Final examen in *Asymptotic Statistics*
Preparation 1 hour

Manuscript notes and handout of the lectures are allowed

1 Cheap Bin

Let Z be a random variable with binomial distribution of parameters $M \in \mathbb{N}_*$ and $0 < \pi < 1$ ($Z \sim \mathcal{B}(M, \pi)$). We recall the following formulas

$$\begin{aligned}
\mathbb{E}(Z) &= M\pi, \\
\mathbb{E}(Z^2) &= M\pi(1 + (M-1)\pi), \\
\mathbb{E}(Z^3) &= M\pi(1 - 3\pi + 3M\pi + 2\pi^2 - 3M\pi^2 + M^2\pi^2), \\
\mathbb{E}(Z^4) &= M\pi(1 - 7\pi + 7M\pi + 12\pi^2 - 18M\pi^2 + 6M^2\pi^2 - 6\pi^3 + 11M\pi^3 - 6M^2\pi^3 + M^3\pi^3).
\end{aligned}$$

For $0 < \pi^* < 1$, let $X_1, \ldots, X_n$ be an i.i.d. sample with common law $\mathcal{B}(1, \sqrt{\pi^*})$. To estimate the parameter π^*, we consider the maximum likelihood estimator $\widehat{\pi}$.

1. Show that $\widehat{\pi} = \overline{X_n}^2$. Here, as usual, $\overline{X_n}$ denotes the empirical mean built on the sample $X_1, \ldots, X_n$. Compute the two first moments of $\widehat{\pi}$. Compute its mean square error :
$$R_{\widehat{\pi}}(\pi^*) := \mathbb{E}[(\widehat{\pi} - \pi^*)^2].$$
2. Modify $\widehat{\pi}$ to build an unbiased estimator $\widehat{\widehat{\pi}}$. Compute $R_{\widehat{\widehat{\pi}}}(\pi^*)$.
3. Show that $\sqrt{n}(\widehat{\pi} - \pi^*)$ and $\sqrt{n}(\widehat{\widehat{\pi}} - \pi^*)$ converge both in distribution towards the same law. What is the limit law ?
4. What estimator should we use ? Why ?
5. Show that the statistical model is LAN. Is $\widehat{\widehat{\pi}}$ optimal ?