Homework

From Particle Systems to Mean Field PDEs

Mean-field limit for the one-dimensional Vlasov–Poisson system

Due date: _____

Reminders

We recall the notation of Chapter 3.

- A mean-field PDE on $\mathbb{R}^d$ is a transport equation of the form

$$\partial_t f + \nabla_z \cdot (\mathcal{K}[f]\, f) = 0, \qquad \mathcal{K}[f](t,z) := \int_{\mathbb{R}^d} K(z,z')\, f(t,\mathrm{d}z'), \tag{1}$$

  where $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is the interaction kernel. The assumptions on the kernel are

(HK1) **Skew-symmetry**: $K(z,z') = -K(z',z)$ for all $z, z' \in \mathbb{R}^d$;

(HK2) **Lipschitz regularity**: $K \in \mathscr{C}^1(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$ and there exists $L \geq 0$ such that $\sup_{z'} |\nabla_z K(z,z')| \leq L$ and $\sup_z |\nabla_{z'} K(z,z')| \leq L$.

- For two probability measures μ, ν with finite first moment, $\Pi(\mu,\nu)$ denotes the set of couplings of μ and ν, that is the probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν, and the Monge–Kantorovich distance is

$$\mathrm{dist}_1(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|\, \pi(\mathrm{d}z\, \mathrm{d}z'). \tag{2}$$

- Under (HK1)–(HK2), Dobrushin's estimate states that two mean-field solutions $\mu_j(t) = Z_j(t,\cdot)_\# \mu_j^{in}$, $j = 1, 2$, satisfy

$$\mathrm{dist}_1\left(\mu_1(t), \mu_2(t)\right) \leq e^{2L|t|}\, \mathrm{dist}_1\left(\mu_1^{in}, \mu_2^{in}\right). \tag{3}$$

  Its proof relies on a Gronwall estimate on the averaged distance between characteristics

$$D[\pi^{in}](t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \left|Z_1(t,\zeta_1) - Z_2(t,\zeta_2)\right| \pi^{in}(\mathrm{d}\zeta_1\, \mathrm{d}\zeta_2), \qquad \pi^{in} \in \Pi(\mu_1^{in}, \mu_2^{in}).$$

Problem

The Coulomb interaction does not satisfy assumption (HK2), so that Dobrushin's estimate does not apply directly to the Vlasov–Poisson system. The goal of this problem is to show that, in dimension one, a particle system nevertheless converges to the Vlasov–Poisson system, by using a uniform (instead of averaged) control of the distance between particles and characteristics.

We consider the one-dimensional Vlasov–Poisson system on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, with a uniform neutralising background:

$$\begin{cases} \partial_t f + v\, \partial_x f + E(t,x)\, \partial_v f = 0, \qquad (x,v) \in \mathbb{T} \times \mathbb{R}, \\ \partial_x E = \rho - 1, \qquad \displaystyle\int_{\mathbb{T}} E(t,x)\, \mathrm{d}x = 0, \qquad \rho(t,x) = \int_{\mathbb{R}} f(t,x,v)\, \mathrm{d}v. \end{cases} \tag{4}$$

The initial datum $f^{in} \geq 0$ is bounded, supported in $\mathbb{T} \times [-R, R]$ for some $R > 0$, and satisfies

$$\int_{\mathbb{T} \times \mathbb{R}} f^{in}\, \mathrm{d}x\, \mathrm{d}v = 1,$$

which is global neutrality. For $x \in \mathbb{T}$, we denote by $|x|_{\mathbb{T}} = \min_{k \in \mathbb{Z}} |x - k| \in [0, \frac{1}{2}]$ the distance to 0 on the torus, and for $z = (x,v) \in \mathbb{T} \times \mathbb{R}$ we set $|z|_\infty = \max(|x|_{\mathbb{T}}, |v|)$.

1. *The electric field.* Let $G : \mathbb{T} \to \mathbb{R}$ be the 1-periodic function defined by

$$G(x) \;=\; \frac{1}{2}\,\mathrm{sign}(x) - x, \qquad x \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right],$$

with the convention $\mathrm{sign}(0) = 0$, and let $\rho \geq 0$ be a probability density on $\mathbb{T}$.

   (a) Show that G is continuous on $\mathbb{T} \setminus \{0\}$, that

$$|G| \leq \frac{1}{2}, \qquad \text{and} \qquad \int_{\mathbb{T}} G \,\mathrm{d}x = 0,$$

   and that $G' = \delta_0 - 1$ in the sense of distributions on $\mathbb{T}$.

   (b) Deduce that $E = G \star \rho$, that is

$$E(x) = \int_{\mathbb{T}} G(x-y)\,\rho(y)\,\mathrm{d}y,$$

   is the unique solution of $\partial_x E = \rho - 1$ with zero mean, and that

$$|E(x)| \leq \frac{1}{2}, \qquad \forall x \in \mathbb{T}.$$

   (c) Show that if moreover $\rho \leq M$, then

$$|E(x) - E(x')| \leq (M+1)\,|x - x'|_{\mathbb{T}}, \qquad \forall x,\, x' \in \mathbb{T}.$$

   (d) Write (4) in the form (1) and identify the interaction kernel K. Which of the assumptions (HK1)–(HK2) fail?

2. *A priori bounds.* We admit that (4) has a solution $f(t) = Z(t,\cdot)_{\#} f^{in}$, where $Z(t,\zeta) = (Y(t,\zeta), W(t,\zeta))$ is the flow of the characteristics

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = W, \qquad \frac{\mathrm{d}W}{\mathrm{d}t} = E(t,Y), \qquad Z(0,\zeta) = \zeta.$$

   (a) Show that the vector field $(v, E(t,x))$ is divergence free in (x,v), and deduce that $f(t, Z(t,\zeta)) = f^{in}(\zeta)$, so that $\|f(t)\|_{L^\infty} = \|f^{in}\|_{L^\infty}$.

   (b) Using $|E| \leq \frac{1}{2}$, show that $f(t)$ is supported in $\mathbb{T} \times \{|v| \leq R + t/2\}$, and deduce that for all $t \in [0,T]$,

$$\rho(t,x) \;\leq\; M_T \;:=\; \|f^{in}\|_{L^\infty}\,(2R+T).$$

3. *The particle system.* Let $m \in \mathbb{N}^*$, $h = 1/m$, and cover $\mathbb{T} \times [-R,R]$ by N disjoint square cells C_k of side h and center $z_k^{in} = (x_k^{in}, v_k^{in})$, $1 \leq k \leq N$, so that $N \leq m\,(2Rm+1)$. We set $\omega_k = \int_{C_k} f^{in}\,\mathrm{d}\zeta$ and consider the weighted particle system

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = v_k, \qquad \frac{\mathrm{d}v_k}{\mathrm{d}t} = E_N(t,x_k), \qquad E_N(t,x) = \sum_{j=1}^{N} \omega_j\, G(x - x_j(t)), \tag{5}$$

with $z_k(0) = z_k^{in}$. We admit that this system has a unique solution with Lipschitz trajectories $z_k(t) = (x_k(t), v_k(t)) \in \mathbb{T} \times \mathbb{R}$, and we define the empirical measure

$$\mu_N(t) = \sum_{k=1}^{N} \omega_k\, \delta_{z_k(t)}.$$

(a) Check that $E_N(t,\cdot) = G \star \rho_N(t)$, where

$$\rho_N(t) = \sum_j \omega_j \delta_{x_j(t)}$$

is the first marginal of $\mu_N(t)$, so that the particle system (5) is the characteristic system for the measure μ_N.

(b) Show that the self-interaction term $j = k$ vanishes in $E_N(t, x_k)$.

4. *The key estimate.* We define the maximal distance between the characteristics and the particles,

$$\delta(t) \ := \ \max_{1\le k\le N} \ \sup_{\zeta\in C_k} \ \big|Z(t,\zeta) - z_k(t)\big|_\infty .$$

(a) Show that $\delta(0) \le h/2$.

(b) Using $f(t) = Z(t,\cdot)_\# f^{in}$ and the definition of ω_j, show that for every $x \in \mathbb{T}$,

$$E(t,x) = \sum_{j=1}^N \int_{C_j} G(x - Y(t,\eta))\, f^{in}(\eta)\,\mathrm{d}\eta,$$

and

$$E_N(t,x) = \sum_{j=1}^N \int_{C_j} G(x - x_j(t))\, f^{in}(\eta)\,\mathrm{d}\eta.$$

(c) Show that G has slope -1 away from 0 and a jump of $+1$ at 0, and deduce that for all $a, b \in \mathbb{T}$,

$$\big|G(a) - G(b)\big| \ \le \ |a-b|_{\mathbb{T}} \ + \ \mathbf{1}_{\{|a|_{\mathbb{T}} \le |a-b|_{\mathbb{T}}\}}.$$

Hint: distinguish whether or not the shortest arc between a and b contains 0.

(d) Let $\zeta \in C_k$ and $\eta \in C_j$, and set $a = Y(t,\zeta) - Y(t,\eta)$, $b = x_k(t) - x_j(t)$. Show that $|a-b|_{\mathbb{T}} \le 2\delta(t)$.

(e) Conclude, using question 2(b), that for all $t \in [0,T]$, $1 \le k \le N$ and $\zeta \in C_k$,

$$\begin{aligned} \big|E\big(t, Y(t,\zeta)\big) - E_N\big(t, x_k(t)\big)\big| \ &\le \ 2\delta(t) + \int_{\{|Y(t,\zeta)-y|_{\mathbb{T}} \le 2\delta(t)\}} \rho(t,y)\,\mathrm{d}y \\ &\le \ (2 + 4M_T)\,\delta(t). \end{aligned}$$

5. *Gronwall argument.* Let $\zeta \in C_k$. By integrating the characteristic equations and the particle equations (5) between 0 and t, show that

$$\big|Z(t,\zeta) - z_k(t)\big|_\infty \ \le \ \delta(0) + C_T \int_0^t \delta(s)\,\mathrm{d}s, \qquad C_T = 2 + 4M_T,$$

and deduce that $\delta(t) \le \frac{h}{2}\, e^{C_T t}$ for all $t \in [0,T]$.

6. *Convergence.* We define the Monge–Kantorovich distance on $\mathbb{T}\times\mathbb{R}$ as in (2), with $|z - z'|$ replaced by $\big(|x-x'|_{\mathbb{T}}^2 + |v-v'|^2\big)^{1/2}$. Consider the coupling

$$\pi(t) \ = \ \sum_{k=1}^N \Big(Z(t,\cdot)_\#\big(f^{in}\,\mathbf{1}_{C_k}\big)\Big) \otimes \delta_{z_k(t)}.$$

(a) Show that $\pi(t) \in \Pi(f(t), \mu_N(t))$.

(b) Deduce that

$$\mathrm{dist}_1\left(f(t),\mu_N(t)\right) \;\le\; \sqrt{2}\,\delta(t) \;\le\; \frac{h}{\sqrt{2}}\,e^{C_T t}, \qquad t\in[0,T].$$

(c) Express this estimate in terms of the number of particles N, and conclude that $\mu_N(t)$ converges to $f(t)$ as $N\to+\infty$, uniformly on $[0,T]$, for the Monge–Kantorovich distance.

7. *Discussion.* Compare with Dobrushin's estimate (3). Which quantity plays the role of the Lipschitz constant L of the kernel? Explain why an averaged control of the distance between particles and characteristics, as in the quantity $D[\pi^{in}](t)$ recalled above, would not be sufficient to estimate the difference of the electric fields in question 4.

0.1 Correction of the problem

1. (a) On $(0,\frac12]$ we have $G(x)=\frac12-x$, and on $(-\frac12,0)$ we have $G(x)=-\frac12-x$. Both expressions tend to 0 at $\pm\frac12$, so that the periodic extension of G is continuous on $\mathbb{T}\setminus\{0\}$, and $|G|\le\frac12$. Since G is odd, $\int_{\mathbb{T}} G\,\mathrm{d}x=0$. Moreover, $G(x)=\frac12-x$ for all $x\in(0,1)$, hence for every smooth 1-periodic test function φ, an integration by parts gives

$$\langle G',\varphi\rangle \;=\; -\int_0^1\Big(\frac12-x\Big)\varphi'(x)\,\mathrm{d}x \;=\; -\Big[\Big(\frac12-x\Big)\varphi(x)\Big]_0^1-\int_0^1\varphi(x)\,\mathrm{d}x \;=\; \varphi(0)-\int_{\mathbb{T}}\varphi\,\mathrm{d}x,$$

that is, $G'=\delta_0-1$ in the sense of distributions on $\mathbb{T}$.

(b) Since derivation commutes with convolution, $\partial_x E = G'\star\rho = \rho-\int_{\mathbb{T}}\rho\,\mathrm{d}y = \rho-1$. By Fubini's theorem, $\int_{\mathbb{T}}E\,\mathrm{d}x=\int_{\mathbb{T}}G\,\mathrm{d}x\int_{\mathbb{T}}\rho\,\mathrm{d}y=0$. If $\tilde E$ is another solution with zero mean, then $\partial_x(E-\tilde E)=0$, so $E-\tilde E$ is constant, and this constant vanishes since both fields have zero mean. Finally, $|E(x)|\le\sup|G|\int_{\mathbb{T}}\rho\,\mathrm{d}y=\frac12$.

(c) If $\rho\le M$, then $\partial_x E=\rho-1\in[-1,M-1]$, so that $|\partial_x E|\le M+1$ almost everywhere. Integrating along the shortest arc between x and x' gives $|E(x)-E(x')|\le(M+1)\,|x-x'|_{\mathbb{T}}$.

(d) With $z=(x,v)$, $z'=(x',v')$ and $K(z,z')=(v,\,G(x-x'))$, we have, since f is a probability measure,

$$\int_{\mathbb{T}\times\mathbb{R}} K(z,z')\,f(t,\mathrm{d}z') \;=\; (v,\,(G\star\rho)(t,x)) \;=\; (v,\,E(t,x)),$$

so that (4) reads $\partial_t f+\nabla_z\cdot(\mathcal{K}[f]\,f)=0$. The second component of K is skew-symmetric since G is odd, but the first one is not: $K(z,z')+K(z',z)=(v+v',0)$, so (HK1) fails. More importantly, G is discontinuous at 0, so that K is not Lipschitz and (HK2) fails.

2. (a) We have $\nabla_{x,v}\cdot(v,E(t,x))=\partial_x v+\partial_v E(t,x)=0$. By Theorem 2.2.5-(iii), the Jacobian of the flow is identically equal to 1 (strictly speaking, Theorem 2.2.5 requires a $\mathscr{C}^1$ field, whereas E is only Lipschitz in x by 1(c); this is part of what we admitted on the flow), so that, as in Theorem 2.3.9, $f(t)=Z(t,\cdot)_{\#}f^{in}$ has density $f(t,z)=f^{in}(Z(t,\cdot)^{-1}(z))$, that is $f(t,Z(t,\zeta))=f^{in}(\zeta)$. Since $Z(t,\cdot)$ is a bijection, $\|f(t)\|_{L^\infty}=\|f^{in}\|_{L^\infty}$.

(b) If ζ belongs to the support of f^{in}, then $|W(0,\zeta)|\le R$ and, since $|E|\le\frac12$,

$$|W(t,\zeta)| \;\le\; |W(0,\zeta)|+\int_0^t|E(s,Y(s,\zeta))|\,\mathrm{d}s \;\le\; R+\frac{t}{2}.$$

By (a), $f(t,x,v)\neq0$ only if $(x,v)=Z(t,\zeta)$ with ζ in the support of f^{in}, hence $f(t)$ is supported in $\mathbb{T}\times\{|v|\le R+t/2\}$. Therefore, for $t\in[0,T]$,

$$\rho(t,x) \;=\; \int_{|v|\le R+t/2} f(t,x,v)\,\mathrm{d}v \;\le\; \|f^{in}\|_{L^\infty}\,(2R+t) \;\le\; M_T.$$

3. (a) By definition of the convolution with a Dirac mass, $(G\star\rho_N(t))(x) = \sum_j \omega_j\, G(x-x_j(t)) = E_N(t,x)$. Hence the particles follow the characteristics $\dot x = v$, $\dot v = (G\star\rho_N)(t,x)$, which are those of the Vlasov–Poisson system with ρ replaced by ρ_N. Note that $\sum_j \omega_j = \int f^{in} = 1$, so that $\mu_N(t)$ is a probability measure, as is $f(t)$.

(b) The term $j=k$ is $\omega_k\, G(x_k - x_k) = \omega_k\, G(0) = 0$, by the convention $\mathrm{sign}(0) = 0$.

4. (a) If $\zeta = (x,v) \in C_k$, which is a square of side h centred at z_k^{in}, then $|x - x_k^{in}|_{\mathbb{T}} \le h/2$ and $|v - v_k^{in}| \le h/2$. Since $Z(0,\zeta) = \zeta$ and $z_k(0) = z_k^{in}$, we get $\delta(0) \le h/2$.

(b) Since $\rho(t)$ is the first marginal of $f(t) = Z(t,\cdot)_{\#} f^{in}$, the definition of the push-forward gives

$$E(t,x) = \int_{\mathbb{T}\times\mathbb{R}} G(x-y)\, f(t, \mathrm{d}y\, \mathrm{d}w) = \int_{\mathbb{T}\times\mathbb{R}} G\big(x - Y(t,\eta)\big)\, f^{in}(\eta)\, \mathrm{d}\eta,$$

and we split the integral over the disjoint cells C_j, which cover the support of f^{in}. The formula for E_N follows from $\omega_j = \int_{C_j} f^{in}(\eta)\, \mathrm{d}\eta$.

(c) By 1(a), G is affine with slope -1 on $\mathbb{T}\setminus\{0\}$, and $G(0^+) - G(0^-) = \frac12 - (-\frac12) = 1$. Let γ be the shortest arc between b and a, of length $\ell = |a-b|_{\mathbb{T}}$. If $0 \notin \gamma$, then G is affine with slope -1 along γ, so that $|G(a) - G(b)| = \ell$. If $0 \in \gamma$, then $G(a) - G(b)$ is the sum of the variation of the affine part, bounded by ℓ, and of the jump at 0, bounded by 1, so that $|G(a) - G(b)| \le \ell + 1$; moreover, since $0 \in \gamma$, the distance from a to 0 satisfies $|a|_{\mathbb{T}} \le \ell$. In both cases,

$$|G(a) - G(b)| \;\le\; |a-b|_{\mathbb{T}} + \mathbf{1}_{\{|a|_{\mathbb{T}} \le |a-b|_{\mathbb{T}}\}}.$$

(d) We have $a - b = \big(Y(t,\zeta) - x_k(t)\big) - \big(Y(t,\eta) - x_j(t)\big)$, hence by the triangle inequality on $\mathbb{T}$ and the definition of $\delta(t)$,

$$|a-b|_{\mathbb{T}} \;\le\; |Y(t,\zeta) - x_k(t)|_{\mathbb{T}} + |Y(t,\eta) - x_j(t)|_{\mathbb{T}} \;\le\; 2\delta(t).$$

(e) Applying (b) at $x = Y(t,\zeta)$ for E and at $x = x_k(t)$ for E_N, we get

$$E\big(t, Y(t,\zeta)\big) - E_N\big(t, x_k(t)\big) \;=\; \sum_{j=1}^{N} \int_{C_j} \big(G(a) - G(b)\big)\, f^{in}(\eta)\, \mathrm{d}\eta,$$

with a, b as in (d). By (c) and (d), $|G(a) - G(b)| \le 2\delta(t) + \mathbf{1}_{\{|Y(t,\zeta) - Y(t,\eta)|_{\mathbb{T}} \le 2\delta(t)\}}$, so that, using $\int f^{in} = 1$ and again the definition of the push-forward,

$$\big|E\big(t,Y(t,\zeta)\big) - E_N\big(t,x_k(t)\big)\big| \;\le\; 2\delta(t) + \int_{\mathbb{T}\times\mathbb{R}} \mathbf{1}_{\{|Y(t,\zeta)-Y(t,\eta)|_{\mathbb{T}} \le 2\delta(t)\}}\, f^{in}(\eta)\, \mathrm{d}\eta \;=\; 2\delta(t) + \int_{\{|Y(t,\zeta)-y|_{\mathbb{T}} \le 2\delta(t)\}} \rho(t,y)\, \mathrm{d}$$

The set $\{y \in \mathbb{T} : |Y(t,\zeta) - y|_{\mathbb{T}} \le 2\delta(t)\}$ has Lebesgue measure at most $4\delta(t)$ and $\rho(t) \le M_T$ by 2(b), which gives the bound $(2 + 4M_T)\,\delta(t)$.

This is the key point of the proof: the discontinuity of the kernel is compensated by the fact that the limit density $\rho(t)$ is bounded, so that only a small proportion of particles, of order $M_T\,\delta(t)$, may have changed sides with respect to the point $Y(t,\zeta)$.

5. Let $\zeta \in C_k$. Integrating the equations for the positions,

$$\big|Y(t,\zeta) - x_k(t)\big|_{\mathbb{T}} \;\le\; \big|Y(0,\zeta) - x_k(0)\big|_{\mathbb{T}} + \int_0^t \big|W(s,\zeta) - v_k(s)\big|\, \mathrm{d}s \;\le\; \delta(0) + \int_0^t \delta(s)\, \mathrm{d}s,$$

and for the velocities, using 4(e),

$$\big|W(t,\zeta) - v_k(t)\big| \;\le\; \delta(0) + \int_0^t \big|E\big(s,Y(s,\zeta)\big) - E_N\big(s,x_k(s)\big)\big|\, \mathrm{d}s \;\le\; \delta(0) + (2+4M_T)\int_0^t \delta(s)\, \mathrm{d}s.$$

Taking the maximum of both estimates gives the claimed inequality with $C_T = \max(1, 2 + 4M_T) = 2 + 4M_T$, and taking the supremum over $\zeta \in C_k$ and $1 \le k \le N$,

$$\delta(t) \;\le\; \delta(0) + C_T \int_0^t \delta(s)\,\mathrm{d}s, \qquad t \in [0,T].$$

The function δ is continuous, as the maximum over finitely many k of the supremum over the compact set $\overline{C_k}$ of a function which is continuous in (t,ζ). Gronwall's lemma and 4(a) then yield $\delta(t) \le \delta(0)\, e^{C_T t} \le \frac{h}{2}\, e^{C_T t}$.

6. (a) For bounded continuous functions φ and ψ on $\mathbb{T} \times \mathbb{R}$, by definition of the push-forward,

$$\int \varphi(z)\,\pi(t)(\mathrm{d}z\,\mathrm{d}z') = \sum_{k=1}^N \int_{C_k} \varphi(Z(t,\zeta))\, f^{in}(\zeta)\,\mathrm{d}\zeta = \int \varphi(Z(t,\zeta))\, f^{in}(\zeta)\,\mathrm{d}\zeta = \int \varphi\,\mathrm{d}f(t),$$

and

$$\int \psi(z')\,\pi(t)(\mathrm{d}z\,\mathrm{d}z') = \sum_{k=1}^N \psi(z_k(t)) \int_{C_k} f^{in}(\zeta)\,\mathrm{d}\zeta = \sum_{k=1}^N \omega_k\, \psi(z_k(t)) = \int \psi\,\mathrm{d}\mu_N(t).$$

Hence $\pi(t)$ has marginals $f(t)$ and $\mu_N(t)$, that is $\pi(t) \in \Pi(f(t), \mu_N(t))$.

(b) Using $\pi(t)$ as a (not necessarily optimal) coupling, and since $\left(|x|_{\mathbb{T}}^2 + |v|^2\right)^{1/2} \le \sqrt{2}\,|z|_\infty$,

$$\mathrm{dist}_1\left(f(t), \mu_N(t)\right) \;\le\; \sum_{k=1}^N \int_{C_k} \sqrt{2}\,\left|Z(t,\zeta) - z_k(t)\right|_\infty f^{in}(\zeta)\,\mathrm{d}\zeta \;\le\; \sqrt{2}\,\delta(t) \sum_{k=1}^N \omega_k \;=\; \sqrt{2}\,\delta(t),$$

and we conclude with question 5.

(c) Since $h = 1/m$ and $N \le m(2Rm+1) \le (2R+1)\,m^2$, we have $h \le \sqrt{(2R+1)/N}$, hence

$$\sup_{t\in[0,T]} \mathrm{dist}_1\left(f(t), \mu_N(t)\right) \;\le\; \sqrt{\frac{2R+1}{2N}}\; e^{C_T T} \;\longrightarrow\; 0, \qquad N \to +\infty.$$

The empirical measure converges to the solution of the Vlasov–Poisson system at the rate $N^{-1/2}$, uniformly on $[0,T]$. Note that the cells with $\omega_k = 0$ can be removed without changing the argument.

7. The constant $C_T = 2 + 4M_T$ plays the role of $2L$ in (3). It is essentially the Lipschitz constant $M_T + 1$ of the limit field E (question 1(c)), which is not a property of the kernel K, which is discontinuous, but of the solution: it depends on the bound $\|\rho(t)\|_{L^\infty} \le M_T$. Only the limit solution needs to be regular, the empirical measure having no bounded density.

An averaged control such as $D[\pi^{in}](t) = \int |Z_1 - Z_2|\,\mathrm{d}\pi^{in}$ is not sufficient, because the function G is not Lipschitz: the quantity $|G(a) - G(b)|$ is not bounded by $C|a-b|$, so that $\int |G(a) - G(b)|$ cannot be controlled by $\int |a-b|$. A small fraction of particles located far from their characteristics may change sides with respect to a given point and produce an error of order one on the field, whatever their number. The uniform control $\delta(t)$ ensures that *all* the pairs (a,b) satisfy $|a-b|_{\mathbb{T}} \le 2\delta(t)$ simultaneously, which is what allows to use the indicator function in question 4(e) and the bound on ρ. This corresponds to working with the Wasserstein distance of order infinity instead of the Monge–Kantorovich distance.