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b)\circ \mathbf{\Phi}(\theta_1^b,\theta_2^b) - \mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a)\circ \mathbf{\Phi}(\theta_1^a,\theta_2^a))^T\|_{L^\infty(\mathbf{L}^\infty)}\|\mathbf{v}^a\|_{L^2(\mathbf{H}^1)}\|\partial_t \mathbf{\Psi}(\theta_1^a,\theta_2^a)\circ\mathbf{\Phi}\|_{L^\infty(\mathbf{L}^\infty)} \\
&\quad +\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b)\circ \mathbf{\Phi}(\theta_1^b,\theta_2^b))^T\|_{L^\infty(\mathbf{L}^\infty)}\|\mathbf{v}^b-\mathbf{v}^a\|_{L^2(\mathbf{H}^1)}\|\partial_t \mathbf{\Psi}(\theta_1^a,\theta_2^a)\circ\mathbf{\Phi}\|_{L^\infty(\mathbf{L}^\infty)} \\
&\quad +\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b)\circ \mathbf{\Phi}(\theta_1^b,\theta_2^b))^T\|_{L^\infty(\mathbf{L}^\infty)}\|\mathbf{v}^b\|_{L^2(\mathbf{H}^1)}\|\partial_t(\mathbf{\Psi}(\theta_1^b,\theta_2^b))\circ\mathbf{\Phi} - \partial_t(\mathbf{\Psi}(\theta_1^a,\theta_2^a))\circ\mathbf{\Phi}\|_{L^\infty(\mathbf{L}^\infty)} \\
&\leq KT^{1/2}(\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathbb{U}_T} + \|\theta^a-\theta^b\|_{\Theta_T}).
\end{aligned}
$$

In the following estimate we use the Sobolev embedding : $\mathbf{H}^{1/2+\varepsilon_0} \hookrightarrow \mathbf{L}^4$ (see [1, Theorem 7.58]). We also use the fact that $\mathcal{J}_{\mathbf{\Psi}}$ is the identity near the boundary, i.e. $\mathcal{E}$ has support in Ω_ε (defined in Lemma 1.5), then $\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}(\theta_1,\theta_2,\mathbf{y})) - I = 0$ for $\mathbf{y}\in\Omega\backslash\Omega_\varepsilon$. Hence,

$$
\begin{aligned}
\left\|\frac{\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}})_{ki}\dfrac{\partial\Psi_l}{\partial x_j}\dfrac{\partial\Psi_m}{\partial x_j} - \delta_{ki}\delta_{lj}\delta_{mj}}{\prod_j r_j^\beta}\right\|_{L^\infty(\mathbf{L}^\infty(\Omega))}
&\leq \left\|\frac{\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}})_{ki}\dfrac{\partial\Psi_l}{\partial x_j}\dfrac{\partial\Psi_m}{\partial x_j} - \delta_{ki}\delta_{lj}\delta_{mj}}{\prod_j r_j^\beta}\right\|_{L^\infty(\mathbf{L}^\infty(\Omega_\varepsilon))} \\
&\leq \left\|\frac{1}{\prod_j r_j^\beta}\right\|_{\mathbf{L}^\infty(\Omega_\varepsilon)}\left\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}})_{ki}\frac{\partial\Psi_l}{\partial x_j}\frac{\partial\Psi_m}{\partial x_j} - \delta_{ki}\delta_{lj}\delta_{mj}\right\|_{L^\infty(\mathbf{L}^\infty(\Omega))}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\|\mathbf{F}^3(\theta_1^a,\theta_2^a,\mathbf{v}^a) - \mathbf{F}^3(\theta_1^b,\theta_2^b,\mathbf{v}^b)\|_{\mathbb{F}_T} \\
&\leq \nu\sum_{j,k,l,m}\left\|\frac{\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a))_{ki}\dfrac{\partial\Psi_l}{\partial x_j}(\theta_1^a,\theta_2^a)\dfrac{\partial\Psi_m}{\partial x_j}(\theta_1^a,\theta_2^a) - \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b))_{ki}\dfrac{\partial\Psi_l}{\partial x_j}(\theta_1^b,\theta_2^b)\dfrac{\partial\Psi_m}{\partial x_j}(\theta_1^b,\theta_2^b)}{\prod_n r_n^\beta}\right\|_{L^\infty(\mathbf{L}^\infty)} \\
&\times\left\|\frac{\partial^2 v_k^a}{\partial y_l\partial y_m}\right\|_{L^2(\mathbf{L}^2_\beta)} + \left\|\frac{\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b))\dfrac{\partial\Psi_l}{\partial x_j}(\theta_1^b,\theta_2^b)\dfrac{\partial\Psi_m}{\partial x_j}(\theta_1^b,\theta_2^b) - \delta_{lm}\delta_{ki}\delta_{li}}{\prod_n r_n^\beta}\right\|_{L^\infty(\mathbf{L}^\infty)}\left\|\frac{\partial^2 v_k^a}{\partial y_l\partial y_m} - \frac{\partial^2 v_k^b}{\partial y_l\partial y_m}\right\|_{L^2(\mathbf{L}^2_\beta)} \\
&+2\nu\sum_{j,k,l}\left\|\frac{\partial}{\partial x_j}(\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a))_{ki})\frac{\partial\Psi_l}{\partial x_j}(\theta_1^a,\theta_2^a) - \frac{\partial}{\partial x_j}(\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b))_{ki})\frac{\partial\Psi_l}{\partial x_j}(\theta_1^b,\theta_2^b)\right\|_{L^\infty(\mathbf{L}^4)}\|\mathbf{v}^a\|_{L^2(\mathbf{W}^{1,4})} \\
&\qquad + \left\|\frac{\partial}{\partial x_j}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b))_{ki}\frac{\partial\Psi_l}{\partial x_j}(\theta_1^b,\theta_2^b)\right\|_{L^\infty(\mathbf{L}^4)}\|\mathbf{v}^a-\mathbf{v}^b\|_{L^2(\mathbf{W}^{1,4})} \\
&+\nu\sum_{j,k,l}\left\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}})_{ki}(\theta_1^a,\theta_2^a)\frac{\partial^2\Psi_l}{\partial x_j^2}(\theta_1^a,\theta_2^a) - \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}})_{ki}(\theta_1^b,\theta_2^b)\frac{\partial^2\Psi_l}{\partial x_j^2}(\theta_1^b,\theta_2^b)\right\|_{L^\infty(\mathbf{L}^4)}\|\mathbf{v}^a\|_{L^2(\mathbf{W}^{1,4})} \\
&\qquad + \left\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}})_{ki}(\theta_1^b,\theta_2^b)\frac{\partial^2\Psi_l}{\partial x_j^2}(\theta_1^b,\theta_2^b)\right\|_{L^\infty(\mathbf{L}^4)}\|\mathbf{v}^a-\mathbf{v}^b\|_{L^2(\mathbf{W}^{1,4})} \\
&+\nu\sum_{j,k}\left\|\frac{\partial^2}{\partial x_j^2}\mathrm{cof}\Big(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a) - \mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b)\Big)_{ki}\right\|_{L^\infty(\mathbf{L}^2)}\|\mathbf{v}^a\|_{L^2(\mathbf{L}^\infty)} + \left\|\frac{\partial^2}{\partial x_j^2}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b))_{ki}\right\|_{L^\infty(\mathbf{L}^2)}\|\mathbf{v}^a-\mathbf{v}^b\|_{L^2(\mathbf{L}^\infty)}.
\end{aligned}
$$

Moreover,

$$\begin{aligned}
&\left\| \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a))_{ki}\frac{\partial\Psi_l}{\partial x_j}(\theta_1^a,\theta_2^a)\frac{\partial\Psi_m}{\partial x_j}(\theta_1^a,\theta_2^a) - \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b))_{ki}\frac{\partial\Psi_l}{\partial x_j}(\theta_1^b,\theta_2^b)\frac{\partial\Psi_m}{\partial x_j}(\theta_1^b,\theta_2^b)\right\|_{L^\infty(\mathbf{L}^\infty)} \\
&\quad \leq \|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a) - \mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b))_{ki}\|_{L^\infty(\mathbf{L}^\infty)}\|\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a)\|_{L^\infty(\mathbf{L}^\infty)}\|\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a)\|_{L^\infty(\mathbf{L}^\infty)} \\
&\qquad + \|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b))\|_{L^\infty(\mathbf{L}^\infty)}\|\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a) - \mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b)\|_{L^\infty(\mathbf{L}^\infty)}\|\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a)\|_{L^\infty(\mathbf{L}^\infty)} \\
&\qquad + \|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b))\|_{L^\infty(\mathbf{L}^\infty)}\|\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b)\|_{L^\infty(\mathbf{L}^\infty)}\|\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a) - \mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b)\|_{L^\infty(\mathbf{L}^\infty)} \\
&\quad \leq KT\|\theta^a - \theta^b\|_{\Theta_T},
\end{aligned}$$

and with similar estimates, we get

$$\|\mathbf{F}^3(\theta_1^a,\theta_2^a,\mathbf{v}^a) - \mathbf{F}^3(\theta_1^b,\theta_2^b,\mathbf{v}^b)\|_{\mathbb{F}_T} \leq KT(\|\theta^a-\theta^b\|_{\Theta_T} + \|\mathbf{v}^a - \mathbf{v}^b\|_{\mathbb{U}_T}).$$

The estimate on $\mathbf{F}^4$ can be decomposed in the following way

$$\begin{aligned}
&\|\mathbf{F}^4(\theta_1^a,\theta_2^a,\mathbf{v}^a) - \mathbf{F}^4(\theta_1^b,\theta_2^b,\mathbf{v}^b)\|_{\mathbb{F}_T} \\
&\leq \sum_{j,k,r}\left\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}})_{ki}(\theta_1^a,\theta_2^a)\frac{\partial}{\partial x_j}(\mathcal{J}_{\mathbf{\Psi}})_{ki}(\theta_1^a,\theta_2^a) - \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}})_{ki}(\theta_1^b,\theta_2^b)\frac{\partial}{\partial x_j}(\mathcal{J}_{\mathbf{\Psi}})_{ki}(\theta_1^b,\theta_2^b)\right\|_{L^\infty(L^\infty)}\|v_k^a v_r^a\|_{L^2(L^2)} \\
&\qquad + \left\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}})_{ki}(\theta_1^b,\theta_2^b)\frac{\partial}{\partial x_j}(\mathcal{J}_{\mathbf{\Psi}})_{ri}(\theta_1^b,\theta_2^b)\right\|_{L^\infty(L^\infty)}\|v_k^a v_r^a - v_k^b v_r^b\|_{L^2(L^2)} \\
&\qquad + \sum_{k,r}\left\|\det(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a))^2\frac{\partial\mathbf{\Phi}}{\partial y_r}(\theta_1^a,\theta_2^a) - \det(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b))^2\frac{\partial\mathbf{\Phi}}{\partial y_r}(\theta_1^b,\theta_2^b)\right\|_{L^\infty(\mathbf{L}^\infty)}\left\|v_k^a\frac{\partial v_r^a}{\partial y_k}\right\|_{L^2(L^2)} \\
&\qquad + \left\|\det(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b))^2\frac{\partial\mathbf{\Phi}}{\partial y_r}(\theta_1^b,\theta_2^b)\right\|_{L^\infty(\mathbf{L}^\infty)}\left\|v_k^a\frac{\partial v_r^a}{\partial y_k} - v_k^b\frac{\partial v_r^b}{\partial y_k}\right\|_{L^2(L^2)}.
\end{aligned}$$

At this point we use estimates (A.17), (A.18), (A.3), (A.2), (A.4), (A.19), (A.8) and the Sobolev embedding $\mathbf{H}^{1/2+\varepsilon_0} \hookrightarrow \mathbf{L}^4$ to obtain

$$\|\mathbf{F}^4(\theta_1^a,\theta_2^a,\mathbf{v}^a) - \mathbf{F}^4(\theta_1^b,\theta_2^b,\mathbf{v}^b)\|_{\mathbb{F}_T} \leq C\left(T\|\theta^a-\theta^b\|_{\Theta_T} + \|v_k^a v_r^a - v_k^b v_r^b\|_{L^2(L^2)} + \left\|v_k^a\frac{\partial v_r^a}{\partial y_k} - v_k^b\frac{\partial v_r^b}{\partial y_k}\right\|_{L^2(L^2)}\right).$$

Hölder inequalities yield

$$\begin{aligned}
\left\|v_k^a\frac{\partial v_r^a}{\partial y_k} - v_k^b\frac{\partial v_r^b}{\partial y_k}\right\|_{L^2(L^2)} &\leq \left\|(v_k^a - v_k^b)\frac{\partial v_r^a}{\partial y_k}\right\|_{L^2(0,T;L^2(\mathscr{F}_0))} + \left\|v_k^b\left(\frac{\partial v_r^a}{\partial y_k} - \frac{\partial v_r^b}{\partial y_k}\right)\right\|_{L^2(0,T;L^2(\mathscr{F}_0))} \\
&\leq T^{1/4}\Big(\left\|(v_k^a - v_k^b)\frac{\partial v_r^a}{\partial y_k}\right\|_{L^4(0,T;L^2(\mathscr{F}_0))} + \left\|v_k^b\left(\frac{\partial v_r^a}{\partial y_k} - \frac{\partial v_r^b}{\partial y_k}\right)\right\|_{L^4(0,T;L^2(\mathscr{F}_0))}\Big) \\
&\leq CT^{1/4}\left(\|v_k^a - v_k^b\|_{L^\infty(0,T;L^{10})}\left\|\frac{\partial v_r^a}{\partial y_k}\right\|_{L^4(0,T;L^{5/2})} + \|v_k^b\|_{L^\infty(0,T;L^{10})}\left\|\frac{\partial v_r^a}{\partial y_k} - \frac{\partial v_r^b}{\partial y_k}\right\|_{L^4(0,T;L^{5/2})}\right).
\end{aligned}$$

To estimate the previous terms, we adapt the proof of [4, p. 298]. We use the Sobolev interpolation

$$\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathbf{H}^{1/4}(\mathscr{F}_0)} \leq C\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathbf{H}^{1/2}(\mathscr{F}_0)}^{1/2}\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathbf{L}^2(\mathscr{F}_0)}^{1/2},$$

and we compute

$$\begin{aligned}
\left\|\frac{\partial v_r}{\partial y_k}\right\|_{L^4(0,T;\mathbf{H}^{1/4}(\mathscr{F}_0))}^4 = \int_0^T\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathbf{H}^{1/4}(\mathscr{F}_0)}^4\mathrm{d}t &\leq C^4\int_0^T\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathbf{H}^{1/2}(\mathscr{F}_0)}^2\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathbf{L}^2(\mathscr{F}_0)}^2\mathrm{d}t \\
&\leq C^4\left\|\frac{\partial v_r}{\partial y_k}\right\|_{L^2(0,T;\mathbf{H}^{1/2}(\mathscr{F}_0))}^2\left\|\frac{\partial v_r}{\partial y_k}\right\|_{L^\infty(0,T;\mathbf{L}^2(\mathscr{F}_0))}^2.
\end{aligned}$$

The same technique can be used on the term $\|v_k^a v_r^a - v_k^b v_r^b\|_{L^2(L^2)}$. Then the Sobolev embeddings $\mathbf{H}^{1/4}(\mathscr{F}_0) \hookrightarrow \mathbf{L}^{5/2}(\mathscr{F}_0)$ and $\mathbf{H}^1(\mathscr{F}_0) \hookrightarrow \mathbf{L}^{10}(\mathscr{F}_0)$ yield the estimate

$$\|\mathbf{F}^4(\theta_1^a,\theta_2^a,\mathbf{v}^a) - \mathbf{F}^4(\theta_1^b,\theta_2^b,\mathbf{v}^b)\|_{\mathbb{F}_T} \leq CT^{1/4}(\|\theta^a-\theta^b\|_{\Theta_T} + \|\mathbf{v}^a - \mathbf{v}^b\|_{\mathbb{U}_T}).$$

The following estimate uses (A.18) and (A.14),

$$\begin{aligned}\|\mathbf{F}^5(\theta_1^a,\theta_2^a,q^a) & -\mathbf{F}^5(\theta_1^b,\theta_2^b,q^b)\|_{\mathbb{F}_T} \\ & \leq \left\| \frac{\mathcal{J}_{\mathbf{\Psi}}^T(\theta_1^b,\theta_2^b) - \mathcal{J}_{\mathbf{\Psi}}^T(\theta_1^a,\theta_2^a)}{\prod_j r_j^\beta} \right\|_{\mathbf{L}^\infty} \|q^a\|_{L^2(H^1_\beta)} + \left\| \frac{I - \mathcal{J}_{\mathbf{\Psi}}^T(\theta_1^b,\theta_2^b)}{\prod_j r_j^\beta} \right\|_{L^\infty} \|q^a - q^b\|_{L^2(H^1_\beta)} \\ & \leq KT(\|\theta^a-\theta^b\|_{\Theta_T} + \|q^a-q^b\|_{\mathbb{P}_T}).\end{aligned}$$

• **Estimate** (3.8)**:** we use the fact that $\mathbf{H}^2(\mathscr{F}_0)$ is an algebra and estimates (A.17), (A.2), (A.9), (A.24), (A.12) and (A.10),

$$\begin{aligned}&\left\| \frac{\partial}{\partial t}\left(\mathbf{G}(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a) - \mathbf{G}(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)\right)\right\|_{L^2(0,T;\mathbf{H}^{3/2}(\partial S_0))} \\ &\leq \sum_{j=1}^2 \|\ddot\theta_j^a - \ddot\theta_j^b\|_{L^2(0,T)} \left\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1^a,\theta_2^a))^T\partial_{\theta_j}\mathbf{\Phi}(\theta_1^a,\theta_2^a,\mathbf{y}) - \partial_{\theta_j}\mathbf{\Phi}(0,0,\mathbf{y})\right\|_{L^\infty(0,T;\mathbf{H}^2(\mathscr{F}_0))} \\ &+ \sum_{j=1}^2 \|\ddot\theta_j^b\|_{L^2(0,T)} \left\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1^a,\theta_2^a,\mathbf{y}))^T\partial_{\theta_j}\mathbf{\Phi}(\theta_1^a,\theta_2^a,\mathbf{y}) - \mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1^b,\theta_2^b,\mathbf{y}))^T\partial_{\theta_j}\mathbf{\Phi}(\theta_1^b,\theta_2^b,\mathbf{y})\right\|_{L^\infty(0,T;\mathbf{H}^2(\mathscr{F}_0))} \\ &+ \sum_{j=1}^2 \|\dot\theta_j^a - \dot\theta_j^b\|_{L^2(0,T)} \left\|\frac{\partial}{\partial t}\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1^a,\theta_2^a))^T\partial_{\theta_j}\mathbf{\Phi}(\theta_1^a,\theta_2^a,\mathbf{y})\right\|_{L^\infty(0,T;\mathbf{H}^2(\mathscr{F}_0))} \\ &+ \sum_{j=1}^2 \|\dot\theta_j^b\|_{L^2(0,T)} \left\|\frac{\partial}{\partial t}\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1^a,\theta_2^a,\mathbf{y}))^T\partial_{\theta_j}\mathbf{\Phi}(\theta_1^a,\theta_2^a,\mathbf{y}) - \frac{\partial}{\partial t}\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1^b,\theta_2^b,\mathbf{y}))^T\partial_{\theta_j}\mathbf{\Phi}(\theta_1^b,\theta_2^b,\mathbf{y})\right\|_{L^\infty(0,T;\mathbf{H}^2(\mathscr{F}_0))} \\ &+ \sum_{j,k=1}^2 \|\dot\theta_j^a\dot\theta_k^a - \dot\theta_j^b\dot\theta_k^b\|_{L^2(0,T)}\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1^a,\theta_2^a,\mathbf{y}))^T\partial_{\theta_j\theta_k}\mathbf{\Phi}(\theta_1^a,\theta_2^a,\mathbf{y})\|_{L^\infty(0,T;\mathbf{H}^2(\mathscr{F}_0))} \\ &+ \sum_{j,k=1}^2 \|\dot\theta_j^b\dot\theta_k^b\|_{L^2(0,T)}\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1^a,\theta_2^a,\mathbf{y}))^T\partial_{\theta_j\theta_k}\mathbf{\Phi}(\theta_1^a,\theta_2^a,\mathbf{y}) - \mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1^b,\theta_2^b,\mathbf{y}))^T\partial_{\theta_j\theta_k}\mathbf{\Phi}(\theta_1^b,\theta_2^b,\mathbf{y})\|_{L^\infty(0,T;\mathbf{H}^2(\mathscr{F}_0))},\end{aligned}$$

and $\|\dot\theta_j^a\dot\theta_k^a - \dot\theta_j^b\dot\theta_k^b\|_{L^2(0,T;\mathbb{R})} \leq T^{1/2}\|\dot\theta_j^a\dot\theta_k^a - \dot\theta_j^b\dot\theta_k^b\|_{L^\infty(0,T;\mathbb{R})}$.

We have proven that $\left\|\partial_t\left(\mathbf{G}(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a) - \mathbf{G}(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)\right)\right\|_{L^2(0,T;\mathbf{H}^{3/2}(\partial S_0))} \leq KT^{1/2}\|\theta^a-\theta^b\|_{\Theta_T}$. With the same technique, we also prove $\|\mathbf{G}(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a) - \mathbf{G}(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)\|_{L^2(0,T;\mathbf{H}^{3/2}(\partial S_0))} \leq KT^{1/2}\|\theta^a-\theta^b\|_{\Theta_T}$ and we get estimate (3.8).

• **Estimate** (3.9)**:** we use the following decomposition

$$\begin{aligned}&[\mathbf{S}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a,\tilde p^a) - \mathbf{S}(\theta_1^b,\theta_2^b,\tilde{\mathbf{u}}^b,\tilde p^b)]_j \\ &= \left[\left(\mathcal{M}_{\theta_1^b,\theta_2^b} - \mathcal{M}_{\theta_1^a,\theta_2^a}\right)\begin{pmatrix}\ddot\theta_1^a\\ \ddot\theta_2^a\end{pmatrix}\right]_j + \left[\left(\mathcal{M}_{0,0} - \mathcal{M}_{\theta_1^b,\theta_2^b}\right)\begin{pmatrix}\ddot\theta_1^a - \ddot\theta_1^b\\ \ddot\theta_2^a - \ddot\theta_2^b\end{pmatrix}\right]_j + \left[\mathbf{M}_{\mathbf{I}}(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a) - \mathbf{M}_{\mathbf{I}}(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)\right]_j \\ &+ \int_{\partial S_0} |\mathcal{J}_{\mathbf{\Phi}}^a\mathbf{t}_0|\left[\tilde p^a I - \nu(\mathcal{G}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a) + \mathcal{G}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a)^T)\right](\mathbf{n}_{\theta_1^a,\theta_2^a}\circ\mathbf{\Phi}^a)\cdot(\partial_{\theta_j}\mathbf{\Phi}(\theta_1^a,\theta_2^a,\gamma_y) - \partial_{\theta_j}\mathbf{\Phi}(\theta_1^b,\theta_2^b,\gamma_y)) \\ &+ \int_{\partial S_0} |\mathcal{J}_{\mathbf{\Phi}}^a\mathbf{t}_0|\left[\tilde p^a I - \nu(\mathcal{G}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a) + \mathcal{G}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a)^T)\right](\mathbf{n}_{\theta_1^a,\theta_2^a}\circ\mathbf{\Phi}^a - \mathbf{n}_{\theta_1^b,\theta_2^b}\circ\mathbf{\Phi}^b)\cdot\partial_{\theta_j}\mathbf{\Phi}(\theta_1^b,\theta_2^b,\gamma_y) \\ &+ \int_{\partial S_0} -\nu|\mathcal{J}_{\mathbf{\Phi}}^a\mathbf{t}_0|\left[\mathcal{G}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a)+\mathcal{G}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a)^T - \mathcal{G}(\theta_1^b,\theta_2^b,\tilde{\mathbf{u}}^b) - \mathcal{G}(\theta_1^b,\theta_2^b,\tilde{\mathbf{u}}^b)^T - \nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b) - \nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b)^T\right] \\ &\qquad\qquad (\mathbf{n}_{\theta_1^b,\theta_2^b}\circ\mathbf{\Phi}^b)\cdot\partial_{\theta_j}\mathbf{\Phi}(\theta_1^b,\theta_2^b,\gamma_y) \\ &+ \int_{\partial S_0} |\mathcal{J}_{\mathbf{\Phi}}^a\mathbf{t}_0|\left[(\tilde p^a - \tilde p^b)I - \nu(\nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b) + \nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b)^T)\right](\mathbf{n}_{\theta_1^b,\theta_2^b}\circ\mathbf{\Phi}^b)\cdot(\partial_{\theta_j}\mathbf{\Phi}(\theta_1^b,\theta_2^b,\gamma_y) - \partial_{\theta_j}\mathbf{\Phi}(0,0,\gamma_y)) \\ &+ \int_{\partial S_0} |\mathcal{J}_{\mathbf{\Phi}}^a\mathbf{t}_0|\left[(\tilde p^a - \tilde p^b)I - \nu(\nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b) + \nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b)^T)\right](\mathbf{n}_{\theta_1^b,\theta_2^b}\circ\mathbf{\Phi}^b - \mathbf{n}_0)\cdot\partial_{\theta_j}\mathbf{\Phi}(0,0,\gamma_y) \\ &+ \int_{\partial S_0} (|\mathcal{J}_{\mathbf{\Phi}}^a\mathbf{t}_0| - 1)\left[(\tilde p^a - \tilde p^b)I - \nu(\nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b) + \nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b)^T)\right]\mathbf{n}_0\cdot\partial_{\theta_j}\mathbf{\Phi}(0,0,\gamma_y) \\ &+ \int_{\partial S_0} (|\mathcal{J}_{\mathbf{\Phi}}^a\mathbf{t}_0| - |\mathcal{J}_{\mathbf{\Phi}}^b\mathbf{t}_0|)\left[\tilde p^b I - \nu(\mathcal{G}(\theta_1^b,\theta_2^b,\tilde{\mathbf{u}}^b) + \mathcal{G}(\theta_1^b,\theta_2^b,\tilde{\mathbf{u}}^b)^T)\right](\mathbf{n}_{\theta_1^b,\theta_2^b}\circ\mathbf{\Phi}^b)\cdot\partial_{\theta_j}\mathbf{\Phi}(\theta_1^b,\theta_2^b,\gamma_y),\end{aligned}$$

and we use the estimate

$$\begin{aligned}\|\mathbf{M}_{\mathbf{I}}(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a) - \mathbf{M}_{\mathbf{I}}(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)\|_{L^2(0,T)} &\leq K(\|(\dot\theta_1^a - \dot\theta_1^b)^2\|_{L^4} + \|(\dot\theta_1^a - \dot\theta_1^b)(\dot\theta_2^a - \dot\theta_2^b)\|_{L^2} + \|(\dot\theta_2^a - \dot\theta_2^b)^2\|_{L^4}) \\ &\leq KT^{1/2}(\|\dot\theta_1^a - \dot\theta_1^b\|_{L^\infty}^2 + \|(\dot\theta_1^a - \dot\theta_1^b)(\dot\theta_2^a - \dot\theta_2^b)\|_{L^\infty} + \|\dot\theta_2^a - \dot\theta_2^b\|_{L^\infty}^2),\end{aligned}$$

and (A.9), (A.21), (A.22), (A.6), (A.27), (A.7), (A.11), (A.23) and (A.16) to conclude and obtain (3.9).

• **Estimate** (3.10)**:** we use the Lipschitz regularity of $\mathbf{f}_{\mathscr{F}}$ and estimate (A.1),

$$\begin{aligned}\|\mathbf{f}_{\mathscr{F}}(t,\mathbf{\Phi}(\theta_1^a,\theta_2^a,\mathbf{y}))-\mathbf{f}_{\mathscr{F}}(t,\mathbf{\Phi}(\theta_1^b,\theta_2^b,\mathbf{y}))\|_{L^2(0,T;\mathbf{L}^2(\mathscr{F}_0))} & \\ \leq C\|\mathbf{f}_{\mathscr{F}}\|_{L^2(0,T;\mathbf{W}^{1,\infty}(\Omega))}\|\mathbf{\Phi}(\theta_1^a,\theta_2^a,\mathbf{y})-\mathbf{\Phi}(\theta_1^a,\theta_2^a,\mathbf{y})\|_{L^\infty(0,T;\mathbf{L}^\infty(\mathscr{F}_0))} & \\ \leq CT\|\theta^a-\theta^b\|_{\Theta_T}. &\end{aligned}$$

□

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