Introduction to collective behavior

I) Introduction

Collective motion or self-organisation is observed in various natural processes such as a fish schools, bird flocks, herds of bulls, cellular dynamics, pedestrian motion

Self organization does not occur by chance but rather due to the numerous, specific interactions among the agents (particles)
The underlying "forces" or phenomena leading to self-organization can be of various type

- physical mechanisms (gravity, electro-magnetic forces, nuclear forces)
- chemical mechanisms (pheromones, Van-der-Waals forces)
- instinctive survival mechanisms (fear, feeding, ...)

The latter is much more complicated to describe from a mathematical point of view.

Self organization systems obey evolution equations which are highly - non linear and non local

Such models take the form of ODE systems, or non-local transport PDEs

Mathematical study of "flocking" models began with Viscek and his collaborators

Ref T. Viscek et al. Novel type of phase transition in a system of self-driven particles, Phys-Rev Letters (1995)

It is a stochastic, time-discrete model

Later Cucker-Smale proposed a deterministic, time continuous model

Ref Cucker and Smale, Emergent behavior in Flocks IEEE Transactions on autonomous control (2007)

, On the mathematics of emergence, Japanese Journal of Mathematics (2007)

Many other models have been proposed later; we in particular mention the three-zone model, based on Reynolds empirical rules

1. Flocking : the "desire" of agents to stay together, for safety, social reasons

② <u>Collision avoidance</u> : agents tend to repel when coming too close

③ <u>Velocity matching</u> : attempt to keep similar velocities and flying directions as its neighbours

II) Some examples of flocking models

We consider the system of N particles with positions and velocities $(x_i(t), v_i(t))_{1 \leq i \leq N}$ and masses $m_i = 1$

We first give some definitions

<u>Definition</u> System $(x_i, v_i)_{1 \leq i \leq N}$ is said to have an <u>asymptotic flocking pattern</u> if the following two conditions are satisfied

1) <u>aggregation</u> : the spatial diameter $D(t)$ of the particle cloud is uniformly bounded in time, meaning

$$\sup_{t \geq 0} D(t) < +\infty \qquad \text{with } D(t) = \max_{1 \leq i,j \leq N} \|x_i(t) - x_j(t)\|$$

2) <u>velocity alignment</u> the velocity diameter $A(t)$ of the particle cloud tends to zero as $t \to +\infty$

$$\lim_{t \to +\infty} A(t) = 0 \quad \text{with} \quad A(t) = \max_{1 \leq i \leq j \leq N} \|(v_i - v_j)(t)\|$$

Another notion exists in the literature

"swarming", which is less restrictive than flocking, requiring only cohesion,

$$\sup_{t \geq 0} \max_i \|x_i(t) - x_c(t)\| < +\infty$$

$$\sup_{t \geq 0} \max_i \|v_i(t) - v_c(t)\| < +\infty$$

where $x_c(t) = \frac{1}{N} \sum x_i(t)$

$v_c(t) = \frac{1}{N} \sum v_i(t)$

We now present two different models

1) The Cucker-Smale model

$$\begin{cases} x_i'(t) = v_i(t) \\ v_i'(t) = \frac{1}{N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|)(v_j - v_i) \end{cases}$$

ψ is the communication strenghts

$\psi_b(r) = \frac{1}{(1 + r^2)^{\beta/2}}$ bounded

$\psi_s = \frac{\alpha}{r^\beta}$ $\alpha > 0$ $\beta \in \mathbb{R}^+$

$$\int_{r_0}^{+\infty} \psi(r)\,dr = +\infty \qquad (\text{long range conditions, heavy tail})$$

$$\int_{0}^{r_0} \psi(r)\,dr = +\infty \qquad (\text{short range conditions})$$

Property
$$v_c(t) = v_c(0)$$
$$x_c(t) = x_c(0) + v_c(0)\,t$$

Since ψ only depends on the relative positions

For simplicity, we assume that $x_c(0) = 0$
$v_c(0) = 0$

We have the following theorem

Theorem (Flocking in the bounded case)

Suppose that ψ is bounded and initial conditions are non-collisional ($x_i^0 \neq x_j^0 \ \forall i,j \ i \neq j$)

Then

(i) if $\beta \in [0,1]$ (long range), one has an unconditional flocking : $\exists\, d_m$ and d_M

$$0 \leq d_m \leq \sum \|x_i(t)\|^2 \leq d_M$$

and
$$\sum \|v_i(t)\|^2 \leq \sum \|v_i^0\|^2 \, e^{-2\psi_b(d_M)t}$$

(ii) if $\beta \in]1, +\infty[$; one has conditional flocking for initial data
$$\|V_i^0\|_2 < \int_{\|X_i^0\|_2}^{+\infty} \psi(r)\,dr$$

such that we get the previous estimate

In the strong singular case; we have the following result

Theorem Suppose that

$$\int_0^{r_0} \psi(r)\,dr = +\infty \quad \forall r_0 > 0$$

and initial data such as $x_i^0 \neq x_j^0 \quad \forall i \neq j$

Then system on $(x_i, v_i)_{1 \le i \le N}$ has a unique global solution such that $x_i(t) \neq x_j(t) \quad \forall i \neq j$

and if $\|V_i^0\|_\infty = \max_{1 \le i \le N} \|v_i^0\| < \frac{1}{2} \int_{2\|x_i^0\|_\infty}^{+\infty} \psi(r)\,dr$

then $\exists$ d_M

$$\|x_i(t) - x_j(t)\| \le d_M$$

$$\|V(t)\|_\infty \le \|V(0)\|_\infty\, e^{-\psi(2 d_M) t}$$

Reference For bounded kernel; we refer to S.Y Ha & J.G. Liu CMS (2009)

For singular kernel; we refer to J.A. Carrillo, Y.P Choi, PM Mucha and J Peszek in Nonlinear Analysis (2017)

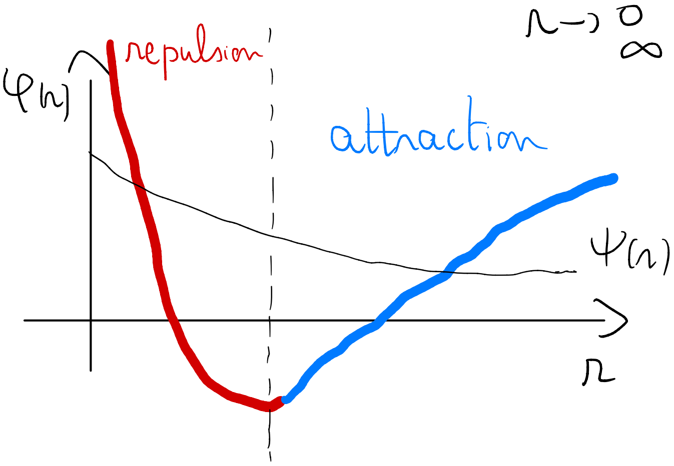
2) Three zone model

$$\begin{cases} x_i'(t) = v_i(t) \\ v_i'(t) = \frac{1}{N} \sum_j \psi(\|x_i - x_j\|)(v_j - v_i) \\ \qquad - \frac{1}{N} \sum_{j \neq i} \nabla_x \varphi(\|x_i - x_j\|) \end{cases}$$

where we suppose that the interacting potential φ is such that

$\varphi \in \mathcal{E}^1(\mathbb{R}_*^+)$

$\varphi(r) > 0 \qquad \lim_{r \to 0} \varphi(r) = +\infty$ ($r \to \infty$)



- repulsion for $r \ll 1$
- alignment for $r \sim 1$
- attraction for $r \gg 1$

For this model ; we have the following result

Theorem Suppose that ψ is bounded and φ satisfies the assumption above.
Then
- For non-collisional initial data, there exists a unique global solution such that

$$0 < d_m \leq \|x_i(t) - x_j(t)\| \leq d_M \quad \forall t \geq 0$$

and $A(t) \xrightarrow[t \to +\infty]{} 0$

Ref Cao - Motsch - Reamy and Theisen, Math Bio Eng (2020)

To conclude this first part of the lectures, let us emphasize that there are different levels of description for this self-organization phenomena

III) Kinetic & fluid description

Kinetic description particles are replaced by a probability distribution function $f(t, x, v) \geq 0$ solution to a mean field model obtained as the limit of the particle model

Consider $f_N(t) = \sum_i \delta(x - x_i(t))\, \delta(v - v_i(t))$

where $(x_i, v_i)_{1 \le i \le N}$ is solution to the particle system, hence f_N is formally solution to

$$\partial_t f_N + v \cdot \nabla_x f_N + \mathrm{div}_v \left(L[f_N]\, f_N + E_N\, f_N \right) = 0$$

where

$$\begin{cases} L(f_N) = \Psi * J_N - (\Psi * \rho_N)\, v \\ E_N = -\nabla \varphi * \rho_N \end{cases}$$

with
$$J_N = \frac{1}{N} \sum_j v_j\, \delta(v - v_j)\, \delta(x - x_j)$$
$$\rho_N = \frac{1}{N} \sum \delta(x - x_i)$$

Replacing f_N by any distribution function and sum and discrete convolutions by integrals and convolution it yields

$$\partial_t f + v \nabla_x f + \mathrm{div}\left(\left(L(f) - \nabla \varphi * \rho \right) f \right) = 0$$

$$L(f) = \Psi * J - (\Psi * \rho)\, v$$

$$J = \int f v \, dv \qquad \rho = \int f \, dv$$

<u>Fluid description</u> we study system of equations for macroscopic equations (ρ, J)

$$\begin{cases} \partial_t \rho + \mathrm{div}_x (J) \\ \partial_t J + \mathrm{div}_x \left(\int f v \otimes v \right) - (\Psi * J \rho - \Psi * \rho \, J) = 0 \end{cases}$$

we need a closure to eliminate f in the previous eq

For instance $f = \rho \, \delta(v - \frac{J}{\rho})$; this ansatz give

$$\begin{cases} \partial_t \rho + \mathrm{div}_x J = 0 \\ \partial_t J + \mathrm{div}_x \left(\frac{J \otimes J}{\rho} \right) = (\Psi * J) \rho - (\Psi * \rho) J \end{cases}$$

In the following we will study these kinetic and fluid models and investigate how there are related.

IV) Mathematical models sharing the same structure

1) FitHugh-Nagumo model

this model appears in neurosciences

it describe the time evolution of a potential membrane $V_i(t)$ of the neuron i and an adaptation variable $W_i(t)$

$$\begin{cases} dV_i = (N(V_i) - W_i + I_{ext})dt + \sqrt{2}\, dB_i \\ dW_i = A(V_i, W_i) \end{cases}$$

$N(V) = V - V^3$

$A(V, W) = aV - bW + c \qquad a, b, c \geq 0$

and I_{ext} describes the interactions between neurons

$$I_{ext}(t) = \frac{1}{N} \sum_j \Psi_{ij} (V_j - V_i)$$

Ψ_{ij} is related to a conductance and is given by the neural networks.

2) Kuramoto model

$$\theta_i'(t) = \nu_i + \frac{1}{N} \sum_j a_{ij} \sin(\theta_j - \theta_i)$$

It can be written as a second order model (θ_i, ω_i)

$$\begin{cases} \theta_i' = \omega_i \\ \omega_i' = \dfrac{1}{N} \sum_j \cos(\theta_j - \theta_i)(\omega_j - \omega_i) \end{cases}$$

About kinetic flocking models

I) Introduction.

We study the time evolution of the distribution function $f \equiv f(t,x,v) \geq 0$ of agents at time $t \geq 0$, position $x \in \mathbb{R}^d$, $d = 1, 2$ or 3 and $v \in \mathbb{R}^d$.

It solves the kinetic flocking model

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v (L[f]\, f) + \beta\, \mathrm{div}_v ((u - v) f) = \sigma \Delta_v f$$

The alignment operator $L[f]$ has the form

$$L[f] = \int_{\mathbb{R}^d \times \mathbb{R}^d} K(x,y)\, f(t,y,w)\, (w - v)\, dw\, dy$$

whereas the term $\beta\,(v - u) f$ describes local alignment where

$$\rho u = \int f v \, dv \quad \text{and} \quad \rho = \int f \, dv$$

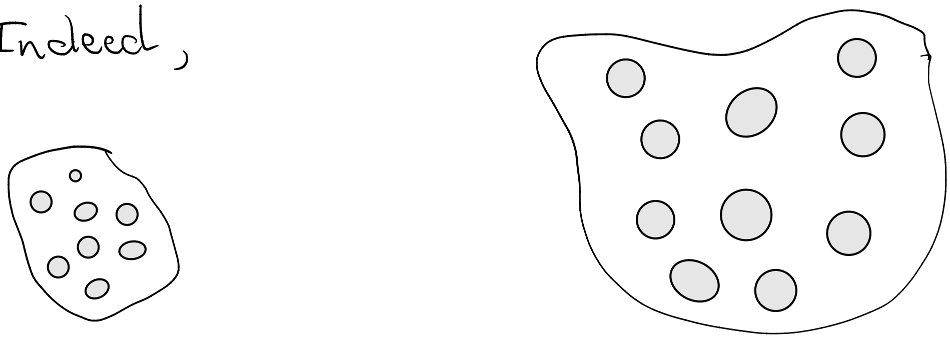
Here we will suppose that K is symmetric

$$K(x,y) = K(y,x) \quad \text{and smooth}$$

<u>Remark</u> the local alignment operator can be interpreted as a limit of the Tadmor-Motsch alignment

operator which consider that K may depend on f

Indeed,



$$\tilde{L}(f) = \frac{\phi * \rho u - (\phi * \rho) v}{\phi * \rho}$$

local interactions are dominant; the small group does not interact too much with the big group located far away.

In the asymptotic limit where $\phi = \beta \delta_0$, we recover our previous model $\tilde{L}(f) = \beta (u - v) f$

We may also consider the Cucker-Smale model with noise, self-propulsion and friction

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v (L[f] f) + \beta \, \mathrm{div}_v ((u - v) f) = \sigma \Delta_v f - \mathrm{div}_v ((a - b|v|^2) v f) \qquad (CS)$$

with $\sigma \geq 0$, $a, b \geq 0$

The main difficulty comes from the non-linear terms describing the alignment, hence we will neglect the other ones.

Our first result concerns the existence of weak solutions for the kinetic flocking equation

II) Existence of solutions

We prove the following theorem

Theorem Assume that $f_0 \geq 0$ and

$$f_0 \in L^\infty \cap L^1(\mathbb{R}^d \times \mathbb{R}^d) \quad \text{and} \quad \int (|x|^2 + |v|^2) f_0 \, dx dv < +\infty$$

Then there exists a weak solution f to the kinetic Flocking system (CS) such that

$$\int_0^{+\infty} \int_{\mathbb{R}^{2d}} f \Big(\partial_t \varphi + v \nabla_x \varphi + L[f] \nabla_v \varphi + \beta (u - v) \nabla_v \varphi + \sigma \Delta_v \varphi + (a - b|v|^2) v \nabla_v \varphi \Big) dx dv dt$$

$$= - \int_{\mathbb{R}^{2d}} f^0 \varphi(0) \, dx dv \qquad \forall \varphi \in \mathcal{C}_c^\infty([0, +\infty[\times \mathbb{R}^{2d})$$

where $u = \dfrac{J}{\rho}$

Remark • the definition of u is ambiguous when ρ vanishes, hence we define it as

$$u(t,x) = \begin{cases} \dfrac{J(t,x)}{\rho(t,x)} & \text{when } \rho(t,x) \neq 0 \\ 0 & \text{else} \end{cases}$$

Since $|J| \leq \left(\int |v|^2 f \, dv \right)^{1/2} \rho^{1/2}$, we have that $J \equiv 0$ when $\rho = 0$, hence this latter definition ensures that $J \equiv \rho u$

- We notice that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (f u)^2 \, dx dv \leq \|f\|_{L^\infty} \int |v|^2 f \, dx dv$$

Then the term $uf \in L^2(\mathbb{R}^{2d})$ and our notion of solution is well defined.

$$\int (f u)^2 \, dv dx = \int_{\mathbb{R}^d} \left(u^2 \int f^2 dv \right) dx$$

$$\leq \|f\|_{L^\infty} \int \rho u^2 dx$$

$$= \|f\|_{L^\infty} \int \frac{J^2}{\rho} dx$$

$$\leq \|f\|_{L^\infty} \int f |v|^2 dx dv$$

1) A priori estimates

We get as a first result that (CS) preserves the non-negativity of the distribution function as the solution of a transport / diffusion equation.

Moreover, we have conservation of mass

$$\int f \, dx dv = \int f_0 \, dx dv$$

Then, we prove the propagation of L^p norms, by multiplying (CS) by $p f^{p-1}$ and integrating in $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$.

$$\frac{d}{dt}\|f\|_{L^p}^p + \frac{4\sigma(p-1)}{p}\int |\nabla_v f^{p/2}|^2\,dxdv$$

$$= -(p-1)\int f^p \,\mathrm{div}_v L[f]\,dxdv$$

$$+\beta(p-1)d\int f^p\,dxdv$$

Now we observe that

$$\mathrm{div}_v L[f] = -d\int K_0(x,y)\,f(y,w)\,dydw$$

hence

$$|\mathrm{div}_v L[f]| \leq d\,\|K_0\|_{L^\infty}\,\|f_0\|_{L^1}$$

Therefore a Gronwall argument gives that

$$\boxed{\frac{d}{dt}\|f\|_{L^p}^p + \frac{4\sigma(p-1)}{p}\int |\nabla f^{p/2}|^2\,dxdv \leq (p-1)\left[\beta + \|K_0\|_{L^\infty}\|f_0\|_{L^1}\right] \times \|f\|_{L^p}^p}$$

Next; we need better integrability of f for large $\|x\|$ and $\|v\|$; hence we study propagation of moments

$$\mathcal{E}(f) = \frac{1}{2}\int (|v|^2+|x|^2)\,f\,dxdv$$

It gives

$$\frac{d}{dt}\mathcal{E}(f) + \beta\int_{\mathbb{R}^{2d}} |v-u|^2 f \,dx dv$$

$$+ \frac{1}{2}\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} K_0(x,y)\, f(x,v)\, f(y,w)\, |v-w|^2 \,dw dy dv dx$$

$$= \sigma d \, \|f_0\|_{L^1} + \int_{\mathbb{R}^{2d}} x.v f \,dx dv$$

$$\leq \sigma d \, \|f_0\|_{L^1} + \mathcal{E}(f)$$

hence $\boxed{\mathcal{E}(f)(t) \leq \left(\sigma d \, \|f_0\|_{L^1} + \mathcal{E}(f_0)\right) e^t}$

Remark. If we add self-propulsion and friction ; we have the same kind of inequality.

. Here we use the symmetry of K

These estimate allows to get some compactness property and to pass to the limit in the equation to prove existence of solutions

The main point is to deal with macroscopic quantities

Proposition Suppose that $\|f(t)\|_{L^\infty} < C$ and $\int f\, |v|^2 dx dv < +\infty$; hence we have

$$\|\rho(t)\|_{L^p} \leq C \qquad \text{for } p\in\left[1, \frac{d+2}{d}\right]$$

$$\|j(t)\|_{L^p} \leq C \qquad \text{for } p\in\left[1, \frac{d+2}{d+1}\right]$$

Proof

$$\rho(t,x) = \int f\,dv$$

$$= \int_{B(0,R)} f\,dv + \int_{B^c(0,R)} f\,dv$$

$$\leq R^d \underbrace{\|f\|_{L^\infty}}_{a} + \frac{1}{R^2}\underbrace{\int v^2 f\,dv}_{b}$$

Optimize R or use scaling arguments, to

$$\varphi(R) = a R^d + \frac{b}{R^2}$$

$$\varphi'(R) = d\,a R^{d-1} - \frac{2b}{R^3}$$

$$R_{opt} \simeq \left(\frac{b}{a}\right)^{\frac{1}{d+2}}$$

$$|\rho(t,x)| \leq C\|f\|_{L^\infty}^{2/(d+2)} \times \left(\int |v|^2 f\right)^{d/(d+2)}$$

We proced in the same way for j.

Unfortunately, it is not enough to get strong compactness on the macroscopic quantities. Hopefully, we prove "averaging lemma"

<u>Proposition</u> Consider $\rho_\varphi = \int f\, \varphi(v)\,dv$ where $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Let $(f^n)_n$ and $(G^n)_n$ where f^n and G^n are bounded in $L^p_{loc}(\mathbb{R}^{2d+1})$ and

$$\partial_t f^n + v\cdot\nabla_x f^n = \nabla_v^k G^n$$

Then $(\rho_\varphi^n)_n$ is relatively compact in $L^p_{loc}(\mathbb{R}^{d+1})$

We refer to the work of Golse, Perthame, Lions and Sentis on averaging lemmas and to Perthame and Souganidis for the previous Proposition

This does not give directly strong compactness on ρ and j; but rather on the approximated sequence $\left(\int f^n \varphi(v)\,dv\right)_n$; where φ is a smooth function.

The last difficulty is to get compactness on

$$u = \frac{J}{\rho}$$

An important step is to define a suitable approximation of solutions

2) Approximated solution

We consider the following problem where the velocity is regularized

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v (L[f] f)$$

$$= \sigma \Delta_v f + \beta \, \mathrm{div}((\chi_\lambda(u_\delta) - v) f)$$

where

$$\chi_\lambda(u) = u \, \mathbb{1}_{\{u \leq \lambda\}}$$

and $u_\delta = \dfrac{J}{\delta + \rho}$

Formally, we see that in the limit $\delta \to 0$

and $t \to +\infty$; we recover the initial problem.

Passing to the limit we prove our main theoreme.

We refer to Karper, Mellet & Trivisa SIMA 2013 for more details.

III) Flocking behavior of the kinetic model

We consider the (CS) model without diffusion

$$\partial_t f + v \nabla_x f + \operatorname{div}\left(L[f]\, f\right) + \beta \operatorname{div}\left((u-v) f\right) = 0$$

$$L[f] = \Psi * \rho u - (\Psi * \rho)\, v$$

Ψ is symmetric

We define the following Lyapunov functionals

$$\mathcal{E}_1(t) = \int f\, |v-u|^2 \, dx dv \qquad \text{relative energy}$$

and

$$\mathcal{E}_2(t) = \int_{\mathbb{R}^{2d}} \rho(t,x)\,\rho(t,y)\,|u(t,x) - u(t,y)|^2\,dx\,dy$$

$\mathcal{E}_1$ measures the local alignment whereas $\mathcal{E}_2$ measures alignment of the macroscopic velocities

We set $$\boxed{\mathcal{E}(t) = \mathcal{E}_1(t) + \frac{1}{2}\mathcal{E}_2(t)}$$

and prove the following theorem

<u>Theorem</u> $\mathcal{E}(t) \leq \mathcal{E}(0)\, e^{-ct}$

where $c = 2 \min(1, \psi_m)$

and $\psi_m = \min_{x,y} \psi(x-y) > 0$

Let us give the main steps for the proof of this result

$$\frac{d\mathcal{E}_1}{dt} = \underbrace{2\int f\,(u-v)\,\partial_t u}_{=0} + \int \partial_t f\,|u-v|^2\,dx\,dv$$

$$= \left. 2\int \nabla_x u\,(u-v)\cdot v f\,dx\,dv\right) = I_1$$

$$\left. -2\int \left[(u-v)\cdot L[f]\, f\,dx\,dv\right.\right) = I_2 \leq 0$$

$$\underbrace{-2\int |u-v|^2 f\,dx\,dv}_{2\mathcal{E}_1}$$

$I_1 = 2\int \operatorname{div} P\cdot u\,dx$ where

P is the stress tensor $P = \int (v-u)\otimes(v-u)\,f\,dv$

Hence we get that

$$\frac{d\mathcal{E}_1}{dt} \leq 2\int \operatorname{div}_x P\cdot u\,dx - 2\,\mathcal{E}_1$$

We next estimate

$$\frac{d\mathcal{E}_2}{dt} = 2\int \partial_t\rho(t,x)\,\rho(t,y)\,\overset{J_1}{\underset{\shortparallel}{|}}u(t,x)-u(t,y)|^2\,dx\,dy$$

$$+2\int \rho(t,x)\,\rho(t,y)\,\overset{J_2}{\underset{\shortparallel}{(}}u(t,x)-u(t,y))\,\partial_t\big(u(t,x)-u(t,y)\big)\,dx\,dy$$

Then we use the equations satisfied by the macroscopic equations

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0 \\ \rho \partial_t u + \rho u \cdot \nabla_x u + \mathrm{div}_x P = \int L(f) f dv \end{cases}$$

and get that

$$J_1 = 4 \int \rho u \cdot \nabla_x u \cdot u \, dx$$
$$- 4 \left(\int \rho u \cdot \nabla_x u \, dx \right) \cdot \int \rho u \, dx$$

$$J_2 = - J_1 - 4 \int \mathrm{div} P \cdot u \, dx$$
$$- 2 \int \Psi(x-y) \rho(t,x) \rho(t,y) |u(t,x) - u(t,y)|^2 dx dy$$

Therefore we have

$$\frac{d \mathcal{E}_2}{dt} = -4 \int \mathrm{div} P \cdot u - 2 \int \Psi(x-y) \rho(t,x) \rho(t,y) |u(t,x) - u(t,y)|^2 dx dy$$

Gathering the two results; it yields

$$\frac{d\mathcal{E}}{dt} \leq -2\,\mathcal{E}_1 - \Psi_m\,\mathcal{E}_2$$

$$\leq -2\,\min(1, \Psi_m)\,\mathcal{E}$$

we conclude by a Grenwall lemma $\square$

About fluid limit of kinetic flocking models

The main goal of this chapter is to show that a singular limit of strong local alignment

$$\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon + \mathrm{div}_v\left(L[f^\varepsilon] f^\varepsilon\right) + \frac{1}{\varepsilon}\,\mathrm{div}((u^\varepsilon - v) f^\varepsilon) = 0$$

$$L[f^\varepsilon] = \Psi * (\rho^\varepsilon u^\varepsilon) - (\Psi * \rho^\varepsilon)\, v$$

$$u^\varepsilon = \frac{\int f^\varepsilon v\, dv}{\int f^\varepsilon\, dv} \qquad \rho^\varepsilon = \int f^\varepsilon\, dv$$

As $\varepsilon \to 0$, it is expected that f^ε converges in a weak sense to a mono-kinetic distribution $\rho(t,x) \otimes \delta(v - u(t,x))$ where (ρ, u) solves the pressureless Euler equations with alignment

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) = (\Psi * (\rho u))\rho - (\Psi * \rho)\rho u \end{cases}$$

The pressureless Euler equations can be applied to describe large-scale structure in astrophysics

It is worth to mention that this system admits a convex entropy

$$\eta(\rho, \rho u) = \rho \frac{|u|^2}{2}$$

which is not strictly convex with respect to ρ.

Hence this entropy is not enough to prove the convergence of ρ^ε towards ρ as $\varepsilon \to 0$.
The strategy will consists in proving the convergence of u^ε towards u thanks to the entropy and the convergence of ρ^ε towards ρ using the second order Wasserstein distance.

Let us first explain the formal derivation of the pressureless Euler equations.

We consider f^ε solution to (CS) and compute the time evolution of $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$

$$\partial_t \rho^\varepsilon + \mathrm{div}_x (\rho^\varepsilon u^\varepsilon) = 0$$

$$\partial_t (\rho^\varepsilon u^\varepsilon) + \mathrm{div}_x \left(\int f^\varepsilon v \otimes v \, dv \right) = (\psi * (\rho^\varepsilon u^\varepsilon))\rho^\varepsilon - (\psi * \rho^\varepsilon) \rho^\varepsilon u^\varepsilon$$

Now we simply write

$$\int f^\varepsilon v \otimes v \, dv = \rho^\varepsilon u^\varepsilon \otimes u^\varepsilon + P^\varepsilon$$

where $P^\varepsilon = \int f \, (v - u^\varepsilon) \otimes (v - u^\varepsilon) \, dv$

Formally

$$\int f^\varepsilon (v - u^\varepsilon) \, dv \sim O(\varepsilon)$$

$$\Rightarrow \quad f^\varepsilon \sim \rho^\varepsilon \, \delta(v - u^\varepsilon)$$

hence it yields the pressureless Euler equations.

In this lecture we suppose that there exists a unique classical solution $(\rho, \rho u)$ to the limit system (true for short time) on the interval $[0, T_*]$.
We prove the following result

<u>Theorem</u> Under the previous assumptions we have for all $t \in [0, T_*]$

$$\int \rho^\varepsilon |u^\varepsilon - u| \, dx + W_2^2(\rho^\varepsilon(t), \rho(t))$$

$$+ \frac{1}{2} \int_0^t \iint \psi(x-y)\rho^\varepsilon(t,x)\rho^\varepsilon(t,y)\left[u^\varepsilon(t,x) - u(t,x) - u^\varepsilon(t,y) - u(t,y)\right]^2$$

$$\leq C(T_*)\,\varepsilon$$

and $f^\varepsilon \longrightarrow \delta(v - u(t,x)) \otimes \rho(t,x)$

in $L^1([0,T_*], \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d))$

$\mathcal{M}$ is the set of Radon measure

We follow the relative entropy method

$J = \rho u$

$$U = \begin{pmatrix} \rho \\ J \end{pmatrix}$$

$$A(U) = \begin{pmatrix} J \\ \frac{J \otimes J}{\rho} \end{pmatrix} \qquad F(U) = \begin{pmatrix} 0 \\ (\Psi * J)\rho - (\Psi * \rho) J \end{pmatrix}$$

The limit eq^{ns} can be written as

$$\partial_t U + \operatorname{div}(A(U)) = F(U)$$

We introduce $\eta(U) = \frac{1}{2}\rho u^2 = \frac{J^2}{2\rho}$

and using that

$$\partial_t u + u \cdot \nabla_x u = \Psi * J - (\Psi * \rho) u$$

we have

$$\frac{d}{dt}\int \eta(U)\,dx = -\frac{1}{2}\int \Psi(x-y)\,\rho(y)\rho(x)\,[u(x) - u(y)]^2\,dx\,dy$$

$$\leq 0$$

Entropy is dissipated

Then we consider the relative entropy

$$\eta(V|U) = \eta(V) - \eta(U) - D\eta(U) \cdot (V-U)$$

$$A(V|U) = A(V) - A(U) - DA(U) \cdot (V-U)$$

Here we simply get that

$$\eta(V|U) = \frac{q}{2}(u-w)^2$$

where $U = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}$ $V = \begin{pmatrix} q \\ qw \end{pmatrix}$

We have the following proposition

<u>Proposition</u> Let U be a smooth solution to the limit problem and $V = \begin{pmatrix} q \\ qw \end{pmatrix}$ be a smooth function

$$\frac{d}{dt}\int \eta(V|U)\,dx + \frac{1}{2}\int \Psi(x-y)\,q(x)\,q(y)\,|w(x)-w(y)-u(x)-u(y)|^2\,dxdy$$

$$= \frac{d}{dt}\int \eta(V)\,dx + \frac{1}{2}\int \Psi(x-y)\,q(x)\,q(y)\,[w(x)-w(y)]^2\,dxdy$$

$$\underset{\parallel}{I(V,U)}$$

$$- \int D\eta(U) \cdot \left[\partial_t V + \mathrm{div}_x (A(V)) - F(V) \right] dx$$

$$- \int \nabla_x D\eta(U) : A(V|U)\, dx$$

$$- \int \psi(x-y)\, q(x) \left(p(y) - q(y)\right) \left(u(y) - u(x)\right) \left(w(x) - u(x)\right) dx dy$$

Then the idea is to take $V = \begin{pmatrix} \rho^\varepsilon \\ \rho^\varepsilon u^\varepsilon \end{pmatrix}$ solution obtained from the kinetic equation.

It gives after some computations that

$$\int \rho^\varepsilon \frac{|u^\varepsilon - u|^2}{2}(t)\, dx + \int_0^t I^\varepsilon(U^\varepsilon, U)(s)\, ds$$

$$\leq O(\varepsilon) + C \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon(s)\, |u^\varepsilon - u|^2\, dx ds$$

$$- \int_0^t \iint \psi(x-y)\, \rho^\varepsilon(x) \left(\rho(y) - \rho^\varepsilon(y)\right) \left(u(y) - u(x)\right) \times \left(u^\varepsilon(x) - u(x)\right) dx dy ds$$

To conclude it remains to evaluate the error between ρ^ε and ρ

We have the following lemma

Lemma

$$W_2^2(\rho^\varepsilon(t), \rho(t)) \leq C e^{T_*} \int_0^t \int \rho^\varepsilon |u^\varepsilon(x) - u(x)|^2 \, dx \, ds + O(\varepsilon)$$

Gathering the results; we get the theorem.