A conservative and entropic method for the non homogeneous Vlasov-Landau equation

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Abstract. In this paper we compute numerical solutions to the Vlasov equation coupled with the Fokker-Planck-Landau collision operator using a phase space grid. On the one hand, the algorithm is based on the conservation of the flux of particles, and the distribution function is reconstructed to preserve energy and to control numerical oscillations. On the other hand, the method preserves the main properties of the collision operators. Some results are presented in one dimension in space and three dimensions in velocity.

1. Introduction

The aim of this paper is to compute the evolution of a collisional plasma constituted of different species of charged particles (ions, electrons...). The Landau or Fokker-Planck-Landau equation is a common kinetic model in plasma physics ([3, 4, 9]). It describes binary collisions between charged particles with long-range Coulomb interaction. The evolution of electrons is given by the distribution function $f(t,x,v)$ which depends on time t, position $x \in \Omega \subset \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. This distribution function is solution to the scaled Fokker-Planck-Landau equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + E \cdot \nabla_v f = \nu_i \, Q_i(f) + \nu_e \, Q_e(f,f), \tag{1.1}$$

where $E = E(t,x)$ is the electric field which can evolve through the Poisson equation

$$\nabla_x \cdot E(t,x) = \int_{\mathbf{R}^3} f(v)dv - \rho_i, \;\; E(t,x) = -\nabla_x \phi(t,x), \tag{1.2}$$

where ρ_i is the ion density. Moreover, ν_i is related to the collision frequency and $Q_i(f)$ describes collisions between two different kinds of particles as ions and electrons

$$Q_i(f) = \nabla_v \cdot (\Phi(v) \nabla_v f) \, . \tag{1.3}$$

Finally, ν_e is related to the electron-electron collision frequency and $Q_e(f,f)$ is

$$Q_e(f,f) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} \Phi(v-v')(\nabla_v f(v) f(v') - \nabla_{v'} f(v') f(v)) dv' \right), \qquad (1.4)$$

where $\Phi(v)$ is the 3×3 matrix

$$\Phi(v) = \frac{1}{|v|^3} \left(|v|^2 I_3 - v \otimes v \right). \qquad (1.5)$$

The main goal of this note is to describe a scheme in the x depending case and in the whole 3D velocity space. The first step consists in the approximation of the Vlasov equation, which is the left hand side of (1.1). We use a method which discretizes it on a phase space grid (see [5, 6, 8]). This kind of approach allows to get an accurate approximation of the distribution function in the phase space, but the non conservation of the energy due to the projection of the interpolation on the grid can be an inconvenient for the long time behaviour of the solution. In this paper, we propose a new scheme which overcomes to these inconvenient: space and velocity derivatives are approximated by a centered finite volume method and an adapted approximation of the electric field E allows us to obtain a numerical scheme that conserves the total energy. As this kind of discretization does not ensure the positivity of the unknown, we introduce slope correctors. The distribution function is reconstructed following the PFC method ([5, 6]) of order two. When the slope correctors act, the total energy is not conserved any more; nevertheless, the variations are not very important compared to others methods such as semi-Lagrangian method.

Finally, we consider collision operators derived from (1.4). We propose conservative schemes based on the ideas of [3] to approximate this different operators. Even if these schemes give interesting properties (conservations, decay of the entropy, positivity of the distribution function), their direct implementation is very expensive in high dimensions. Thus, we adopt the multigrid method used in [1] to reduce the computational cost.

The rest of the paper is organized as follows. In the next section, we draw up the main properties of the solution to the system (1.1)-(1.2)-(1.4) and formally derive the operator intended to model collisions between different species. We then present in section 3 a finite volume scheme for the discretization of the Vlasov-Poisson equation. Finally, some numerical results are presented in section 4 to illustrate the efficiency of the method.

2. Description of the kinetic model

In this section, we briefly recall some classical estimates on the system (1.1)-(1.2)-(1.4). First, the Vlasov-Poisson system (1.1)-(1.2) without collision, preserves

mass, momentum and total energy. It also conserves the kinetic entropy

$$H(t) = \int_{\mathbf{R}^3 \times \mathbf{R}^3} f(t)\, \log(f(t)) dx dv = H(0).$$

Next, let us recall the expression of the Fokker-Planck-Landau (FPL) operator which describes collisions between two electrons

$$Q_e(f,f) = \nabla_v \cdot \left(\int_{\mathbf{R}^3} \Phi(v-v')(\nabla_v f(v) f(v') - \nabla_{v'} f(v') f(v)) dv' \right), \qquad (2.6)$$

where $\Phi(v)$ is the 3×3 matrix (1.5). The algebraic structure of the FPL operator leads to physical properties such as the conservations of mass, momentum and energy and the decay of the entropy $H(t)$. Moreover, the equilibrium states of the FPL operator, *i.e.* the distribution functions f which satisfy $Q_e(f,f) = 0$, are given by the Maxwellians (see [3, 9] for more details).

Finally, the operator Q_i describes collisions between two different species (for instance electrons and ions), and can be derived from the two species form of the full Landau operator (1.4). If we assume that the temperature of ions T_i is negligeable compared to the temperature of electrons T, we may consider that the distribution function of ions is given by a Dirac measure in velocity

$$f_i(t,x,v) = \rho_i(t,x)\, \delta_0(v - u_i(t,x)), \qquad (2.7)$$

where the density ρ_i and the mean velocity u_i are given or satisfy hydrodynamic equations. The so-obtained operator then reads

$$Q_i(f) = \rho_i \nabla_v \cdot (\Phi(v-u_i)\nabla_v f), \qquad (2.8)$$

where $\Phi(v)$ is the 3×3 matrix given by (1.5). The linear collision operator (2.8) satisfies the conservations of mass and energy.

The equilibrium states are given by the isotropic functions $\{f(|v-u_i|^2)\}$, and each convex function Ψ of f is an entropy for Q_i (see [2] for more details).

3. The numerical method

For the approximation of the linear operator, we adopt a conservative and entropic scheme based on the ideas of [3] (see [2] for details). The FPL operator is discretized following the method introduced in [3] which respects the properties of the operator; the multigrid method is employed to reduce the computational cost (see [1]).

In this subsection, we introduce a new scheme for the discretization of the Vlasov-Poisson equation. Thanks to a discretization based on a finite volume method, joint with an adapted approximation of the electric field, we will prove that this scheme preserves the total energy. We consider a cartesian grid in one dimension in space and velocity for the sake of simplicity.

We introduce the mesh points $(x_{i+1/2})_{i\in I}, (v_{j+1/2})_{j\in\mathbb{Z}}$ of the computational domain $[x_{min}, x_{max}]\times\mathbb{R}$. We will denote by $\Delta x = x_{i+1/2} - x_{i-1/2}$ and $\Delta v = v_{j+1/2} - v_{j-1/2}$ the space and velocity step, and by $C_{i,j} = [x_{i-1/2}, x_{i+1/2}]\times[v_{j-1/2}, v_{j+1/2}]$ the control volume. Finally, $t^n = n\Delta t$ is the time discretization, $n\in\mathbb{N}$.

We assume that $f_{i,j}^n$ is an average approximation of f on the control volume $C_{i,j}$ at time t^n

$$f_{i,j}^n = \frac{1}{\Delta x\,\Delta v}\int_{C_{i,j}} f(t^n, x, v)dxdv.$$

Then, we approximate the average of f at the time t^{n+1} by integrating the Vlasov equation on a control volume and using a backward Euler scheme in time

$$\begin{aligned} f_{i,j}^{n+1} &= f_{i,j}^n - \frac{\Delta t}{\Delta x\,\Delta v}\int_{v_{j-1/2}}^{v_{j+1/2}} v\left(f^n(x_{i+1/2}, v) - f^n(x_{i-1/2}, v)\right)dv \\ &\quad -\frac{\Delta t}{\Delta x\Delta v}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{E}^n(x)\left(f^n(x, v_{j+1/2}) - f^n(x, v_{j-1/2})\right)dx, \quad (3.9) \end{aligned}$$

where $\tilde{E}^n(x)$ is an approximation of the electric field obtained by the Poisson equation (1.2).

The main step to get an accurate solution is to reconstruct the distribution function in each direction. For that purpose, we fix the velocity $v\in[v_{j-1/2}, v_{j+1/2})$, which we assume positive, and consider the function $f_h(x,v), x\in[x_{i-1/2}, x_{i+1/2})$ as an approximation of the distribution function $f(t^n, x, v), x\in[x_{i-1/2}, x_{i+1/2})$. Thus, the high order approximation $f_h(x,v)$ is obtained via a reconstruction by a primitive $F_h(x,v), x\in[x_{i-1/2}, x_{i+1/2})$ such that

$$F_h(x_{i+1/2}, v) = \int_0^{x_{i+1/2}} f_h(x,v)dx, \quad F_h(x_{i+1/2}, v) - F_h(x_{i-1/2}, v) = \Delta x f_{i,j}^n.$$

We first build an approximation of F_h using the stencil $\left\{x_{i-1/2}, x_{i+1/2}, x_{i+3/2}\right\}$. Then we get

$$F_h(x,v) = F_h(x_{i-1/2}, v) + (x - x_{i-1/2})f_{i,j}^n + \frac{(x-x_{i-1/2})(x-x_{i+1/2})}{2\Delta x}(f_{i+1,j}^n - f_{i,j}^n).$$

By differentiation, we obtain a second order accurate approximation of the distribution function on the interval $[x_{i-1/2}, x_{i+1/2}]$

$$f_h(x,v) = \frac{\partial F_h}{\partial x}(x,v) = f_{i,j}^n + \frac{(x-x_i)}{\Delta x}(f_{i+1,j}^n - f_{i,j}^n).$$

As this kind of approximation can generate spurious oscillations, we introduce a slope corrector, ensuring the positivity of the distribution function on the interval $(x_{i-1/2}, x_{i+1/2})$

$$\epsilon_i^+ = \min\left(1, 2f_{i,j}^n/(f_{i+1,j}^n - f_{i,j}^n)\right), \quad \text{if} \quad (f_{i+1,j}^n - f_{i,j}^n) > 0. \qquad (3.10)$$

Now, we proceed in the same manner to reconstruct f_h in the x direction when $v < 0$ using the stencil $\{x_{i-3/2}, x_{i-1/2}, x_{i+1/2}\}$ and in the v direction depending on the sign of $\tilde{E}^n(x)$.

When the slope correctors do not occur in the approximation of the velocity derivative (it is the case when the distribution function is sufficiently smooth as a function of v) we obtain a classical centered scheme. With an adapted approximation of the electric field, $\tilde{E}_i^n = (E_i^{n+1} + E_i^n)/2$, the so-obtained scheme preserves the total energy. On the other hand, the electric field at time t^n is determined through the following approximation to the Poisson equation

$$-D_x^\star E_i^n = \rho_i^n - \rho_i, \quad E_i^n = -D_x \phi_i^n, \tag{3.11}$$

where D_x is a discrete finite difference operator whereas $D_x^\star$ stands for its formal adjoint, which represents an approximation of $-\partial_x$ (for example, we can consider D_x as the usual uncentered discrete operator). Finally, E_i^{n+1} is a prediction of the electric field at time t^{n+1} obtained from the discretization to the Poisson equation (3.11) and to the continuity equation

$$\rho^{n+1} = \rho_i^n + \Delta t\, D_x^\star J_i^n, \tag{3.12}$$

where J_i^n and ρ_i^n are respectively an approximation of the current density $j(t,x)$ and of the charge density $\rho(t,x)$

$$J_i^n = \Delta v \sum_{j\in\mathbb{Z}} v_j f_{i,j}^n, \quad \rho_i^n = \Delta v \sum_{j\in\mathbb{Z}} f_{i,j}^n. \tag{3.13}$$

The approximation of the Vlasov-Poisson equation obtained from this algorithm satisfies the conservation of mass and total energy (see [2] for the proof).

4. Numerical simulations

4.1. Linear Landau damping

In this section, we investigate the Landau damping in the linear context. To that purpose, we apply the above approximation to discretize the full system (1.1)-(1.2)-(1.4) where the collision operators are given by (2.6) and (2.8) with $\rho_i = 1$ and $u_i = 0$. Besides, the collision parameters are chosen as follows: $\nu_e = \nu/Z$ and $\nu_i = \nu$, where Z is the number of charge. The initial condition is chosen as a periodic perturbation of the global equilibrium

$$f_0(x,v) = \frac{1}{(2\pi)^{3/2}} \exp\left(-|v|^2/2\right)(1 + A\cos(kx)),\ x \in [0, 2\pi/k],\ v \in \mathbb{R}^3, \tag{4.14}$$

where A is the amplitude of the perturbation, and k denotes the wave number. In this subsection we only consider linear regimes and A is taken equal to $A = 10^{-5}$.

To capture the Landau damping, the size of the velocity domain must be chosen greater than the phase velocity $v_\phi = \omega/k$ (with $\omega^2 \approx 1 + 3\,k^2$). Then, we set $v_{max} = 5.75$ where the velocity grid extends from $-v_{max}$ to v_{max}. We use $N_v = 32$ grid points in each direction for the velocity grid whereas the number of spatial grid points is $N_x = 50$. The boundary conditions for the distribution function are periodic in x whereas the distribution function is truncated to zero for large velocities.

We are interested in the evolution of the square root of the electric energy approximated by

$$\mathcal{E}_h(t) = \left(\sum_i \Delta x E_i^2(t) \right)^{1/2}. \tag{4.15}$$

On Fig. 1, we plot (4.15) in log scale and its associated damping line (which is determined numerically) as a function of time. We study the influence of the collisions and consider values of ν equal to $\nu = 0$, 0.01 and 0.05 whereas $k = 0.3$ and $Z = 1$. The value of $\nu = 0$ corresponds to the Vlasov-Poisson equation. The amplitude of the electric field is damped exponentially in time behaviour and we observe the influence of the collisions on the electric energy: the damping rate becomes stronger when the collisions are more important, *i.e.* when ν is bigger.

On Fig. 2, we study the influence of Z (with $\nu = 0.01$) on the damping of (4.15). We observe that the damping rate is not very sensible to Z. We recover the fact that e-i collisions play a more important role than e-e ones in the damping of electronic plasma waves.

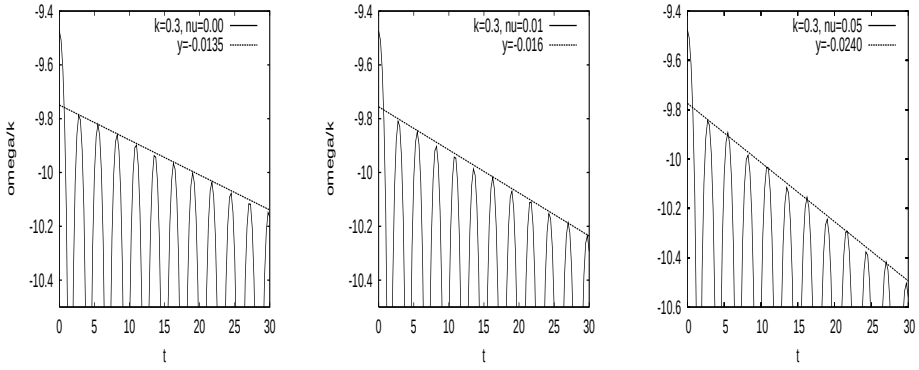
The results of these tests case can be compared to estimated results. In fact, when the amplitude perturbation A is small enough, we can linearize the initial model. The solution of the so-obtained linearized model is a plasma wave the frequency and damping of which can be computed. For the damping rate, one finds $\gamma = \gamma_L + \gamma_C$, where γ_L comes from the Landau damping and γ_C from collisional effects

$$\gamma_L = -\sqrt{\frac{\pi}{8}} \frac{1}{k^3} \exp(-1/(2k^2) - 3/2 - 3k^2), \quad \gamma_C = -\frac{1}{3}\sqrt{\frac{2}{\pi}}\nu \left(1 + \frac{5}{2}k^2 + \frac{4\sqrt{2}}{5Z}k^2 \right). \tag{4.16}$$

One notes that our numerical results are in very good agreement with formulas (4.16), even for subtle phenomenas such as the variation of Z.

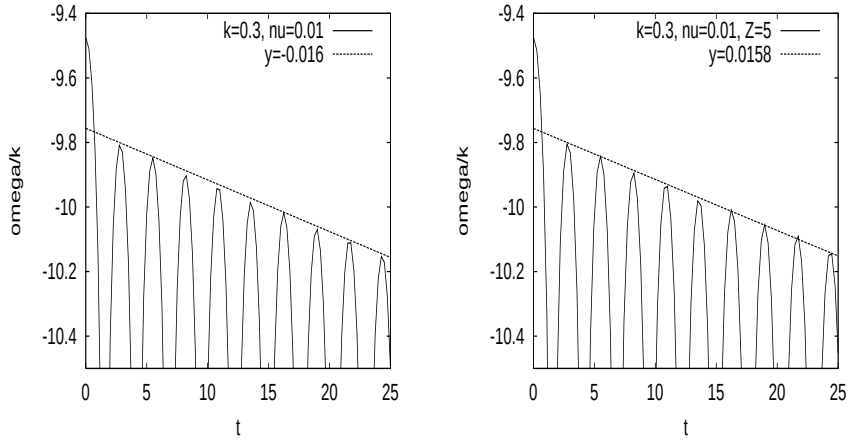
4.2. Ion acoustic waves

The aim of this subsection is the study of the ion acoustic waves (see [4, 7]. To that purpose, we only consider the Vlasov equation coupled with the Fokker-Planck-Landau operator (1.4) which modelizes the ion-ion collisions. The initial condition is the same as (4.14).



(a) (b) (c)

Figure 1: Study of the influence of ν. $k = 0.3$ and $\nu = 0, 0.01, 0.05$.



(a) (b)

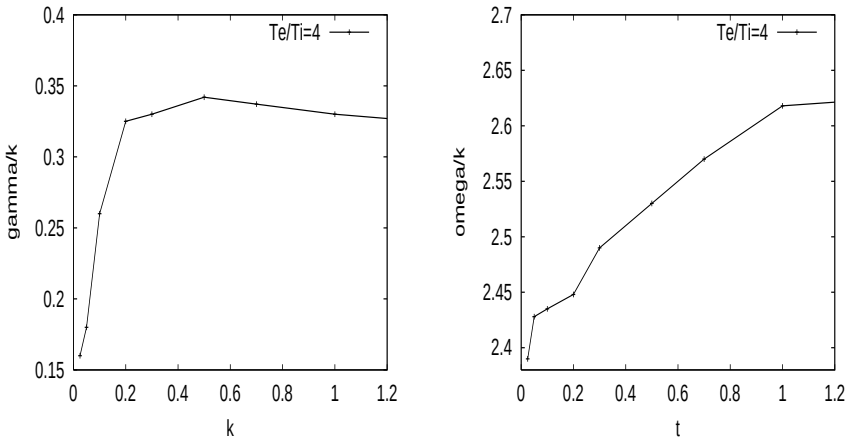
Figure 2: Study of the influence of Z. $k = 0.3$, $\nu = 0.01$ and $Z = 1, 5$.

As the ionic mean free path is bigger than the electron one, we can consider a constant electron temperature; in this case, the electric field is written $E = -T_e/T_i \nabla_x(\log \rho)$, where T_e and T_i are respectively the electron and ion temperature, and ρ the ion density (see [7]). We will take the ratio T_e/T_i equal to 4, $\nu = Z = 1$ and k varies from 0.05 to 3.

As in the previous subsection, we only consider linear regimes and take $A = 10^{-3}$. The size of the velocity domain extends from $-v_{max}$ to $v_{max} = 6$ and use $N_v = 16$ grid points in each direction for the velocity grid whereas the number of spatial grid points is $N_x = 50$. Let us remark that the algorithm presented in section 3 can not be applied in this case. For the sake of simplicity, the electric field E at time t^n at x_i is approximated by $E_i^n = -T_e/T_i D_c(\log \rho^n)_i$ where D_c is the usual discrete centered operator.

We are interested in the evolution of (4.15). Its amplitude is expected to be exponentially and periodically decreasing. Then, we compute numerically the damping rate γ and the frequency ω and compare our results to the results obtained in [7].

On Fig. 3, the quantities $|\gamma|/k$ and ω/k are plotted as a functions of k. One should note that the behavior of our curves is similar to [7]. Indeed, for small k, $|\gamma|/k$ is proportional to k, and for large values of k, $|\gamma|/k$ becomes stabilized and decreases slowly. Concerning the frequency, we observe that ω/k slowly increases before becoming nearly constant for large k, as in [7]. From a quantitative point of view, our results for the damping rate are in very good agreement with the values presented in [7], especially for large k (the collisionless limit ($k \to +\infty$) is well reached). However, we note that our numerical values of ω/k are not in good agreement with [7]. The quantities γ/k and ω/k are reported on 1 and 2.



(a) (b)

Figure 3: Study of the influence of k on the frequency ω and the damping γ.

	k=0.025	k=0.05	k=0.1	k=0.2	k=0.3	k=0.5	k=0.7	k=1
Num.	0.16	0.18	0.26	0.325	0.33	0.342	0.3371	0.33
[7]	0.093	0.165	0.258	0.331	0.344	0.342	0.337	0.33

Table 1: Comparison of γ/k with [7]

	k=0.025	k=0.05	k=0.1	k=0.2	k=0.3	k=0.5	k=0.7	k=1
Num.	2.39	2.428	2.435	2.448	2.49	2.53	2.57	2.618
[7]	2.38	2.39	2.42	2.59	2.66	2.73	2.76	2.78

Table 2: Comparison of ω/k with [7]

5. Conclusion

We have developped a new numerical simulation code for realistic and collisional plasmas in one dimension in the physical space and in three dimensions in velocity space. This code takes into account two different species of particles (*e.g.* electrons and ions). A new discretization of the Vlasov-Poisson equation, from which we proved the important properties of conservation (mass and total energy) has been written. In the applications to Landau damping, our results are in good agreement with the estimations. On the other hand, we accurately recover the results of [7] in the applications of ion acoustic waves.

Acknowledgements: This work was partially performed at the Commissariat à l'énergie atomique (CEA-CESTA). The authors are also grateful to B. Dubroca and V. Tikhonchuk for suggesting the problem and for very fruitful discussions.

References

[1] C. Buet, S. Cordier, P. Degond, M. Lemou, Fast algorithms for numerical, conservative, and entropy approximations of the Fokker-Planck-Landau equation, J. Comput. Phys. **133** (1997), 310–322.

[2] N. Crouseilles, F. Filbet, Numerical approximation of collisional plasmas by high order methods, submitted to J. Comput. Phys. (2004).

[3] P. Degond, B. Lucquin-Desreux, An entropy scheme for the Fokker-Planck collision operator of plasma kinetic theory, Numer. Math. **68** (1994), 239–262.

[4] J. P. Delcroix, A. Bers, Physique des plasmas, Savoirs Actuels, InterEditions, CNRS Editions, (1994).

[5] F. Filbet, L. Pareschi, Numerical method for the accurate solution of the Fokker-Planck-Landau equation in the non homogeneous case, J. Comput. Phys. **179** (2002), 1–26.

[6] F. Filbet, E. Sonnendrücker, P. Bertrand, Conservative Numerical schemes for the Vlasov equation, J. Comput. Phys. **172** (2001), 166–187.

[7] M. Ono, R. M. Kulsrud, Frequency and damping of ion acoustic waves, The Physics of Fluids **10** (1975), 1287–1293.

[8] M. Shoucri, G. Knorr, Numerical integration of the Vlasov equation, J. Comput. Phys. **14** (1974), 84–92.

[9] C. Villani, A review of mathematical topics in collisional kinetic theory, Handbook of Fluid Mechanics, S. Frielander and D. Serre, Editors, (2003).