ANALYSIS OF SPECTRAL METHODS FOR THE HOMOGENEOUS BOLTZMANN EQUATION

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Abstract. The development of accurate and fast algorithms for the Boltzmann collision integral and their analysis represent a challenging problem in scientific computing and numerical analysis. Recently, several works were devoted to the derivation of spectrally accurate schemes for the Boltzmann equation, but very few of them were concerned with the stability analysis of the method. In particular there was no result of stability except when the method is modified in order to enforce the positivity preservation, which destroys the spectral accuracy. In this paper we propose a new method to study the stability of homogeneous Boltzmann equations perturbed by smoothed balanced operators which do not preserve positivity of the distribution. This method takes advantage of the "spreading" property of the collision, together with estimates on regularity and entropy production. As an application we prove stability and convergence of spectral methods for the Boltzmann equation, when the discretization parameter is large enough (with explicit bound).

Keywords. Boltzmann equation; spectral methods; numerical stability; asymptotic stability; Fourier-Galerkin method.

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Contents

1. Introduction

This work is the sequel of several papers devoted to the approximation of the Boltzmann equation using fast spectral methods [27, 11, 21, 14]. The present paper is devoted to the stability and convergence analysis of general spectral algorithms.

In a microscopic description of rarefied gas, the particles move by a constant velocity until they undergo binary collisions. In statistical physics, the properties of the gas are described by a density function in phase space, $f(t,x,v)$, called the *distribution function*, which gives the fraction of particles per unit volume in phase space at time t. The distribution function satisfies the Boltzmann equation, a non-linear integro-differential equation, which describes the combined effect of the free flow and binary collisions between the particles.

The main difficulty in the approximation of the Boltzmann equation is due to the multidimensional structure of the collisional integral, since the integration runs on a highly-dimensional unflat manifold. In addition the numerical integration requires great care since the collision integral is at the basis of the macroscopic properties of the equation. Further difficulties are represented by the presence of stiffness, like the case of small mean free path or the case of large velocities [10].

For such reasons realistic numerical simulations are based on Monte-Carlo techniques. The most famous examples are the Direct Simulation Monte-Carlo (DSMC) methods by Bird [1] and by Nanbu [22]. These methods guarantee efficiency and preservation of the main physical properties. However, avoiding statistical fluctuations in the results becomes extremely expensive in presence of non-stationary flows or close to continuum regimes.

Among deterministic approximations, Discrete Velocity Models (DVM) are based on a Cartesian grid in velocity and on a discrete collision mechanism on the points of the grid that preserves the main physical properties. Unfortunately DVM are not competitive with Monte-Carlo methods in terms of computational cost and their accuracy seems to be less than first order [23, 24, 25, 9].

Another class of numerical methods, based on the use of spectral techniques in the velocity space, has been developed. The methods were first derived in [26], inspired from spectral methods in fluid mechanics [6] and by previous works on the use of Fourier transform techniques for the Boltzmann equation [2]. They are based on approximating in the velocity space the distribution function by a periodic function, and on its representation by Fourier series. The spectral method has been further developed in [27, 29] where evolution equations for the Fourier modes were explicitly derived and spectral accuracy of the method was proven. Strictly speaking these methods are not conservative, since they only preserve mass, whereas momentum and energy are approximated with spectral accuracy. Moreover, the spectral method has been applied also to non homogeneous situations [11], to the Landau equation [10, 28], where fast algorithms can be readily derived, and to the case of granular gases [13]. Independently A. Bobylev & S. Rjasanow [3, 4] have also constructed fast algorithms based on a Fourier transform approximation of the distribution function, but the method is not spectrally accurate (only second order).

In [21] a fast spectral method was proposed for a class of particle interactions including pseudo-Maxwell molecules in dimension 2 and, most importantly, hard spheres in dimension 3, on the basis of the previous spectral method together with a suitable semi-discretization of the collision operator. This method permits to reduce the computational cost from $O(n^2)$ to $O(n \log_2 n)$ without loosing the spectral accuracy, thus making the method competitive with Monte-Carlo.

However an important drawback of the spectral methods up to now had been the lack of proof of stability. Indeed as compared to discrete velocity methods the difficulties are somehow opposite: consistency results are easily obtained, whereas the lack of positivity preservation of the scheme is a major issue when one studies its stability properties. The only paper concerned with the issue of stability for spectral methods applied to the Boltzmann collision operator is [29], but in the latter the author introduce some "filters" on the Fourier modes in order to restore the positivity-preservation of the scheme, which breaks the spectral accuracy.

In this paper we give the first stability result for the spectral methods applied to the Boltzmann collision operator. Moreover we propose a method which is likely to have other utilizations in collisional kinetic theory:

- we write the Galerkin approximation on the first N Fourier modes of the evolution equation as a *a smooth balanced perturbation* of the original equation, in the sense of a

perturbation by some small and mass-preservation (although not positivity-preserving) error term;
- we prove existence and uniqueness of smooth solution for small times, conditionally to a bound on the L^1 norm;
- we use the mixing structure [30, 19] of the collision process to show appearance of positivity after a small time (depending on the size of the box of truncation and the approximation parameter N);
- we use the mass conservation to deduce uniform bounds on the L^1 norm, and therefore regularity bounds growing at most exponentially in time;
- we perform a detailed analysis of the unperturbed truncated problem, showing uniform in time regularity and asymptotic convergence to equilibrium;
- finally we use that the equilibrium is unchanged by the smooth balanced perturbation, and that it is non-linearly stable for the perturbed periodized Boltzmann equation, in order to prove global in time stability and convergence to equilibrium for the perturbed Boltzmann equation (we connect the previous point for initial times together with the stability of equilibrium for asymptotic times).

Hence our paper introduces a general method on how to exploit fine mixing properties of the collision process in the study of stability of a particular class of perturbed Boltzmann equation, with the application in mind to the stability of spectral methods.

The outline of this paper is as follows. The Boltzmann equation and its basic features are presented in Section 2. In Section 3 we explain the truncation and periodization associated with spectral methods and fast spectral methods and we formulate the problem of stability of these methods in the general framework of the stability properties of the Boltzmann equation with respect to a smooth balanced perturbation. Section 4 is devoted to the proof of the main stability result in the general framework. Section 5 is devoted to the study of the asymptotic behavior of the truncated problem on the basis of the entropy production theory. Finally in Section 6 we apply the latter result to the spectral method and establish some stability and convergence results of the numerical solution.

2. The Boltzmann equation

The Boltzmann equation describes the behavior of a dilute gas of particles when the only interactions taken into account are binary elastic collisions. It reads for $x \in \Omega$, $v \in \mathbb{R}^d$ where $\Omega \in \mathbb{R}^d$ is the spatial domain ($d \geq 2$):

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f)$$

where $f := f(t, x, v)$ is the time-dependent particles distribution function in the phase space. The Boltzmann collision operator Q is a quadratic operator local in (t, x). The time t and position x act only as parameters in Q and therefore will be omitted in its description

$$Q(f, f)(v) = \int_{v_\star \in \mathbb{R}^d} \int_{\sigma \in \mathbb{S}^{d-1}} B(|v - v_\star|, \cos\theta) \, \left(f'_\star f' - f_\star f\right) \, d\sigma \, dv_\star. \tag{2.1}$$

We used the shorthand $f = f(v)$, $f_\star = f(v_\star)$, $f' = f(v')$, $f'_\star = f(v'_\star)$. The velocities of the colliding pairs $(v, v_\star)$ and $(v', v'_\star)$ are related by

$$\begin{cases} v' = v - \dfrac{1}{2}\big((v - v_\star) - |v - v_\star| \, \sigma\big), \\ v'_\star = v - \dfrac{1}{2}\big((v - v_\star) + |v - v_\star| \, \sigma\big), \end{cases}$$

with $\sigma \in \mathbb{S}^{d-1}$. The collision kernel B is a non-negative function which by physical arguments of invariance only depends on $|v - v_\star|$ and $\cos\theta = u \cdot \sigma$, where $u = (v - v_\star)/|v - v_\star|$ is the normalized relative velocity.

In this work we are concerned with *short-range interaction* models. More precisely we assume that B is locally integrable. Here are the hypothesis on the collision kernel:

$$B(|u|, \cos\theta) = \Phi(|u|)\, b(\cos\theta), \tag{2.2}$$

with

$$\Phi(z) = z^\gamma, \quad z \in \mathbb{R}_+, \quad \text{for some } \gamma \in (0, +\infty) \tag{2.3}$$

and b smooth such that

$$\int_0^\pi b(\cos\theta) \sin^{d-2}\theta \, d\theta < +\infty. \tag{2.4}$$

These assumptions are satisfied for the so-called *hard spheres model* $B(u) = |u|$, and it is known as *Grad's angular cutoff assumption* when it is (artificially) extended to interactions deriving from a power-law potentials. As an important benchmark model for the numerical simulation we therefore introduce the so-called *variable hard spheres model* (VHS), which writes

$$B(|u|, \cos\theta) = C_\gamma\, |u|^\gamma,$$

for some $\gamma \in (0, 1]$ and a constant $C_\gamma > 0$.

For this class of models, one can split the collision operator as

$$Q(f, f) \,=\, Q^+(f, f) \,-\, L(f)\, f,$$

with

$$Q^+(f, f) \,=\, \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|v - v_\star|, \cos\theta) f' f'_\star \, d\sigma \, dv_\star,$$

and

$$L(f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|v - v_\star|, \cos\theta) f_\star \, d\sigma \, dv_\star.$$

Boltzmann's collision operator has the fundamental properties of conserving mass, momentum and energy: at the formal level

$$\int_{\mathbb{R}^d} Q(f, f)\, \phi(v)\, dv = 0, \qquad \phi(v) = 1, v, |v|^2,$$

and it satisfies well-known Boltzmann's H theorem

$$-\frac{d}{dt} \int_{\mathbb{R}^d} f \log f \, dv = -\int_{\mathbb{R}^d} Q(f, f) \log(f)\, dv \geq 0.$$

The functional $-\int f \log f$ is the *entropy* of the solution. Boltzmann's H theorem implies that any equilibrium distribution function, *i.e.*, any function which is a maximum of the entropy, has the form of a locally Maxwellian distribution

$$M(\rho, u, T)(v) = \frac{\rho}{(2\pi T)^{d/2}} \exp\left(-\frac{|u - v|^2}{2T}\right),$$

where ρ, u, T are the *density*, *macroscopic velocity* and *temperature* of the gas, defined by

$$\rho = \int_{v \in \mathbb{R}^d} f(v)\, dv, \quad u = \frac{1}{\rho} \int_{v \in \mathbb{R}^d} v\, f(v)\, dv, \quad T = \frac{1}{d\rho} \int_{v \in \mathbb{R}^d} |u - v|^2\, f(v)\, dv.$$

For further details on the physical background and derivation of the Boltzmann equation we refer to Cercignani, Illner, Pulvirenti [8] and Villani [31].

3. Formulation of a general stability result

In this section we remind the basic principles leading to the periodized truncations of the Boltzmann collision operator arising in spectral methods. Then, we present the main result of this paper: the stability of the spatially homogeneous Boltzmann equation with respect to a smooth balanced perturbation, preserving mass and smoothness but not non-negativity of the solution. This stability means that we are able to construct global solutions and estimate the error between perturbed and unperturbed solutions.

Any deterministic numerical method requires to work on a *bounded* velocity space. This therefore supposes a non physical truncation (associated with limit conditions) of this velocity space, which we shall discuss below.

3.1. General framework.

We consider the spatially homogeneous Boltzmann equation written in the following general form

$$\frac{\partial f}{\partial t} = Q(f, f), \tag{3.1}$$

where $Q(f, f)$ is given by

$$Q(f, f) = \int_{\mathcal{C}} \mathcal{B}(y, z) \left[f' f'_\star - f_\star f \right] dy\, dz, \quad v \in \mathbb{R}^d \tag{3.2}$$

with

$$v' = v + \Theta'(y, z), \qquad v'_\star = v + \Theta'_\star(y, z), \qquad v_\star = v + \Theta_\star(y, z).$$

In the equations above, $\mathcal{C}$ is some given (unbounded) domain for y, z, and Θ, Θ', $\Theta'_\star$ are suitable functions, to be defined later. This general framework emphasizes the translation invariance property of the collision operator, which is crucial for the spectral methods. We will be more precise in the next paragraphs for some changes of variables allowing to reduce the classical operator (2.1) to the form (3.2).

A problem associated with deterministic methods which use a fixed discretization in the velocity domain is that the velocity space is approximated by a finite region. Physically the domain for the velocity is $\mathbb{R}^d$, and the property of having compact support is not preserved by the collision operator. In general the collision process indeed spreads the support by a factor $\sqrt{2}$ in the elastic case (see [30, 19] and also [18] for similar properties in the inelastic case). As a consequence, for the continuous equation in time, the function f is immediately positive in the whole domain $\mathbb{R}^d$. Thus, at the numerical level, some non physical condition has to be imposed to keep the support uniformly bounded. In order to do this there are two main strategies:

- One can remove the physical binary collisions that will lead outside the bounded velocity domains. This means a possible increase of the number of *local invariants*, *i.e.*, the functions φ such that
$$(\varphi'_\star + \varphi' - \varphi_\star - \varphi)$$
is zero everywhere on the domain. If this is done properly (*i.e.*, without removing too many collisions), the scheme remains conservative and without spurious invariants. However, this truncation breaks down the convolution-like structure of the collision operator, which requires the translation invariance in velocity. Indeed the modified collision kernel depends on v through the boundary conditions. This truncation is the starting point of most schemes based on Discrete Velocity Models.
- One can add some non physical binary collisions by *periodizing* the function and the collision operator. This implies the loss of some local invariants (some non physical collisions are added). Thus the scheme is not conservative anymore, although it still

preserves the mass if the periodization is done carefully. However in this way the structural properties of the collision operator are maintained and thus they can be exploited to derive fast algorithms. This periodization is the basis of spectral methods.

Therefore, we consider the space homogeneous Boltzmann equation in a bounded domain in velocity $\mathcal{D}_L = [-L, L]^d$ with $0 < L < \infty$. We truncate the integration in y and z in (3.2) since periodization would yield infinite result if not: we set y and z to belong to some truncated domain $\mathcal{C}_R \subset \mathcal{C}$ (the parameter R refers to its size and will be defined later). For a compactly supported function with support included in $\mathcal{B}_S$, the ball centered at 0 with radius $S > 0$, one has to prescribe suitable relations (depending on the precise change of variable and truncation chosen) between S, R and L in order to retain all possible collisions and at the same time prevent intersections of the regions where f is different from zero (this is the so-called *dealiasing condition*). Then the *truncated* collision operator reads

$$Q^R(f,f) = \int_{\mathcal{C}_R} \mathcal{B}(y,z) \left(f'_\star f' - f_\star f \right) dy\, dz \tag{3.3}$$

for $v \in \mathcal{D}_L$ (the expression for $v \in \mathbb{R}^d$ is deduced by periodization). By making some changes of variable on v, one can easily prove for the two choices of variables y, z of the next subsections, that for any function φ periodic on $\mathcal{D}_L$ the following weak form is satisfied:

$$\int_{\mathcal{D}_L} Q^R(f,f)\,\varphi(v)\,dv = \frac{1}{4}\int_{\mathcal{D}_L}\int_{\mathcal{C}_R} \mathcal{B}(y,z)\, f_\star f \, \left(\varphi'_\star + \varphi' - \varphi_\star - \varphi\right)\, dy\, dz\, dv. \tag{3.4}$$

Now, we use the representation Q^R to derive spectral methods. Hereafter, we use just one index to denote the d-dimensional sums with respect to the vector $k = (k_1, .., k_d) \in \mathbb{Z}^d$, hence we set

$$\sum_{k=-N}^{N} := \sum_{k_1,\dots,k_d=-N}^{N}.$$

The approximate function f_N is represented as the truncated Fourier series

$$f_N(v) = \sum_{k=-N}^{N} \hat{f}_k\, e^{i\frac{\pi}{L}k\cdot v}, \tag{3.5}$$

with

$$\hat{f}_k = \frac{1}{(2L)^d}\int_{\mathcal{D}_L} f(v)\, e^{-i\frac{\pi}{L}k\cdot v}\, dv.$$

In a Fourier-Galerkin method the fundamental unknowns are the coefficients $\hat{f}_k(t)$, $k = -N, \dots, N$. We obtain a set of ODEs for the coefficients $\hat{f}_k$ by requiring that the residual of (3.3) be orthogonal to all trigonometric polynomials of degree less than N. Hence for $k = -N, \dots, N$

$$\int_{\mathcal{D}_L} \left(\frac{\partial f_N}{\partial t} - Q^R(f_N, f_N) \right) e^{-i\frac{\pi}{L}k\cdot v}\, dv = 0. \tag{3.6}$$

By substituting expression (3.5) in (3.4) we get

$$Q^R(f_N, f_N) = Q^{R,+}(f_N, f_N) - L^R(f_N)\, f_N$$

with

$$L^R(f_N)\, f_N = \sum_{l=-N}^{N} \sum_{m=-N}^{N} \beta(m,m)\, \hat{f}_l\, \hat{f}_m\, e^{i\frac{\pi}{L}(l+m)\cdot v}, \tag{3.7}$$

$$Q^{R,+}(f_N, f_N) = \sum_{l=-N}^{N} \sum_{m=-N}^{N} \beta(l,m)\, \hat{f}_l\, \hat{f}_m\, e^{i\frac{\pi}{L}(l+m)\cdot v}, \tag{3.8}$$

where

$$\beta(l,m) = \int_{\mathcal{C}_R} \mathcal{B}(y,z)\, e^{i\frac{\pi}{L}\left(l\cdot\Theta'(y,z)+m\cdot\Theta'_\star(y,z)\right)}\, dy\, dz. \tag{3.9}$$

The *spectral equation* is the projection of the collision equation in $\mathbb{P}_N$, the $(2N+1)^d$-dimensional vector space of trigonometric polynomials of degree at most N in each direction, *i.e.*,

$$\frac{\partial f_N}{\partial t} = \mathcal{P}_N\, Q^R(f_N, f_N),$$

where $\mathcal{P}_N$ denotes the orthogonal projection on $\mathbb{P}_N$ in $L^2(\mathcal{D}_L)$.

3.2. **The truncation associated with classical spectral methods.** In the classical spectral method [27], a simple change of variables in (2.1) permits to write

$$Q(f,f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \mathcal{B}^{\text{class}}(g,\omega)\big(f(v')f(v'_\star) - f(v)f(v_\star)\big)\, d\omega\, dg, \tag{3.10}$$

with $g = v - v_\star \in \mathbb{R}^d$, $\omega \in \mathbb{S}^{d-1}$, and

$$\begin{cases} v' = v - \frac{1}{2}(g - |g|\omega), \\ v'_\star = v - \frac{1}{2}(g + |g|\omega), \\ v_\star = v + g. \end{cases} \tag{3.11}$$

Then, we set $\mathcal{C} := \mathbb{R}^d \times \mathbb{S}^{d-1}$ and

$$\Theta'(g,\omega) := -\frac{1}{2}(g - |g|\omega), \quad \Theta'_\star(g,\omega) := -\frac{1}{2}(g + |g|\omega), \quad \Theta_\star(g,\omega) := g.$$

Finally the collision kernel $\mathcal{B}^{\text{class}}$ is defined by

$$\mathcal{B}^{\text{class}}(g,\omega) = 2^{d-1}\left(1 - (\hat{g}\cdot\omega)\right)^{d/2-1} B\big(|g|, 2(\hat{g}\cdot\omega)^2 - 1\big). \tag{3.12}$$

Thus, the Boltzmann operator (3.10) is now written in the form (3.2). We consider the bounded domain $\mathcal{D}_L = [-L, L]^d$, for the distribution f, and the bounded domain $\mathcal{C}_R = \mathcal{B}_R \times \mathbb{S}^{d-1}$ for some $R > 0$. The truncated operator reads in this case

$$Q^R(f,f)(v) = \int_{\mathcal{B}_R\times\mathbb{S}^{d-1}} \mathcal{B}^{\text{class}}(g,\omega)\, \big(f(v'_*)f(v') - f(v_*)f(v)\big)\, d\omega\, dg. \tag{3.13}$$

3.3. **The truncation associated with fast spectral methods.** Here we shall approximate the collision operator starting from a representation which conserves more symmetries of the collision operator when one truncates it in a bounded domain. This representation was used in [3, 15] to derive finite differences schemes and it is close to the classical Carleman representation (cf. [7]). The basic identity we shall need is (for $u \in \mathbb{R}^d$)

$$\frac{1}{2}\int_{\mathbb{S}^{d-1}} F(|u|\sigma - u)\, d\sigma = \frac{1}{|u|^{d-2}} \int_{\mathbb{R}^d} \delta(2\, y\cdot u + |y|^2)\, F(y)\, dy. \tag{3.14}$$

Using (3.14) the collision operator (2.1) can be written as

$$Q(f,f)(v) = 2^{d-1}\int_{x\in\mathbb{R}^d}\int_{y\in\mathbb{R}^d} \mathcal{B}^{\text{fast}}(y,z)\,\delta(y\cdot z) \qquad \big(f(v+z)f(v+y) - f(v+y+z)f(v)\big)\,dy\,dz, \tag{3.15}$$

with

$$\mathcal{B}^{\text{fast}}(y,z) = 2^{d-1}\,B\left(|y+z|, -\frac{y\cdot(y+z)}{|y|\,|y+z|}\right)\,|y+z|^{-(d-2)}.$$

Thus, the collision operator is now written in the form (3.2) with $\mathcal{C} := \mathbb{R}^d\times\mathbb{R}^d$,

$$\mathcal{B}(y,z) = \mathcal{B}^{\text{fast}}(y,z)\,\delta(y\cdot z),$$

and

$$v'_\star = v + \Theta'_\star(y,z), \quad v' = v+\Theta'(y,z), \quad v_\star = v+\Theta_\star(y,z).$$

with

$$\Theta'_\star(y,z) := z, \quad \Theta'(y,z) := y, \quad \Theta_\star(y,z) := y+z.$$

Now we consider the bounded domain $\mathcal{D}_L = [-L,L]^d$, $(0<L<\infty)$ for the distribution f, and the bounded domain $\mathcal{C}_R = \mathcal{B}_R\times\mathcal{B}_R$ for some $R>0$. The (truncated) operator now reads

$$Q^R(f,f)(v) = \int_{\mathcal{C}_R} \mathcal{B}^{\text{fast}}(y,z)\,\delta(y\cdot z)\,\big(f(v+z)f(v+y) - f(v+y+z)f(v)\big)\,dy\,dz, \tag{3.16}$$

for $v\in\mathcal{D}_L$. This representation of the collision kernel yields better decoupling properties between the arguments of the operator and allows to lower significantly the computation cost of the method by using the fast Fourier transform (see [21, 14]).

Let us make a crucial remark about the choice of R. When f has support included in $\mathcal{B}_S$, $S>0$, it is usual (see [27, 21]) to search for the minimal period L (in order to minimize the computational cost) which prevents interactions between different periods of f during *one* collision process. From now on, we shall always assume that we can take L and R large enough such that, when needed, $R\geq\sqrt{2}\,L$. Hence all the torus is covered (at least once) in the integration of the variables (g,ω) or (y,z).

3.4. **A common abstract formulation for the stability of spectral methods.** From now on, Q^R shall denote a periodized truncated collision operator as in (3.13) or (3.16). As we shall see, using this formulations, both classical and fast spectral methods fall into the following framework:

$$\begin{cases} \dfrac{\partial f}{\partial t} = Q^R(f,f) + P_\varepsilon(f), \quad v\in\mathcal{D}_L, \\[2ex] f(0,v) = f_{0,\varepsilon}(v), \quad v\in\mathcal{D}_L, \end{cases} \tag{3.17}$$

where P_ε is a "smooth balanced perturbation", which means that it satisfies the following (balanced law)

$$\int_{\mathcal{D}_L} P_\varepsilon(f)\,dv = 0 \tag{3.18}$$

and preserves the smoothness of the distribution function, *i.e.*, there exist constants C_0, $C_k>0$ such that

$$\begin{cases} \|P_\varepsilon(f)\|_{L^1} \;\leq\; C_0\,\|f\|_{L^1}\,\|f\|_{L^1} \\[2ex] \|P_\varepsilon(f)\|_{H^k_{\text{per}}} \;\leq\; C_k\,\|f\|_{L^1}\,\|f\|_{H^k_{\text{per}}}, \quad k\geq 0, \end{cases} \tag{3.19}$$

where $\|\cdot\|_{H^k_{\rm per}}$ is the usual norm of the Sobolev space of periodic functions $H^k_{\rm per}(\mathcal{D}_L)$.

Moreover the perturbation is supposed to be small in the following sense: there exists a function $\varphi(\varepsilon)$ such that for any $p \geq 0$,

$$\|P_\varepsilon(f)\|_{H^p_{\rm per}} \ \leq \ \varphi(\varepsilon), \tag{3.20}$$

where $\varphi(\varepsilon)$ depends on $\|f\|_{H^{p+k}_{\rm per}}$ for some $k > 0$, and goes to zero as ε goes to zero.

Finally in order to prove global existence with uniform regularity bounds, we shall require additional assumptions on the relation between the equilibrium distributions of the perturbed and unperturbed (periodized) Boltzmann equations, and about the stability of the unperturbed equation (see the following statement).

Let us therefore write the unperturbed equation for reference:

$$\left\{ \begin{array}{l} \dfrac{\partial f}{\partial t} = Q^R(f,f), \quad v \in \mathcal{D}_L, \quad t > 0 \\ f(0,v) = f_0(v), \quad v \in \mathcal{D}_L. \end{array} \right. \tag{3.21}$$

Let us state the general stability theorem:

Theorem 3.1. *Let us consider a perturbed Boltzmann equation (3.17) in the torus $\mathcal{D}_L$, where Q^R is defined by (3.13) or (3.16), and for a sequence of smooth balanced perturbations $\big(P_\varepsilon = P_\varepsilon(f)\big)_{\varepsilon>0}$ which satisfy* (3.18)-(3.19)-(3.20).

Assume that the constant functions are equilibria of the perturbed equation (3.17) *(as for equation* (3.21)*) and that they are nonlinearly locally stable in any $H^k_{\rm per}(\mathcal{D}_L)$ for equation* (3.21).

We assume that f_0 is a non-negative function, non zero everywhere, belonging to $H^k_{\rm per}(\mathcal{D}_L)$ with $k \in \mathbb{N}$ and $k > d/2$. We consider a sequence of smooth balanced perturbations $f_{0,\varepsilon}$ of the initial datum for the perturbed problem (3.17) *(non necessarily positive) such that*

$$\int_{\mathcal{D}_L} f_{0,\varepsilon} = \int_{\mathcal{D}_L} f_0 \quad \text{ and } \quad \|f_0 - f_{0,\varepsilon}\|_{H^k_{\rm per}} \ \leq \ \psi(\varepsilon),$$

with $\psi(\varepsilon)$ goes to zero when ε goes to zero.

Then, there exists $\varepsilon_0 > 0$ depending only on the collision kernel B, the truncation R, the constants in (3.19)-(3.20) *for the perturbation, and the $L^1(\mathcal{D}_L)$ and $H^k_{\rm per}(\mathcal{D}_L)$ norms on f_0, such that for any $\varepsilon \in (0, \varepsilon_0)$,*

- *(i) there exists a unique global smooth solution f_ε to* (3.17)*;*
- *(ii) for any $p < k$, this solution belongs to $H^p_{\rm per}(\mathcal{D}_L)$ for all times with uniform bounds as time goes to infinity;*
- *(iii) this solution remains "essentially non-negative" uniformly in time, in the sense that there is $\eta(\varepsilon) > 0$ (with $\eta(\varepsilon) \to 0$ as ε goes to 0) such that the non-positive part is $\eta(\varepsilon)$-small:*
$$\forall\, t \geq 0, \quad \|f_\varepsilon^-(t,\cdot)\|_{L^\infty} \leq \eta(\varepsilon)$$
where f_ε^- denotes $|f_\varepsilon|\,\mathbf{1}_{\{f_\varepsilon \leq 0\}}$;
- *(iv) this solution f_ε converges in $H^p_{\rm per}(\mathcal{D}_L)$ for any $p < k$, uniformly on any $[0,T]$, $T > 0$, to the solution f of the unperturbed equation* (3.21) *when the parameter ε goes to zero;*
- *(v) the solution f_ε to* (3.17) *converges in $H^k_{\rm per}(\mathcal{D}_L)$ as time goes to infinity to the constant equilibrium distribution in the torus prescribed by its mass, and it is "asymptotically uniformly positive", that is for t larger than some fixed explicit time.*

We split the proof into two main steps: first in Section 4 we prove existence, uniqueness and smoothness of a solution on an arbitrary bounded time interval (as the size of perturbation goes to 0). The main difficulty is to prove that non-negativity of the distribution function is

recovered in a certain sense. Then, in Section 5 we study the asymptotic behavior and establish global stability in time. The main issue is there to prove regularizing properties of the gain operator $Q^{R,+}(f,f)$ and entropy production estimates on Q^R. Finally in Section 6 we apply the previous general results to spectral methods, and prove their stability and convergence.

4. Proof of stability on an arbitrary bounded time interval

In this section we first give some technical lemmas and next establish a result showing existence and uniqueness of a smooth solution on an arbitrary time interval to the perturbed equation (3.17), depending on an assumption of smallness on the size of the perturbation. Then, in Lemma 4.5 we prove the control of negative values of $f(t)$.

4.1. **Preliminary results.**

We start this section by a classical result of L^p estimates on the Boltzmann operator $Q^R(g,h)$ given by

$$Q^R(g,h) := \int_{\mathcal{C}_R} \mathcal{B}(y,z)\left(g'_\star\, h' \,-\, g_\star\, h\right) dy\, dz.$$

Lemma 4.1. *Let the collision kernel B satisfy the assumption* (2.2)-(2.3)-(2.4)*. Then, the periodized Boltzmann operator Q^R (defined by (3.13) or (3.16)) satisfies: for all $p \in [0,\infty]$ there exists a constant $C_p(R,B) > 0$ such that*

$$\|Q^R(g,h)\|_{L^p},\ \|Q^R(h,g)\|_{L^p} \ \leq\ C_p(R,B)\,\|g\|_{L^1}\,\|h\|_{L^p}. \tag{4.1}$$

Proof. The proof is exactly similar to the case of the usual Boltzmann collision operator for a collision kernel bounded with compact support, see for instance [20] for a recent proof. □

Now, we prove smoothness of the solution to the perturbed problem (3.17) on a fixed time interval under the assumption of an *a priori* bound on the L^1 norm of the solution.

Lemma 4.2. *Let us consider a collision kernel B which satisfies the assumptions* (2.2)-(2.3)-(2.4) *and a sequence of smooth balanced perturbations $\left(P_\varepsilon = P_\varepsilon(f^\varepsilon)\right)_{\varepsilon>0}$ which satisfy* (3.18)-(3.19)-(3.20)*, and let $T > 0$ be the length of the time interval. Assume that $f_0 \in H^k(\mathcal{D}_L)$ for $k \in \mathbb{N}$ and that $f(t)$ is a (non necessarily positive) solution to* (3.17) *with initial datum f_0, which satisfies the L^1-estimate*

$$\forall\, t \in [0,T], \quad \|f(t)\|_{L^1} \leq M. \tag{4.2}$$

Then, there exists a constant $C_k(M) > 0$, only depending on M, R, T and $\|f_0\|_{H^k_{\rm per}}$ such that

$$\forall\, t \in [0,T], \quad \|f(t)\|_{H^k_{\rm per}} \,\leq\, C_k(M). \tag{4.3}$$

Proof of Lemma 4.2. We proceed by induction on $k \geq 0$. For the first stage $k = 0$, we apply Lemma 4.1 with $p = 2$ and $g = h = f$ and we use assumption (3.19) on the perturbation:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f(t)\|^2_{L^2} \;&\leq\; \|Q^R(f,f) \,+\, P_\varepsilon(f)\|_{L^2}\,\|f(t)\|_{L^2} \\ &\leq\; (C_2(R,B) \,+\, C_0)\,\|f(t)\|_{L^1}\,\|f(t)\|^2_{L^2}.\end{aligned}$$

Hypothesis (4.2) provides a control on $\|f(t)\|_{L^1}$ and we can apply Gronwall's lemma to get the result (4.3) at stage $k = 0$.

Let us now assume that (4.3) holds at stage $k \geq 0$ and let us prove that it also holds at stage $(k+1)$.

Let us first recall a formula on the derivatives of Q^R: from the bilinearity of Q^R and the translation invariance property of the periodized Boltzmann collision operator, one has

$$\nabla_v Q^R(f,f) = Q^R(\nabla_v f, f) + Q^R(f, \nabla_v f),$$

which yields a Leibniz formula at any order $s \in \mathbb{N}$:

$$\begin{aligned} \left\|Q^R(f,f)\right\|^2_{H^s_{\rm per}} &= \sum_{|\nu|\le s} \left\|\partial^\nu Q^R(f,f)\right\|^2_{L^2} \\ &\le C \sum_{|\nu|\le s} \sum_{|\mu|\le|\nu|} \left(\begin{array}{c}\nu\\ \mu\end{array}\right) \left\|Q^R(\partial^\mu f, \partial^{\nu-\mu} f)\right\|^2_{L^2}. \end{aligned} \tag{4.4}$$

Now, using (4.4) with $s = k+1$ we have

$$\left\|Q^R(f,f)\right\|^2_{H^{k+1}_{\rm per}} \le C \left\|Q^R(f,f)\right\|^2_{H^k_{\rm per}} + C \sum_{|\nu|=k+1} \sum_{|\mu|\le k+1} \left(\begin{array}{c}\nu\\ \mu\end{array}\right) \left\|Q^R(\partial^\mu f, \partial^{\nu-\mu} f)\right\|^2_{L^2}. \tag{4.5}$$

From Lemma 4.1 with $p = 2$ and $g = \partial^\mu f$, $h = \partial^{\nu-\mu} f$ together with the hypothesis (4.3), we get

$$\left\|Q^R(f,f)\right\|_{H^k_{\rm per}} \le C(k,R) \|f\|_{H^k_{\rm per}} \|f\|_{H^k_{\rm per}} \le C_2(k,R,B)\, C_k(M)^2. \tag{4.6}$$

Then we split the last term of (4.5) in two parts for $\mu \neq 0$ and $\mu = 0$. For $\mu \neq 0$, we again apply Lemma 4.1 with $p = 2$ and $g = \partial^\mu f$, $h = \partial^{\nu-\mu} f$ and use the fact that both derivatives $|\mu| = \sum_{i=1}^d |\mu_i| \le k$ and $|\nu - \mu| \le k$:

$$\sum_{|\nu|=k+1} \sum_{\substack{|\mu|\le k+1\\ \mu\neq 0}} \left(\begin{array}{c}\nu\\ \mu\end{array}\right) \left\|Q^R(\partial^\mu f, \partial^{\nu-\mu} f)\right\|^2_{L^2} \le C_3(k,R,B)\, C_k(M)^2. \tag{4.7}$$

Finally, for $\mu = 0$ we apply Lemma 4.1 with $p = 2$ and $g = f$, $h = \partial^\nu f$:

$$\sum_{|\nu|=k+1} \left\|Q^R(f, \partial^\nu f)\right\|^2_{L^2} \le C_4(k,R,B) \|f\|_{L^1} \|f\|_{H^{k+1}_{\rm per}}. \tag{4.8}$$

Then, gathering inequalities from (4.6) to (4.8) and using the assumption (3.19) on the smooth balanced perturbation P_ε, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|f\|^2_{H^{k+1}_{\rm per}} &\le \left\|Q^R(f,f) + P_\varepsilon(f)\right\|_{H^{k+1}_{\rm per}} \|f\|_{H^{k+1}_{\rm per}} \\ &\le C\,(C_2+C_3)\, C_k(M)^2 \|f\|_{H^{k+1}_{\rm per}} + C\,(C_4+C)\, \|f\|_{L^1} \|f\|^2_{H^{k+1}_{\rm per}}. \end{aligned}$$

Finally using the control (4.2) on $\|f(t)\|_{L^1}$ we apply Gronwall's lemma to get (4.3) at stage $k+1$: there exists a constant $C_{k+1}(M)$, only depending on M, R, T and $\|f_0\|_{H^{k+1}_{\rm per}}$ such that

$$\forall\, t \in [0,T], \quad \|f(t)\|_{H^{k+1}_{\rm per}} \le C_{k+1}(M).$$

□

Then, we establish existence and uniqueness of a smooth solution for the perturbed problem (3.17) on a small time interval $[0, \bar\tau]$, $\bar\tau > 0$.

Proposition 4.3. *Let us consider a collision kernel B which satisfies the assumptions* (2.2)-(2.3)-(2.4) *and a sequence of smooth balanced perturbations $\big(P_\varepsilon = P_\varepsilon(f)\big)_{\varepsilon>0}$ which satisfy* (3.18)-(3.19)-(3.20). *We assume that $f_0 \in H^k_{\rm per}(\mathcal{D}_L)$, for $k \in \mathbb{N}$ and set*

$$M = 2\,\|f_0\|_{L^1}. \tag{4.9}$$

Then, there exists $\bar\tau = \bar\tau(M) > 0$ such that for all $\varepsilon > 0$ the perturbed Boltzmann equation (3.17) *admits a unique solution (non necessarily positive) on the time interval $[0, \bar\tau]$, where the solution f satisfies*

$$\forall\, t \in [0,\bar\tau], \quad \|f(t)\|_{L^1} \le M. \tag{4.10}$$

Moreover, there exists a constant $C_k(M) > 0$, only depending on M, R, and $\|f_0\|_{H^k_{\rm per}}$ such that

$$\forall\, t \in [0, \bar\tau], \quad \|f(t)\|_{H^k_{\rm per}} \;\leq\; C_k(M). \tag{4.11}$$

Proof of Proposition 4.3. First, we apply Lemma 4.1 with $p = 1$ and $g = h = f$

$$\left\|Q^R(f,f)\right\|_{L^1} \leq C_1(R,B)\,\|f\|_{L^1}\,\|f\|_{L^1}.$$

Moreover, using assumption (3.19) on the perturbation, there exists $C > 0$ such that for all $\varepsilon > 0$

$$\|P_\varepsilon(f)\|_{L^1} \leq C\,\|f\|^2_{L^1}.$$

Therefore, we obtain a constant $C > 0$, only depending on R and the collision kernel B such that

$$\frac{d}{dt}\|f(t)\|_{L^1} \leq C\,\|f\|^2_{L^1}.$$

This implies that

$$\|f(t)\|_{L^1} \;\leq\; \frac{\|f_0\|_{L^1}}{1 \,-\, C\,\|f_0\|_{L^1}\,t}.$$

Now, setting $M = 2\,\|f_0\|_{L^1}$ and from the latter inequality, we show that there exists $\bar\tau < 1/(2\,C\,\|f_0\|_{L^1})$ such that

$$\forall\, t \,\in\, [0, \bar\tau], \quad \|f(t)\|_{L^1} \leq M,$$

which gives (4.10).

From the estimate in $L^1(\mathcal{D}_L)$ on the function $f(t)$ on the time interval $[0, \bar\tau]$, we prove existence and uniqueness of a solution by Cauchy-Lipschitz theorem in $L^1(\mathcal{D}_L)$ (because of the truncation on $\mathcal{D}_L$ the collision kernel is a bounded bilinear function from $L^1(\mathcal{D}_L) \times L^1(\mathcal{D}_L)$ to $L^1(\mathcal{D}_L)$). Finally, from the bound (4.10) and the smoothness assumption $f_0 \in H^k_{\rm per}(\mathcal{D}_L)$ on the initial datum, we are able to apply Lemma 4.2, which proves that there exists a constant $C_k(M) > 0$, only depending on M, R, T and $\|f_0\|_{H^k_{\rm per}}$ such that

$$\|f(t)\|_{H^k_{\rm per}} \;\leq\; C_k(M).$$

This concludes the proof. □

By iterating Proposition 4.3 and Lemma 4.2, we observe that uniform control on the $L^1(\mathcal{D}_L)$ norm on an arbitrarily large time interval $[0, T]$ together with smoothness on the initial datum will ensure existence and uniqueness of a smooth solution on this time interval $[0, T]$. Furthermore we observe that the control on the $L^1(\mathcal{D}_L)$ norm is obvious for the classical Boltzmann equation thanks to the positivity and mass preservations. Therefore, we shall now focus on the control of positivity of the solution, showing that the solution remains "almost positive" for arbitrarily large time interval, hence allowing to produce uniform control on the $L^1(\mathcal{D}_L)$ norm for arbitrarily large time interval.

We first state a technical lemma which takes advantage of the mixing property of $Q^{R,+}$ in order to show spreading of the support of a characteristic function of a ball.

Lemma 4.4. *Let us consider a collision kernel B which satisfies the assumptions* (2.2)-(2.3)-(2.4) *and a truncated operator Q^R defined by (3.13) or (3.16). Then for all $0 < r < \sqrt{2}\,L$, we have*

$$Q^{R,+}(\mathbf{1}_{\mathcal{B}(v,r)}, \mathbf{1}_{\mathcal{B}(v,r)}) \geq C_0\,\mathbf{1}_{\mathcal{B}(v,\mu\,r)}$$

for some explicit $\mu = \mu(R, L) > 1$ and $C_0 > 0$.

Remark: Note that for $r \geq \sqrt{2}\,L$, one has $\mathbf{1}_{\mathcal{B}(v,r)} = 1$ on the torus $[-L, L]^d$ and there is nothing to prove.

Proof of Lemma 4.4. The invariance by translation allows to reduce the proof to the case $v = 0$. The invariance by rotations implies that $I_r := Q^{R,+}(\mathbf{1}_{\mathcal{B}(0,r)}, \mathbf{1}_{\mathcal{B}(0,r)})$ is radially symmetric. More precisely, taking a $\mathcal{C}^\infty$ radially symmetric function ϕ such that $\phi > 0$ on $\mathcal{B} = \mathcal{B}(0,r)$ and $\phi \leq \mathbf{1}_{\mathcal{B}}$ on $\mathbb{R}^d$, we have

- the function $v \longrightarrow Q^{R,+}(\phi,\phi)(v)$ is continuous,
- for all $v \in \mathbb{R}^d$, $I_r(v) \geq Q^+(\phi,\phi)(v)$,
- for all $v \in \mathcal{B}$, $Q^{R,+}(\phi,\phi)(v) > 0$.

As a consequence, for any ball $\mathcal{B}' = \mathcal{B}(0,r')$ strictly included in $\mathcal{B}$, there exists $\kappa_{r'} > 0$ such that $I_r \geq \kappa_{r'}\, \mathbf{1}_{\mathcal{B}'}$.

In order to conclude, we just need to estimate the support of I_r close to the ball $\mathcal{B}$.

Let us fix $r' \in (0,r)$ and choose $v', v'_\star \in \mathcal{B}'$ such that $|v'| = |v'_\star| = r'$ and $|v' - v'_\star| = \min\{R; \sqrt{2}r'\}$. Then taking σ in $\mathcal{B}'$ such that $v, v_\star, v', v'_\star$ is a square, we find $|v| = \sqrt{2}\, r'$ if $\sqrt{2}\, r' \leq R$ and

$$|v| = \left[(r')^2 - \frac{R^2}{4}\right]^{1/2} + \frac{R}{2}$$

else. We define

$$\mu_0 = \min_{R/\sqrt{2} \leq r' \leq \sqrt{2}\, L} \left[1 - \frac{R^2}{4(r')^2}\right]^{1/2} + \frac{R}{2r'} = \min_{R/(2\sqrt{2}L) \leq y \leq 1/\sqrt{2}} \left(\sqrt{1-y^2} + y\right) \in (1, \sqrt{2}).$$

This concludes the proof: we deduce that for any v such that $|v| \in (r', \mu_0\, r')$, we have

$$I_r(v) \geq Q^+(\phi,\phi)(v) > 0$$

since ϕ is strictly positive in the neighborhood of the $v', v'_\star$ associated to v constructed above. Hence we deduce by taking $r' < r$ close to r that for any $0 < \mu < \mu_0$, we have

$$I_r \geq C(\mu, r)\, \mathbf{1}_{\mathcal{B}(0,\mu\, r)}$$

for some constant $C(\mu, r) > 0$ depending continuously on r. We can choose $\mu = (1 + \mu_0)/2$ for instance, and, for this choice of μ, we take

$$C_0 \;=\; \min_{0 \leq r \leq \sqrt{2}} C(\mu, r) \;>\; 0.$$

□

Finally we establish the following positivity result on the solution to the perturbed problem (3.17):

Lemma 4.5. *Let us consider a collision kernel B which satisfies the assumptions* (2.2)-(2.3)-(2.4)*, a truncated operator Q^R defined by (3.13) or (3.16), and a sequence of smooth balanced perturbations $\left(P_\varepsilon = P_\varepsilon(f)\right)_{\varepsilon > 0}$ which satisfy* (3.18)-(3.19)-(3.20)*.*

We assume that f_0 is a non-negative function such that $f_0 \in H^k_{\rm per}(\mathcal{D}_L)$ with $k \in \mathbb{N}$ and $k > d/2$. Moreover, we define $M = 2\, \|f_0\|_{L^1}$ and $f_{0,\varepsilon}$ a smooth balanced perturbation of f_0, which is non necessarily positive and such that

$$\int_{\mathcal{D}_L} f_{0,\varepsilon} = \int_{\mathcal{D}_L} f_0 \quad \text{ and } \quad \|f_0 - f_{0,\varepsilon}\|_{H^k_{\rm per}} \;\leq\; \psi(\varepsilon)$$

where $\psi(\varepsilon)$ goes to zero when ε goes to zero. We also set $\bar\tau, C_k(M) > 0$ the constant defined in Lemma 4.2 such that from Proposition 4.3 we have

$$\forall\, t \in [0, \bar\tau], \quad \|f(t)\|_{H^k_{\rm per}} \;\leq\; C_k(M).$$

Then, there exists $\hat\tau \in (0, \bar\tau)$ which only depends on M, R and the collision kernel B, and there exists $\hat\varepsilon > 0$ which only depends on $\hat\tau$, $C_k(M)$, such that for all ε such that $0 < \varepsilon < \hat\varepsilon$ and

for any smooth solution $H^k_{\rm per}(\mathcal{D}_L)$ to the perturbed Boltzmann equation (3.17) *with perturbed initial datum $f_{0,\varepsilon}$, we have*

$$\forall\, v \in \mathcal{D}_L, \quad f_\varepsilon(\hat\tau, v) > 0.$$

Moreover there exists $\eta(\varepsilon)$, which goes to 0 as ε goes to zero, such that the non-positive part of f satisfies

$$\|f^-(t)\|_{L^\infty} \,\le\, \eta(\varepsilon), \quad t \in [0, \hat\tau], \tag{4.12}$$

Proof of Lemma 4.5. Let $\bar\tau > 0$ be the length of the time interval for which there exists a smooth solution to the perturbed Boltzmann equation (3.17) with perturbed initial datum $f_{0,\varepsilon}$ such that (in the following we omit the subscript ε for the solution)

$$\|f(t)\|_{H^k_{\rm per}} \le C_k(M), \quad t \,\in\, [0, \bar\tau].$$

We split the proof into three steps: first, we give a classical estimate on the loss term $L^R(f)$, second we establish an estimate of f^- with respect to the amplitude of the perturbation $P_\varepsilon(f)$, and third we use the spreading properties of $Q^{R,+}(f,f)$ to prove that there exists $\hat\tau \in (0, \bar\tau)$ such that

$$f(\hat\tau, v) > 0, \qquad v \in \mathcal{D}_L.$$

Step 1. Applying Proposition 4.3 for the control of $\|f(t)\|_{L^1}$ on the time interval $[0, \bar\tau]$, we get

$$\|L^R(f)\|_{L^\infty} \,\le\, C(R,B)\,\|f\|_{L^1} \,\le\, C(R,B)\,M, \tag{4.13}$$

which gives for all $t \,\in\, [0, \tau_0]$, with $\tau_0 = \min\{\bar\tau, \ln 2/(M\,C(R,B))\}$

$$2 \,\ge e^{M\,C(R,B)\,t} \ge\, e^{-\int_0^t L^R(f(s))\,ds} \,\ge\, e^{-\,M\,C(R,B)\,t} \,\ge\, \frac{1}{2}. \tag{4.14}$$

Step 2. Let us split f as $f = f^+ - f^-$, with $f^\pm = \max\{0, \pm f\}$ and use the monotonicity of $Q^{R,+}$ for nonnegative distribution functions; it follows that

$$\begin{aligned} Q^{R,+}(f,f) \;&=\; Q^{R,+}\left(f^+ - f^-, f^+ - f^-\right) \\ &\ge\; -\left[Q^{R,+}\left(f^+, f^-\right) + Q^{R,+}\left(f^-, f^+\right)\right]. \end{aligned} \tag{4.15}$$

On the time interval $[0, \tau_0]$, we apply Proposition 4.3 to estimate $\|f(t)\|_{L^1}$ and since $\|f^+(t)\|_{L^1} \le \|f(t)\|_{L^1}$; we get from Lemma 4.1

$$\|Q^{R,+}\left(f^+, f^-\right)\|_{L^\infty},\ \|Q^{R,+}\left(f^-, f^+\right)\|_{L^\infty} \,\le\, C_\infty(R,B)\,M\,\|f^-(t)\|_{L^\infty}, \tag{4.16}$$

which yields using (4.15)

$$Q^{R,+}(f,f) \,\ge\, -2\,C_\infty(R,B)\,M\,\|f^-(t)\|_{L^\infty}. \tag{4.17}$$

Thus from the Duhamel representation of the solution f, we have for $v \in \mathcal{D}_L$,

$$\begin{aligned} f(t,v) \;&=\; f_{0,\varepsilon}(v)\,e^{-\int_0^t L^R(f(s))(v)\,ds} \\ &\quad + \int_0^t \left[Q^{R,+}(f(s),f(s)) + P_\varepsilon(f(s))\right](v)\,e^{-\int_s^t L^R(f(u))(v)\,du}\,ds, \\ &\ge\; -2\,\psi(\varepsilon) - \int_0^t \left[Q^{R,+}(f(s),f(s)) + P_\varepsilon(f(s))\right](v)\,e^{-\int_s^t L^R(f(u))(v)\,du}\,ds. \end{aligned}$$

Hence, we get from the lower estimate (4.17) of $Q^{R,+}(f,f)$ and the smallness assumption (3.20) of the perturbation $P_\varepsilon(f)$, for all $v \in \mathcal{D}_L$

$$f^-(t,v) \,=\, \max\{0, -f(t,v)\} \,\le\, 2\,\psi(\varepsilon) + 2\int_0^t \left(2\,C_\infty(R,B)\,M\,\|f^-(s)\|_{L^\infty} + \varphi(\varepsilon)\right)\,ds.$$

Finally, we take the supremum in $v \in \mathcal{D}_L$ and apply Gronwall's lemma to get for any $0 \leq \tau \leq \tau_0$

$$\|f^-(t)\|_{L^\infty} \leq \big(2\,\psi(\varepsilon) + 2\,\tau_0\,\varphi(\varepsilon)\big)\, e^{4\,C_\infty(R,B)\,M\,t} := \eta(\varepsilon), \quad t \in [0,\tau_0],$$

which proves (4.12) on the time interval $[0,\tau_0]$.

Step 3. Let us prove that there exists $\hat\tau \in (0,\tau_0)$ such that

$$f(\hat\tau, v) > 0, \quad v \in \mathcal{D}_L.$$

We start again with the Duhamel representation of the solution

$$\begin{aligned} f(t,v) &= f_{0,\varepsilon}(v)\, e^{-\int_0^t L^R(f(s))(v)\,ds} \\ &\quad + \int_0^t \left[Q^{R,+}(f(s),f(s)) + P_\varepsilon(f(s))\right](v)\, e^{-\int_s^t L^R(f(u))(v)\,du}\,ds, \end{aligned}$$

but we now take into account the fact that the first term is essentially positive and we use the spreading property of the operator $Q^{R,+}$ (Lemma 4.4).

On the one hand since the initial datum is smooth enough ($k > d/2$ is large enough such that $f_{0,\varepsilon}$ is Hölder), there exists an explicit $\delta > 0$ depending on $C_k(M)$ such that for ε small enough, there exists $v_0 \in \mathcal{D}_L$ such that

$$f_{0,\varepsilon}(v) \geq \frac{\eta}{2}\,\mathbf{1}_{\mathcal{B}(v_0,\delta)}(v) - \psi(\varepsilon), \quad \text{with } \eta = \frac{\|f_0\|_{L^1}}{(2L)^d}.$$

On the other hand, using the lower bound on the gain operator (4.17) and the estimate (4.12) of $f^-(t)$, it gives

$$Q^{R,+}(f,f) \geq -2\,C_\infty(R,B)\,M\,\eta(\varepsilon)$$

Finally, using (4.14) and the smallness assumption on $P_\varepsilon(f)$, it first yields for any $0 < \tau < \tau_0$

$$f(t,v) \geq A_0\,\mathbf{1}_{\mathcal{B}(v_0,\delta)}(v) - \varphi_1(\varepsilon),$$

with

$$A_0 := \frac{\eta}{4}, \qquad \varphi_0(\varepsilon) := 2\,\tau_0\left(2\,C_\infty(R,B)\,M\,\eta(\varepsilon) + \varphi(\varepsilon)\right) + \psi(\varepsilon). \tag{4.18}$$

Now we choose $\varepsilon_1 > 0$ small enough such that

$$0 < \frac{A_0}{2} \leq A_0 - \varphi_0(\varepsilon_1) \quad \Longleftrightarrow \quad \varphi_0(\varepsilon_1) \leq \frac{A_0}{2}.$$

Thus, we get on any time interval $[0,\tau] \subset [0,\tau_0]$

$$f^+(t,v) \geq \frac{A_0}{2}\,\mathbf{1}_{\mathcal{B}(v_0,\delta)}(v).$$

Hence, using the spreading properties of the operator $Q^{R,+}(f,f)$ of Lemma 4.4 and the monotonicity of $Q^{R,+}$ for nonnegative distribution functions, it follows that

$$\begin{aligned} Q^{R,+}(f^+,f^+) &\geq \frac{A_0^2}{4}\,Q^{R,+}\big(\mathbf{1}_{\mathcal{B}(v_0,\delta)}, \mathbf{1}_{\mathcal{B}(v_0,\delta)}\big) \\ &\geq \frac{A_0^2}{4}\,C_0\,\mathbf{1}_{\mathcal{B}(v_0,\mu\,\delta)}. \end{aligned}$$

Next, we again use the uniform bounds previously established in (4.16) on $Q^{R,+}(f^+,f^-)$ and $Q^{R,+}(f^-,f^+)$:

$$\begin{aligned} Q^{R,+}(f,f) &\geq Q^{R,+}(f^+,f^+) - Q^{R,+}(f^-,f^+) - Q^{R,+}(f^+,f^-) \\ &\geq \frac{A_0^2}{4}\,C_0\,\mathbf{1}_{\mathcal{B}(v_0,\mu\,\delta)} - 2\,C_\infty(R,B)\,M\,\big(\eta(\varepsilon) + \varphi_0(\varepsilon)\big). \end{aligned}$$

Finally, from the smallness assumption (3.20) of the perturbation $P_\varepsilon(f)$, it yields for $t \in [\tau/2, \tau]$ and with (4.14)

$$\begin{aligned} f(t,v) &\geq -\psi(\varepsilon) + \int_0^t \left[Q^{R,+}(f(s),f(s)) + P_\varepsilon(f(s))\right](v)\, e^{-\int_s^t L^R(f(u))(v)\,du}\,ds, \\ &\geq \frac{\tau}{2}\frac{A_0^2}{4} C_0\, \mathbf{1}_{\mathcal{B}(v_0,\mu\,\delta)} - 2\,\tau\,\left[2\,C_\infty(R,B)\,M\left(\eta(\varepsilon)+\varphi_0(\varepsilon)\right) + \varphi(\varepsilon)\right] - \psi(\varepsilon) \\ &= A_1\,\mathbf{1}_{\mathcal{B}(v_0,\mu\,\delta)} - \varphi_1(\varepsilon) \end{aligned}$$

with

$$A_1 = \frac{\tau}{8} A_0^2\, C_0, \qquad \varphi_1(\varepsilon) := 2\,\tau_0\,\left[2\,C_\infty(R,B)\,M\left(\eta(\varepsilon)+\varphi_0(\varepsilon)\right) + \varphi(\varepsilon)\right] - \psi(\varepsilon).$$

Now, we proceed by induction: assume that there exists $(A_j, \varepsilon_j, \varphi_j)$ such that on the time interval $[\tau - \tau/2^j, \tau] \subset [0, \tau_0]$ and for $\varepsilon \in (0, \varepsilon_j)$, we have

$$f(t,v) \;\geq\; A_j\;\mathbf{1}_{\mathcal{B}(v_0,\mu^j\,\delta)}(v) \;-\; \varphi_j(\varepsilon),$$

where

$$A_j := \left(\frac{\tau}{8}\right)^{2^j-1} A_0^{2^j}\, C_0^{2^{j-1}-1}$$

and $\varphi_j(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Using the same method as before, we first set ε_{j+1} such that

$$f^+(t,v) \;\geq\; \frac{A_j}{2}\mathbf{1}_{\mathcal{B}(v_0,\mu^j\,\delta)}$$

and prove that for $t \in [\tau - \frac{\tau}{2^{j+1}}, \tau]$

$$Q^{R,+}(f,f) \;\geq\; \left(\frac{A_j}{2}\right)^2 C_0\,\mathbf{1}_{\mathcal{B}(v_0,\mu^{j+1}\,\delta)} - 2\,C_\infty(R,B)\,M\left(\eta(\varepsilon)+\varphi_j(\varepsilon)\right).$$

Then, from the Duhamel formula and the smallness assumption of $P_\varepsilon(f)$, we finally get the following lower bound

$$f(t,v) \;\geq\; A_{j+1}\,\mathbf{1}_{\mathcal{B}(v_0,\mu^{j+1}\,\delta)} \;-\; \varphi_{j+1}(\varepsilon),$$

with

$$A_{j+1} = \frac{\tau}{8} A_j^2\, C_0, \qquad \varphi_{j+1}(\varepsilon) := 2\,\tau_0\,\left[2\,C_\infty(R,B)\,M\left(\eta(\varepsilon)+\varphi_j(\varepsilon)\right) + \varphi(\varepsilon)\right] - \psi(\varepsilon).$$

Since $\mu > 1$, the ball $\mathcal{B}(v_0, \mu^j\,\delta)$ eventually recovers the periodic box $[-L, L]^d$ *i.e.*, for some J large enough: $[-L, L]^d \subset \mathcal{B}(v_0, \mu^J\,\delta)$, and for all $t \in [\tau - \frac{\tau}{2^J}, \tau]$, by applying J's times the previous induction we get for $\varepsilon \in (0, \varepsilon_J)$:

$$f(t,v) \;\geq\; A_J\,\mathbf{1}_{\mathcal{B}(v_0,\mu^J\,\delta)}(v) \;-\; \varphi_J(\varepsilon).$$

Finally, up to reducing ε further, we have proved that there exists $(\hat\tau, \hat\varepsilon)$ which only depend on the collision kernel B, the initial datum f_0, L and the perturbation function $\varphi = \varphi(\varepsilon)$ such that for all $0 < \varepsilon < \hat\varepsilon$,

$$\forall\, v \in \mathcal{D}_L, \quad f(\hat\tau, v) > 0.$$

□

4.2. **Existence and regularity on a bounded time interval.**

Proposition 4.6. *Let us consider a fixed time $T > 0$, a collision kernel B which satisfies the assumptions* (2.2)-(2.3)-(2.4)*, a truncation Q^R defined by (3.13) or (3.16), and a sequence of smooth balanced perturbations $\big(P_\varepsilon = P_\varepsilon(f)\big)_{\varepsilon>0}$ which satisfy* (3.18)-(3.19)-(3.20)*.*

We assume that f_0 is a non-negative function, not zero everywhere, and such that $f_0 \in H^k_{\rm per}(\mathcal{D}_L)$ with $k \in \mathbb{N}$ and $k > d/2$. We define $M = 2\,\|f_0\|_{L^1}$ and $(f_{0,\varepsilon})_{\varepsilon>0}$ a sequence of smooth perturbations of f_0 (which is non necessarily positive) such that

$$\int_{\mathcal{D}_L} f_{0,\varepsilon} = \int_{\mathcal{D}_L} f_0 \quad \text{and} \quad \|f_0 - f_{0,\varepsilon}\|_{H^k_{\rm per}} \;\leq\; \psi(\varepsilon)$$

where $\psi(\varepsilon)$ goes to zero when ε goes to zero.

Then, there exists $\hat\varepsilon > 0$, which only depends on the $H^k_{\rm per}(\mathcal{D}_L)$ and $L^1(\mathcal{D}_L)$ norms of f_0, such that for all $\varepsilon \in (0, \hat\varepsilon)$,

- *(i) there is a unique smooth solution $f_\varepsilon = f_\varepsilon(t,\cdot)$ on $[0,T]$ to the perturbed equation* (3.17) *with initial datum $f_{0,\varepsilon}$;*
- *(ii) this belongs to $H^k_{\rm per}(\mathcal{D}_L)$ (with bound growing at most exponentially);*
- *(iii) there is some explicit $\eta(\varepsilon) > 0$ (with $\eta(\varepsilon) \to 0$ as ε goes to 0) such that the non-positive part is $\eta(\varepsilon)$-small:*

$$\forall\, t \in [0,T], \quad \|f_\varepsilon^-(t,\cdot)\|_{L^\infty} \leq \eta(\varepsilon)$$

where f_ε^- denotes $\max\{0, -f_\varepsilon\}$;

- *(iv) this solution satisfies for any $p < k$*

$$\forall\, t \in [0,T], \quad \|f(t,\cdot) - f_\varepsilon(t,\cdot)\|_{H^p_{\rm per}} \;\leq\; \bar\varphi(\varepsilon)$$

where $f(t,\cdot)$ is the solution of unperturbed periodized Boltzmann equation (3.21), and $\bar\varphi(\varepsilon)$ is another explicit function which goes to zero as ε goes to zero. Hence up to reducing ε, the perturbed solution remains close to the unperturbed solution on the finite time interval on which we have constructed it.

Proof. We set

$$M_k(T) := \|f_0\|_{H^k_{\rm per}}\, e^{C\,M\,T}. \tag{4.19}$$

First, applying Proposition 4.3 we have proven that there exists a small $\bar\tau > 0$ such that the perturbed Boltzmann equation (3.17) admits an unique smooth solution on the time interval $[0, \bar\tau]$ with

$$\|f_\varepsilon(t)\|_{L^1} \;\leq\; M$$

and

$$\|f_\varepsilon(t)\|_{H^k_{\rm per}} \;\leq\; M_k(\bar\tau) \;\leq M_k(T).$$

Moreover from Lemma 4.5, there exist $\hat\tau \leq \bar\tau$ and $\hat\varepsilon > 0$, only depending on M, $M_k(\bar\tau) < M_k(T)$, R and the collision kernel B such that for all $0 < \varepsilon < \hat\varepsilon$,

$$\forall\, v \in \mathcal{D}_L, \quad f_\varepsilon(\hat\tau, v) > 0,$$
$$\|f^-(t)\|_{L^\infty} \;\leq\; \eta(\varepsilon), \quad t \in [0, \hat\tau].$$

and

$$\forall\, t \in [0, \hat\tau], \quad \|f(t)\|_{H^k_{\rm per}} \leq \|f_0\|_{H^k_{\rm per}}\, e^{C\,M\,2\,\hat\tau}.$$

Then, from the preservation of mass under the action of Q^R and P_ε:

$$\int_{\mathcal{D}_L} Q^R(f_\varepsilon, f_\varepsilon)(v)\,dv \;=\; \int_{\mathcal{D}_L} P_\varepsilon(f_\varepsilon)(v)\,dv \;=\; 0,$$

we have that

$$\int_{\mathcal{D}_L} f_\varepsilon(\hat\tau, v)\, dv \;=\; \int_{\mathcal{D}_L} f_0(v)\, dv.$$

Since f_0 is a nonnegative function, it gives that at time $\hat\tau$

$$\|f_\varepsilon(\hat\tau)\|_{L^1} \;=\; \|f_0\|_{L^1},$$

and on the time interval $t \in [0, \hat\tau]$ we have

$$\|f_\varepsilon(t)\|_{L^1} \;\le\; M \quad \text{and} \quad \|f_\varepsilon(t)\|_{H^k_{\rm per}} \;\le\; M_k(T).$$

Therefore, we consider the perturbed Boltzmann equation (3.17) starting from $f_\varepsilon(\hat\tau)$ as initial data. On the time interval $[\hat\tau, 2\,\hat\tau]$, we apply Proposition 4.3 and get that

$$\|f_\varepsilon(t)\|_{L^1} \;\le\; 2\,\|f_\varepsilon(\hat\tau)\|_{L^1} \;=\; 2\,\|f_0\|_{L^1} \;=\; M, \quad \forall\, t \,\in\, [\hat\tau, 2\,\hat\tau]$$

and

$$\begin{aligned}
\|f_\varepsilon(t)\|_{H^k_{\rm per}} &\le \|f_\varepsilon(\hat\tau)\|_{H^k_{\rm per}}\, e^{C\,M\,(2\,\hat\tau\,-\,\hat\tau)} \\
&\le \|f_0\|_{H^k_{\rm per}}\, e^{C\,M\,2\,\hat\tau} \\
&\le \|f_0\|_{H^k_{\rm per}}\, e^{C\,M\,T} \;=\; M_k(T), \quad \forall\, t \,\in\, [\hat\tau, 2\,\hat\tau].
\end{aligned}$$

Moreover, since $\hat\tau$ only depends on M, $M_k(T)$, B and R, we can again apply Lemma 4.5 on the time interval $[\hat\tau, 2\,\hat\tau]$, which yields that

$$f_\varepsilon(2\,\hat\tau) > 0.$$

We finally proceed by induction to prove existence and uniqueness of a smooth solution f_ε of the perturbed Boltzmann equation (3.17) on the time interval $[0, T]$, which proves assertions (i), (ii) and (iii).

To prove (iv), we compute the difference between the solution $f(t)$ to the unperturbed problem (3.21) and the solution $f_\varepsilon(t)$ to (3.17):

$$\frac{\partial(f - f_\varepsilon)}{\partial t} = \frac{1}{2}\left(Q^R(f - f_\varepsilon, f + f_\varepsilon) + Q^R(f + f_\varepsilon, f - f_\varepsilon)\right) + P_\varepsilon(f_\varepsilon).$$

Then, using the smoothness of f and f_ε, we have from Lemma 4.2 for any $p < k$

$$\|Q^R(f - f_\varepsilon, f + f_\varepsilon)\|_{H^p_{\rm per}},\ \|Q^R(f + f_\varepsilon, f - f_\varepsilon)\|_{H^p_{\rm per}} \;\le\; C_p(M)\,\|f + f_\varepsilon\|_{H^p_{\rm per}}\,\|f - f_\varepsilon\|_{H^p_{\rm per}}$$

and since the perturbation is small (assumption (3.20))

$$\|P_\varepsilon(f)\|_{H^p_{\rm per}} \;\le\; \varphi(\varepsilon),$$

it yields that for all $t \in [0, T]$

$$\|f(t) - f_\varepsilon(t)\|_{H^p_{\rm per}} \le \bar\varphi(\varepsilon),$$

for some function $\bar\varphi(\varepsilon)$ going to zero as ε goes to zero. □

5. Asymptotic behavior and global in time stability

In this section we shall study the asymptotic behavior of the (unperturbed) periodized Boltzmann equation (3.21) based on a regularity study (in the spirit of [20]) and the entropy – entropy production theory (mainly relying on the method developed in [32]). Finally on the basis of these results we shall prove a global in times stability result for the perturbed equation (3.17).

5.1. **Regularity study of the periodized Boltzmann equation.** Let us prove the following result

Proposition 5.1. *Let us consider $0 \leq f_0 \in L^1(\mathcal{D}_L)$ such that $f_0 \in H^k_{\rm per}(\mathcal{D}_L)$ for some $k \geq 0$. Then there is a constant $C > 0$ depending on the L^1 and $H^k_{\rm per}(\mathcal{D}_L)$ norms of f_0 such that the unique global non-negative solution $(f(t))_{t\geq 0}$ to the periodized equation (3.21) satisfies*

$$\forall\, t \geq 0, \quad \|f(t)\|_{H^k_{\rm per}} \leq C.$$

We proceed as in [20]. In particular we shall extend Lions regularity result on Q^+ to the truncated case $Q^{R,+}$ [16, 17]. Hence we shall first prove the regularity property on the gain operator when the collision kernel is smooth and compactly supported, avoiding cancellations at zero relative velocities. Then, we shall include the non-smooth part of the kernel using the loss operator.

We shall split the collision kernel into a smooth and a non-smooth part. As a convention, we shall use subscripts "s" for smooth and "ns" for the non-smooth parts. In terms of the classical truncation (3.13) we set

$$\left\{ \begin{array}{l} \mathcal{B}^R_s(|g|,\cos\theta) = \mathcal{B}(|g|,\cos\theta)\ \chi^R_\eta(|g|)\ \zeta_\eta(|g|)\ \Theta_\eta\left(\cos\theta\right), \\ \mathcal{B}^R_{\rm ns}(|g|,\cos\theta) = \mathcal{B}^R(|g|,\cos\theta) - \mathcal{B}^R_s(|g|,\cos\theta), \end{array} \right.$$

where $\chi^R_\eta(g)$ is the mollified $\mathscr{C}^\infty$ version of $\mathbf{1}_{|g|\leq R}$, $\zeta_\eta(g)$ is the mollified $\mathscr{C}^\infty$ version of $\mathbf{1}_{|g|\geq \eta}$, and Θ_η is a $\mathscr{C}^\infty$ function on $[-1,1]$ which is 1 on $-1+2\eta \leq u \leq 1-2\eta$, and 0 in $[-1,-1+\eta)$ and $(1-\eta,1]$ (the parameter η is the mollification parameter).

In terms of the "fast" truncation (3.16) we set

$$\left\{ \begin{array}{l} \mathcal{B}^R_s(|y|,|z|) = \mathcal{B}(|y|,|z|)\ \chi^R_\eta(|z|)\ \chi^R_\eta(|y|)\ \zeta_\eta(|z+y|)\ \Theta_\eta\left(\frac{|y|}{\sqrt{|y|^2+|z|^2}}\right), \\ \\ \mathcal{B}^R_{\rm ns}(|y|,|z|) = \mathcal{B}^R(|y|,|z|) - \mathcal{B}^R_s(|y|,|z|), \end{array} \right.$$

with the same notations.

We deduce the following decomposition of the collision operator:

$$Q^{R,+} = Q^{R,+}_{\rm s} + Q^{R,+}_{\rm ns},$$

where for instance, with the variables from the "fast" truncation

$$Q^{R,+}_{\rm s}(f,f) = \int_{y\in\mathbb{R}^d}\int_{z\in\mathbb{R}^d} \mathcal{B}^R_{\rm s}(|y|,|z|)\,\delta(y\cdot z)\,f(v+z)\,f(v+y)\,dy\,dz. \tag{5.1}$$

Under the assumption that both Φ and b defined in (2.2) are smooth, the regularized truncature introduced above ensures that there exist two functions Φ^R_η and b^R_η such that

$$\left\{ \begin{array}{l} B^R_\eta(|z|,\cos\theta) = \Phi^R_\eta(|z|)\,b^R_\eta(\cos\theta) \\ \Phi^R_\eta \in \mathscr{C}^\infty_0(\mathbb{R}^d\backslash\{0\},\mathbb{R}), \quad b^R_\eta \in \mathscr{C}^\infty_0([-1,1],\mathbb{R}). \end{array} \right. \tag{5.2}$$

In the following lemma we shall prove the regularity property of $Q^{R,+}_{\rm s}$.

Lemma 5.2. *Let $B^R_\eta(|v-v_\star|,\cos\theta)$ satisfy the assumption (2.2)-(2.4) and (5.2). Then, for all $r \in \mathbb{R}^+$*

$$\left\|Q^{R,+}_{\rm s}(f,f)\right\|_{H^{r+\frac{d-1}{2}}_{\rm per}} \quad \leq \quad C_{\rm reg}(r,\mathcal{B}^R_{\rm s})\ \|f\|_{L^1}\ \|f\|_{H^r_{\rm per}},$$

where the constant $C_{\rm reg}(r,\mathcal{B}^R_{\rm s})$ only depends on r and on the collision kernel.

Proof. We closely follow the proof given by Lions [16, 17] and simplified and then reformulated in [33, 20]. Again the preservation of the translation invariance by the truncation is fundamental. Starting from the collision operator in the form (5.1) and performing a change of variable we get for $v \in \mathcal{D}_L$

$$Q_{\rm s}^{R,+}(f,f) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \tilde{\mathcal{B}}_{\rm s}^R(|v'_\star - v'|, |v'-v|)\,\delta\left((v'_\star - v)\cdot(v'-v)\right)\, f(v'_\star)\, f(v')\, dv'_\star\, dv',$$

where $\tilde{\mathcal{B}}_{\rm s}^R$ only depends on $\mathcal{B}_{\rm s}^R$. Then we set [34]

$$Tg(y) = \int_{y+y^\perp} \tilde{\mathcal{B}}_{\rm s}^R(|z|,|y|)\, g(z)\, dz, \quad \tau_z g(\cdot) = g(\cdot - z),$$

where

$$y^\perp = \left\{ z \in \mathbb{R}^d, \quad z^t \cdot y = 0 \right\}$$

and easily get for $v \in \mathcal{D}_L$

$$Q_{\rm s}^{R,+}(f,f) \;=\; \int_{\mathbb{R}^d} f(v')\; (\tau_{v'} \circ T \circ \tau_{-v'})\, f(v)\, dv'.$$

Now, we want to estimate Sobolev norms of $Q_{\rm s}^{R,+}$ as a function defined in the torus $\mathcal{D}_L$. Applying the Fubini theorem with the discrete and Lebesgue measures and the Cauchy-Schwarz inequality, it leads to estimate the Sobolev norms of the Radon transform T on the torus $\mathcal{D}_L$

$$\left\|Q_{\rm s}^{R,+}(f,f)\right\|^2_{H^{r+\frac{d-1}{2}}_{\rm per}} \;\leq\; \|f\|_{L^1} \int_{\mathbb{R}^d} |f(v')|\, \|\, \tau_{v'} \circ T \circ \tau_{-v'}\, f\, \|^2_{H^{r+\frac{d-1}{2}}_{\rm per}}\, dv'. \tag{5.3}$$

On the one hand, since the kernel $\tilde{\mathcal{B}}_{\rm s}^R$ is compactly supported in y and z, the operator T maps periodic functions g to a compactly supported function Tg with supp $(Tg) \subset \mathcal{B}_R \subset \mathcal{D}_L$. Then, we can consider Tg as a function in the whole space $\mathbb{R}^d$.

On the other hand, using the regularized truncations χ_η^R and Θ_η and the smoothness of Φ and b in (2.2), it yields that $B_\eta^R = \Phi_\eta^R(|v-v_\star|)\, b_\eta^R(\cos\theta)$ with

$$\Phi_\eta^R \in \mathscr{C}_o^\infty(\mathbb{R},\mathbb{R}), \quad b_\eta^R \in \mathscr{C}_o^\infty([-1,1],\mathbb{R}).$$

Then, we can directly apply the result in [20], where the authors proved the following regularity estimates on the Radon transform T for smooth kernels

$$\|Tg\|_{H^{r+\frac{d-1}{2}}} \;\leq\; C_{\rm reg}(r,\mathcal{B}_{\rm s}^R)\, \|g\|_{H^r}$$

for a function g defined in $\mathbb{R}^d$. However in the proof of the latter inequality, we can replace g by the smooth and compactly supported function $g\,\chi_\eta^R$ for which supp $\left(\chi_\eta^R\, g\right) \subset \mathcal{B}_R \subset \mathcal{D}_L$. Thus, for all g defined in the torus $\mathcal{D}_L$, we get

$$\|Tg\|_{H^{r+\frac{d-1}{2}}_{\rm per}} \;=\; \|Tg\|_{H^{r+\frac{d-1}{2}}} \;\leq\; C_{\rm reg}(r,\mathcal{B}_{\rm s}^R)\, \|g\|_{H^r_{\rm per}}. \tag{5.4}$$

Finally gathering (5.3) and (5.4), we obtain the result

$$\left\|Q_{\rm s}^{R,+}(f,f)\right\|_{H^{r+\frac{d-1}{2}}_{\rm per}} \;\leq\; C_{\rm reg}(r,\mathcal{B}_{\rm s}^R) \|f\|_{L^1}\, \|f\|_{H^r_{\rm per}}.$$

□

Corollary 5.3. *Let $B_\eta^R(|v-v_\star|,\cos\theta)$ satisfy the assumption (2.2)-(2.4) and (5.2). Then, for all $p \in (1,\infty)$*

$$\|Q_{\rm s}^{R,+}(f,f)\|_{L^q} \;\leq\; C_{\rm reg}(r,\mathcal{B}_{\rm s}^R)\; \|f\|_{L^1}\; \|f\|_{L^p},$$

with

$$q = \begin{cases} \dfrac{p}{2 - \frac{1}{d} + p\left(\frac{1}{d} - 1\right)} & \text{if } p \in (1, 2] \\ p\, d & \text{if } p \in [2, \infty). \end{cases}$$

Proof. It is a direct consequence of Sobolev embedding and interpolation between L^p spaces. □

Now we extend the regularity of $Q^{R,+}$ to general non-smooth kernels.

Lemma 5.4. *Let B be a collision kernel satisfying (2.2)-(2.4). Then, for all $p > 1$, there exist constants C, κ and $q < p$ (q only depending on p and d), such that for all $\delta > 0$, and for all measurable function f*

$$\left\|Q^{R,+}(f,f)\right\|_{L^p} \quad \leq \quad C\,\delta^{-\kappa}\,\|f\|_{L^1}\,\|f\|_{L^q} \;+\; \delta\,\|f\|_{L^1}\,\|f\|_{L^p}.$$

Proof. We use a decomposition approach and split the operator $Q^{R,+}$ as the sum of a smooth part and a non-smooth part

$$Q^{R,+} = Q^{R,+}_{\rm s} + Q^{R,+}_{\rm ns},$$

where $Q^{R,+}_{\rm s}$ is given by (5.1). Then, applying Corollary 5.3 we have for all $p \in (1, \infty)$, there exist $q < p$, namely (the role of q and p are exchanged here with respect to Corollary 5.3)

$$q = \begin{cases} \dfrac{(2d-1)p}{d + (d-1)p} & \text{if } p \in (1, 2d] \\ \dfrac{p}{d} & \text{if } p \in [2d, \infty). \end{cases}$$

and $C_{\rm reg}(\eta, \mathcal{B}^R_{\rm s})$, depending on the the regularization parameter η and blowing-up polynomially when $\eta \to 0$ such that

$$\left\|Q^{R,+}_{\rm s}(f,f)\right\|_{L^p} \quad \leq \quad C_{\rm reg}(\eta, \mathcal{B}^R_{\rm s})\;\|f\|_{L^1}\;\|f\|_{L^q}. \tag{5.5}$$

Now, we need to estimate the remainder $Q^{R,+}_{\rm ns} = Q^{R,+} - Q^{R,+}_{\rm s}$. To this aim, we split it as

$$Q^{R,+}_{\rm ns}(f,f) = Q^{R,+}_1(f,f) + Q^{R,+}_2(f,f) + Q^{R,+}_3(f,f) + Q^{R,+}_4(f,f),$$

with

$$\begin{aligned}
Q^{R,+}_1(f,f) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{B}(|y|,|z|)\,\delta(y\cdot z)\,\chi^R(|z|)\,\left[\chi^R(|y|) - \chi^R_\eta(|y|)\right]\,\zeta_\eta\,\Theta_\eta\,f'\,f'_\star\,dy\,dz,\\
Q^{R,+}_2(f,f) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{B}(|y|,|z|)\,\delta(y\cdot z)\,\chi^R_\eta(|y|)\,\left[\chi^R(|z|) - \chi^R_\eta(|z|)\right]\,\zeta_\eta\,\Theta_\eta\,f'\,f'_\star\,dy\,dz,\\
Q^{R,+}_3(f,f) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{B}(|y|,|z|)\,\delta(y\cdot z)\,\chi^R_\eta(|z|)\,\chi^R_\eta(|y|)\,\zeta_\eta\,\left[1 - \Theta_\eta\left(\frac{|y|}{\sqrt{|y|^2+|z|^2}}\right)\right]\,f'\,f'_\star\,dy\,dz\\
Q^{R,+}_4(f,f) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{B}(|y|,|z|)\,\delta(y\cdot z)\,\chi^R_\eta(|z|)\,\chi^R_\eta(|y|)\,\Theta_\eta\,(1-\zeta_\eta)\,f'\,f'_\star\,dy\,dz.
\end{aligned}$$

On the one hand, we give a first estimate in $L^1(\mathcal{D}_L)$ applying directly the estimate in Lemma 4.1:

$$\|Q^{R,+}_\alpha\|_{L^1} \quad \leq \quad C_1(R,B)\,\|f\|_{L^1}\|f\|_{L^1}, \quad \alpha \in \{1,2,3,4\}. \tag{5.6}$$

On the other hand, we treat for instance the operator $Q_1^{R,+}(f,f)$ and have for a fixed $v \in \mathcal{D}_L$

$$\begin{aligned}
&\left|Q_1^{R,+}(f,f)(v)\right| \\
&\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{B}(|y|,|z|)\,\delta(y\cdot z)\,\chi^R(|z|)\,\left|\chi^R(|y|)-\chi_\eta^R(|y|)\right|\,|f'|\,|f'_\star|\,dy\,dz \\
&\leq \int_{\mathcal{D}_L} f'\left(\int_{\mathbb{R}^d} \mathcal{B}(|v-v'_\star|,|v-v'|)\,\delta((v-v'_\star)\cdot(v-v'))\right. \\
&\qquad \left.\times\left|\chi^R(|v-v'_\star|)-\chi_\eta^R(|v-v'_\star|)\right|\,|f'_\star|\,dv'_\star\right)dv' \\
&\leq \|f\|_{L^1}\,\|f\|_{L^\infty}\sup_{y\in\mathbb{R}^d}\left(\int_{z\in\ y^\perp}\mathcal{B}^R(|z|,|y|)\,\left|\chi^R(|z|)-\chi_\eta^R(|z|)\right|\,dz\right) \\
&\leq C_\infty(R,B)\,\eta\,\|f\|_{L^1}\,\|f\|_{L^\infty}.
\end{aligned}$$

Using similar techniques, we prove that for $\alpha\in\{1,2,3\}$

$$\|Q_\alpha^{R,+}\|_{L^\infty} \;\leq\; C_\infty(R,B)\,\eta\,\|f\|_{L^1}\|f\|_{L^\infty}. \tag{5.7}$$

For the fourth term, we have using the cancellation of the collision kernel B at small relative velocities as $|v-v_*|^\gamma$:

$$\|Q_4^{R,+}\|_{L^\infty} \;\leq\; C_\infty(R,B)\,\eta^\gamma\,\|f\|_{L^1}\|f\|_{L^\infty}. \tag{5.8}$$

Finally, by the Riesz-Thorin interpolation Theorem, from (5.7-5.8) and (5.6), we deduce that for $p\in[1,+\infty]$ there exist $C_p(R,B)>0$ and $\beta\in(0,1]$ such that

$$\|Q_\alpha^{R,+}(f,f)\|_{L^p} \;\leq\; C_p(R,B)\,\eta^\beta\|f\|_{L^1}\,\|f\|_{L^p}. \tag{5.9}$$

To sum up we have obtained for all $p>1$ and $\eta>0$, there exist $C>0$, $q<p$, $\kappa_0>0$ and $\beta\in(0,1)$ such that

$$\|Q^{R,+}(f,f)\|_{L^p} \;\leq\; C\,\eta^{-\kappa_0}\,\|f\|_{L^1}\,\|f\|_{L^q} \;+\; \eta^\beta\,\|f\|_{L^1}\,\|f\|_{L^p}.$$

The conclusion follows by choosing η small enough. □

Proof of Proposition 5.1. Now the proof of the propagation of regularity bounds is done exactly as in [20, Section 4 and Subsections 5.1 & 5.2] (except for the simplification that there is no moments estimates to take care of). □

5.2. **Entropy – entropy production inequalities.** The periodized equation (3.21) preserves non-negativity, and for a non-negative distribution f one can formally compute an H theorem (see [14]):

$$\frac{d}{dt}H(f(t)) = -D(f(t)) \leq 0$$

with

$$H(f) = \int_{\mathcal{D}_L} f\,\log f\,dv$$

and

$$D(f) = -\int_{\mathcal{D}_L} Q^R(f,f)\,\log f\,dv = \frac{1}{4}\int_{\mathcal{D}_L\times\mathcal{C}_R}(f'f'_\star - ff_\star)\,\log\left(\frac{f'f'_\star}{ff_\star}\right)\,\mathcal{B}(y,z)\,dv\,dy\,dz.$$

Then we can state the result which relates the entropy functional H and the entropy production functional D:

Proposition 5.5. *We consider the periodized Boltzmann collision operator for some truncation parameter $R > \sqrt{2}\,L$, and we assume that the collision kernel satisfies $B \geq b_0\,|v - v_\star|^\gamma$, $\gamma > 0$, for $|v - v_\star| \leq R$. Then for any $\eta, \alpha > 0$ there is $k \in \mathbb{N}$ and $M, K > 0$ (depending only on $\eta, \alpha, b_0, \gamma, R$) such that*

$$D(f) \geq K\,H(f|m_\infty)^{1+\eta}, \qquad m_\infty = \frac{\rho}{|\mathcal{D}_L|}, \quad \rho = \int_{\mathcal{D}_L} f\,dv,$$

for any $\alpha \leq f \in L^1(\mathcal{D}_L)$ with $H^k_{\rm per}(\mathcal{D}_L)$ norm bounded by M.

Remarks:

1. Note that this is a functional inequality independent of the flow of the Boltzmann equation itself.

2. In the case of the classical Boltzmann equation with $v \in \mathbb{R}^d$ the entropy production functional controls the relative entropy according to the Maxwellian equilibrium. Here the equilibrium is a constant, defined by the mass of f divided by the total volume of the torus.

We shall adapt the method developed in [32, Proof of Theorem 2.1]. In the first step, we treat the case of a collision kernel B which is uniformly bounded from below. In this case we prove the equivalent of the so-called Cercignani conjecture in the context of the Boltzmann operator periodized in the velocity space.

Remark: Note that the assumption $R \geq \sqrt{2}\,L$ allows to replace in the bound from below, when needed, the truncation by the integration over the whole torus.

Lemma 5.6. *Let us consider a collision kernel B which satisfies $B \geq b_0 > 0$, a truncation Q^R defined by (3.13) or (3.16), together with $R \geq \sqrt{2}\,L$. Then there is an explicit constant K such that for any $0 \leq f \in L^1(\mathcal{D}_L)$ we have*

$$D(f) \geq K\,H(f|m_\infty).$$

Proof. We proceed in several steps.

Step 1. Since the entropy production functional is monotonous in terms of the collision kernel $\mathcal{B}$, it is no restriction to replace $\mathcal{B}$ by 1 in the sequel for the estimate from below. Moreover it is always possible to bound from below the truncation $|v - v'| \leq R$ and $|v - v'_\star| \leq R$ (in case we performed the truncation for the fast spectral method) by the classical truncation $|v - v_\star| \leq R$.

Step 2. Using Jensen's inequality on the sphere integration (coming back to the classical truncation by the previous remark) and the joint convexity of the function $(X, Y) \mapsto (X - Y)\,(\log X - \log Y)$ on $\mathbb{R}_+ \times \mathbb{R}_+$ we compute

$$D(f) \geq C \int_{\mathcal{D}_L} \int_{\mathcal{B}(v,R)} (F - G)\,\log \frac{F}{G}\,dv_\star\,dv \;=:\; \bar{D}(f)$$

where $F = f\,f_\star$ and

$$G = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} f'\,f'_\star\,d\sigma.$$

Let us study more precisely the function G. As it was already observed by Boltzmann himself, the function G only depends on $v + v_\star$ and $(|v|^2 + |v_\star|^2)/2$. Moreover, here it is also periodic on the torus $\mathcal{D}_L$ since f is periodic. It implies (when f is smooth, but we can always use mollifications to relax this assumption here) that it in fact only depends on $v + v_\star$.

Step 3. Let us denote by S_t the semi-group of the heat equation on $L^1(\mathcal{D}_L)$ (and for brevity we keep the same notation for its semi-group in $L^1(\mathcal{D}_L^2)$). Then the semi-group is compatible with the symmetries in the sense that:

$$S_t(f\, f_\star) \,=\, S_t(f)\; S_t(f_\star)$$

and S_tG only depends on $v + v_\star$ (this follows from a straightforward computation using the explicit formula for the Green kernel of S_t).

Step 4. Then we have the following computation as in [32]:

$$\left.\frac{d}{dt}\right|_{t=0}\left[S_t\left((F-G)\,\log\frac{F}{G}\right)-(S_tF-S_tG)\,\log\frac{S_tF}{S_tG}\right]=\left|\frac{\nabla F}{F}-\frac{\nabla G}{G}\right|^2\,(F+G),$$

where ∇ denotes the gradient with respect to $(v, v_\star) \in \mathbb{R}^{2d}$.

Then we bound from below the truncation $\mathbf{1}_{B(v,R)}(v_\star)$ by the integration over the whole torus for $v_\star$ (since R is large enough), and we compute

$$\begin{aligned}
-\left.\frac{d}{dt}\right|_{t=0}\bar{D}(S_tf) \;&=\; \int_{\mathcal{D}_L\times\mathcal{D}_L}\left|\frac{\nabla F}{F}-\frac{\nabla G}{G}\right|^2\,(F+G)\,dv\,dv_\star\\
&\quad-\int_{\mathcal{D}_L\times\mathcal{D}_L}\Delta\left((F-G)\,\log\frac{F}{G}\right)\,dv\,dv_\star\\
&=\; \int_{\mathcal{D}_L\times\mathcal{D}_L}\left|\frac{\nabla F}{F}-\frac{\nabla G}{G}\right|^2\,(F+G)\,dv\,dv_\star.
\end{aligned}$$

We deduce by the semi-group property that for all $t > 0$

$$-\frac{d}{dt}\bar{D}(S_tf)\geq\int_{\mathcal{D}_L\times\mathcal{D}_L}\left|\frac{\nabla S_tF}{S_tF}-\frac{\nabla S_tG}{S_tG}\right|^2\,(S_tF+S_tG)\,dv\,dv_\star$$

and therefore

$$\bar{D}(f)\geq\int_0^{+\infty}\left(\int_{\mathcal{D}_L\times\mathcal{D}_L}\left|\frac{\nabla S_tF}{S_tF}-\frac{\nabla S_tG}{S_tG}\right|^2\,(S_tF+S_tG)\,dv\,dv_\star\right)\,dt.$$

Step 5. We now use the fact that the operator

$$P:\left\{\begin{array}{lcl}\mathbb{R}^d\times\mathbb{R}^d&\mapsto&\mathbb{R}^d\\(A,B)&\mapsto&(A-B)\end{array}\right.$$

is bounded from $\mathcal{D}_L \times \mathcal{D}_L$ to $\mathcal{D}_L$. Hence

$$\left|\frac{\nabla S_tF}{S_tF}-\frac{\nabla S_tG}{S_tG}\right|^2\geq C_1\,\left|\frac{P\nabla S_tF}{S_tF}\right|^2$$

since $P\nabla S_tG = \nabla_v G - \nabla_{v_\star} G = 0$ from the fact that G only depends on $v + v_\star$. We deduce

$$\bar{D}(f)\geq C\int_0^{+\infty}\left(\int_{\mathcal{D}_L\times\mathcal{D}_L}\left|\frac{\nabla_vS_tf}{S_tf}-\left(\frac{\nabla_{v_\star}S_tf}{S_tf}\right)_\star\right|^2\,(S_tf\,S_tf_\star+S_tG)\,dv\,dv_\star\right)\,dt$$

and thus (dropping the term S_tG)

$$\bar{D}(f)\geq C\int_0^{+\infty}\left(\int_{\mathcal{D}_L\times\mathcal{D}_L}\left|\frac{\nabla_vS_tf}{S_tf}-\left(\frac{\nabla_{v_\star}S_tf}{S_tf}\right)_\star\right|^2\,S_tf\,S_tf_\star\,dv\,dv_\star\right)\,dt.$$

Step 6. From now on the proof departs slightly more from [32]: it is simpler since we are in the torus and we have more symmetries. Let us show the following functional inequality: for any smooth non-negative function h,

$$\int_{\mathcal{D}_L\times\mathcal{D}_L}\left|\frac{\nabla_v h}{h}-\left(\frac{\nabla_{v_\star} h}{h}\right)_\star\right|^2 h\,h_\star\,dv\,dv_\star \geq C_2\,I(h|m_\infty)$$

where

$$I(h|g)=\int_{\mathcal{D}_L} h\,\left|\nabla_v\log\frac{h}{g}\right|^2\,dv.$$

The proof only amounts to Jensen's inequality on the variable $v_\star$: since $\int_{\mathcal{D}_L} h\,dv=\rho$,

$$\int_{\mathcal{D}_L\times\mathcal{D}_L}\left|\frac{\nabla_v h}{h}-\left(\frac{\nabla_{v_\star} h}{h}\right)_\star\right|^2 h\,h_\star\,dv\,dv_\star \geq \frac{C}{\rho}\int_{\mathcal{D}_L}\left|\int_{\in\mathcal{D}_L}\left(\frac{\nabla_v h}{h}-\left(\frac{\nabla_{v_\star} h}{h}\right)_\star\right)h_\star\,dv_\star\right|^2 h\,dv.$$

Then as

$$\int_{\mathcal{D}_L}\nabla h_\star\,dv_\star\;=\;0$$

we deduce

$$\int_{\mathcal{D}_L\times\mathcal{D}_L}\left|\frac{\nabla_v h}{h}-\left(\frac{\nabla_{v_\star} h}{h}\right)_\star\right|^2 h\,h_\star \geq C\int_{\mathcal{D}_L}\left|\frac{\nabla_v h}{h}\right|^2 h\,dv\;=\;C\,I(h|m_\infty).$$

Step 7. So far we have proved

$$D(f)\geq\bar{D}(f)\geq C_3\int_0^{+\infty} I(S_t f|m_\infty)\,dt.$$

Then a trivial computation shows that

$$\frac{d}{dt}H(S_t f|m_\infty)=-I(S_t f|m_\infty).$$

Moreover from the explicit formula for S_t we have

$$H(S_t f|m_\infty)\xrightarrow{t\to+\infty}0$$

and thus we finally obtain

$$D(f)\geq\bar{D}(f)\geq C_3\;(H(S_0 f|m_\infty)-0)\geq C_3\,H(f|m_\infty).$$

□

Now we are ready to prove Proposition 5.5. Since we deal with a bounded velocity domain we do not care about possible decay of the collision kernel at large relative velocity (as for soft potentials) and the only cancellation we have to treat is for zero relative velocities.

Proof of Proposition 5.5. We only mention the difference as compared to the previous proof.

The reduction to a collision kernel uniformly bounded from below studied in Lemma 5.6 is done as in [32][Theorem 4.1]: one write for some small $\delta>0$

$$B(|v-v_\star|)\;\geq\;\delta^\gamma\,\left(B_0-\mathbf{1}_{\mathcal{B}(0,\delta)}(|v-v_\star|)\right)$$

where $B_0\;\geq\;b_0\;>\;0$ and we deduce

$$D(f)\;\geq\;\delta^\gamma\,\Big(D_0(f)\;-\;\tilde{D}_\delta(f)\Big)$$

where D_0 is the entropy production functional corresponding to B_0, and

$$\tilde{D}_\delta\;=\;\frac{1}{4}\int_{\mathcal{D}_L\times\mathcal{D}_L}\int_{\mathbb{S}^{d-1}}(f'f'_\star-ff_\star)\,\log\frac{f'f'_\star}{ff_\star}\,\mathbf{1}_{\mathcal{B}(0,\delta)}(|v-v_\star|)\,d\sigma\,dv\,dv_\star.$$

Then we have the following Lemma, which is proved exactly similarly as [32, Theorem 4.2]. It is even simpler since Maxwellians are replaced by constant functions and the study of the tail is not needed (we omit the proof for brevity).

Lemma 5.7. *For any $\varepsilon \in (0,1)$ and $\alpha > 0$, there are constants of smoothness k, M and some corresponding constant $C_{\rm diag} > 0$ such that*

$$\tilde{D}_\delta \leq C_{\rm diag}\, H(f|m_\infty)^{1-\varepsilon}\, \delta^{d/4}$$

for any $\alpha \leq f \in L^1(\mathcal{D}_L)$ with $H^k_{\rm per}(\mathcal{D}_L)$ norm bounded by M.

But since we have

$$D_0(f) \geq C\, H(f|m_\infty)$$

from Lemma 5.6, it is straightforward to get the result by choosing correctly the parameter δ. □

Now we can proceed to the proof of Theorem 3.1.

5.3. **Proof of the global in time stability.** In this subsection we shall turn to the question of obtaining uniform bounds as well as global existence, in order to conclude the proof of Theorem 3.1. Indeed in Section 4 the smallness assumption on the truncation parameter ε *a priori* depends on T and could go to 0 as T goes to infinity, since it depends on regularity bounds growing exponentially in times.

In order to overcome this difficulty, we shall combine the following arguments:

- for the unperturbed problem (3.21) we have a Liapunov structure and the solution converges to a unique prescribed equilibrium from the regularity and entropy production studies;
- the equilibrium distribution of the unperturbed problem (3.21) (that is the constant functions on the torus) are also equilibrium distribution of the perturbed problem (3.17);
- by taking the size of the perturbation small enough (measured in terms of ε) it is possible to construct a solution to the perturbed problem on an arbitrarily large time interval $[0,T]$, on which moreover the perturbed solution remains close to the unperturbed solution (3.21), say in Sobolev norms;
- finally the constant equilibrium functions are non-linearly stable for the perturbed problem, with a stability domain independent on the size of the perturbation.

Hence we shall deduce that as soon as the time for which the perturbed solution departs from the unperturbed solution is larger that the time-scale of relaxation to equilibrium for the unperturbed problem (3.21), the perturbed solution shall be trapped by the stability domain of the equilibrium before instability due to the perturbation can develop. Let us formalize these arguments in a proof:

Proof of the global stability and asymptotic behavior in Theorem 3.1. Let us consider some initial datum $0 \leq f_0 \in L^1$ which belongs to $H^k_{\rm per}(\mathcal{D}_L)$, and some smooth balanced perturbations $f_{0,\varepsilon}$ of it. These perturbations have the same mass, and therefore are corresponding to the same equilibrium (this is the reason for this assumption).

Since the smooth balanced perturbation preserves the constant equilibrium of (3.21) and is nonlinearly locally stable in any $H^k_{\rm per}(\mathcal{D}_L)$, we fix a $\eta > 0$ such that the constant distribution $m_\infty = \rho/|\mathcal{D}_L|$ associated to the mass ρ of f_0 on the torus has attraction domain with size η in $H^k_{\rm per}(\mathcal{D}_L)$ for the perturbed problem (3.17).

On the one hand, we show that there exists a unique solution $f(t)$ to (3.21) and from Proposition 5.1 we obtain uniform regularity bounds for all $t \geq 0$

$$\|f(t)\|_{H^k_{\rm per}} \leq C.$$

Moreover, from Proposition 5.5 there exists a time T_0 such that the solution $f(t)$ is $\eta/2$-close to the equilibrium in $H^p_{\rm per}(\mathcal{D}_L)$ $(p < k)$ for $t \geq T_0$ (using the Csiszár-Kullback inequality in the torus, see [5, Theorem 1] for instance):

$$\|f(t) - m_\infty\|_{H^p_{\rm per}} \ \leq \ \eta/2.$$

On the other hand, applying Proposition 4.6 with $T = T_0$, we prove that there exists $\hat{\varepsilon}$, which only depends on the $H^k_{\rm per}(\mathcal{D}_L)$ and $L^1(\mathcal{D}_L)$ norms of f_0 such that for all ε such that $0 < \varepsilon < \hat{\varepsilon}$, there exists a unique smooth solution $f_\varepsilon(t) \in H^k_{\rm per}(\mathcal{D}_L)$ to (3.17) on $[0, T_0]$ such that

$$\forall\, t \in [0, T_0], \quad \|f_\varepsilon(t)\|_{H^k_{\rm per}} \ \leq \ C(T_0),$$

and (for any $p < k$)

$$\|f(t) - f_\varepsilon(t)\|_{H^p_{\rm per}} \ \leq \ \bar{\varphi}_{T_0}(\varepsilon),$$

where $f(t)$ is solution to (3.21) and $\bar{\varphi}_{T_0}(\varepsilon)$ goes to zero when ε goes to zero.

Then, we fix a perturbation parameter $\hat{\varepsilon}$ small enough such that for $\varepsilon \in (0, \hat{\varepsilon})$ the perturbed solution f_ε satisfies

$$\|f(t) - f_\varepsilon(t)\|_{H^p_{\rm per}} \ \leq \ \eta/2.$$

Finally, at time T_0 the perturbed solution f_ε belongs to the stability domain of the constant distribution for the perturbed problem and it is trapped.

Therefore there exists a unique global smooth solution f_ε, which is uniformly bounded for all $t \geq 0$ and such that for any $p < k$

$$\|f_\varepsilon(t)\|_{H^p_{\rm per}} \ \leq \ \max\left(C(T_0), C + \eta\right).$$

This achieves the proof of (i), (ii), (iii) and (iv).

□

6. Application: stability and convergence of spectral methods

In this section we consider the following spectral approximation of (3.21)

$$\frac{\partial f_N}{\partial t} = \mathcal{P}_N\, Q^R(f_N, f_N),$$

where $\mathcal{P}_N$ denotes the orthogonal projection on $\mathbb{P}_N$ in $L^2(\mathcal{D}_L)$ (the space of trigonometric polynomials with degree less at most N in each direction).

The goal of this section is to prove the following theorem:

Theorem 6.1. *Consider any nonnegative initial datum $f_0 \in H^k_{\rm per}(\mathcal{D}_L)$, with $k > d/2$, which is not zero everywhere. Then there exists $N_0 \in \mathbb{N}$ (depending on the mass and $H^k_{\rm per}(\mathcal{D}_L)$ norm of f) such that for all $N \geq N_0$:*

(i) there is a unique global solution $f_N = f_N(t, \cdot)$ to the following problem

$$\left\{ \begin{array}{l} \dfrac{\partial f_N}{\partial t} \ = \ \mathcal{P}_N\, Q^R(f_N, f_N), \\ \\ f_N(t = 0) = \mathcal{P}_N f_0; \end{array} \right. \tag{6.1}$$

(ii) for any $p < k$, there exists $C > 0$ such that

$$\forall\, t \geq 0, \quad \|f_N(t, \cdot)\|_{H^k_{\rm per}} \ \leq \ C;$$

(*iii*) *this solution is everywhere positive for time large enough, and the mass of its negative values can be made uniformly (in times) L^∞ small as $N \to \infty$;*
(*iv*) *this solution f_N converges to $f(t)$ the solution to (3.21) with the spectral accuracy, uniformly in time;*
(*v*) *this solution converges exponentially fast to a constant solution on the torus prescribed by the mass conservation law.*

To prove Theorem 6.1, we want to apply Theorem 3.1 with the perturbation

$$P_N^R(f_N) := \mathcal{P}_N\, Q^R(f_N, f_N) \,-\, Q^R(f_N, f_N),$$

which preserves the mass:

$$\int_{\mathcal{D}_L} P_N^R(f_N)\, dv \,=\, \int_{\mathcal{D}_L} \big(\mathcal{P}_N\, Q^R(f_N, f_N) \,-\, Q^R(f_N, f_N)\big)\, dv \,=\, 0,$$

In the next Lemma, we prove a consistency and smoothness result for this approximation.

Lemma 6.2. *Consider a nonnegative function $f \in H^k_{\rm per}(\mathcal{D}_L)$, with $k > d/2$, which is not zero everywhere. Then, there exists $C > 0$ depending only on the collision kernel B and the truncation such that for all $p \in [0, k]$ we have*

$$\|P_N^R(f)\|_{H^p_{\rm per}} \,\le\, C\, \|f\|_{L^1}\, \|f\|_{H^p_{\rm per}}. \tag{6.2}$$

Moreover, for all $p \in [0, k]$

$$\|P_N^R(f)\|_{H^p_{\rm per}} \,\le\, C\, \|f\|_{L^1}\, \frac{\|f\|_{H^k_{\rm per}}}{N^{k-p}}. \tag{6.3}$$

Proof. First, we split the operator P_N as

$$\|P_N^R(f)\|_{H^p_{\rm per}} \le \|Q^R(f,f)\|_{H^p_{\rm per}} + \|\mathcal{P}_N\, Q^R(f,f)\|_{H^p_{\rm per}}.$$

As in the proof of Lemma 4.2 we get that for all $p \in [0, k]$

$$\|Q^R(f,f)\|^2_{H^p_{\rm per}} \,\le\, C_p(R,B)\, \|f\|^2_{L^1}\, \|f\|^2_{H^p_{\rm per}}. \tag{6.4}$$

Concerning the interpolation error estimate, the following result holds. If $u \in H^p_{\rm per}(\mathcal{D}_L)$ for some $p \ge 1$, then

$$\|u - \mathcal{P}_N u\|_{H^p_{\rm per}} \le \frac{C}{N^{k-p}}\, \|u\|_{H^k_{\rm per}}. \tag{6.5}$$

Then, taking $p = k$ in the latter inequality and from (6.4) we obtain

$$\|\mathcal{P}_N Q^R(f,f)\|^2_{H^p_{\rm per}} \,\le\, \|Q^R(f,f)\|^2_{H^p_{\rm per}} \,\le\, C_p(R,B)\, \|f\|^2_{L^1}\, \|f\|^2_{H^p_{\rm per}}. \tag{6.6}$$

Gathering (6.4) and (6.6), we finally get

$$\|P_N^R(f)\|_{H^p_{\rm per}} \,\le\, C_p(R,B)\, \|f\|_{L^1}\, \|f\|_{H^p_{\rm per}}.$$

Moreover, using again the error estimate (6.5), it yields

$$\begin{aligned} \|P_N^R(f)\|_{H^p_{\rm per}} \,&\le\, C\, \frac{\|Q^R(f,f)\|_{H^k_{\rm per}}}{N^{k-p}} \\ &\le\, C(R,B)\, \|f\|_{L^1}\, \frac{\|f\|_{H^k_{\rm per}}}{N^{k-p}}. \end{aligned}$$

□

Let us now perform a linearized study of the perturbed equation (6.1) by classical Fourier-basis decomposition. The only equilibrium distributions of the equation (3.21) are the constant, prescribed by the mass conservation. Let us consider the linearized version of the perturbed equation (6.1) around such a constant m_∞:

$$\frac{\partial f}{\partial t} = m_\infty \, \mathcal{L}^{N,R}(f)$$

where

$$\mathcal{L}^{N,R}(f) = \mathcal{P}_N \left[Q^R(f,1) + Q^R(1,f) \right].$$

Let us prove the following lemma:

Lemma 6.3. *The operator $\mathcal{L}^{N,R}$ is bounded and self-adjoint in $L^2(\mathcal{D}_L)$. Moreover it is non-negative, its null space is given by the constant functions, and it has a spectral gap $\lambda > 0$. As a consequence, the constant are nonlinearly locally stable in any $H^k_{\rm per}(\mathcal{D}_L)$ for the equation* (6.1)*, with a stability domain independent on N.*

Proof. The boundedness is trivial. Then, the periodized operator Q^R is translation invariant, which implies that the Fourier modes

$$e_k(v) = \frac{\exp\left[i \, \frac{\pi}{L} \, (k \cdot v)\right]}{|\mathcal{D}_L|}$$

for $k \in \mathbb{Z}^d$ are trivially eigenfunctions of $\mathcal{L}^{N,R}$. This provides a complete orthonormal eigenbasis in $L^2(\mathcal{D}_L)$. A trivial computation yields

$$\mathcal{L}^{N,R}(e_k) = a_k \, e_k \, \mathbf{1}_{|k| \le N} \quad \text{with} \quad a_k := - \int_{\mathcal{C}_R} \left[1 + e_k(y+z) - e_k(y) - e_k(z) \right] \mathcal{B}(y,z) \, dy \, dz.$$

In particular we deduce that $\mathcal{L}^{N,R} = \mathcal{L}^{N,R} \, \mathcal{P}_N$, and the self-adjointness comes from the following identity obtained by the usual changes of variables:

$$\int_{\mathcal{D}_L} \mathcal{L}^{N,R}(f) \, g \, dv = -\frac{1}{4} \int_{\mathcal{D}_L \times \mathcal{C}_R} \left[\mathcal{P}_N f' + \mathcal{P}_N f'_\star - \mathcal{P}_N f - \mathcal{P}_N f_\star \right] \\ \times \left[\mathcal{P}_N g' + \mathcal{P}_N g'_\star - \mathcal{P}_N g - \mathcal{P}_N g_\star \right] \mathcal{B}(y,z) \, dy \, dz \, dv.$$

Another formula for a_k is readily deduced from the previous representation:

$$a_k = -\frac{1}{4} \int_{\mathcal{D}_L \times \mathcal{C}_R} \left| (e_k)' + (e_k)'_\star - (e_k) - (e_k)_\star \right|^2 \mathcal{B}(y,z) \, dy \, dz \, dv.$$

One sees from the second representation that $a_k = a_{-k} \le 0$ for any $k \in \mathbb{Z}^d$, and from the first representation and Lebesgue theorem it is easily seen that for $|k| \to \infty$ the coefficients a_k converge as

$$a_k \xrightarrow[|k| \to \infty]{} a_\infty = - \int_{\mathcal{C}_R} \mathcal{B}(y,z) \, dy \, dz \ \in (-\infty, 0).$$

Hence we deduce that $a_k \in [a_\infty, 0]$ for any $k \in \mathbb{Z}^d$, with asymptotic convergence towards a_∞ for large k. Moreover the null space can be computed: for some smooth periodic function ϕ, the equation

$$\phi' + \phi'_\star - \phi - \phi_\star = 0$$

implies that the third-order derivative of ϕ is zero, and the periodicity then imposes that it is a constant. Thus the null space is spanned by e_0. It concludes the proof of the existence of a spectral gap

$$\lambda_N := \min \left\{ |a_k|, \ k \in [|-N, N|]^d, \ k \neq 0 \right\} > 0$$

which is uniformly bounded from below as $N \to +\infty$, since

$$\lambda_N \to \lambda_\infty := \min\left\{|a_k|,\ k \in \mathbb{Z}^d,\ k \neq 0\right\} > 0.$$

The non-linear stability in L^2 comes from the fact that for the perturbation $h = f - m_\infty$, we have the following control on the bilinear part:

$$\|\mathcal{P}_N Q^R(h,h)\|_{H^k_{\rm per}} \leq C_{\mathcal{B},R}\,\|h\|^2_{L^2}$$

for some given constant $C_{\mathcal{B},R} > 0$ independent on N.

Finally using the eigenbasis of the Fourier modes, a similar study can be performed in any Sobolev space $H^k_{\rm per}(\mathcal{D}_L)$. □

Remark: Exact computations could be made for particular physical collision kernels $\mathcal{B}$ (in a similar way as the computation of the kernel modes $\beta(l,m)$ in [11, 12, 21, 14]).

6.1. **Proof of Theorem 6.1.** Consider the numerical solution f_N given by solving (6.1). We can formulate the problem as a perturbation of the truncated Boltzmann equation. Indeed setting

$$P_N(f_N) = -(Id - \mathcal{P}_N)Q^R(f_N, f_N),$$

the problem (6.1) can be written as

$$\frac{\partial f_N}{\partial t} = Q^R(f_N, f_N) + P_N(f_N).$$

Then, applying Lemma 6.2, the perturbation P_N satisfies the assumptions of Theorem 3.1. Moreover since $f_0 \in H^k_{\rm per}(\mathcal{D}_L)$, we have straightforwardly

$$\|f_N(0)\|_{H^p_{\rm per}} \leq \|f_0\|_{H^p_{\rm per}}, \quad \|f_N(0) - f_0\|_{H^p_{\rm per}} \to 0.$$

Hence, we can directly apply Theorem 3.1 to the perturbation P_N, which proves that there exists N_0 large enough and only depending on f_0, the kernel B and the truncation, such that for all $N \geq N_0$, the perturbed system admits a unique uniformly smooth solution, which converges to a constant, and satisfies all the points in Theorem 3.1.

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