

Chapter 1

Kinetic theory

Kinetic theory provides a mesoscopic description of many-particle systems, bridging the gap between microscopic particle dynamics and macroscopic continuum models. Rather than following every particle individually, one introduces a distribution function f describing the statistical state of the system in phase space. This framework plays a central role in plasma physics, rarefied gas dynamics, collective behaviour, and many other applications. Particular emphasis will be placed on the Vlasov–Poisson and Vlasov–Maxwell systems, which serve as guiding examples throughout these notes.

The state of the system is encoded in the *distribution function* $f(t, x, v)$, where $x \in \mathbb{R}^n$ is the position variable, $v \in \mathbb{R}^n$ is the velocity variable, with $n = 1, 2$, or 3 . More precisely,

$$f(t, x, v) dx dv = \text{number of particles in the phase-space volume } dx dv.$$

The distribution function satisfies the following transport equation:

$$\partial_t f + \nabla_x \cdot (v f) + \frac{1}{m} \nabla_v \cdot (F(t, x, v) f) = 0,$$

where $m > 0$ represents the mass of one particle.

When the force field satisfies $\nabla_v \cdot F = 0$ (as is the case for the Lorentz force), this simplifies to

$$\partial_t f + v \cdot \nabla_x f + \frac{1}{m} F(t, x, v) \cdot \nabla_v f = 0.$$

The principal macroscopic observables are defined as velocity moments of f :

$$\begin{pmatrix} n \\ nu \\ e \end{pmatrix} (t, x) = \int_{\mathbb{R}^n} \begin{pmatrix} 1 \\ v \\ m \frac{|v|^2}{2} \end{pmatrix} f(t, x, v) dv,$$

representing the particle density, momentum density, and kinetic energy density, respectively.

1.1 Electrostatic interactions

One of the most important examples of a kinetic model arises when particles interact through an electric field generated by their own charge distribution. The resulting coupling between

transport and Poisson's equation leads to the Vlasov–Poisson system, which is a cornerstone of plasma physics and galactic dynamics.

Considering particles of charge q and mass m , the Coulomb force acting on a particle is

$$F(t, x) = q E(t, x) = -q \nabla_x \phi(t, x),$$

where the electrostatic potential ϕ is determined self-consistently by the particle distribution through Poisson's equation:

$$-\epsilon_0 \Delta_x \phi = q n.$$

In this equation, the constant ϵ_0 is the vacuum permittivity. Combining these with the transport equation yields the *Vlasov–Poisson system*:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \frac{q}{m} \nabla_x \phi \cdot \nabla_v f = 0, \\ -\epsilon_0 \Delta_x \phi = q n, \\ f(t = 0) = f^{in}, \end{cases}$$

where the charge density is

$$n(t, x) = \int_{\mathbb{R}^n} f(t, x, v) dv.$$

and f^{in} is a given initial datum.

Properties of the Vlasov–Poisson system.

The Vlasov–Poisson system enjoys the following fundamental properties:

Proposition 1.1.1. *Consider (f, E) a smooth solution of the Vlasov–Poisson system. Then, we have*

- **Preservation of positivity.** *If $f^{in} \geq 0$, then $f(t) \geq 0$ for all $t \geq 0$.*
- **Preservation of L^p norms** ($1 \leq p \leq \infty$). *Since the phase-space divergence vanishes,*

$$\nabla_{(x,v)} \cdot (v, E(t, x)) = 0,$$

the flow is incompressible in phase space and therefore

$$\|f(t)\|_{L^p} = \|f^{in}\|_{L^p}, \quad \forall t \geq 0.$$

More generally, we may define Casimir invariants: for every continuous function $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}$, such that $\beta(0) = 0$, the quantity

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \beta(f(t, x, v)) dv dx$$

is conserved for all $t \geq 0$.

- **Conservation of energy.** *The total energy*

$$\mathcal{E}(t) := \frac{m}{2} \int_{\mathbb{T}^n \times \mathbb{R}^n} |v|^2 f(t, x, v) \, dv \, dx + \frac{\epsilon_0}{2} \int_{\mathbb{T}^n} |\nabla_x \phi(t, x)|^2 \, dx$$

is conserved for all $t \geq 0$.

Proof. First, let us consider $\beta \in \mathcal{C}^1(\mathbb{R})$ with $\beta(0) = 0$, and assume f decays fast enough in v that all the integrals below converge and that integrations by parts in v produce no boundary term as $|v| \rightarrow \infty$. Since f is smooth, the chain rule applied to the Vlasov equation gives

$$\partial_t \beta(f) + v \cdot \nabla_x \beta(f) + \frac{1}{m} F(t, x) \cdot \nabla_v \beta(f) = \beta'(f) \left[\partial_t f + v \cdot \nabla_x f + \frac{1}{m} F \cdot \nabla_v f \right] = 0,$$

where $F = -q \nabla_x \phi$. Integrating over $\mathbb{T}^n \times \mathbb{R}^n$,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}^n \times \mathbb{R}^n} \beta(f) \, dx \, dv &= - \int_{\mathbb{R}^n} \left(\int_{\mathbb{T}^n} v \cdot \nabla_x \beta(f) \, dx \right) dv \\ &\quad - \int_{\mathbb{T}^n} \left(\int_{\mathbb{R}^n} \frac{1}{m} F(t, x) \cdot \nabla_v \beta(f) \, dv \right) dx. \end{aligned}$$

For fixed v ,

$$\int_{\mathbb{T}^n} v \cdot \nabla_x \beta(f) \, dx = v \cdot \int_{\mathbb{T}^n} \nabla_x \beta(f) \, dx = 0,$$

since the integral of a gradient over the periodic torus vanishes. For fixed x ,

$$\frac{1}{m} \int_{\mathbb{R}^n} F(t, x) \cdot \nabla_v \beta(f) \, dv = \frac{1}{m} F(t, x) \cdot \int_{\mathbb{R}^n} \nabla_v \beta(f) \, dv = 0,$$

since F does not depend on v and $\beta(f) \rightarrow \beta(0) = 0$ as $|v| \rightarrow \infty$. Hence

$$\frac{d}{dt} \int_{\mathbb{T}^n \times \mathbb{R}^n} \beta(f(t, x, v)) \, dx \, dv = 0,$$

that is,

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \beta(f(t)) \, dx \, dv = \int_{\mathbb{T}^n \times \mathbb{R}^n} \beta(f^{in}) \, dx \, dv, \quad \forall t \geq 0. \quad (1.1)$$

Actually this results is also valid for any continuous function $\beta \in \mathcal{C}(\mathbb{R})$, with $\beta(0) = 0$. This can be proven using a classical regularizing technique : for any $\varepsilon > 0$, we build $\beta_\varepsilon \in \mathcal{C}^1(\mathbb{R})$ an approximation of β , then we get that for all $\varepsilon > 0$ and $t \geq 0$

$$\frac{d}{dt} \int_{\mathbb{T}^n \times \mathbb{R}^n} \beta_\varepsilon(f(t)) \, dx \, dv = 0,$$

hence using the Heine Theorem and the density of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$, we prove that for all $t \geq 0$

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \beta_\varepsilon(f(t)) \, dx \, dv \rightarrow \int_{\mathbb{T}^n \times \mathbb{R}^n} \beta(f(t)) \, dx \, dv, \quad \text{as } \varepsilon \rightarrow 0.$$

Positivity. Apply (1.1) to $\beta(s) = (s^-)^2$, where $s^- := \max(-s, 0)$, that is,

$$\beta(s) = \begin{cases} s^2, & \text{for } s \leq 0 \\ 0, & \text{for } s \geq 0. \end{cases}$$

Then $\beta \in \mathcal{C}^1(\mathbb{R})$, nonnegative and $\beta(s) = 0$, if and only if $s \geq 0$.

Since $f^{in} \geq 0$, $\beta(f^{in}) \equiv 0$, so (1.1) gives

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} (f(t, x, v)^-)^2 dx dv = 0, \quad \forall t \geq 0.$$

As f is continuous and the integrand is nonnegative, this forces $f(t, x, v)^- = 0$ for every (x, v) , *i.e.*

$$f(t, x, v) \geq 0, \quad \forall t \geq 0, (x, v) \in \mathbb{T}^n \times \mathbb{R}^n.$$

L^p norms and Casimir invariants. Identity (1.1) holds for every $\beta \in \mathcal{C}^1(\mathbb{R})$ with $\beta(0) = 0$; this is precisely the conservation of the Casimir functionals

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \beta(f(t)) dv dx.$$

Taking $\beta(s) = |s|^p$ (suitably mollified near 0 so as to be $\mathcal{C}^1$ when $1 \leq p < 2$) yields, for every $1 \leq p < \infty$,

$$\|f(t)\|_{L^p} = \|f^{in}\|_{L^p}, \quad 1 \leq p < \infty.$$

For $p = \infty$, recall the standard fact that for $g \in L^{p_0} \cap L^\infty$ for some finite p_0 ,

$$\|g\|_{L^\infty} = \lim_{p \rightarrow \infty} \|g\|_{L^p}.$$

Applying this to $g = f(t)$ and to $g = f^{in}$ both lying in $L^1 \cap L^\infty$, using positivity and the case $p = 1$ above and passing to the limit $p \rightarrow \infty$ in the previous identity gives

$$\|f(t)\|_{L^\infty} = \|f^{in}\|_{L^\infty}.$$

Conservation of energy. We first write the total energy as

$$\mathcal{E}(t) := \mathcal{E}_{kin}(t) + \mathcal{E}_{pot}(t)$$

with

$$\mathcal{E}_{kin}(t) = \frac{m}{2} \iint_{\mathbb{T}^n \times \mathbb{R}^n} |v|^2 f dv dx, \quad \mathcal{E}_{pot}(t) = \frac{\epsilon_0}{2} \int_{\mathbb{T}^n} |\nabla_x \phi|^2 dx.$$

Using the Vlasov equation and integrating by parts (in x , by periodicity; in v , using decay of f),

$$\begin{aligned} \frac{d\mathcal{E}_{kin}}{dt} &= \frac{m}{2} \iint_{\mathbb{T}^n \times \mathbb{R}^n} |v|^2 \partial_t f dx dv \\ &= -\frac{m}{2} \iint_{\mathbb{T}^n \times \mathbb{R}^n} |v|^2 v \cdot \nabla_x f dx dv + \frac{q}{2} \iint_{\mathbb{T}^n \times \mathbb{R}^n} |v|^2 \nabla_x \phi \cdot \nabla_v f dx dv. \end{aligned}$$

The first term vanishes since, for fixed v ,

$$\int_{\mathbb{T}^n} v \cdot \nabla_x f \, dx = v \cdot \int_{\mathbb{T}^n} \nabla_x f \, dx = 0.$$

In the second term,

$$\int_{\mathbb{R}^n} |v|^2 \nabla_v f \, dv = -2 \int_{\mathbb{R}^n} v f \, dv$$

by parts in v , so, denoting the momentum density

$$nu(t, x) := \int_{\mathbb{R}^n} v f \, dv,$$

we have

$$\frac{d\mathcal{E}_{\text{kin}}}{dt} = -q \int_{\mathbb{T}^n} \nabla_x \phi \cdot (nu) \, dx. \quad (1.2)$$

Multiplying the Poisson equation $-\epsilon_0 \Delta_x \phi = q n$ by ϕ and integrating by parts on $\mathbb{T}^n$,

$$\mathcal{E}_{\text{pot}}(t) = \frac{\epsilon_0}{2} \int_{\mathbb{T}^n} |\nabla_x \phi|^2 \, dx = \frac{q}{2} \int_{\mathbb{T}^n} n \phi \, dx.$$

Furthermore, by self-adjointness of $-\Delta_x$ on $\mathbb{T}^n$ and the relation $-\epsilon_0 \Delta_x \partial_t \phi = q \partial_t n$ obtained by differentiating the Poisson equation in t , we have

$$\int_{\mathbb{T}^n} \partial_t n \phi \, dx = -\frac{\epsilon_0}{q} \int_{\mathbb{T}^n} (\Delta_x \partial_t \phi) \phi \, dx = -\frac{\epsilon_0}{q} \int_{\mathbb{T}^n} \partial_t \phi \Delta_x \phi \, dx = \int_{\mathbb{T}^n} n \partial_t \phi \, dx,$$

hence

$$\frac{d\mathcal{E}_{\text{pot}}}{dt} = \frac{q}{2} \int_{\mathbb{T}^n} (\partial_t n \phi + n \partial_t \phi) \, dx = q \int_{\mathbb{T}^n} \partial_t n \phi \, dx.$$

Finally, integrating the Vlasov equation in v gives the continuity equation

$$\partial_t n + \nabla_x \cdot (nu) = 0$$

the force term integrates to zero in v , being itself a v -divergence. Substituting and integrating by parts in x ,

$$\frac{d\mathcal{E}_{\text{pot}}}{dt} = -q \int_{\mathbb{T}^n} \nabla_x \cdot (nu) \phi \, dx = q \int_{\mathbb{T}^n} (nu) \cdot \nabla_x \phi \, dx. \quad (1.3)$$

Adding (1.2) and (1.3), it yields that

$$\frac{d\mathcal{E}}{dt} = -q \int_{\mathbb{T}^n} \nabla_x \phi \cdot (nu) \, dx + q \int_{\mathbb{T}^n} (nu) \cdot \nabla_x \phi \, dx = 0,$$

so we have $\mathcal{E}(t) = \mathcal{E}(0)$ for all $t \geq 0$. $\square$

Another important property of the solution to the Vlasov–Poisson system concerns the time propagation of velocity moments. For $p \geq 1$, the p th-order velocity moment

$$m_p(t) := \int_{\mathbb{T}^n \times \mathbb{R}^n} |v|^p f(t, x, v) \, dx \, dv$$

is finite for all t whenever it is finite initially. This control is crucial in the existence proof of solutions for the Vlasov–Poisson system [23, 21].

Illustration of some physical phenomena in one dimension.

For some physically relevant regimes, spatial variations occur predominantly along a single direction, and one works on a periodic spatial domain $\mathbb{T} = \mathbb{R}/(L\mathbb{Z})$ for some period $L > 0$. Setting $x \in \mathbb{T}$ and $v \in \mathbb{R}$, the *one-dimensional Vlasov–Poisson system* reads

$$\begin{cases} \partial_t f + v \partial_x f - \partial_x \phi \partial_v f = 0, & (x, v) \in \mathbb{T} \times \mathbb{R}, \\ -\partial_{xx} \phi = n - 1, & x \in \mathbb{T}, \end{cases} \quad (1.4)$$

where all physical parameters are taken equal to one and the charge density is

$$n(t, x) = \int_{\mathbb{R}} f(t, x, v) dv.$$

The right-hand side $n - 1$ in Poisson's equation models a uniform neutralising background of ions, which enforces overall charge neutrality.

A *homogeneous equilibrium* is a spatially uniform stationary solution $f = f_{eq}(v) \geq 0$, normalised so that

$$\int_{\mathbb{R}} f_{eq}(v) dv = 1.$$

Such a state trivially satisfies (1.4) since $n_{eq} = 1$ and $\partial_x \phi_{eq} = 0$.

To study the response of the system to small perturbations, write

$$f(t, x, v) = f_{eq}(v) + \varepsilon g(t, x, v)$$

with $\varepsilon \ll 1$ and expand (1.4) to first order in ε . The zeroth-order terms cancel, yielding the *linearised Vlasov–Poisson system*:

$$\begin{cases} \partial_t g + v \partial_x g - \partial_x \phi_g f'_{eq}(v) = 0, \\ -\partial_{xx} \phi_g = \int_{\mathbb{R}} g dv. \end{cases} \quad (1.5)$$

Seeking plane-wave solutions $g(t, x, v) = \hat{g}(v) e^{i(kx - \omega t)}$ and $\phi_g(t, x) = \hat{\phi} e^{i(kx - \omega t)}$, the first equation gives

$$i(kv - \omega) \hat{g}(v) - ik \hat{\phi} f'_{eq}(v) = 0, \quad \text{that is} \quad \hat{g}(v) = \frac{\hat{\phi} f'_{eq}(v)}{v - \omega/k},$$

while the Poisson equation reads $k^2 \hat{\phi} = \int_{\mathbb{R}} \hat{g} dv$. Substituting $\hat{g}$ and dividing by $\hat{\phi} \neq 0$ yields the *Vlasov dispersion relation*:

$$\mathcal{D}(k, \omega) := 1 - \frac{1}{k^2} \int_{\mathbb{R}} \frac{f'_{eq}(v)}{v - \omega/k} dv = 0, \quad \omega \in \mathbb{C}. \quad (1.6)$$

The roots $\omega = \omega(k)$ determine the spectrum of linear perturbations. The sign of $\text{Im}(\omega)$ is decisive:

- $\text{Im}(\omega) < 0$: the mode *decays* (collisionless Landau damping).

- $\text{Im}(\omega) > 0$: the mode *grows exponentially* (linear instability).
- $\text{Im}(\omega) = 0$: the mode *propagates undamped* (marginal stability, rare for generic data).

A sharp criterion for linear stability of a homogeneous equilibrium was established by Penrose [26] : f_{eq} is linearly stable for all wave numbers $k \neq 0$ if and only if, for every local minimum $v_* \in \mathbb{R}$ of f_{eq} ,

$$\mathcal{I}(v_*) = \int_{\mathbb{R}} \frac{f_{eq}(v) - f_{eq}(v_*)}{|v - v_*|^2} dv < 0.$$

A Maxwellian, which has a unique global maximum and no local minimum in its interior, satisfies this criterion for every k , consistent with Landau damping. By contrast, distributions with a secondary hump (such as two-beam or bump-on-tail profiles) violate the Penrose criterion for a band of wave numbers $|k| < k_c$, making those modes linearly unstable.

Beyond the linear regime, the dynamics are governed by nonlinear effects: particle trapping, filamentation of f in phase space, and the eventual formation of *BGK modes* (Bernstein–Greene–Kruskal waves), which are exact nonlinear travelling-wave solutions of (1.4).

We now present numerical simulations of the Landau damping, two-stream instability and bump on the tail for the one-dimensional Vlasov–Poisson system (1.4). In each experiment we track the *potential energy*

$$\mathcal{E}_{pot}(t) = \frac{1}{2} \int_{\mathbb{T}} |\partial_x \phi(t, x)|^2 dx$$

as a scalar diagnostic, together with snapshots of the distribution function $f(t, x, v)$ in phase space. The simulations employ a semi-Lagrangian Strang-split scheme on a uniform grid in (x, v) , which preserves positivity and conserves the L^2 norm of f to machine precision.

Landau damping. Let f_{eq} be the standard Maxwellian,

$$f_{eq}(v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right),$$

and perturb it by a small monochromatic density wave:

$$f_0(x, v) = (1 + \alpha \cos(kx)) f_{eq}(v), \quad (x, v) \in \mathbb{T} \times \mathbb{R},$$

with wave number $k = 0.5$ and amplitude $\alpha = 0.01$. The perturbation is purely spatial: every velocity shell is modulated uniformly, so the initial electric field $E_0 \sim \alpha \sin(kx)$ is small and the linear theory applies at early times.

Landau damping is a *collisionless* damping mechanism: an electrostatic wave transfers energy to resonant particles – those whose velocity equals the wave phase velocity $v_\phi = \omega_r/k$ – without any dissipation. For a Maxwellian, there are always more slow particles than fast ones in the vicinity of v_ϕ , so the net effect is a transfer of energy from the wave to the particles, causing the wave to decay.

In the linear regime, the electric field amplitude behaves as

$$E_k(t) \sim A_k e^{-\gamma_L t} \cos(\omega_r t + \varphi_k),$$

where ω_r is the real Langmuir frequency and $\gamma_L > 0$ is the *Landau damping rate*. For $k = 0.5$ and the Maxwellian equilibrium, numerical root-finding in the dispersion relation (1.6) gives $\omega_r \approx 1.416$ and $\gamma_L \approx 0.154$. The potential energy therefore decays as

$$\mathcal{E}_{pot}(t) \sim C e^{-2\gamma_L t}, \quad t \rightarrow +\infty.$$

At long times the linear approximation breaks down due to phase mixing: f develops increasingly fine filaments in the (x, v) -plane, resulting in arbitrarily small-scale oscillations that prevent return to equilibrium in the strong sense. The distribution function converges only *weakly* to f_{eq} , while macroscopic observables (density, electric field) do decay. The rigorous nonlinear theory was established by Mouhot and Villani (2011) for analytic perturbations on the torus [24].

The potential energy $\mathcal{E}_{pot}(t)$ exhibits exponential decay at the predicted Landau rate $2\gamma_L \approx 0.308$ for moderate times, in excellent agreement with the linear theory. At large times the filamentation of f becomes visible as thin oscillating layers in the (x, v) -plane; see Figure 1.1.

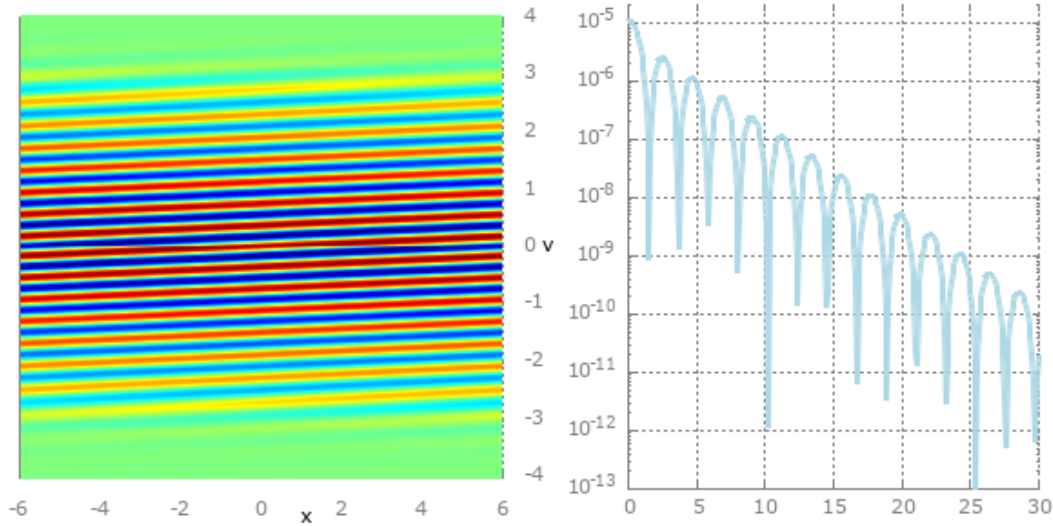


Figure 1.1: Landau damping. *Left*: difference between the distribution function $f(t, x, v)$ and $f_{eq}(v)$ in the (x, v) -plane at a late time showing the characteristic fine-scale filamentation produced by phase mixing. *Right*: time evolution of $\mathcal{E}_{pot}(t)$ on a semi-logarithmic scale; the dashed line indicates the theoretical decay rate $e^{-2\gamma_L t}$ predicted by linear theory.

Two-stream instability. Consider now the symmetric two-beam equilibrium configuration

$$f_{eq}(v) := \frac{1}{2\sqrt{2\pi T_0}} \left[\exp\left(-\frac{|v - v_0|^2}{2T_0}\right) + \exp\left(-\frac{|v + v_0|^2}{2T_0}\right) \right],$$

with drift velocity $v_0 = 1.5$, beam temperature $T_0 = 0.3$, and choose an initial datum such that, for $(x, v) \in \mathbb{T} \times \mathbb{R}$,

$$f_0(x, v) = (1 + 0.01 \cos(kx)) f_{eq}(v),$$

with the wave number $k = 0.5$. This models two equally dense, counter-propagating beams of charged particles, each with a Maxwellian velocity spread of width $\sqrt{T_0}$ centred at $\pm v_0$.

The two-stream instability arises from the violation of the Penrose criterion. The equilibrium distribution f_{eq} has a local minimum at $v = 0$, and the Penrose integral at $v_* = 0$,

$$\mathcal{I}(v_*) = \int_{\mathbb{R}} \frac{f_{eq}(v) - f_{eq}(v_*)}{|v - v_*|^2} dv,$$

is positive for sufficiently large drift v_0 , signalling linear instability for wave numbers $|k| < k_c$. The most unstable mode $k_{\max}$ and its growth rate $\gamma_{\max} = \text{Im}(\omega)$ are determined by the dispersion relation (1.6).

During the linear phase the electric field grows as $e^{\gamma_{\max} t}$, so that $\mathcal{E}_{pot}(t) \sim e^{2\gamma_{\max} t}$. As the amplitude increases, particles from each beam become trapped in the potential wells of the opposite-beam wave, forming closed orbits in the (x, v) -plane known as *phase-space holes* or “cat’s eyes”. The instability saturates when the bounce frequency of trapped particles equals the linear growth rate, at which point $\mathcal{E}_{pot}$ reaches a maximum. The subsequent dynamics involve merging of phase-space vortices and eventual relaxation toward a BGK-type nonlinear state.

The potential energy $\mathcal{E}_{pot}(t)$ grows exponentially at the predicted linear rate before saturating. The distribution function develops the characteristic “cat’s eye” pattern of rotating phase-space islands; see Figure 1.2.

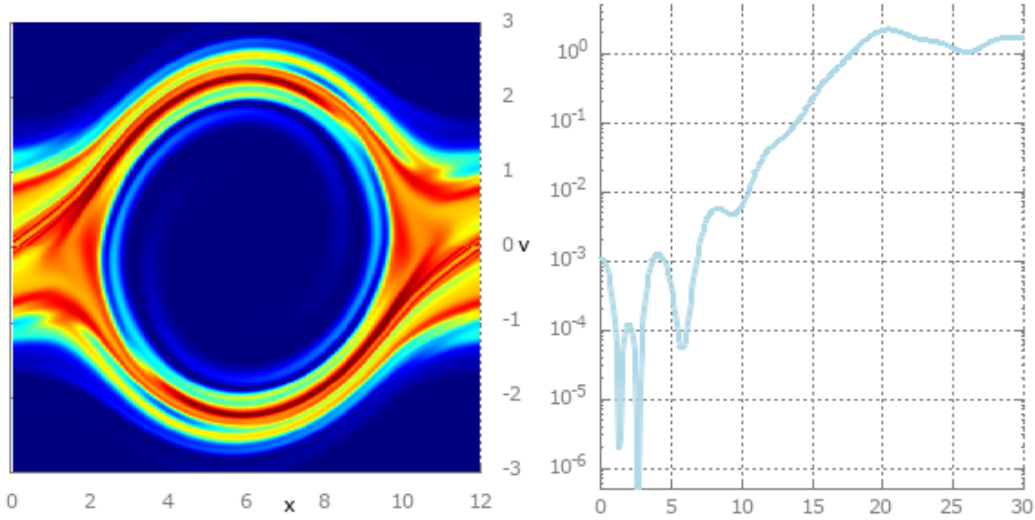


Figure 1.2: Two-stream instability. *Left*: distribution function $f(t, x, v)$ at a late time, exhibiting the characteristic “cat’s eye” phase-space vortices produced by particle trapping. *Right*: time evolution of $\mathcal{E}_{pot}(t)$ showing exponential linear growth followed by nonlinear saturation and oscillation around a BGK-like state.

Bump-on-tail instability. Consider the following superposition of a bulk Maxwellian and a shifted beam for $(x, v) \in \mathbb{T} \times \mathbb{R}$:

$$f_{eq}(v) = \frac{n_0}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right) + \frac{n_b}{\sqrt{2\pi T_b}} \exp\left(-\frac{|v - v_b|^2}{2T_b}\right),$$

with $n_0 = 0.9$, $v_b = 4.5$, $T_b = 0.5$, $n_b = 0.1$. This equilibrium thus consists of a cold bulk (density 0.9, thermal velocity 1) and a fast, tenuous beam with a density $n_b = 0.1$, centred at the large velocity v_b that sits far into the tail of the background Maxwellian. The bump-on-tail instability is a variant of the two-stream instability. The distribution f_{eq} has a local minimum at some $v_* \in (0, v_b)$ followed by a local maximum near v_b : the Penrose criterion is violated for wave numbers $|k| \in [k_{\min}, k_{\max}]$ determined by the beam parameters, making electrostatic modes in that range linearly unstable.

The free energy for the instability comes from the positive slope $\partial_v f_{eq} > 0$ between v_* and v_b : resonant particles in this velocity range are accelerated by the electric field, extracting kinetic energy from the beam and amplifying Langmuir waves at the resonant phase velocity $v_\phi = \omega_r/k \in (v_*, v_b)$. This mechanism is responsible for the emission of Langmuir waves observed in beam–plasma experiments and in space plasmas (e.g. type III solar radio bursts excited by electron beams from solar flares).

We choose the following initial datum

$$f_0(x, v) = f_{eq}(v) + 0.001 \cos(kx) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right),$$

with $k = 0.5$. During the linear phase, $\mathcal{E}_{pot}(t)$ grows exponentially.

We display the time evolution of $\mathcal{E}_{pot}(t)$ and a late-time snapshot of $f(t, x, v)$. The potential energy shows a clear exponential growth phase, followed by oscillations and saturation as the beam is quasilinearly eroded. In phase space, the flattening of the velocity distribution near v_b and the formation of small-scale structures are clearly visible; see Figure 1.3.

1.2 Electromagnetic interactions

The electrostatic approximation becomes insufficient when magnetic effects play a significant role. In that case, particles evolve under the full Lorentz force and the self-consistent fields satisfy Maxwell’s equations. This leads to the Vlasov–Maxwell system, one of the fundamental models of collisionless plasma dynamics.

The Lorentz force acting on a particle of charge q and velocity v is

$$F = q(E + v \times B).$$

Coupling this with Maxwell’s equations yields the *Vlasov–Maxwell system*:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{q}{m} (E + v \times B) \cdot \nabla_v f = 0, \\ \nabla \cdot B = 0, \quad \partial_t B + \nabla \times E = 0, \\ \epsilon_0 \nabla \cdot E = q n, \quad \frac{1}{c^2} \partial_t E - \nabla \times B = -\mu_0 q J, \end{cases}$$

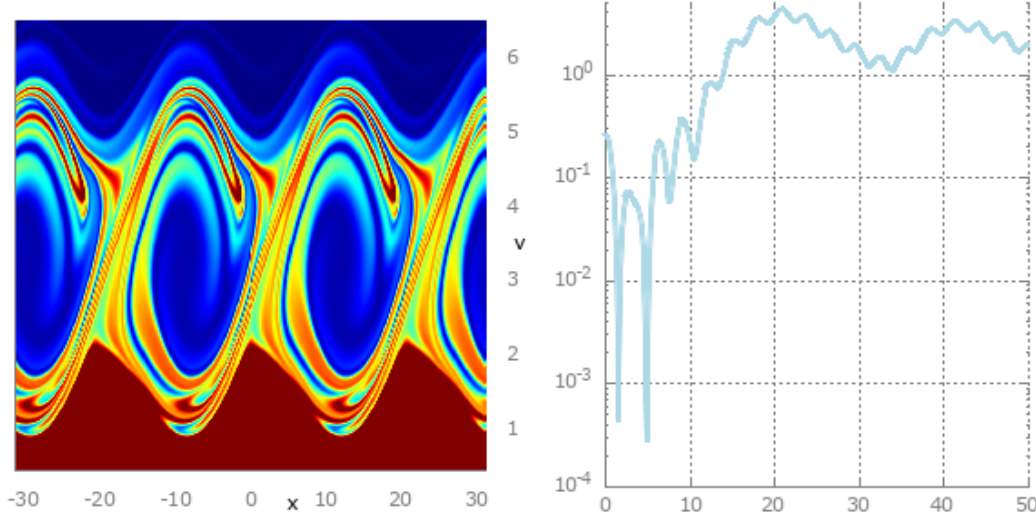


Figure 1.3: Bump-on-tail instability. *Left*: distribution function $f(t, x, v)$ at large time, showing quasilinear plateau formation near the beam velocity v_b and residual small-scale structures. *Right*: time evolution of $\mathcal{E}_{pot}(t)$, exhibiting exponential growth driven by the positive velocity gradient of the beam, followed by saturation as the bump is eroded.

where the charge and current densities are

$$q n(t, x) := \int_{\mathbb{R}^n} f(t, x, v) dv, \quad q J(t, x) := \int_{\mathbb{R}^n} v f(t, x, v) dv.$$

In these equations, the constants ϵ_0 and μ_0 are respectively the vacuum permittivity and magnetic permeability, while

$$c^2 = \frac{1}{\epsilon_0 \mu_0}$$

is the speed of light in vacuum.

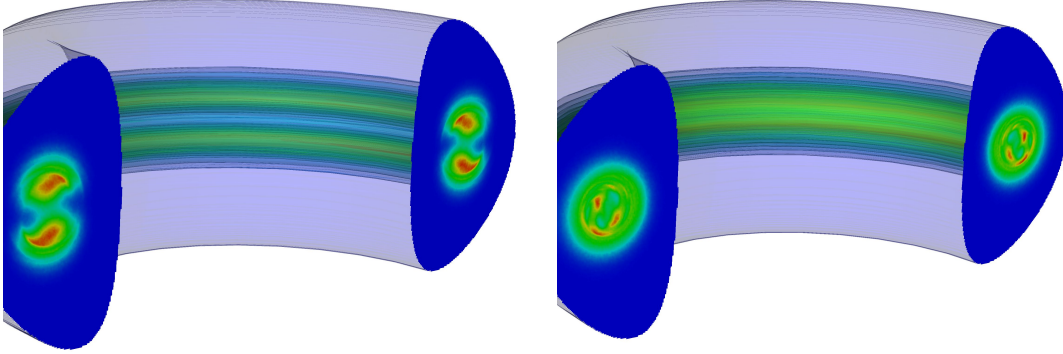
Example 1.2.1. *Magnetic confinement in a tokamak*¹

1.3 The aggregation equation

Transport equations also arise in many contexts outside plasma physics. Aggregation models describe the collective motion of interacting agents, particles, or organisms. Depending on the interaction kernel, these models may exhibit clustering, pattern formation, or the emergence of coherent structures.

The *aggregation equation* arises in granular media models, swarming models for animal collective behaviour, equilibrium problems in self-assembly, and mean-field games in socioeconomics,

¹PhD K. H. Trinh on Particle-In-Cells simulations.



among other applications:

$$\begin{cases} \partial_t n + \nabla_x \cdot (\mathcal{K}[n] n) = 0, \\ \mathcal{K}[n](t, x) = \int_{\mathbb{R}^n} K(x, x') n(t, x') dx', \end{cases}$$

where the kernel K encodes the pairwise interaction law. A common and physically motivated choice is

$$K(x, y) = -\nabla_x W(x - y),$$

where $W : \mathbb{R}^n \rightarrow \mathbb{R}$ is an interaction potential. Here $K(x, y)$ represents the force exerted on a particle at x by an infinitesimal mass located at y .

These lecture notes provide a self-contained introduction to the mathematical foundations of kinetic theory, with a particular emphasis on collisionless systems and mean-field limits. First, Chapter 2 develops an analytical toolbox for transport equations: the method of characteristics, conservative and non-conservative formulations, flow maps, and measure-valued solutions. Then, Chapter 3 studies the rigorous derivation of mean-field PDEs from N-particle systems. It presents the general formalism for interacting particle systems satisfying Newton's third law, constructs the mean-field characteristic flow via a Picard fixed-point argument, proves Dobrushin's stability estimate in the Wasserstein-1 distance, and justifies the mean-field limit of empirical measures. Finally, Chapter 4 addresses numerical methods tailored to transport equations, focusing on particle methods (including the Particle-In-Cell algorithm) and semi-Lagrangian schemes. Convergence analysis in Wasserstein distance and practical considerations for high-dimensional kinetic simulations are discussed.

The presentation combines rigorous mathematical analysis with physical intuition and numerical perspectives.