Chapter 2

Transport equations

Transport equations form the mathematical foundation of kinetic theory. Their solutions are propagated along characteristic curves determined by an underlying velocity field. Understanding these characteristic trajectories provides both an intuitive interpretation of the dynamics and a powerful analytical tool. In this chapter we introduce the method of characteristics and derive several fundamental properties that will be repeatedly used later.

2.1 The free transport equation

We begin with the simplest transport model, corresponding to particles moving freely without external forces. Although elementary, this equation already contains the essential geometric ideas of transport theory and serves as a useful prototype for more general kinetic equations.

Consider the Cauchy problem

$$\partial_t f \;+\; v \cdot \nabla_x f \;=\; 0,$$

with initial datum $f(0, x, v) \;=\; f^{in}(x, v)$. The characteristic curve of this equation passing through the point $y \in \mathbb{R}^n$ at time $t = 0$ is the straight line $\gamma(t) = y + tv$, and the solution is constant along each characteristic:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t, \gamma(t), v) \;=\; 0.$$

Theorem 2.1.1. *For $f^{in} \in \mathscr{C}^1(\mathbb{R}^n \times \mathbb{R}^n)$, the Cauchy problem*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = 0, \\ f(0, x, v) \;=\; f^{in}(x, v) \end{cases}$$

has a unique solution $f \in \mathscr{C}^1(\mathbb{R}_+ \times \mathbb{R}^{2n})$ given by

$$f(t, x, v) \;=\; f^{in}(x - tv,\, v). \tag{2.1}$$

Proof. We first prove uniqueness, and then verify that the formula defines a solution.

Uniqueness. Let f be a $\mathscr{C}^1$ solution and fix $y, v \in \mathbb{R}^n$. Along the characteristic $\gamma(t) = y + tv$,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t,\, y+tv,\, v) \;=\; \partial_t f + v \cdot \nabla_x f \;=\; 0.$$

Hence

$$f(t,\, y+tv,\, v) = f(0, y, v) = f^{in}(y, v).$$

With the change of variable $x = y + tv$, this gives $f(t, x, v) = f^{in}(x - tv, v)$, hence the solution is uniquely determined by f^{in}.

Existence. Define

$$f(t, x, v) = f^{in}(x - tv, v).$$

Clearly $f(0, x, v) = f^{in}(x, v)$, and a direct computation gives

$$\partial_t f + v \cdot \nabla_x f = -v \cdot (\nabla_x f^{in})(x - tv, v) + v \cdot (\nabla_x f^{in})(x - tv, v) = 0. \quad \square$$

The formula (2.1) says that the distribution function at time t is obtained by *shifting each velocity class v by tv*: particles with velocity v started at position $x - tv$ and arrived at position x.

An example of phase mixing. Phase mixing is a damping mechanism due to shearing in the phase space $\mathbb{T} \times \mathbb{R}$ by the free-transport dynamics [24, 25],

$$\partial_t f \;+\; v \cdot \nabla_x f \;=\; 0,$$

leading to fast decay to equilibrium of macroscopic quantities such as the charged density n, momentum nu and kinetic energy e

$$\begin{cases} n(t,x) \;=\; \displaystyle\int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}v, \\ nu(t,x) \;=\; \displaystyle\int_{\mathbb{R}} v f(t,x,v)\,\mathrm{d}v, \\ e(t,x) \;=\; \displaystyle\frac{1}{2}\int_{\mathbb{R}} |v|^2 f(t,x,v)\,\mathrm{d}v. \end{cases}$$

Indeed, the transport dynamics or shearing of each elementary mode $e^{ik\cdot x}$, for $k \in \mathbb{Z} \setminus \{0\}$, creates fast oscillation $e^{-ikt\cdot v}$ in v, which leads to rapid decay of macroscopic quantities, thanks to velocity averaging. We now make this mechanism precise.

We first identify $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and, for $g \in L^1(\mathbb{T})$, write its Fourier coefficients as

$$\hat{g}_k \;=\; \frac{1}{2\pi}\int_0^{2\pi} g(x)\, e^{-ikx}\,\mathrm{d}x, \qquad k \in \mathbb{Z},$$

so that formally

$$g(x) = \sum_{k\in\mathbb{Z}} \hat{g}_k\, e^{ikx}.$$

Applying this to $f^{in}(\cdot, v)$ for each fixed v and using the explicit solution (2.1),

$$f(t,x,v) = f^{in}(x - tv, v) = \sum_k \hat{f}^{in}_k(v)\, e^{ik(x-tv)};$$

so that the Fourier coefficients of $f(t,\cdot,v)$ are

$$\hat{f}_k(t,v) \;=\; \hat{f}_k^{in}(v)\, e^{-ikvt}, \qquad k \in \mathbb{Z}. \tag{2.2}$$

Therefore, for $k \neq 0$, the factor e^{-ikvt} oscillates in v at a frequency $|k|t$ growing linearly in time: this is precisely the "fast oscillation" created by shearing. Consequently, the k-th Fourier coefficient of a macroscopic (velocity-averaged) quantity

$$m_\varphi(t,x) \;:=\; \int_{\mathbb{R}} \varphi(v)\, f(t,x,v)\,\mathrm{d}v, \qquad \varphi(v) \in \{1,\, v,\, \tfrac{1}{2}v^2\},$$

is given by

$$\hat{m}_{\varphi,k}(t) \;=\; \int_{\mathbb{R}} \varphi(v)\, \hat{f}_k^{in}(v)\, e^{-ikvt}\,\mathrm{d}v, \tag{2.3}$$

that is, the Fourier transform in v of $\varphi\,\hat{f}_k^{in}$ evaluated at the frequency kt. Now we observe that since $k \neq 0$ is fixed while $t \to +\infty$, this is an oscillatory integral at diverging frequency, and *velocity averaging* – integration against the fast oscillation e^{-ikvt} – is exactly what drives its decay. The mode $k = 0$ is not affected:

$$\hat{m}_{\varphi,0}(t) = \int_{\mathbb{R}} \varphi(v) \hat{f}_0^{in}(v)\,\mathrm{d}v$$

is constant in time, consistently with the conservation of the spatial average of any velocity moment along the free flow.

Theorem 2.1.2 (Phase mixing: decay of macroscopic quantities)**.** *Let $f^{in} \in L^1(\mathbb{T}\times\mathbb{R})$ and let*

$$f(t,x,v) = f^{in}(x - tv, v)$$

be the corresponding solution of the free transport equation.

(i) **Qualitative decay.** *For every $\varphi(v) = v^m$, $m \in \{0,1,2\}$, such that $\varphi\,\hat{f}_k^{in} \in L^1(\mathbb{R})$ for every $k \neq 0$, one has*

$$\hat{m}_{\varphi,k}(t) \xrightarrow[t\to+\infty]{} 0 \qquad \text{for every } k \in \mathbb{Z}\setminus\{0\}.$$

(ii) **Quantitative, exponential decay.** *Suppose in addition that f^{in} satisfies the following uniform analyticity assumption: there exist $\sigma > 0$ and $C_0 > 0$ such that, for every $k \in \mathbb{Z}$, the function $v \mapsto \hat{f}_k^{in}(v)$, $v \in \mathbb{R}$, extends to a holomorphic function on the strip $S_\sigma := \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \sigma\}$, continuous on $\overline{S_\sigma}$, satisfying*

$$\big|\hat{f}_k^{in}(u + i\eta)\big| \;\le\; \frac{C_0}{(1+u^2)^2}, \qquad \forall\, u \in \mathbb{R},\ |\eta| \le \sigma,\ k \in \mathbb{Z}. \tag{2.4}$$

Then there exists $C_1 = C_1(C_0,\sigma) > 0$, independent of k and t, such that for all $m \in \{0,1,2\}$,

$$\big|\hat{m}_{\varphi,k}(t)\big| \;\le\; C_1\, e^{-|k|\sigma t}, \qquad \forall\, k \in \mathbb{Z}\setminus\{0\},\ \forall\, t \ge 0,\ \forall\, \varphi(v) = v^m. \tag{2.5}$$

Proof. We start to prove *(i)* by considering

$$\hat{m}_{\varphi,k}(t) = \int_{\mathbb{R}} (\varphi\, \hat{f}_k^{in})(v)\, e^{-i(kt)v}\, \mathrm{d}v,$$

which is, for fixed $k \neq 0$, the Fourier transform on $\mathbb{R}$ of the L^1 function $\varphi\, \hat{f}_k^{in}$, evaluated at the frequency kt. Since $k \neq 0$ is fixed and $t \to +\infty$, the frequency $kt \to \pm\infty$, and the Riemann–Lebesgue lemma gives $\hat{m}_{\varphi,k}(t) \to 0$.

Let us now prove the second item (ii). First let us observe that on the real axis, $|e^{-ikvt}| = 1$ for every $v \in \mathbb{R}$: the oscillating factor never decays in modulus, it only rotates in $\mathbb{C}$. Hence, the decay obtained in (i) comes entirely from a fine cancellation inside the integral, about which Riemann–Lebesgue says nothing quantitative. To obtain an explicit rate, we exploit the analyticity of $\hat{f}_k^{in}$ to *deform the contour of integration off the real axis*. Indeed, writing $z = u - i\eta \in \mathbb{C}$ with $\eta > 0$,

$$e^{-ikzt} \;=\; e^{-ik(u-i\eta)t} \;=\; e^{-ikut} \cdot e^{-k\eta t},$$

and the new factor $e^{-k\eta t}$ is *real* and exponentially decaying in t for $k, \eta, t \geq 0$ – it appears only because the integration variable has been given an imaginary part. This is a consequence of the Cauchy's theorem, which guarantees that this deformation does not change the value of the integral, as long as the path stays within the strip of holomorphy S_σ and no singularity is crossed (Figure 2.1). In short: analyticity converts a purely oscillatory integral into one carrying an explicit exponential prefactor, without ever changing its value – this is the same mechanism, in its most elementary form, that underlies the contour-deformation arguments used to establish Landau damping for the Vlasov–Poisson system [24, 25].

More precisely, we fix $k > 0$ and $t \geq 0$ (the case $k < 0$ follows by the symmetric argument, shifting the contour upwards instead of downwards). Since $\varphi(z) = z^m$ is a polynomial and $\hat{f}_k^{in}$ is holomorphic on S_σ, the function

$$z \;\longmapsto\; g_k(z) := \varphi(z)\, \hat{f}_k^{in}(z)\, e^{-ikzt}$$

is holomorphic on S_σ and continuous on $\overline{S_\sigma}$. For $z = u + i\eta$ with $\eta \in [-\sigma, \sigma]$ and $m \leq 2$, $|\varphi(z)| = |z|^m \leq (u^2 + \sigma^2)^{m/2} \leq C_\sigma(1 + u^2)$, so by (2.4),

$$\begin{aligned} |g_k(u + i\eta)| \;&\leq\; C_\sigma\,(1 + u^2) \cdot \frac{C_0}{(1 + u^2)^2} \cdot |e^{-ik(u+i\eta)t}| \\ &=\; \frac{C_\sigma C_0}{1 + u^2}\, e^{k\eta t}, \qquad \eta \in [-\sigma, \sigma]. \end{aligned}$$

Since $k > 0$ and $t \geq 0$, the factor $e^{k\eta t}$ is bounded by 1 for $\eta \in [-\sigma, 0]$: this is the direction we shift into. On the strip $\eta \in [-\sigma, 0]$, the latter bound therefore reads

$$|g_k(u + i\eta)| \leq \frac{C_\sigma\, C_0}{1 + u^2}, \tag{2.6}$$

uniformly in η, by an $L^1(\mathrm{d}u)$ function.

Let Γ_T be the boundary of the rectangle with vertices $-T$, T, $T - i\sigma$, $-T - i\sigma$, traversed in this cyclic order (top edge left to right, right edge downward, bottom edge right to left, left edge upward), as illustrated in Figure 2.1.

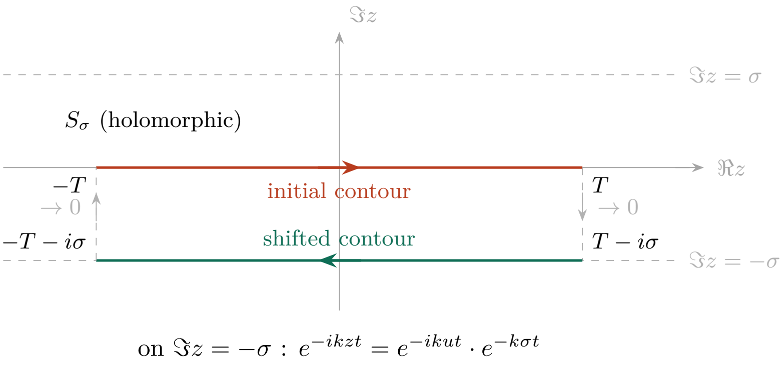

Figure 2.1: The rectangular contour Γ_T used in the proof, for $k > 0$. The top edge (red), on the real axis, carries the original integral $\hat{m}_{\varphi,k}(t)$; the bottom edge (teal), on $\Im z = -\sigma$, carries the shifted integral which produces the decay factor $e^{-k\sigma t}$. The two vertical edges (dashed) vanish as $T \to +\infty$, and only serve to close the contour so that Cauchy's theorem can be applied.

The bottom edge $\{\Im z = -\sigma\}$ lies on the boundary of the open strip S_σ, so we first apply Cauchy's theorem to the slightly smaller rectangle $R_T^\varepsilon := [-T, T] \times [-\sigma + \varepsilon, 0]$, which is compactly contained in S_σ for every $\varepsilon \in (0, \sigma)$: since g_k is holomorphic there,

$$\oint_{\partial R_T^\varepsilon} g_k(z)\,\mathrm{d}z \;=\; 0.$$

Letting $\varepsilon \to 0$, the bound (2.6) uniformly with respect to η) and the continuity of g_k on $\overline{S_\sigma}$ allow us to pass to the limit by dominated convergence, giving

$$\oint_{\Gamma_T} g_k(z)\,\mathrm{d}z = 0.$$

Summing the four edges parametrized as above, with $\mathrm{d}z = \mathrm{d}u$ on the horizontal edges and $\mathrm{d}z = i\,\mathrm{d}\eta$ on the vertical ones, it gives

$$\int_{-T}^{T} g_k(u)\,\mathrm{d}u \;-\; \int_{-T}^{T} g_k(u - i\sigma)\,\mathrm{d}u \;-\; i\int_{-\sigma}^{0} g_k(T + i\eta)\,\mathrm{d}\eta \;+\; i\int_{-\sigma}^{0} g_k(-T + i\eta)\,\mathrm{d}\eta \;=\; 0.$$

By (2.6) with $\eta \in [-\sigma, 0]$, we show that the two vertical contributions are bounded as

$$\left|\int_{-\sigma}^{0} g_k(T + i\eta)\,\mathrm{d}\eta\right| \;+\; \left|\int_{-\sigma}^{0} g_k(-T + i\eta)\,\mathrm{d}\eta\right| \le 2\,\sigma\,\frac{C_\sigma C_0}{1 + T^2} \to 0, \qquad \text{as } T \to +\infty.$$

Letting $T \to +\infty$, it yields

$$\int_{\mathbb{R}} g_k(u)\,\mathrm{d}u \;=\; \int_{\mathbb{R}} g_k(u - i\sigma)\,\mathrm{d}u,$$

the left-hand side being exactly $\hat{m}_{\varphi,k}(t)$ by (2.3). On the right-hand side, $e^{-ik(u-i\sigma)t} = e^{-ikut}\,e^{-k\sigma t}$, so

$$\hat{m}_{\varphi,k}(t) \;=\; e^{-k\sigma t}\int_{\mathbb{R}} \varphi(u - i\sigma)\,\hat{f}_k^{in}(u - i\sigma)\,e^{-ikut}\,\mathrm{d}u.$$

Using again (2.6) at $\eta = -\sigma$, the remaining integral is bounded by

$$\left| \int_{\mathbb{R}} \varphi(u - i\sigma)\, \hat{f}_k^{in}(u - i\sigma)\, e^{-ikut}\, \mathrm{d}u \right| \le \int_{\mathbb{R}} \frac{C_\sigma C_0}{1 + u^2}\, \mathrm{d}u = \pi C_\sigma C_0 =: C_1$$

where $C_1 > 0$ is independent of k and t. Hence, for $k > 0$ we have

$$|\hat{m}_{\varphi,k}(t)| \le C_1\, e^{-k\sigma t} = C_1\, e^{-|k|\sigma t}.$$

For $k < 0$, the same argument with the contour shifted to $\Im z = +\sigma$ instead (where now $|e^{-ikzt}| = e^{k\eta t} = e^{-|k|\eta t} \le 1$ for $\eta \in [0, \sigma]$, since $k < 0$) gives $|\hat{m}_{\varphi,k}(t)| \le C_1\, e^{-|k|\sigma t}$ as well. This proves (2.5). $\square$

Corollary 2.1.3 (Exponential relaxation to the homogeneous state)**.** *Suppose $f^{in}(x,v) = \sum_{|k| \le K} \hat{f}_k^{in}(v)\, e^{ikx}$ is a trigonometric polynomial in x of finite degree $K \ge 1$ satisfying the analyticity assumption (2.4) of Theorem 2.1.2 for every $|k| \le K$. Then, for every $\varphi(v) = v^m$, $m \in \{0, 1, 2\}$,*

$$\left\| m_\varphi(t, \cdot) \;-\; \hat{m}_{\varphi,0}(0) \right\|_{L^\infty(\mathbb{T})} \;\le\; 2K\, C_1\, e^{-\sigma t} \qquad \forall\, t \ge 0,$$

where $\hat{m}_{\varphi,0}(0) = \frac{1}{2\pi} \int_0^{2\pi} m_\varphi(0,x)\, \mathrm{d}x$ is the (conserved) spatial average of m_φ. In particular, the density $n = m_1$, momentum $nu = m_v$ and kinetic energy $e = m_{v^2/2}$ converge, uniformly in x, to their spatially homogeneous equilibrium values exponentially fast, at the rate σ given by the width of analyticity of the initial datum in v.

Proof. Since f^{in} has only finitely many nonzero Fourier modes $|k| \le K$, so does $f(t, \cdot, v)$ by (2.2), and

$$m_\varphi(t,x) - \hat{m}_{\varphi,0}(t) \;=\; \sum_{1 \le |k| \le K} \hat{m}_{\varphi,k}(t)\, e^{ikx}.$$

Since $\hat{m}_{\varphi,0}(t) = \hat{m}_{\varphi,0}(0)$ is constant in time (the $k = 0$ mode of (2.2) does not depend on t), Theorem 2.1.2(ii) gives, for every $x \in \mathbb{T}$,

$$\left| m_\varphi(t,x) - \hat{m}_{\varphi,0}(0) \right| \;\le\; \sum_{1 \le |k| \le K} \left| \hat{m}_{\varphi,k}(t) \right| \;\le\; \sum_{1 \le |k| \le K} C_1\, e^{-|k|\sigma t} \;\le\; 2K\, C_1\, e^{-\sigma t},$$

using $|k| \ge 1$ for every term in the sum. Taking the supremum over $x \in \mathbb{T}$ gives the claim. $\square$

Remark 2.1.4. *The Maxwellian $f_0(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$ is entire and satisfies $|f_0(u + i\eta)| = \frac{1}{\sqrt{2\pi}} e^{\eta^2/2} e^{-u^2/2} \le \frac{e^{\sigma^2/2}}{\sqrt{2\pi}} e^{-u^2/2}$ for $|\eta| \le \sigma$, which decays faster than any polynomial in u and so satisfies (2.4) for every $\sigma > 0$ (with $C_0 = C_0(\sigma)$). Hence a single-mode perturbation of a Maxwellian, $f^{in}(x,v) = f_0(v)(1 + \varepsilon \cos x)$ – a typical choice for the numerical illustration below – satisfies the hypotheses of Corollary 2.1.3 for every $\sigma > 0$: the corresponding decay rate can be taken arbitrarily large, only limited in practice by the constant $C_1(\sigma) \to \infty$ as $\sigma \to \infty$, reflecting the fact that faster decay requires resolving finer filamentation in v (consistently with the filamentation observed in Figures 2.2–2.3).*

Consider now, as a concrete illustration of Corollary 2.1.3, an initial datum f^{in} such that

$$\frac{1}{\mathrm{mes}(\mathbb{T})}\int_{\mathbb{T}}\int_{\mathbb{R}}\begin{pmatrix}1\\ v\\ \frac{|v|^2}{2}\end{pmatrix} f^{in}(x,v)\,\mathrm{d}v\,\mathrm{d}x=\begin{pmatrix}1\\ 0\\ \frac{1}{2}\end{pmatrix},$$

i.e., $\hat{n}_0(0)=1$, $\widehat{(nu)}_0(0)=0$, $\hat{e}_0(0)=\frac{1}{2}$ in the notation (2.3). By the conservation of the $k=0$ mode noted above, these are exactly the (time-independent) equilibrium values $\hat{m}_{\varphi,0}(0)$ appearing in Corollary 2.1.3: the macroscopic quantities are therefore guaranteed to converge to this space homogeneous state exponentially fast, as soon as f^{in} satisfies the analyticity assumption of the Corollary (e.g. for the Maxwellian-based data of the Remark above).

In the following figure we explore numerically the time evolution of the distribution function and the trend to equilibrium of macroscopic quantities n, nu and energy e.

This is the simplest example of phase mixing where macroscopic quantities converge to equilibrium exponentially fast while the distribution function is a solution of a reversible equation.

2.2 Non-conservative equation

We now consider a more general transport equation in which the transported quantity is not necessarily conserved along the flow. Such equations naturally appear after changes of variables, linearization procedures, or when source terms are incorporated into the dynamics.

Setting $z=(x,v)\in\mathbb{R}^d$ and letting $V(t,z)\in\mathbb{R}^d$ be a time-dependent vector field, we consider the Cauchy problem

$$\begin{cases}\partial_t f+V(t,z)\cdot\nabla_z f=0,\\ f(0,z)=f^{in}(z).\end{cases}$$

We impose the following assumptions on the velocity field:

(H1) $V\in\mathscr{C}([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$.

(H2) $\nabla_z V\in\mathscr{C}([0,T]\times\mathbb{R}^d;\mathbb{R}^{d\times d})$.

(H3) *Linear growth:* there exists $\kappa>0$ such that

$$\|V(t,z)\|\le\kappa(1+\|z\|),\qquad\forall(t,z)\in[0,T]\times\mathbb{R}^d.$$

Definition 2.2.1 (Characteristic curves). *Given $(t,z)\in[0,T]\times\mathbb{R}^d$, the* characteristic curve *of the vector field V issued from z at time t is the solution $s\mapsto\gamma(s)$ of*

$$\begin{cases}\dfrac{\mathrm{d}\gamma}{\mathrm{d}s}(s)\;=\;V(s,\gamma(s)),\\ \gamma(t)=z.\end{cases}$$

We denote this solution by $Z(s,t,z):=\gamma(s)$.

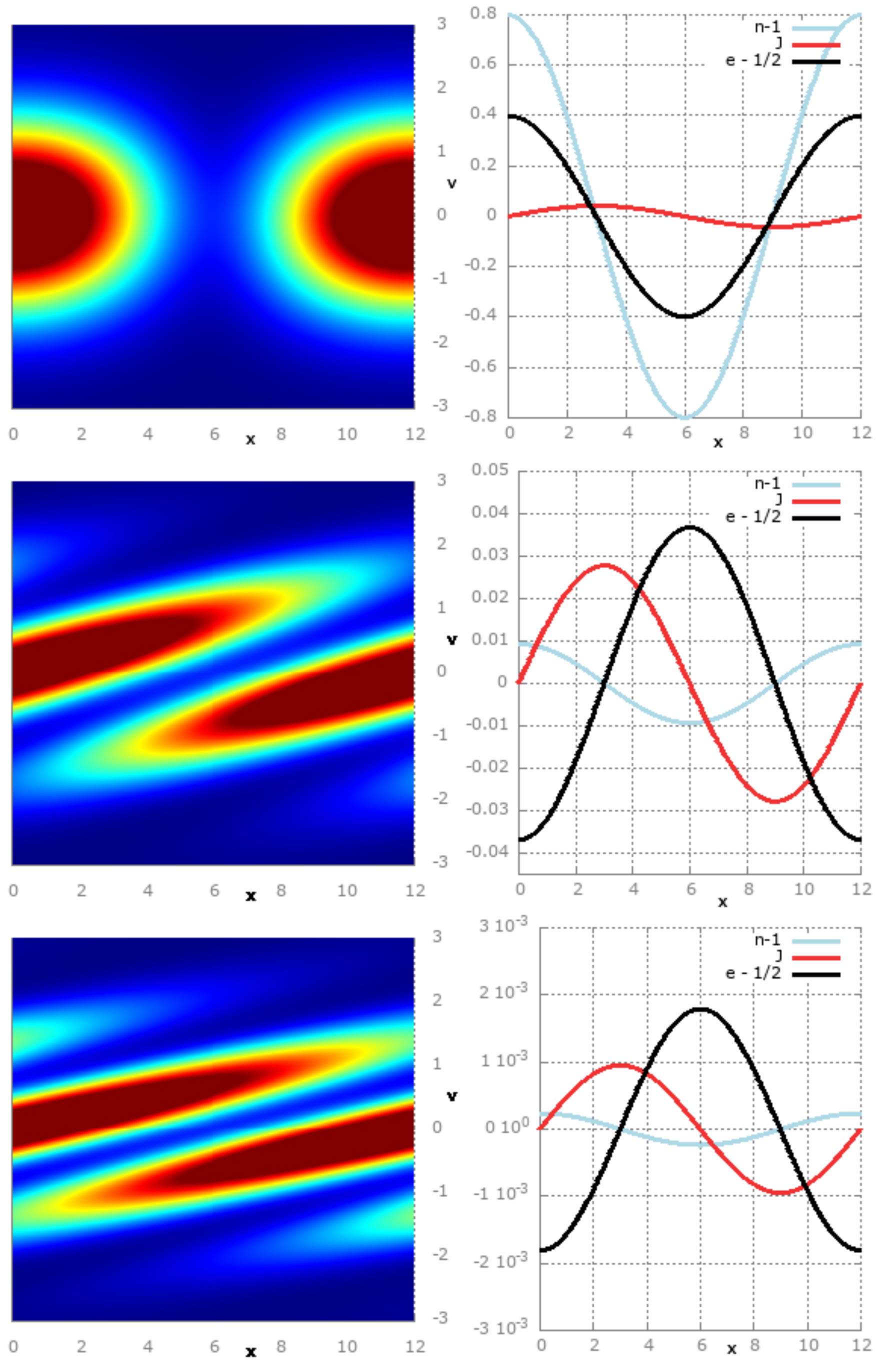

Figure 2.2: Phase mixing. *Left:* difference between the distribution function $f(t,x,v)$ and $f_{eq}(v)$ in the (x,v)-plane at time $t = 0.1$, 5.7 and 7.7, showing the characteristic fine-scale filamentation produced by phase mixing. *Right:* time evolution of the difference between the density n, momentum J and energy e and their equilibrium values 1, 0 and 1/2.

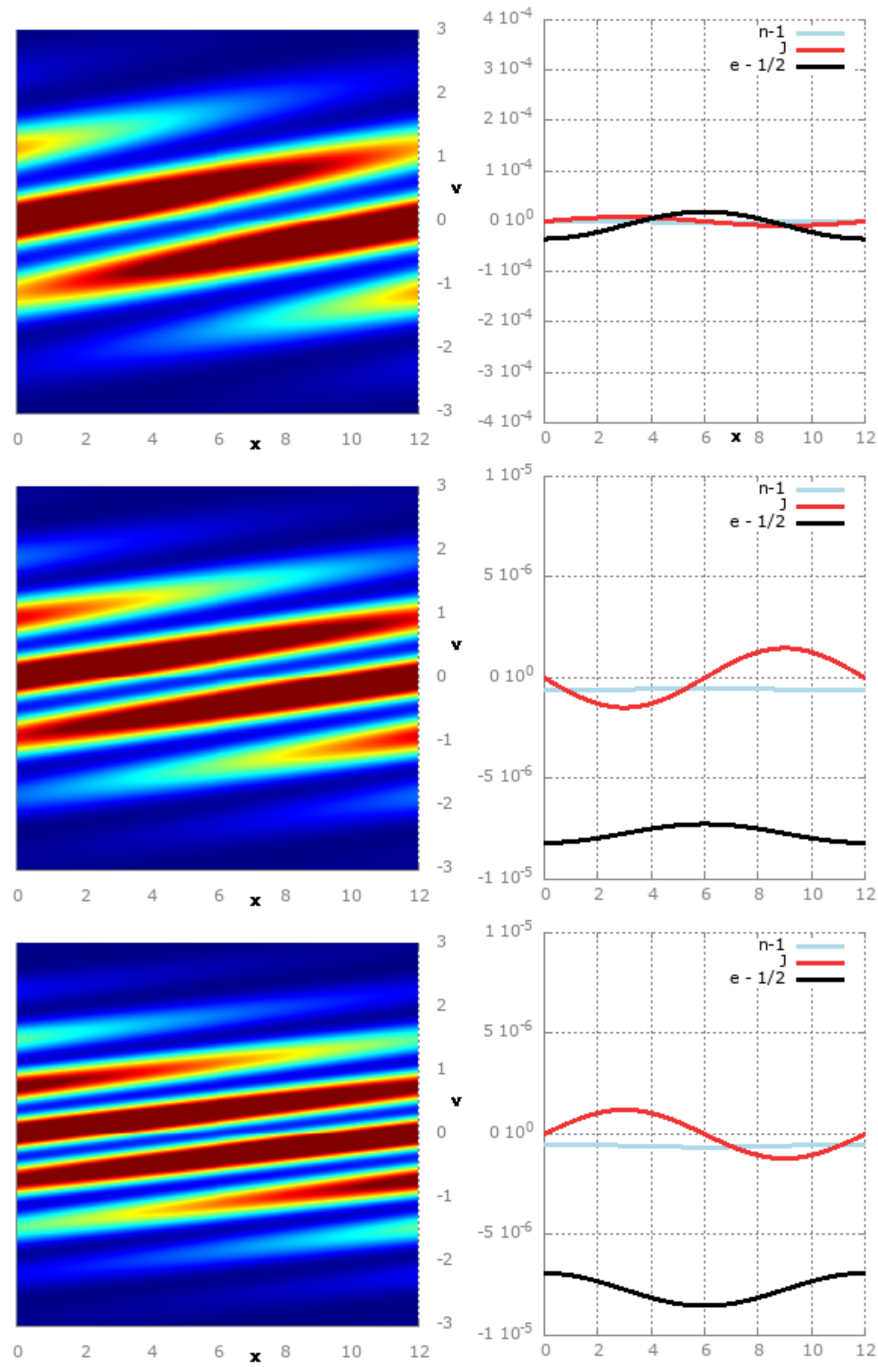

Figure 2.3: Phase mixing. *Left:* difference between the distribution function $f(t, x, v)$ and $f_{eq}(v)$ in the (x, v)-plane at time 9.7, 12.2 and 15.7, showing the characteristic fine-scale filamentation produced by phase mixing. *Right:* time evolution of the difference between the density n, momentum J and energy e and their equilibrium values 1, 0 and 1/2.

Theorem 2.2.2 (Cauchy–Lipschitz for characteristics)**.** *Under assumptions* (H1)–(H3)*, for each* $(t, z) \in [0, T] \times \mathbb{R}^d$ *there exists a unique global solution*

$$s \mapsto Z(s, t, z) \in \mathscr{C}^1([0, T]; \mathbb{R}^d).$$

Moreover, $Z \in \mathscr{C}^1([0, T]^2 \times \mathbb{R}^d; \mathbb{R}^d)$.

Proof. The velocity field V satisfies assumptions (H1)–(H2), so by the Cauchy–Lipschitz theorem there exists a unique maximal $\mathscr{C}^1$ solution defined on some interval $I(t, z) \subset [0, T]$. For any $s \in I(t, z)$, integrating the ODE gives

$$Z(s, t, z) - z = \int_t^s V(\tau, Z(\tau, t, z))\ \mathrm{d}\tau,$$

hence

$$|Z(s, t, z)| \leq |z| + \int_t^s |V(\tau, Z(\tau, t, z))|\ \mathrm{d}\tau.$$

Applying assumption (H3) yields

$$|Z(s, t, z)| \leq |z| + \kappa \int_t^s (1 + |Z(\tau, t, z)|)\ \mathrm{d}\tau.$$

Gronwall's lemma then gives the global bound

$$|Z(s, t, z)|\ \leq\ (|z|\ +\ \kappa T)\, e^{\kappa T},$$

which shows that the solution cannot blow up, so $\overline{I(t, z)} = [0, T]$. Continuous and differentiable dependence on the parameters (t, z) follows from another application of the Gronwall inequality to the variational equation. $\square$

The flow map satisfies the following additional structural properties.

Theorem 2.2.3. *The flow* $Z(s, t, z)$ *satisfies:*

(i) ***Flow (semigroup) property:*** *for all* t_1, t_2, $t_3 \in [0, T]$ *and* $z \in \mathbb{R}^d$*, we have*

$$Z(t_3, t_2, Z(t_2, t_1, z))\ =\ Z(t_3, t_1, z).$$

(ii) ***Diffeomorphism:*** $Z(s, t, \cdot)$ *is a* $\mathscr{C}^1$*-diffeomorphism of* $\mathbb{R}^d$ *with inverse*

$$Z(s, t, \cdot)^{-1}\ =\ Z(t, s, \cdot).$$

(iii) ***Jacobian:*** *the determinant* $J(s, t, z) = \det(D_z Z(s, t, z))$ *satisfies*

$$\partial_s J = (\nabla_z \cdot V)\,(s, Z(s, t, z))\, J, \quad J(t, t, z) = 1,$$

which integrates to

$$J(s, t, z) = \exp\!\left(\int_t^s (\nabla_z \cdot V)(\tau, Z(\tau, t, z))\, \mathrm{d}\tau\right).$$

Proof. Let us start with the semigroup property for fixed t_1, $t_2 \in [0,T]$ and $z \in \mathbb{R}^d$, we know that both maps

$$t_3 \mapsto Z(t_3, t_2, Z(t_2, t_1, z))$$

and

$$t_3 \mapsto Z(t_3, t_1, z)$$

are integral curves of the vector field V. At $t_3 = t_2$ the first curve takes the value $Z(t_2, t_2, Z(t_2, t_1, z)) = Z(t_2, t_1, z)$, while the second takes the value $Z(t_2, t_1, z)$ as well.

Therefore, both integral curves pass through the same point $Z(t_2, t_1, z)$ at time t_2, and by uniqueness for the ODE of Definition 2.2.1, they coincide for all $t_3 \in [0,T]$. This is exactly the semigroup identity.

Let us now prove that $Z(s,t,.)$ is a diffeomorphism. Setting $t_3 = t_1$ in (i) yields that

$$Z(t_1, t_2, Z(t_2, t_1, z)) \;=\; Z(t_1, t_1, z) = z.$$

Hence $z \mapsto Z(t_2, t_1, z)$ is invertible with inverse $Z(t_1, t_2, \cdot)$, which belongs to $\mathscr{C}^1(\mathbb{R}^d)$ by Theorem 2.2.2.

Finally, let us define $J(s,t,z) = \det(D_z Z(s,t,z))$. Using that $Z \in \mathscr{C}^1([0,T]^2 \times \mathbb{R}^d)$, we can differentiate with respect to the last variable and take the determinant, which commute smoothly. Then, we use the Jacobi identity for the differential of the determinant (Jacobi formula): for $A \in \mathrm{GL}_d(\mathbb{R})$,

$$B \mapsto (D \det A) \cdot B = \det(A)\ \mathrm{tr}(A^{-1}B).$$

Differentiating J with respect to s and using $\partial_s(D_z Z) = D_z(\partial_s Z) = D_z(V(s, Z(s,t,z)))$, we obtain

$$\begin{aligned}
\partial_s J(s,t,z) &= J(s,t,z)\ \mathrm{tr}\Big((D_z Z)^{-1}(s,t,z)\ \nabla_z V(s, Z(s,t,z))\, D_z Z(s,t,z)\Big)\\
&= J(s,t,z)\ \mathrm{tr}(D_z V(s, Z(s,t,z)))\\
&= J(s,t,z)\,(\nabla_z \cdot V)(s, Z(s,t,z)),
\end{aligned}$$

where we used the cyclic property of the trace $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any A, $B \in \mathcal{M}_d(\mathbb{C})$. The initial condition $J(t,t,z) = 1$ follows from $D_z Z(t,t,z) = I_d$. □

Remark 2.2.4. *The Jacobian $J(s,t,z) = \det(D_z Z(s,t,z))$ measures the local volume distortion of the flow. When $\nabla_z \cdot V = 0$ we have $J \equiv 1$: the flow is* incompressible *and the non-conservative equation coincides with the conservative form*

$$\begin{cases} \partial_t f + \nabla_z \cdot (V(t,z)\, f) = 0, \\ f(0,z) = f^{in}(z), \end{cases}$$

which implies conservation of total mass:

$$\int_{\mathbb{R}^d} f(t)\, \mathrm{d}z \;=\; \int_{\mathbb{R}^d} f^{in}\, \mathrm{d}z.$$

When $J \not\equiv 1$ the flow is compressible and mass is not preserved.

We now use the flow to construct the solution of the non-conservative equation.

Theorem 2.2.5. *Under assumptions* (H1)–(H2)*, the unique* $\mathscr{C}^1$ *solution of the non-conservative transport equation is*

$$f(t,z) = f^{in}(Z(0,t,z))\,,$$

and satisfies $\|f(t)\|_{L^\infty} = \|f^{in}\|_{L^\infty}$.

Proof. We follow the same strategy as in the free transport case.

Uniqueness. Let $f(t,z)$ be a $\mathscr{C}^1$ solution. For any $(t,z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f(t, Z(t,0,z)) &= \Big(\partial_t f + \partial_s Z(s,0,z)\big|_{s=t} \cdot \nabla_z f\Big)(t, Z(t,0,z)) \\ &= \left(\partial_t f + V \cdot \nabla_z f\right)(t, Z(t,0,z)) = 0, \end{aligned}$$

since f is a solution. Hence $f(t, Z(t,0,z)) = f(0,z) = f^{in}(z)$. Setting $y = Z(t,0,z)$ and inverting via $z = Z(0,t,y)$ yields

$$f(t,y) = f^{in}(Z(0,t,y)),$$

which proves uniqueness.

Existence. The formula $f(t,z) = f^{in}(Z(0,t,z))$ defines a $\mathscr{C}^1$ function. We verify that it solves the PDE by differentiating and using the identity

$$\partial_t Z(s,t,z) + \left(V(t,z) \cdot \nabla_z\right) Z(s,t,z) = 0.$$

To establish this identity, we differentiate the flow property provided in Theorem 2.2.3 (i), as

$$Z(t_3, t_2, Z(t_2, t_1, z)) = Z(t_3, t_1, z)$$

with respect to t_2:

$$\left(\partial_t \;+\; V(t_2, Z(t_2,t_1,z)) \cdot \nabla_z\right) Z(t_3, t_2, Z(t_2,t_1,z)) \;=\; 0.$$

Setting $t_2 = t_1 = t$ and $t_3 = s$ yields the claimed identity. ☐

2.3 Conservative equation

Conservative formulations play a central role in applications because they directly express the conservation of mass or probability. They are particularly well adapted to weak formulations and numerical discretizations based on conservation principles.

Let $V(t,z) \in \mathbb{R}^d$ be a time-dependent vector field and consider the Cauchy problem

$$\begin{cases} \partial_t \mu + \nabla_z \cdot (V(t,z)\,\mu) = 0, \\ \mu(0,z) = \mu^{in}(z). \end{cases}$$

When μ^{in} is only a positive Radon measure or an $L^1(\mathbb{R}^d)$ function, we resort to a notion of weak solution. In the context of kinetic theory, it is natural to work with measure-valued solutions, since the distribution functions of particle systems are nonnegative by definition.

Definition 2.3.1 (Weak solution)**.** *A* weak solution *of the Cauchy problem for the conservative transport equation is an element* $\mu \in \mathscr{C}([0,T]; w\text{-}\mathcal{M}(\mathbb{R}^d))$ *that satisfies the initial condition* $\mu(0) = \mu^{in}$ *and, for every test function* $\varphi \in \mathscr{C}^1_c((0,T)\times\mathbb{R}^d)$*,*

$$\int_0^T \int_{\mathbb{R}^d} \left(\partial_t\varphi + V(t,z)\cdot\nabla_z\varphi\right)(t,z)\,\mu(t,\mathrm{d}z)\,\mathrm{d}t = 0.$$

Definition 2.3.2 (Push-forward of measures)**.** *Let* μ *be a positive Radon measure on* $\mathbb{R}^d$ *and let* $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ *be a measurable map. The* push-forward *of* μ *by* Φ *is the measure* $\Phi_{\#}\mu$ *defined by*

$$(\Phi_{\#}\mu)(B) = \mu(\Phi^{-1}(B))$$

for every Borel set $B \subset \mathbb{R}^d$*. Equivalently, for any bounded measurable function* ψ*,*

$$\int_{\mathbb{R}^d} \psi(y)\,(\Phi_{\#}\mu)(\mathrm{d}y) = \int_{\mathbb{R}^d} \psi(\Phi(x))\,\mu(\mathrm{d}x).$$

Theorem 2.3.3 (Measure solution via push-forward)**.** *Let* $V \equiv V(t,z)$ *satisfy* (H1)–(H2)*, and let* $\mu^{in} \in \mathcal{M}_+(\mathbb{R}^d)$ *be a positive Radon measure on* $\mathbb{R}^d$*. Then the unique weak solution in the sense of Definition 2.3.1 is*

$$\mu(t) \;=\; Z(t,0,\cdot)_{\#}\,\mu^{in},$$

that is, for every test function $\varphi \in \mathscr{C}(\mathbb{R}^d)$*,*

$$\int_{\mathbb{R}^d} \varphi(z)\,\mu(t,\mathrm{d}z) \;=\; \int_{\mathbb{R}^d} \varphi(Z(t,0,y))\,\mu^{in}(\mathrm{d}y).$$

Proof. Existence. Set $\mu(t) = Z(t,0,\cdot)_{\#}\mu^{in}$. Since $Z(t,0,\cdot)$ is a measurable map, $\mu(t) \in \mathcal{M}_+(\mathbb{R}^d)$. If μ^{in} has finite total mass, then

$$\int_{\mathbb{R}^d} \mu(t,\mathrm{d}z) \;=\; \int_{\mathbb{R}^d} \mu^{in}(\mathrm{d}z),$$

by the definition of push-forward. Now let $\varphi \in \mathscr{C}^1_c((0,T)\times\mathbb{R}^d)$ and define

$$\mathcal{I}(t) = \int_{\mathbb{R}^d} \varphi(t, Z(t,0,y))\,\mu^{in}(\mathrm{d}y).$$

Differentiating under the integral sign (justified by the compact support of φ):

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}t}(t) &= \int_{\mathbb{R}^d} \left(\partial_t\varphi + V\cdot\nabla_z\varphi\right)(t, Z(t,0,y))\,\mu^{in}(\mathrm{d}y) \\
&= \int_{\mathbb{R}^d} \left(\partial_t\varphi(t,z) + V(t,z)\cdot\nabla_z\varphi(t,z)\right)\mu(t,dz).
\end{aligned}$$

Since φ is compactly supported in time, integrating over $[0,T]$ gives

$$0 = \mathcal{I}(T) - \mathcal{I}(0) \;=\; \int_0^T \int_{\mathbb{R}^d} \left(\partial_t\varphi + V\cdot\nabla_z\varphi\right)(t,z)\,\mu(t,\mathrm{d}z)\,\mathrm{d}t,$$

so μ is a weak solution.

Uniqueness. Let μ be any weak solution and define

$$\nu(t) \ := \ Z(0,t,\cdot)_{\#}\,\mu(t).$$

Fix $\psi \in \mathscr{C}^1_c(\mathbb{R}^d)$. For any $\chi \in \mathscr{C}^\infty_c(0,T)$, we compute

$$-\int_0^T \chi'(t)\left(\int_{\mathbb{R}^d} \psi(x)\,\nu(t,dx)\right)\mathrm{d}t = -\int_0^T \chi'(t)\left(\int_{\mathbb{R}^d} \psi(Z(0,t,y))\,\mu(t,\mathrm{d}y)\right)\mathrm{d}t.$$

Using the key identity

$$(\partial_t + V(t,z)\cdot\nabla_z)\,\psi(Z(0,t,z)) = 0$$

obtained from Theorem 2.2.5, and the fact that μ is a weak solution, one obtains

$$\frac{d}{dt}\int_{\mathbb{R}^d} \psi(x)\,\nu(t,dx) = 0 \quad \text{in } \mathcal{D}'(0,T).$$

Hence $\nu(t) = \nu(0) = \mu^{in}$, which gives $\mu(t) = Z(t,0,\cdot)_{\#}\mu^{in}$.

To justify the differentiation and the integration by parts, we need the support of $(t,y) \mapsto \psi(Z(0,t,y))$ to be compact. This follows from the linear growth bound (H3): for all $s,t \in [0,T]$ and $y \in \mathbb{R}^d$,

$$|Z(s,t,y)| \leq (|y| + \kappa T)\,e^{\kappa T},$$

which ensures that the support remains bounded. □

We finally characterise smooth solutions when the initial datum is a function.

Theorem 2.3.4 (Classical solution of the conservative equation)**.** *Let V satisfy* (H1)–(H3) *and let $f^{in} \in \mathscr{C}^1(\mathbb{R}^d)$. Then the conservative transport equation*

$$\begin{cases} \partial_t f + \nabla_z\cdot(V(t,z)\,f) = 0, \\ f|_{t=0} = f^{in} \end{cases}$$

has a unique $\mathscr{C}^1$ solution given by

$$f(t,z) \ = \ f^{in}(Z(0,t,z)) \ J(0,t,z),$$

where $J(s,t,z) = \det(D_z Z(s,t,z))$ is the Jacobian of the flow.

Proof. Uniqueness. Let g be a solution with zero initial datum and set $u(t) = g(t, Z(t,0,y))$. Differentiating and using the transport equation in conservative form gives

$$u'(t) \ = \ -u(t)\,(\mathrm{div}_z\, V)(t, Z(t,0,y)), \quad \text{with} \quad u(0) = 0.$$

This is a linear ODE with a damping/amplifying coefficient and zero initial datum, so $u(t) = 0$ for all t, hence $g \equiv 0$.

Existence. Setting

$$f(t,z) \ = \ f^{in}(Z(0,t,z))\,J(0,t,z)$$

and differentiating with respect to t and z, it shows that f satisfies the PDE using the evolution equation for J from Theorem 2.2.3 (*iii*). □