Mathematical methods for Image Processing

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Plan

1. Hands-on session with examples: Image denoising using the H^1 regularization

Exercise 1, question 1

Calculate the adjoint operator D_x^* (and D_y^*) of D_x (and D_y). With,

$$(D_x^* w)_{m,n} = w_{m-1,n} - w_{m,n}$$

Exercise 1, question 1

Let w and $w' \in \mathbb{R}^{N^2}$

$$
\begin{aligned}
\langle w, D_x w' \rangle &= \sum_{m,n=1}^{N} w_{m,n} \left(w'_{m+1,n} - w'_{m,n} \right), \\
&= \sum_{m,n=1}^{N} w_{m,n} w'_{m+1,n} - \sum_{m,n=1}^{N} w_{m,n} w'_{m,n}, \\
&= \sum_{m=0}^{N-1} \sum_{n=1}^{N} w_{m-1,n} w'_{m,n} - \sum_{m,n=1}^{N} w_{m,n} w'_{m,n}, \\
&= \sum_{m,n=1}^{N} \left(w_{m-1,n} - w_{m,n} \right) w'_{m,n}, \\
&= \langle D_x^* w, w' \rangle.
\end{aligned}
$$

With,

$$(D_x^* w)_{m,n} = w_{m-1,n} - w_{m,n}$$

Exercise 1, question 2

Calculate a close form formula for: - the linear operator $A : \mathbb{R}^{N^2} \longrightarrow \mathbb{R}^{N^2}$; - the constant image $B \in \mathbb{R}^{N^2}$; - and the constant C (independent of w), such that for all $w \in \mathbb{R}^{N^2}$,

$$E(w) = \langle Aw, w \rangle + \langle B, w \rangle + C,$$

where $\langle ., . \rangle$ stands for the usual inner product in $\mathbb{R}^{N^2}$.

Exercise 1, question 2

$$
\begin{aligned}
E(w) &= \sum_{i,j=0}^{N-1} |\nabla w_{i,j}|^2 + \lambda \sum_{i,j=0}^{N-1} (w_{i,j} - v_{i,j})^2, \\
&= \|D_x w\|^2 + \|D_y w\|^2 + \lambda \|w - v\|^2 \\
&= \langle D_x w, D_x w \rangle + \langle D_y w, D_y w \rangle + \lambda \langle w - v, w - v \rangle \\
&= \langle D_x^* D_x w, w \rangle + \left\langle D_y^* D_y w, w \right\rangle + \lambda \langle w, w \rangle - 2\lambda \langle w, v \rangle + \lambda \langle v, v \rangle \\
&= \left\langle D_x^* D_x w + D_y^* D_y w + \lambda w, w \right\rangle + \langle w, -2\lambda v \rangle + \lambda \|v\|^2
\end{aligned}
$$

So

$$A = D_x^* D_x + D_y^* D_y + \lambda Id$$

$$B = -2\lambda v$$

$$C = \lambda \|v\|^2$$

Exercise 1, question 3

Check that A is self-adjoint. Check that the operator A is a convolution[1] with a kernel $h \in \mathbb{R}^{N^2}$ and give a closed form expression for h.

[1] We remind that the convolution is defined by :

$$\forall m, n = 0..N-1, (h * w)_{m,n} = \sum_{m',n'=0}^{N-1} h_{m',n'} w_{m-m',n-n'}.$$

Exercise 1, question 3

We have

$$\begin{aligned} A^* &= (D_x^* D_x + D_y^* D_y + \lambda Id)^* \\ &= (D_x^* D_x)^* + (D_y^* D_y)^* + \lambda Id^* \\ &= D_x^* (D_x^*)^* + D_y^* (D_y^*)^* + \lambda Id \\ &= D_x^* D_x + D_y^* D_y + \lambda Id = A \end{aligned}$$

Exercise 1, question 3

Let $m, n \in \{0, \dots, N-1\}$, we have

$$\begin{array}{rcl} (D_x^* D_x w)_{m,n} & = & (D_x w)_{m-1,n} - (D_x w)_{m,n} \\ & = & (w_{m-1+1,n} - w_{m-1,n}) - (w_{m+1,n} - w_{m,n}) \\ & = & 2w_{m,n} - w_{m-1,n} - w_{m+1,n} \end{array}$$

Using a similar calculus for $(D_y^* D_y w)_{m,n}$, we finally get

$$(Aw)_{m,n} = (4+\lambda) w_{m,n} - w_{m-1,n} - w_{m+1,n} - w_{m,n-1} - w_{m,n+1}$$

Identifying this formula with

$$(h * w)_{m,n} = \sum_{m',n'=0}^{N-1} h_{m',n'} w_{m-m',n-n'}.$$

We get

$$h_{m',n'} = \begin{cases} 4+\lambda & , \text{ if } (m',n') = (0,0) \\ -1 & , \text{ if } (m',n') = (-1,0) \text{ or } (m',n') = (1,0) \\ -1 & , \text{ if } (m',n') = (0,-1) \text{ or } (m',n') = (0,1) \\ 0 & , \text{ otherwise} \end{cases}$$

Exercise 1, question 4

Check that the discrete Fourier transform[1] of h has the form

$$\hat{h}_{k,l} = \lambda + 4 - 2\left(cos(\frac{2\pi k}{N}) + cos(\frac{2\pi l}{N})\right) \qquad \forall k, l = 0..N-1.$$

[1] We remind that the Discrete Fourier Transform is defined by :

$$\forall k, l = 0..N-1, \hat{h}_{k,l} = \sum_{m,n=0}^{N-1} h_{m,n}\ e^{-2i\pi\frac{km+ln}{N}}.$$

Exercise 1, question 4

Let $k, l \in \{0, \dots, N-1\}$, we have

$$\begin{aligned}
\hat{h}_{k,l} &= \sum_{m,n=0}^{N-1} h_{m,n}\ e^{-2i\pi\frac{km+ln}{N}} \\
&= (4+\lambda)\ e^{-2i\pi\frac{k\ 0+l\ 0}{N}} - (\ e^{-2i\pi\frac{-k}{N}} +\ e^{-2i\pi\frac{k}{N}}) - (\ e^{-2i\pi\frac{-l}{N}} +\ e^{-2i\pi\frac{l}{N}}) \\
&= (4+\lambda) - 2\cos(2\pi\frac{k}{N}) - 2\cos(2\pi\frac{l}{N})
\end{aligned}$$

Notice: We always have $\hat{h}_{k,l} \in \mathbb{R}$ and $\hat{h}_{k,l} > 0$.

Exercise 1, question 5

Calculate the gradient and the Hessian of E : $\nabla E(w)$ and $\nabla^2 E(w)$.

Exercise 1, question 5

Let w and $w' \in \mathbb{R}^{N^2}$, we have

$$\begin{aligned} E(w+w') &= \langle A(w+w'), w+w' \rangle + \langle B, w+w' \rangle + C \\ &= \langle Aw, w \rangle + \langle Aw', w \rangle + \langle Aw, w' \rangle + \langle Aw', w' \rangle \\ &\quad + \langle B, w \rangle + \langle B, w' \rangle + C \\ &= E(w) + \langle 2Aw + B, w' \rangle + \frac{1}{2} \langle 2Aw', w' \rangle \end{aligned}$$

Identifying this formula with the second order Taylor expansion

$$E(w+w') = E(w) + \langle \nabla E(w), w' \rangle + \frac{1}{2} \left\langle \nabla^2 E(w) w', w' \right\rangle + o(\|w'\|^2),$$

we get

$$\nabla E(w) = 2Aw + B$$

and

$$\nabla^2 E(w) = 2A$$

Exercise 1, question 6

Deduce from the previous question upper bounds $\alpha > 0$ and $L > 0$ such that for all w and $w' \in \mathbb{R}^{N^2}$

$$\alpha \|w'\|_2^2 \leq \left\langle \nabla^2 E(w) w', w' \right\rangle \leq L \|w'\|_2^2.$$

Make the connection with the hypotheses guarantying the convergence of the gradient algorithm.

Exercise 1, question 6

From the preceding questions, $\nabla^2 E(w)$ is positive definite and we have

$$\alpha \|w'\|_2^2 \leq \left\langle \nabla^2 E(w) w', w' \right\rangle \leq L \|w'\|_2^2.$$

for

$$\alpha = 2 \min_{k,l} \hat{h}_{k,l} = 2\lambda$$

and

$$L = 2 \max_{k,l} \hat{h}_{k,l} = 2(\lambda + 8)$$

Exercise 1, question 6

We remind the second fundamental theorem of calculus, we have for w and $w' \in \mathbb{R}^{N^2}$, and any smooth $f : \mathbb{R}^{N^2} \longrightarrow \mathbb{R}$

$$f(w') - f(w) = \int_0^1 \langle \nabla f(tw + (1-t)w'), w' - w \rangle dt.$$

Applying it to every partial derivative of E, we obtain for all $m, n = 0..N-1$

$$\begin{aligned} \frac{\partial E}{\partial w_{m,n}}(w') - \frac{\partial E}{\partial w_{m,n}}(w) &= \int_0^1 \sum_{m',n'} \frac{\partial^2 E}{\partial w_{m,n} \partial w_{m',n'}}(tw + (1-t)w') \\ &\qquad (w' - w)_{m',n'} dt \\ &= 2A(w' - w)_{m,n} \end{aligned}$$

Summarizing, we have

$$\nabla E(w') - \nabla E(w) = 2A(w' - w)$$

Exercise 1, question 6

$$\nabla E(w') - \nabla E(w) = 2A(w' - w)$$

Therefore

$$\begin{aligned} \|\nabla E(w') - \nabla E(w)\| &\leq 2 \max_{k,l} \hat{h}_{k,l} \, \|w' - w\| \\ &= L \, \|w' - w\| \end{aligned}$$

and

$$\begin{aligned} \langle \nabla E(w') - \nabla E(w), w' - w \rangle &= \langle 2A(w' - w), w' - w \rangle \\ &\geq 2 \min_{k,l} \hat{h}_{k,l} \|w' - w\|^2 \\ &= \alpha \, \|w' - w\|^2 \end{aligned}$$

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Exercise 1, question 7

Deduce, from the preceding questions, an algorithm based on the Fast Fourier Transform minimizing E.

Exercise 1, question 7

We know that,

$$\nabla E(w^*) = 2Aw^* + B = 0$$

That is

$$Aw^* = h * w^* = -\frac{1}{2}B = \lambda v.$$

Taking the Fourier transform of $h * w^*$ and using that for all $k, l = 0..N-1$,

$$\widehat{h * w^*}_{k,l} = \hat{h}_{k,l}\widehat{w^*}_{k,l},$$

we obtain

$$\widehat{w^*}_{k,l} = \lambda \frac{\hat{v}_{k,l}}{\hat{h}_{k,l}}$$

Exercise 1, question 7

Algorithm 1 Exact Algorithm (used for the comparizon)

Entry: v, λ
Output: Exact minimizer : w^*

Compute the Fast Fourier Transform $\hat{v}$ of v
for $k, l = 0..N-1$ **Do**
  Compute $\widehat{w^*}_{k,l} = \lambda \hat{v}_{k,l} / \hat{h}_{k,l}$
end for
Compute the Fast inverse Fourier Transform w^* of $\widehat{w^*}$

Exercise 1, Conclusion

We have an optimization problem:

- Well conditionned when λ is large
- Poorly conditionned when λ is small
- We have an exact algorithm for computing its true solution

We can use this problem to

- **compare iterative algorithm**
- **illustrate the conditionning problem**