Chap. 3 : Analysis of kinetic models

In this chapter, we will give a brief introduction to transport equations and the theory of characteristics. Then, we use this basic tools to prove existence of solutions for a non-linear kinetic equations applied to describe collective behavior.
The last part of this chapter is devoted to quantitative estimates on kinetic equations to describe flocking phenomena at the kinetic level.

I) Transport equations

Example : We first consider the simple free transport equation

$$\partial_t f + v \cdot \nabla_x f = 0 \qquad v \in \mathbb{R}^d, x \in \mathbb{R}^d$$

Definition 1 The characteristic curve of the transport operator $\partial_t + v \cdot \nabla_x$ passing through $y \in \mathbb{R}^d$ at time $t=0$ is the set

$$\{ (t, \gamma(t)) \in \mathbb{R}^+ \times \mathbb{R}^d \}$$

where γ is solution to $\dot{\gamma}(t) = v \quad \gamma(0) = y$

In the constant coefficient case considered here, one has obviously

$$\gamma(t) = y + tv,$$

so that, $\{ (t, \gamma(t)) \} = \{ (t, y + tv) \,;\, t \in \mathbb{R} \}$ is the straight line in the affine space $\mathbb{R}^{d+1}$ with direction defined by the vector $(1, v) \in \mathbb{R}^{d+1}$ and passing through the point $(0, y)$

The interest of the notion of characteristic curve for solving free transport is explained in the following observation:

Let $f \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^d)$ be a solution of the free transport equation and γ be the solution of the system of characteristic curve

The map $t \to f(t, \gamma(t))$ is of class $\mathcal{C}^1$ on $\mathbb{R}^+$ and

$$\frac{d}{dt} f(t,\gamma(t)) = \left(\partial_t f + \sum_{k=1}^{d} \dot{\gamma}_k(t)\, \partial_{x_k} f\right)(t,\gamma(t))$$
$$= (\partial_t f + v.\nabla_x f)(t,\gamma(t)) = 0$$

Therefore $f(t,\gamma(t)) = \text{const}$

<u>Theorem 1</u> For each $f^{in} \in \mathcal{C}^1(\mathbb{R}^d)$, the Cauchy problem for the transport equation $\begin{cases} \partial_t f + v.\nabla_x f = 0 \\ f(0,x) = f^{in} \end{cases}$

has a unique solution $f \in \mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R}^d)$, given by

$$f(t,x) = f^{in}(x - vt)$$

<u>Proof</u> If f is a solution to the transport equation, we have

$$f(t,\gamma(t)) = f^{in}(\gamma(0))$$
$$f(t, y+vt) = f^{in}(y) \quad \text{or} \quad f^{in}(x-vt) = f(t,x)$$

This proves the uniqueness of the theorem.

Conversely the distribution $f(t,x) = f^{in}(x-vt)$ is solution to the transport equation. $\square$

Now let us consider transport equations with variable coefficients: let $V \equiv V(t,x) \in \mathbb{R}^d$ be a time dependent vector field defined on $[0,T] \times \mathbb{R}^d$ for some $T>0$

We are concerned with the Cauchy problem

$$\begin{cases} \partial_t f + V(t,x)\nabla_x f = 0 & x\in\mathbb{R}^d \quad t\geq 0 \\ f(0,x) = f^{in}(x) & x\in\mathbb{R}^d \end{cases}$$

We suppose that

(H1) $\quad V\in \mathcal{C}([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ and $\nabla_x V\in\mathcal{C}([0,T]\times\mathbb{R}^d, M_d(\mathbb{R}))$

Moreover, we assume that there exists $\kappa>0$

$$|V(t,x)| \leq \kappa(1+|x|) \qquad \forall (t,x)\in[0,T]\times\mathbb{R}^d$$

Definition 2 Let γ be the solution to

$$\begin{cases} \dot\gamma(s) = V(s,\gamma(s)) \\ \gamma(t) = x \end{cases} \qquad (1)$$

The set $\{(s,\gamma(s))\ ;\ s\in[0,T]\}$ is called the characteristic curve passing through x at time $s=t$.

We then prove the following theorem

Theorem 2 Assume that V satisfies (H1) and (H2), then for each $t\in[0,T]$ the ODE system (1) has a unique solution $s\longrightarrow\gamma(s)$ is of class $\mathcal{C}^1$

The solution is henceforth denoted by $\gamma(s) = X(s,t,x)$ and the map X is

(i) $X\in\mathcal{C}^1([0,T]^2\times\mathbb{R}^d,\mathbb{R}^d)$

(ii) $\partial_s\partial_{x_j}X = \partial_{x_j}\partial_s X \in \mathcal{C}([0,T]^2\times\mathbb{R}^d,\mathbb{R}^d)$

(iii) If $V\in\mathcal{C}^k$ and $\nabla V\in\mathcal{C}^k$ for some $k\geq 1$, then one has $X\in\mathcal{C}^{k+1}([0,T]^2\times\mathbb{R}^d,\mathbb{R}^d)$

Proof: see the lecture notes of Golse

As explained in the case of constant coefficient case, solving the transport equation by the method of characteristics involves the change of variable $\gamma(0) \longrightarrow \gamma(t)$. This change of variable is trivial in that case but it is more tricky for non-constant coefficient case

<u>Theorem 3</u> Assume V satisfies (H1) and (H2). Then the map X satisfies the flow property

(i) $X(t_3, t_2, X(t_2, t_1, x)) = X(t_3, t_1, x) \quad \forall x \in \mathbb{R}^d \ \forall t_1, t_2, t_3$

and the map $x \longrightarrow X(s, t, x)$ is a C^1-diffeomorphism of $\mathbb{R}^d$ into itself

(ii) Moreover, $J(s, t, x) = \det\left(D_x X(s, t, x)\right)$ is the solution of the Cauchy problem

$$\begin{cases} \partial_s J = \operatorname{div}_x \left(V(s, X(s, t, x)) \right) J(s, t, x) \\ J(t, t, x) = 1 \end{cases}$$

<u>Proof</u> By definition of the map X, observe that both maps

$$t_3 \longrightarrow X(t_3, t_2, X(t_2, t_1, x))$$
and
$$t_3 \longrightarrow X(t_3, t_1, x)$$

are integral curves of the vector field V passing through $X(t_2, t_1, x)$ for $t_3 = t_2$

By uniqueness of the solution of the ODE with V

so that, both coincide. This proves (i)

Now, (i) implies

$$X(t,s,.)\circ X(s,t,.) = X(s,t,.)\circ X(t,s,.) = Id_{\mathbb{R}^d}$$

$\forall (s,t)\in[0,T]$. Hence it proves that $x \to X(s,t,x)$ is a C^1 diffeomorphism with inverse

$$X(s,t,.)^{-1} = X(t,s,.)$$

To prove (ii); we remind the property

$$(D\det)(A).B = \det(A)\,\mathrm{trace}(A^{-1}B)$$

for all $A\in GL_d(\mathbb{C})$ and $B\in M_d(\mathbb{C})$. Thus

$$\begin{aligned}
\partial_s J(s,t,x) &= \partial_s\big(\det(D_x X(s,t,x))\big)\\
&= J(s,t,x)\,\mathrm{Tr}\big(D_xX(s,t,x)^{-1} D_x\partial_s X(s,t,x)\big)\\
&= J(s,t,x)\,\mathrm{Tr}\big(D_xX(s,t,x)^{-1} D_x\big(V(s,X(s,t,x))\big)\big)\\
&= J(s,t,x)\,\mathrm{Tr}\big(D_xX(s,t,x)^{-1} D_xV(s,X(s,t,x))\,D_xX(s,t,x)\big)\\
&= J(s,t,x)\,\mathrm{Tr}\big(D_xV(s,X(s,t,x))\,D_xX(s,t,x)\,D_xX(s,t,x)^{-1}\big)\\
&= J(s,t,x)\,\mathrm{Tr}\big(D_xV(s,X(s,t,x))\big)\\
&= J(s,t,x)\,\mathrm{div}_x V(s,X(s,t,x))
\end{aligned}$$

$\square$

From this latter result, we can solve the

Theorem 4 Assume that the vector field V satisfies the conditions (H1) and (H2) and let $f^{in} \in \mathcal{E}'(\mathbb{R}^d)$. Then the Cauchy problem

$$\begin{cases} \partial_t f + V(t,x) \nabla_x f = 0 \\ f(t=0) = f^{in} \end{cases}$$

has a unique solution $f \in \mathcal{E}'([0,T] \times \mathbb{R}^d)$ given by

$$f(t,x) = f^{in}(X(0,t,x))$$

Proof The proof is close to the one with constant coefficient. If $f \in \mathcal{E}'([0,T] \times \mathbb{R}^N)$ is a solution, hence the map

$$t \in [0,T] \longrightarrow f(t, X(t,0,y)) \text{ is also } \mathcal{E}'$$

and we have

$$\frac{d}{dt} f(t, X(t,0,y)) = \left(\partial_t f + \nabla_x f \cdot \partial_s X(t,0,y)\right)$$
$$= \left(\partial_t f + V \cdot \nabla_x f\right)(t, X(t,0,y)) = 0$$

That is $f(t, X(t,0,y)) = f(0,y) = f^{in}(y)$

Using that $z \longrightarrow X(0,t,z)$ is C^1-diffeomorphism and setting $x = X(t,0,y)$ so, that $y = X(0,t,x)$ we have $f(t,x) = f(0, X(0,t,x)) = f^{in}(X(0,t,x)) \Rightarrow$ uniqueness

To prove existence, we verify that

$f(t,x) = f^{in}(X(0,t,x))$ is a solution using the previous theorem.

We first observe that $\forall t_3, t_2, t_1$ and $x \in \mathbb{R}^d$

$$X(t_3, t_2, X(t_2, t_1, x)) = X(t_3, t_1, x)$$

By differentiating with respect to t_2; we get

$$\partial_t X(t_3, t_2, X(t_2, t_1, x)) + D_x X(t_3, t_2, X(t_2, t_1, x))\, \partial_s X(t_2, t_1, x) = 0,$$

that is,

$$\partial_t X(t_3, t_2, X(t_2, t_1, x)) + D_x X(t_3, t_2, X(t_2, t_1, x))\, V(t_2, X(t_2, t_1, x)) = 0$$

Set $t_2 = t_1 = t$ and $t_3 = s$ yields

$$\partial_t X(s, t, x) + D_x X(s, t, x)\, V(t, x) = 0$$

Now, we may verify that $f^{in}(X(0,t,x))$ is indeed solution to the transport equation

$$\partial_t \left(f^{in}(X(0,t,x))\right) = \nabla f^{in}(X(0,t,x)) \cdot \partial_t X(0,t,x)$$

$$\partial_{x_j} \left(f^{in}(X(0,t,x))\right) = \nabla f^{in}(X(0,t,x)) \cdot \partial_{x_j} X(0,t,x)$$

Hence, we have

$$(\partial_t f + V(t,x) \nabla_x f) = \nabla f^{in}(X(0,t,x)) \left(\partial_t X + V(t,x)\, D_x X\right)(0,t,x) = 0$$

Then $f(t,x) = f^{in}(X(0,t,x))$ is a solution. $\square$

In the following we will consider conservative transport equations as

$$\partial_t f + \mathrm{div}_x (V(t,x) f) = 0,$$

where V satisfies the same assumption as before.

We define the notion of weak solutions

<u>Definition 3</u> Let $V \in \mathcal{C}([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$ and $f^{in} \in L^1(\mathbb{R}^d)$

A weak solution of the Cauchy problem for the conservative transport equation

$$\begin{cases} \partial_t f + \mathrm{div}_x(V(t,x) f) = 0 \\ f(t=0) = f^{in} \end{cases}$$

is $f \in L^\infty([0,T], L^1(\mathbb{R}^d))$ that satisfies the equality

$$\int_0^T \int_{\mathbb{R}^d} (\partial_t \phi + V(t,x) \nabla_x \phi) f(t,x) \, dx dt = -\int_{\mathbb{R}^d} f^{in}(x) \phi(0,x) \, dx$$

$$\forall \phi \in \mathcal{C}^1_c([0,T[\times\mathbb{R}^d)$$

This notion of weak solution is related with the classical notion of solution as follows

<u>Proposition 1</u> Let $f \in \mathcal{C}^1([0,T]\times\mathbb{R}^d)$. Then the two notions are equivalent

- $\partial_t f + \mathrm{div}_x(V f) = 0$
- $\int_0^T \int_{\mathbb{R}^d} (\partial_t \phi + V \nabla_x \phi) f(t,x) \, dx dt = -\int_{\mathbb{R}^d} f^{in}(x) \phi(0,x) \, dx$

  $\forall \phi \in \mathcal{C}^1_c([0,T[\times\mathbb{R}^d)$

<u>Proof</u> exercise

We then have the following theorem of existence of weak solutions

Theorem 5 Let V satisfy (H1) and (H2) and

$$V \in \mathcal{E}^1([0,T]\times\mathbb{R}^d) \qquad \nabla_x V \in \mathcal{E}^1([0,T]\times\mathbb{R}^d, M_d(\mathbb{R}))$$

Then the Cauchy problem

$$\begin{cases} \partial_t f + \mathrm{div}(Vf) = 0 \\ f(t=0) = f^{in} \in \mathcal{E}^1(\mathbb{R}^d) \end{cases}$$

has a unique solution $f(t,x) = f^{in}(X(0,t,x))\, J(0,t,x)$

In particular $f(t,.) \in L^1(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} f(t)\, dx = \int_{\mathbb{R}^d} f^{in}\, dx$$

Proof We proceed as in the case of transport equation

Step 1 uniqueness

Since the equation is linear, the uniqueness reduces to the following statement : let $g \in \mathcal{E}^1$ satisfy

$$\begin{cases} \partial_t g + \mathrm{div}_x(V(t,x)\, g) = 0 \\ g(t=0) = 0 \end{cases}$$

then $g \equiv 0$

Expanding the second term on the left hand side of the transport equation

$$(\partial_t + V(t,x).\nabla_x)\, g = -g\, \mathrm{div}_x V(t,x)$$

is of class $\mathcal{E}^1$ and satisfies

$$\begin{cases} \dfrac{d}{dt} g(t, X(t,0,y)) = (\partial_t g + V.\nabla_x g)(t, X(t,0,y)) \\ \qquad\qquad\qquad\qquad = -g(t,X(t,0,y))\,(\mathrm{div}_x V)(t, X(t,0,y)) \\ g(0, X(0,0,y)) = 0. \end{cases}$$

This is a linear ODE with variable amplification or damping rate and this means that

$$g(t, X(t,0,y)) = 0 \qquad 0 < t < T \qquad y \in \mathbb{R}^d$$

Since $y \to X(t,0,y)$ is a $\mathcal{C}^1$ diffeomorphism, we get

$$g(t,x) = 0 \qquad \forall t \in [0,T] \text{ and } x \in \mathbb{R}^d$$

Step 2 Existence

We simply verify that

$$f(t,x) = f^{in}(X(0,t,x))\, J(0,t,x)$$

is a weak solution.

$\forall \phi \in \mathcal{C}^1_c([0,T)\times\mathbb{R}^d)$; we have

$$\beta = \int_0^T\int_{\mathbb{R}^d} f(t,x)\,(\partial_t\phi + V(t,x)\nabla_x\phi)(t,x)\,dx\,dt$$

$$= \int_0^T\int_{\mathbb{R}^d} f^{in}(X(0,t,x))\,(\partial_t\phi + V(t,x)\nabla_x\phi)(t,x)\;J(0,t,x)\,dx\,dt$$

We then perform the change of variable

$$y = X(0,t,x) \qquad x = X(t,0,y)$$
$$dy = J(0,t,x)\, dx$$

It yields that

$$\beta = \int_0^T \int_{\mathbb{R}^d} f^{in}(y) \left[\partial_t \phi + V \cdot \nabla_x \phi\right](t, X(t,0,y))\, dy\, dt$$

$$= \int_{\mathbb{R}^d} f^{in}(y) \int_0^T \left(\partial_t \phi + V \cdot \nabla_x \phi\right)(t, X(t,0,y))\, dt\, dy$$

$$= \int_{\mathbb{R}^d} f^{in}(y) \int_0^T \frac{d}{dt} \phi(t, X(t,0,y))\, dt\, dy$$

$$= -\int_{\mathbb{R}^d} f^{in}(y)\, \phi(0,y)\, dy$$

Then f is a weak solution and $f \in \mathcal{E}'$, hence we deduce that f is a classical solution. $\square$

<u>Remark</u> From this latter result and computing directly the derivative of f; we show that J satisfies

$$\partial_t J(0,t,x) + \operatorname{div}_x \left(V(t,x)\, J(0,t,x)\right) = 0$$

II) The Cauchy problem for the kinetic Cucker-Smale equation.

Let us now consider $f^{in} \equiv f^{in}(x,v)$ and solve the following equation

$$
(CS)\begin{cases}
\partial_t f + v\cdot\nabla_x f + \mathrm{div}_v\big(L[f]\,f\big) = 0 & t>0\ x\in\mathbb{R}^d \text{ and } v\in\mathbb{R}^d\\
L[f] = (\psi * J)(t,x) - (\psi * \rho)(t,x)\, v \\
J(t,x) = \int f(t,x,v)\, v\, dv \\
\rho(t,x) = \int f(t,x,v)\, dv \\
f(t=0) = f^{in}
\end{cases}
$$

Here we will suppose that $\psi(s) = \dfrac{\alpha}{(1+s^2)^{\beta/2}}$ $\alpha > 0$ and $\beta \geq 0$

We want to prove that this equation has a weak solution. The difficulty here comes from the fact that this equation is non-linear, hence we cannot apply directly the result of the previous section.

<u>Definition 4</u> We say that $f \in L^\infty([0,T], L^1(\mathbb{R}^d))$ is a weak solution to the (CS) equation if $\forall \phi \in \mathcal{C}_c^1([0,T)\times\mathbb{R}^d)$

$$\int_0^T \int_{\mathbb{R}^{2d}} f(t,x)\left[\partial_t\phi + v.\nabla_x\phi + L[f]\nabla_v\phi\right](t,x,v)\,dx\,dv\,dt$$
$$= -\int_{\mathbb{R}^{2d}} f^{in}(x,v)\,\phi(0,x,v)\,dx\,dv$$

We prove the following theorem

<u>Theorem 6</u> : Suppose that $f^{in} \in L^\infty \cap L^1(\mathbb{R}^d\times\mathbb{R}^d)$ and $f^{in} \geq 0$ such that $\int_{\mathbb{R}^d\times\mathbb{R}^d} f^{in}(x,v)\left[x^2+v^2\right]dx\,dv < +\infty$

Then, there exists a weak solution to the (C-S) system.

The strategy is based on the following arguments:

- we prove a priori estimates on the solution, which are necessary conditions

- we consider a regularized approximation of the equation in order that our a priori estimates hold

- we pass to the limit in the regularized problem to get existence of solution

<u>Remark</u> The regularized problem is in general non linear, hence a linearization process is applied to prove the existence of solutions for the regularized problem.

1) A priori estimates

In this subsection; we suppose that there exist a smooth solution to (CS) and find necessary conditions to be satisfied.

The first one is about non-negativity

Proposition 2 Suppose that we know the solution to (CS) and suppose that $f^{in} \geq 0$. Then $f(t) \geq 0 \quad \forall t \geq 0$

<u>Proof</u> • there are several proofs of this result. Let us give the most classical. Define $f^- = \max(-f, 0)$

$$= \begin{cases} -f & \text{if } f \leq 0 \\ 0 & \text{else} \end{cases}$$

$$\partial_t f^- = \begin{cases} -\partial_t f & \text{if } f \leq 0 \\ 0 & \text{else} \end{cases}$$

hence we also have the same for $\partial_{x_i} f^-$ and $\partial_{v_j} f^-$

$$0 = \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v (L[f] f) = \partial_t f + v \nabla_x f + L[f] \nabla_v f + f \,\mathrm{div}_v (L[f])$$

$$\Rightarrow \quad -(\partial_t f + v \nabla_x f + L[f] \nabla_v f + f \,\mathrm{div}_v L[f]) = 0$$

$$\Rightarrow \quad \partial_t f^- + v \cdot \nabla_x f^- + L[f] \nabla_v f^- + f^- \,\mathrm{div}\, L[f] = 0$$

$$\Rightarrow \quad \partial_t f^- + \mathrm{div}_x (v f^-) + \mathrm{div}_v (L[f] f^-) = 0$$

By conservation of mass

$$\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}^d} f^-(t,x,v)\, dx dv = 0$$

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f^-(t,x,v)\, dx dv = \int (f^{in})^-(x,v)\, dx dv = 0$$

$$\Rightarrow \quad f^-(t,x,v) = 0 \quad \Rightarrow f(t) \geq 0.$$

$\square$

Following the same strategy, we can prove L^p estimates

<u>Proposition 3</u> Suppose that $f(t)$ is a smooth solution to (CS) and suppose that $f^{in} \geq 0$ and

$$\int (f^{in})^p (x,v)\, dx dv < +\infty$$

Then we have $\exists C > 0$, depending on p, Ψ and $\|f^{in}\|_{L^1}$

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f^p(t,x,v)\, dx dv \leq \int (f^{in})^p (t,x,v)\, dx dv \; e^{pCt}$$

<u>Proof</u> We proceed in the same way to prove the propagation of L^p norms.

For any $p \geq 1$

$$\frac{1}{p} \partial_t f^p = f^{p-1} \partial_t f$$

and the same hold for other derivatives

$$\left(\partial_t f + v\, \nabla_x f + L[f]\, \nabla_v f + f\, \mathrm{div}_v\, L[f]\right) f^{p-1} = 0$$

$$\frac{1}{p}\left(\partial_t f^p + v.\nabla_x f^p + L[f]\, \nabla_v f^p\right) + f^p\, \mathrm{div}_v\, L[f] = 0$$

Integrating with respect to $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$; we have

$$\frac{1}{p}\frac{d}{dt}\|f(t)\|_{L^p}^p = \frac{1-p}{p}\int_{\mathbb{R}^d\times\mathbb{R}^d} f^p\, \mathrm{div}_v\, L[f]\, dx dv$$

Now let us remind that $L[f] = \Psi * J - \Psi * \rho\, v$

$$\mathrm{div}_v\, L[f] = -\Psi * \rho\, d$$

and using the Young inequality of the convolution product

$$\|\Psi * \rho\|_{L^\infty} \leq \|\Psi\|_{L^\infty}\|\rho\|_{L^1} \leq \|\Psi\|_{L^\infty}\|f^{in}\|_{L^1}$$

Therefore it yields

$$\frac{1}{p}\frac{d}{dt}\|f(t)\|_{L^p}^p \leq \frac{p-1}{p}\|\Psi\|_{L^\infty}\|f^{in}\|_{L^1}\|f(t)\|_{L^p}^p$$

there exists a constant $C>0$, depending on p, Ψ and $\|f^{in}\|_{L^1}$ such that

$$\|f(t)\|_{L^p} \leq \|f^{in}\|_{L^p}\, e^{Ct}$$

□

In the same way ; we prove an entropy estimate

Proposition 4 Consider f a solution to (CS) such that f^{in} satisfies

$$H(f^{in}) = \int_{\mathbb{R}^d\times\mathbb{R}^d} f^{in} \log(f^{in})\, dx dv < +\infty$$

Then for all time $t \geq 0$

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t)\log(f(t))\, dx dv \leq H(f^{in}) + d\|\Psi\|_{L^\infty}\|f^{in}\|_{L^1}^2\, t$$

Proof

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t)\log(f(t))\, dx dv = \int_{\mathbb{R}^d\times\mathbb{R}^d} \partial_t f \log(f(t)) + \int_{\mathbb{R}^d\times\mathbb{R}^d}\partial_t f\, dx dv$$

$$= -\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\mathrm{div}_x(vf) + \mathrm{div}_v(L[f]f)\right)\log f\, dx dv$$

$$= \int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\mathrm{div}_x(vf) + L[f].\nabla_v f\right) dx dv$$

hence

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t)\log(f(t))\,dx\,dv = -\int_{\mathbb{R}^d\times\mathbb{R}^d} f\,\mathrm{div}_v L[f]\,dx\,dv = d\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t)\,\Psi*\rho\,dv\,dx$$

$$= d\int_{\mathbb{R}^d} \rho(t)\,\Psi*\rho(t)\,dx$$

Now observe that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \rho(x)\Psi(\|x-y\|)\rho(y)\,dy\,dx \leq \|\Psi\|_{L^\infty}\|\rho\|_{L^1}^2$$

then we get that

$$H(f)(t) \leq H(f^{in}) + d\,\|\Psi\|_{L^\infty}\|f^{in}\|_{L^1}^2\, t,$$

where $H(f) = \int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,v)\log(f(x,v))\,dx\,dv$ $\square$

Next; we need better integrability of f for large $\|x\|$ and $\|v\|$, hence we study propagation of moments

We set $\mathcal{E}(f) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} (|v|^2+|x|^2) f(x,v)\,dx\,dv$

and get the following result

Proposition 5 Consider a solution to (C-S) such that $\mathcal{E}(f^{in}) < +\infty$ then we have for all $t \geq 0$

$$\mathcal{E}(f(t)) \leq \mathcal{E}(f^{in})\,e^t$$

Proof

$$\frac{d}{dt}\mathcal{E}(f)(t) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} (|x|^2+|v|^2)\,\partial_t f\,dx\,dv$$

$$= +\int_{\mathbb{R}^d\times\mathbb{R}^d} x.v\,f(t) + v.L[f]\,f\,dx\,dv$$

$$= +\int_{\mathbb{R}^d\times\mathbb{R}^d} x.v\,f\,dx\,dv + I(t)$$

where
$$I(t) = \int_{\mathbb{R}^d\times\mathbb{R}^d} f\, L(f)\cdot v \, dx dv$$
$$= \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} f(x,v)\, \psi(\|x-y\|)\,(w-v)\, f(y,w)\cdot v \, dx\, dy\, dw\, dv$$
$$= -\frac{1}{2}\int_{\mathbb{R}^{4d}} f(x,v) f(y,w)\, \psi(\|x-y\|)\, \|v-w\|^2 \, dx dv dy dw \leq 0$$

It yields
$$\frac{d}{dt}\mathcal{E}(f)(t) \leq \mathcal{E}(f(t)) \Rightarrow \mathcal{E}(f)(t) \leq \mathcal{E}(f^{in})\, e^{t} \qquad \square$$

We also have the following estimate on $\|f(t)\|_{L^\infty}$.

Proposition 6 Suppose that $\|f^{in}\|_{L^\infty} < +\infty$ and consider f the solution to (C-S). Then, $\|f(t)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} \leq \|f^{in}\|_{L^\infty}\, e^{d\|\psi\|_{L^\infty}\|f^{in}\|_{L^1} t}$ $\forall t \geq 0$

Proof We know that
$$0 = \partial_t f + v\cdot\nabla_x f + \mathrm{div}_v(L(f) f) = \partial_t f + v\cdot\nabla_x f + L[f]\nabla_v f - d\,\psi * \rho\, f$$
We want to show that $0 \leq f(t) \leq \underbrace{\|f^{in}\|_{L^\infty} e^{ct}}_{:=\beta(t)}$ $c = d\,\|\psi\|_{L^\infty}\|f^{in}\|_{L^1}$

then we define
$$\varphi(t) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \left((f(t)-\beta(t))^+\right)^2 dx dv$$

$$\varphi'(t) = \int_{\mathbb{R}^d\times\mathbb{R}^d} (\partial_t f - \beta'(t))\,(f(t)-\beta(t))^+ \, dx dv$$
$$= \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(-v\cdot\nabla_x f - \mathrm{div}_v(L[f] f) - \beta'(t)\right)(f(t)-\beta(t))^+ \, dx dv$$

Now observe that

- $\nabla_x f\,(f(t)-\beta(t))^+ = \nabla_x(f-\beta(t))\,(f(t)-\beta(t))^+ = \nabla_x \dfrac{(f(t)-\beta(t))^2}{2}$
- $\mathrm{div}_v(L[f]f)\,(f-\beta(t))^+ = L[f]\,\nabla_v \dfrac{((f(t)-\beta(t))^+)^2}{2} - d\,\psi*\rho\, f(t)\,(f(t)-\beta(t))^+$
$$= L[f]\,\nabla_v \frac{((f(t)-\beta(t))^+)^2}{2} - d\,\psi*\rho\,\frac{((f(t)-\beta(t))^+)^2}{2} - \frac{d\,\psi*\rho}{2}\,(f(t)+\beta(t))\,(f(t)-\beta(t))^+$$

Therefore

$$\varphi'(t) = \int_{\mathbb{R}^d_x \times \mathbb{R}^d} \left(\frac{d\,\Psi * \rho}{2} \left(f(t) + \beta(t) \right) - C\beta(t) \right) (f(t) - \beta(t))^+ \,dx\,dv$$

$$\leq \frac{C}{2} \int_{\mathbb{R}} \left((f(t) - \beta(t))^+ \right)^2 dx\,dv = C\,\varphi(t)$$

$$\Longrightarrow \quad \varphi(t) \leq \varphi(0)\, e^{ct} \quad \text{with } \varphi(0) = 0$$

therefore $\varphi(t) = 0 \quad \forall t \geq 0.$ $\square$

Now we are ready to get estimate on the macroscopic quantities ρ and J

Proposition 7 Suppose that f is such that $f \geq 0$

$$\|f\|_{L^\infty} \leq M \quad \text{and} \quad \int |v|^2 f(x,v)\,dx\,dv < +\infty$$

Then we have

$$\|\rho\|_{L^p} \leq C \qquad \text{For } p \in \left[1, \frac{d+2}{d}\right]$$

$$\|J\|_{L^p} \leq C \qquad \text{For } p \in \left[1, \frac{d+1}{d}\right]$$

Proof

$$\rho(x) = \int_{\mathbb{R}^d} f(x,v)\,dv$$

$$= \int_{B(0,R)} f(x,v)\,dv + \int_{B^c(0,R)} f(x,v)\,dv \qquad \forall R > 0$$

$$\leq R^d \|f\|_{L^\infty} + \frac{1}{R^2} \int_{B^c(0,R)} v^2 f\,dv$$

We set $a = \|f\|_{L^\infty}$ and $b = \int_{\mathbb{R}^d} |v|^2 f\,dv$

the function $\varphi(r) = a r^d + \dfrac{b}{r^2} \geq 0 \qquad \varphi(r) \xrightarrow[\substack{r \to 0 \\ r \to +\infty}]{} +\infty$

$$\varphi'(r) = d\,a\,r^{d-1} - \frac{2b}{r^3} = \frac{d\,a\,r^{d+2} - 2b}{r^3}$$

φ has a minimum for $R_{opt} = \left(\dfrac{2b}{da} \right)^{\frac{1}{d+2}}$

For this particular $R = R_{opt}$; we get

$$|\rho(x)| \leq C \|f\|_{L^\infty}^{\frac{2}{d+2}} \left(\int_{\mathbb{R}^d} |v|^2 f \, dv \right)^{\frac{d}{d+2}}$$

Therefore for $p = \frac{d+2}{d}$

$$\|\rho\|_{L^p}^p \leq C^p \|f\|_{L^\infty}^{2/d} \int_{\mathbb{R}^{2d}} |v|^2 f \, dv \, dx < +\infty$$

We proceed in the same manner for J and obtain the expected result □

2) Compactness arguments

From these a priori estimates ; we can apply classical compactness argument to identify a limit of a sequence of approximated solutions.

a) Preliminaries

We remind some basic tools classically used to prove existence of solutions to linear and non-linear PDEs
We start withe the Ascoli-Arzela theorem

Theorem Suppose that Ω is a bounded open set of $\mathbb{R}^n$. A subset $\mathcal{F}$ of $\mathcal{C}(\bar{\Omega})$ equipped with the maximum norm is pre-compact (its closure is compact) if and only if :

(i) there exists a constant M such that

$$\|f\|_{L^\infty} \leq M \quad \forall f \in \mathcal{F}$$

(ii) For every $\varepsilon > 0$, there exists $\delta > 0$ such that if x and $x+h \in \bar{\Omega}$ and $|h| < \delta$ $\quad |f(x+h) - f(x)| < \varepsilon \quad \forall f \in \mathcal{F}$.

The following theorem known as Kolmogorov-Riesz or Fréchet-Kolmogorov theorem gives analogous conditions to the ones in the Ascoli theorem for a set to be pre-compact in $L^p(\mathbb{R}^n)$

<u>Theorem</u> Let $1 \leq p < +\infty$, A subset $\mathcal{F} \subset L^p(\mathbb{R}^n)$ is precompact if and only if

(i) there exists M such that $\|f(t)\|_{L^p} \leq M \quad \forall f \in \mathcal{F}$

(ii) $\forall \varepsilon > 0 \ \exists R > 0 \quad \left(\int_{|x|>R} |f(x)|^p dx \right)^{1/p} < \varepsilon \quad \forall f \in \mathcal{F}$

(iii) $\forall \varepsilon > 0 \ \exists \delta > 0$ such that if $|h| < \delta$

$$\left(\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx \right)^{1/p} < \varepsilon \quad \forall f \in \mathcal{F}$$

The tightness condition (ii) prevents the functions from escaping to infinity.

Unfortunately, the equicontinuity condition (iii) is not always easy to verify. As we explain next, weak compactness is easier to verify.

Let X be a Banach space and X^* be the dual space of bounded linear functionals on X. We denote the duality between X^* and X by $\langle \cdot, \cdot \rangle : X^* \times X \longrightarrow \mathbb{R}$

<u>Definition</u> A sequence $(x_n)_n$ in X converges weakly to $x \in X$ $x_n \rightharpoonup x$ when $\langle \omega, x_n \rangle \longrightarrow \langle \omega, x \rangle \quad \forall \omega \in X^*$

A sequence $(w_n)_n$ converges weakly-star when $\langle w_n, x\rangle \longrightarrow \langle w, x\rangle$ $\forall x \in X$; we say $w_n \overset{*}{\rightharpoonup} w$.

If X is reflexive $(X^{**} = X)$, then both notions of weak convergence are equivalents.

Example: Ω open set of $\mathbb{R}^n$ and $1 \leq p < +\infty$, then $L^{p^*}(\Omega) = (L^p(\Omega))^*$ with $\frac{1}{p} + \frac{1}{p^*} = 1$

A subset E of a Banach space X is weakly or weakly-$*$ precompact if every sequence in E has a subsequence that converges weakly or weakly-$*$ in X
The following Banach-Alaoglu theorem characterizes weak-star pre-compact subsets of a Banach space
It can be thought as a generalization of the Heine-Borel theorem to infinite dimensional spaces.

Theorem A subset of a Banach space is weak-$*$ precompact if and only if it is bounded.

If X is reflexive, then bounded sets are weakly pre-compact

To construct an approximation solution; we will regularize our data. Hence we remind some properties on the convolution product

Let f, g measurable functions; we have $f * g = g * f$

$$f*(g*h) = (f*g)*h$$

If $f, g \in \mathcal{E}_c(\mathbb{R}^n)$ $\quad \mathrm{supp}(f*g) \subset \mathrm{supp}\, f + \mathrm{supp}\, g$

If neither g nor f have a compact support, then we need some conditions on their growth or decay at infinity to ensure the convolution exists.

We remind the Young's inequality

Theorem Suppose that $1 \leq p, q, r \leq +\infty$ and $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

If $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, then $f*g \in L^r(\mathbb{R}^n)$ and

$$\| f*g \|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

Consider $\eta \in \mathcal{E}_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \eta \, dx = 1$ and $\eta \geq 0$ $\quad \mathrm{supp}\, \eta \subset \overline{B}(0,1)$

then for any $\varepsilon > 0$ define $\eta^\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right)$

η^ε is $\mathcal{E}^\infty$ function supported in $\overline{B}(0,\varepsilon)$. It is called a mollifier

Hence for any $f \in L^1_{loc}(\mathbb{R}^n)$; we define $f^\varepsilon = f * \eta^\varepsilon$

Theorem Suppose that $f \in L^p_{loc}(\mathbb{R}^n)$ for $1 \leq p \leq +\infty$ and $\varepsilon > 0$

Define f^ε as before, then

$f^\varepsilon \in \mathcal{E}^\infty(\mathbb{R}^n)$; $\quad f^\varepsilon \longrightarrow f$ pointwise almost everywhere as $\varepsilon \to 0$

and $f^\varepsilon \xrightarrow[\varepsilon \to 0]{} f$ in $L^p_{loc}(\mathbb{R}^n)$

3) The approximate (CS) system

Let $\eta \in \mathcal{E}_c^\infty(\mathbb{R}^d)$ $\quad \eta(x) = \eta(-x) \quad$ supp $\eta \subset \overline{B}(0,1)$ and $\int_{\mathbb{R}^d} \eta \, dx = 1$

and set $\eta^\varepsilon(x) = \frac{1}{\varepsilon^d} \eta\left(\frac{x}{\varepsilon}\right)$

The approximate (C-S) system is defined as follows

$$(CS^\varepsilon) \begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \operatorname{div}_v \left(L^\varepsilon(f_\varepsilon) f_\varepsilon \right) = 0 \\ L^\varepsilon(f^\varepsilon) = \eta_\varepsilon \underset{x}{*} \eta_\varepsilon *_x L(f_\varepsilon) \\ f_\varepsilon(t=0) = \eta_\varepsilon \underset{xv}{*} \left(\mathbb{1}_{\{\varepsilon|x|<1\}}(x) \, \mathbb{1}_{\{\varepsilon|v|<1\}}(v) \, f^{in}(x,v) \right) =: f_\varepsilon^{in} \end{cases}$$

The initial data is now smooth $\mathcal{E}^\infty$ and compactly supported.

We also have regularized the field $L^\varepsilon(f)$ using the convolution product. This can be necessary when Ψ is singular.

<u>Remark</u> Here $\Psi(s) = \dfrac{\alpha}{(1+s^2)^{\beta/2}}$

$$\Psi'(s) = -\frac{\alpha\beta}{2}(2s) \times \frac{1}{(1+s^2)^{\frac{\beta+2}{2}}}$$

Then $\Psi \in W^{m,\infty}(\mathbb{R}) \quad \forall m \geq 0$; so it is not really needed.

<u>Proposition 8</u> For each $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^{2d})$ satisfying

$$f^{in} \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^{2d}} \frac{|x|^2 + |v|^2}{2} f^{in}(x,v) \, dx dv < +\infty$$

Then there exists a smooth solution f^ε to the approximate problem (CS^ε), which satisfies $\forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$;

$$0 \leq f_\varepsilon(t,x,v) \leq \|f^{in}\|_{L^\infty} e^{Ct} \quad \text{with } C = d \, \|\Psi\|_{L^\infty} \|f^{in}\|_{L^1}$$

$$\mathcal{E}(f_\varepsilon(t)) \leq \mathcal{E}(f^{in}) \, e^t \qquad \forall t \geq 0$$

<u>Idea of the proof</u>: the existence of solution is obtained using a fixed point argument and linearizing the regularized problem. Then it is a simple exercise to verify that this

regularized problem satisfies the a priori estimates already obtained on the original system $\square$

4) Convergence to the (CS) system (proof of theorem 6)

Now let $\varepsilon \to 0^+$ and pass to the limit in (CS^ε), using the a priori estimates on f_ε and $L^\varepsilon(f^\varepsilon)$ that are uniform with respect to ε.

Step 1 Uniform estimates. First one has

$$0 \leq f_\varepsilon(t,x,v) \leq \|f^{in}\|_{L^\infty} e^{ct} \quad \text{For a.e. } (x,v) \in \mathbb{R}^{2d}$$

We use the Young inequality

$$\|f_\varepsilon^{in}\|_{L^\infty} \leq \|\eta^\varepsilon\|_{L^1} \|f^{in}\|_{L^\infty} = \|f^{in}\|_{L^\infty}$$

Next, we also have

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x|^2 + |v|^2}{2} f_\varepsilon(t) \, dx dv \leq \mathcal{E}(f_\varepsilon^{in}) \, e^t$$

and $\mathcal{E}(f_\varepsilon^{in}) \leq \mathcal{E}(f^{in}) \|\eta^\varepsilon\|_{L^1} = \mathcal{E}(f^{in})$

It yields that $\|\rho(t)\|_{L^{\frac{d+2}{d}}} < C e^{ct} \qquad \forall t \geq 0$

$$\|J(t)\|_{L^{\frac{d+1}{d}}} < C e^{ct} \qquad \forall t \geq 0$$

Step 2: Weak compactness. By the Banach-Alaoglu theorem, there exist sub-sequences of $(f_\varepsilon, \rho_\varepsilon, J_\varepsilon)$ still denoted $(f_\varepsilon, \rho_\varepsilon, J_\varepsilon)$ for the sake of simplicity such that

$$f_\varepsilon \xrightarrow{\ast} f \quad \text{in } L^\infty((0,T) \times \mathbb{R}^d \times \mathbb{R}^d)$$

$$\rho_\varepsilon \xrightharpoonup{\ *\ } \rho \quad \text{in} \quad L^\infty\left((0,T), L^{\frac{d+2}{2}}(\mathbb{R}^d)\right)$$

$$J_\varepsilon \xrightharpoonup{\ *\ } J \quad \text{in} \quad L^\infty\left((0,T), L^{\frac{d+1}{d}}(\mathbb{R}^d)\right)$$

We have identified the limit, now it remains to prove that this limit is a weak solution to (CS).

On the one hand since

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f_\varepsilon(t)\, |v|^2\, dx dv < +\infty$$

we conclude that $\rho \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^d))$

and $\displaystyle\int_{\mathbb{R}^d} f(t,x,v)\, dv = \rho(t,x)$ for a.e. $x \in \mathbb{R}^d$ and $t > 0$

where the equality above follows from the tightness in the variable v on the sequence f_ε.

Indeed; we know that for any $\Theta \in \mathcal{C}_c^\infty(\mathbb{R}^d)$

$$I_1^\varepsilon = \left| \int_{\mathbb{R}^d} \left(\rho^\varepsilon(x) - \rho(x)\right) \Theta(x)\, dx \right| \xrightarrow[\varepsilon\to 0]{} 0$$

Let us choose $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ $\quad 0 \leq \chi \leq 1 \quad \chi_{|B_1} = 1$

and set $\chi_R(v) = \chi\left(\frac{v}{R}\right)$, we also know that for any $R > 0$

$$I_2^{\varepsilon,R} = \left| \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(f^\varepsilon(x,v) - f(x,v)\right) \chi_R(v)\, \Theta(x)\, dx dv \right| \xrightarrow[\varepsilon\to 0]{} 0$$

Therefore; we have

$$\left| \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x,v)\, dv - \rho(x) \right) \Theta(x)\, dx \right| \leq \left| \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(f(x,v) - f^\varepsilon(x,v) \right) \Theta(x)\, dx dv \right| +$$

$$\left| \int_{\mathbb{R}^d} \left(\rho^\varepsilon(x) - \rho(x) \right) \Theta(x)\, dx \right|$$

Now observe that

$$I_3^{\varepsilon,R} = \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f^{\varepsilon}(x,v)\, \Theta(x) \left(1 - \chi_R(v)\right) dx\, dv \right| \leq \|\Theta\|_{L^\infty} \int_{|v| \geq R} f^{\varepsilon}(x,v)\, dx\, dv$$

$$\leq \frac{\|\Theta\|_{L^\infty}}{R^2} \int |v|^2 f^{in}(x,v)\, dx\, dv\; e^t$$

and

$$I_4^R = \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,v)\, \Theta(x) \left(1 - \chi_R(v)\right) dv\, dx \right| \leq \frac{\|\Theta\|_{L^\infty}}{R^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 f^{in}(x,v)\, dx\, dv\; e^t$$

where the latter estimates are uniform with respect to $\varepsilon > 0$

Hence for any $\delta > 0$ and choosing $R > 0$ large enough these terms can be bounded by

$$I_3^{\varepsilon,R} + I_4^R \leq \delta/2$$

For such a $R > 0$; we can choose ε small enough such that

$$I_1^{\varepsilon} + I_2^{\varepsilon,R} \leq \delta/2$$

Using that $\forall \delta > 0$;

$$\left| \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x,v)\, dv - \rho(x) \right) \Theta(x)\, dx \right| \leq I_1^{\varepsilon} + I_2^{\varepsilon,R} + I_3^{\varepsilon,R} + I_4^R \leq \delta$$

we prove our result.

<u>Remark</u>: The key point is to prove that mass cannot escape at infinity when we take the weak limit. In the same manner, we prove that

$$J(x) = \int f(x,v)\, v\, dv.$$

Step 3 Passing to the limit in the nonlinear terms

It is easy to show that in the distribution sense

$$\partial_t f^{\varepsilon} + v \cdot \nabla_x f^{\varepsilon} \xrightarrow{\ *\ } \partial_t f + v \cdot \nabla_x f$$

The difficulty comes from the non linear term

$$\int_0^T \int_{\mathbb{R}^{2d}} L^\varepsilon[f^\varepsilon] f^\varepsilon \cdot \nabla_v \phi(t,x,v)\, dx\, dv\, dt \qquad \phi \in \mathcal{C}_c^\infty([0,+\infty[\times \mathbb{R}^{2d})$$

Let us show that the term $L^\varepsilon[f^\varepsilon] \nabla_v \phi$ converges strongly

Indeed

$$L^\varepsilon[f^\varepsilon] = \Psi \underset{x}{*} J^\varepsilon - \Psi \underset{x}{*} \rho^\varepsilon v$$

and let us observe that $\Psi *_x J^\varepsilon$ and $\Psi * \rho^\varepsilon$ are uniformly bounded in L^1, but also

$$\partial_{x_j}(\Psi * J^\varepsilon) = \partial_{x_j}\Psi * J^\varepsilon \implies \|\partial_{x_j}\Psi * J^\varepsilon\|_{L^\infty} \leq \|\Psi'\|_{L^\infty} \|J^\varepsilon\|_{L^1}$$

and the same occurs for $\|\partial_{x_j}\Psi * \rho^\varepsilon\|_{L^\infty} \leq \|\Psi'\|_{L^\infty} \|\rho^\varepsilon\|_{L^1}$

Moreover we have

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}_x J^\varepsilon = 0 \\ \partial_t J^\varepsilon + \mathrm{div}_x \int v \otimes v f^\varepsilon dv = \Psi * J^\varepsilon \rho^\varepsilon - \Psi * \rho^\varepsilon J^\varepsilon \end{cases}$$

Therefore

$$\partial_t(\Psi * \rho^\varepsilon) = \Psi * \partial_t \rho^\varepsilon = -\Psi * \mathrm{div}_x J^\varepsilon = -\nabla_x \Psi * J^\varepsilon$$

$$\begin{aligned} \partial_t(\Psi * J^\varepsilon) = \Psi * \partial_t J^\varepsilon &= -\Psi * \left(\mathrm{div}_x \int f^\varepsilon v \otimes v\, dv - \Psi * J^\varepsilon \rho^\varepsilon - \Psi * \rho^\varepsilon J^\varepsilon\right) \\ &= -{}^t\nabla_x \Psi \int f^\varepsilon v \otimes v\, dv + \Psi * (\Psi * J^\varepsilon \rho^\varepsilon) \\ &\quad - \Psi * (\Psi * \rho^\varepsilon J^\varepsilon) \end{aligned}$$

and

$$\|\partial_t \Psi * \rho^\varepsilon\|_{L^\infty} \leq \|\Psi'\|_{L^\infty} \|J^\varepsilon\|_{L^1}$$

$$\|\partial_t \Psi * J^\varepsilon\|_{L^\infty} \leq \|\Psi'\|_{L^\infty} \|v^2 f^\varepsilon\|_{L^1} + 2\|\Psi\|_{L^\infty}^2 \|J^\varepsilon\|_{L^1} \|\rho^\varepsilon\|_{L^1}$$

From the Frechet-Kolmogorov theorem; there exists a subsequence such that

$$\Psi * \rho^\varepsilon \xrightarrow[\varepsilon \to 0]{} \Psi * \rho \quad \text{in } L^1((0,T)\times\mathbb{R}^d)$$

$$\Psi * J^\varepsilon \xrightarrow[\varepsilon \to 0]{} \Psi * J \quad \text{in } L^1((0,T)\times\mathbb{R}^d)$$

and the term

$$L(\beta^\varepsilon)\,\phi(x,v) \xrightarrow[\varepsilon \to]{} L(\beta)\,\phi(x,v) \quad \text{in } L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$$

Since this term converges strongly; we can pass to the limit in the non linear term and prove that $\mathrm{div}_v(L(\beta^\varepsilon)\,\beta^\varepsilon)$ converges in the distribution sense to $\mathrm{div}_v\, L(\beta)\,\beta$.

This concludes the proof of our main theorem (Theorem 6)

Appendix Proof of Proposition 8

Observe that the regularized problem has now smooth data so we can expect to have smooth solution as we have shown in the first section.

The key point is that the equation is non-linear.

Let us show how to prove existence of smooth solutions by linearization.

Here ε is fixed (so we drop it) and consider a smooth initial datum.
$f^{in} \in \mathcal{C}^1_c(\mathbb{R}^d \times \mathbb{R}^d)$

We construct a sequence of solutions for linear problems

We first set $L[f] = 0$ and solve the free transport equation

$$\begin{cases} \partial_t f_1 + v \cdot \nabla_x f_1 = 0 \\ f_1(0) = f^{in} \end{cases}$$

This equation has an explicit solution which obviously satisfies the estimates provided in Proposition 7.
Hence we define $L[f_1] = \psi * J_1 - \psi * \rho_1 v$
where $\rho_1 = \int f_1 \, dv$ and $J_1 = \int f_1 v \, dv$
From the regularity of f_1, both ρ_1 and J_1 are $\mathcal{C}^1$

hence $L[f_1]$ is also $\mathcal{C}_1(\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^d)$ and for $z=(x,v)\in\mathbb{R}^2$

$$|L[f_1](z)| \leq \|\Psi\|_{L^\infty}\|\rho_1\|_{L^1}|v| + \|\Psi\|_{L^\infty}\|J_1\|_{L^1}$$
$$\leq 2\|\Psi\|_{L^\infty}\left(\|f^{in}\|_{L^1}+\mathcal{E}(f^{in})\right)(1+|z|)$$

Thus ; the following linear equation has a smooth solution f_2

$$\begin{cases} \partial_t f_2 + v.\nabla_x f_2 + \mathrm{div}_v\left(L[f_1]f_2\right)=0, \\ f_2(0)=f^{in}, \end{cases}$$

given by the theory of characteristic curves.

From the latter estimate, we easily show that this solution satisfies the estimate of Proposition 8

Now for $n\geq 2$; we consider

$$\begin{cases} \partial_t f_{n+1} + v.\nabla_x f_{n+1} + \mathrm{div}_v\left(L[f_n]f_{n+1}\right)=0 \\ f_{n+1}(0)=f^{in} \end{cases}$$

which has a smooth solution $f_{n+1}\in\mathcal{C}^1(\mathbb{R}^+\times\mathbb{R}^{2d})$ using the theory of characteristics. Moreover, this solution satisfies the estimate of Proposition 8

f^{in} is compactly supported, hence considering the curves $Z(t) = (X(t), V(t))$

$$\begin{cases} \dfrac{dX}{dt} = V(t) \\ \dfrac{dV}{dt} = \Psi * J_n - \Psi * \rho_n V \\ X(0) = x_0, \quad V(0) = v_0 \end{cases}$$

where $(x_0, v_0) \in \operatorname{supp} f^{in}$;

We have that $|X(t)| + |V(t)| \leq |X(0)| + |V(0)| + \int_0^t \left[\|\Psi\|_{L^\infty} \left(\|f^{in}\|_{L^1} + \mathcal{E}(f^{in}) \right) + 1 \right] (1 + |V(s)|) \, ds$

$$\leq |X(0)| + |V(0)| + C \int_0^t (1 + |X(s)| + |V(s)|) \, ds$$

By the Gronwall lemma it yields that

$|X(t)| + |V(t)| \leq |X(0)| + |V(0)| + C e^{Ct}$, where $C > 0$ does not depend on $n \in \mathbb{N}$.

Therefore $\operatorname{Supp}(f_{n+1}) \subset B(0, R(t))$, where

$R(t) \leq R_{in} + C e^{Ct}$

For all time $t \geq 0$, f_{n+1} is compactly supported with a support independent of $n \in \mathbb{N}$.

Then it is now easy to estimate the derivatives

$$\partial_t (\partial_{v_j} f_{n+1}) + v \cdot \nabla_x (\partial_{v_j} f_{n+1}) + \operatorname{div}_v (L[f_n]) \, \partial_{v_j} f_{n+1} + L[f_n] \cdot \nabla_v \partial_{v_j} f_{n+1} = -\left(\partial_{x_j} f_{n+1} + \Psi * \rho_n \, \partial_{v_j} f_{n+1} \right)$$

that is,

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} |\partial_{v_j} f_{n+1}|\,dx\,dv \leq \left(1+\|\Psi\|_{L^\infty}\|f^{in}\|_{L^1}\right)\left(\|\partial_{v_j} f_{n+1}\|_{L^1}+\|\partial_{x_j} f_{n+1}\|_{L^1}\right)$$

We then estimate the spatial derivatives

$$\partial_t(\partial_{x_j} f_{n+1}) + v.\nabla_x(\partial_{x_j} f_{n+1}) + \mathrm{div}_v(L[f_n])\,\partial_{x_j} f_{n+1} + L[f_n]\,\nabla_v(\partial_{x_j} f_{n+1}) = +\,\partial_{x_j}(\Psi * \rho_n)\, f_{n+1} - \partial_{x_j}(\Psi * J_n - \Psi * \rho_n v)\,\nabla_v f_{n+1}$$

From the support estimate the right-hand side can be bounded

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} |\partial_{x_j} f_{n+1}|\,dx\,dv \leq C\,(1+R(t))\,\|\nabla_v f_{n+1}\|_{L^1}$$

On any finite time interval, we get that

$$\|\nabla_x f_{n+1}(t)\|_{L^1} + \|\nabla_v f_{n+1}(t)\|_{L^1} \leq C_t$$

To conclude the proof let us show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges. We will prove that it is a Cauchy sequence in L^1

We have $\quad g_{n+1} = f_{n+1} - f_n$

$$\partial_t g_{n+1} + \mathrm{div}_x(v g_{n+1}) + \mathrm{div}_v(L(f_n)\, g_{n+1}) = -\,\mathrm{div}_v(L(g_n)\, f_n)$$

hence

$$\frac{d}{dt}\|g_{n+1}(t)\|_{L^1} \leq \|\Psi * (\rho_n - \rho_{n-1})\|_{L^\infty}\|f_n\|_{L^1} \quad +$$

$$\|\Psi * (J_n - J_{n-1}) - \Psi * (\rho_n - \rho_{n-1}) v \|_{L^\infty} \| \nabla_v f_n \|_{L^1}$$

From the estimate on the support of $(f_n)_{n\in\mathbb{N}}$ uniformly with respect to $n \in \mathbb{N}$ and the L^1 estimate of $\nabla_v f_n$ we get

$$\frac{d}{dt} \| g_{n+1}(t) \| \leq C \int_0^t \| g_n(s) \| \, ds$$

$\Rightarrow$ $\forall n \in \mathbb{N}$ and $t \in [0,T]$

$$\| g_{n+1}(t) \|_{L^1} \leq \frac{(Ct)^n}{n!} \sup_{[0,T]} \| f_1 \|_{L^1}$$

The sequence $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T], L^1(\mathbb{R}^d \times \mathbb{R}^d))$, then it converges

Passing to the limit; we get that the limit f is solution to the regularized equation.

and the solution is $\mathcal{C}^1$ and given by the theory of characteristics

It is easy to prove uniqueness in $\mathcal{C}([0,T], L^1(\mathbb{R}^{2d}))$

The proof of Proposition 8 is now complete

Appendix : Complementary materials

Sometimes it is not possible to get strong compactness directly on the Velocity field $L[f]$, hence we may use the following result.

Proposition Consider $\rho_\varphi = \int f \varphi(v) dv$ where $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Let $(f^n)_n$ and $(G^n)_n$ where f^n and G^n are bounded in $L^p_{loc}(\mathbb{R}^{2d+1})$ and

$$\partial_t f^n + v \cdot \nabla_x f^n = \nabla_v^k G^n$$

Then $(\rho_\varphi^n)_n$ is relatively compact in $L^p_{loc}(\mathbb{R}^{d+1})$

We refer to the work of Golse, Perthame, Lions and Sentis on averaging lemmas and to Perthame and Souganidis for the previous Proposition

This does not give directly strong compactness on ρ and j ; but rather on the approximated sequence $\left(\int f^n \varphi(v) dv\right)_n$; where φ is a smooth function.