Cartan prolongations of distributions: singularities hidden behind contact systems

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Abstract

E. Cartan proposed a special way of producing more involved distributions from simpler ones. In his work [**?**] he was prolonging rank-2 distributions; in the most clear way his procedure was (much later) presented in the paper [**?**]. Cartan's prolongations of rank-2 distributions yield new rank-2 ones that are far from being generic (most often, their growth vectors start from $[2, 3, 4, \dots]$). And precisely because of that they are very useful for distributions, called traditionally Goursat, that generate 1-flags.
In fact, any r such prolongations started on $T\mathbb{R}^2$ **produce**, along with singularityless contact distributions [called also Cartan's and originally coming from the jet space $J^r(1,\,1)$], also **all existing singularities** of Goursat flags of that length r. Namely, that resulting universal Goursat distribution D lives on a huge 'monster' $(r+2)$-manifold Mon_r and the standard contact geometry is realized by D at generic points of Mon_r, while the singularities of Goursat materialize in D at the remaining points of Mon_r.

The procedure of [**?**] and [**?**] has been generalized in [**?**] to multi-dimensional, or *generalized* Cartan prolongations (gCp), that produce more involved rank-$(m+1)$ distributions from simpler, also rank-$(m+1)$, ones. The outcomes of gCp's are not generic, neither, making that operation an ideal tool for describing *special m-flags*: distributions having their flags growing in ranks always by m and having a very regular substructure – an involutive subflag similar to the one possessed automatically by 1-flags. (Multi-flags were brought into the mathematical usage by A. Kumpera in 1998; cf. p. 160 in [**?**].)

Special m-flags, likewise Goursat flags, appear to be but the outcomes of sequences of gCp's started from the tangent bundle to $\mathbb{R}^{m+1}$ and living on, bigger than for Goursat, monster manifolds. For any r the relevant manifold is stratified, in function of the geometry of the universal flag structure it bears, into *singularity classes*. The only generic stratum materializes contact Cartan distributions originally known from the jet spaces $J^r(1, m)$. And adjacent to it is plenty of thinner and thinner strata built of singular (for that flag structure) points of different degrees of degeneration. For length 4 there are 14 singularity classes in width 2 and 15 such classes from the width 3 onwards. To give here an idea of the awaiting crowd, let us for example fix the length $r = 7$. Then the numbers of different singularity classes of special m-flags, for $m \in \{1, 2, \ldots, 6\}$, are as follows:

m	1	2	3	4	5	6
#	32	365	715	855	876	877

The value 32 is just 2^{7-2} and belongs still to Goursat world. For inst., the third value in the table, 715, is the evaluation at $r = 7$ of the formula

$$2^{r-1} + \big(2^{r-2} - 1\big)4^0 + \big(2^{r-3} - 1\big)4^1 + \cdots + \big(2^1 - 1\big)4^{r-3}$$

for the number of singularity classes of length $r \geq 4$ of special 3-flags (for $r = 4$ one gets, mentioned above, 15). This is just an example. For a general width $m \geq 2$, one should count the # of words $j_1.j_2 \ldots j_r$ over the alphabet $\{1, 2, \ldots, m, m+1\}$, $r \geq m+1$, s. t. $j_1 = 1$ and, for $l = 1, 2, \ldots, r-1$, if $j_{l+1} > \max(j_1, \ldots, j_l)$, then $j_{l+1} = 1 + \max(j_1, \ldots, j_l)$ (the rule of the least possible *new* jumps upwards in the admissible words). We hope that, in the coming years, these classes could become an active object of research.

References

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