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MODELING AND NUMERICAL SIMULATION OF SPACE CHARGE DOMINATED BEAMS IN THE PARAXIAL APPROXIMATION

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This work is devoted to the modeling of space charge dominated particle beams in the paraxial approximation with several types of external focusing fields (uniform, periodic and alternating gradient). A solid mathematical background for numerical beam simulation is established. The Kapchinsky–Vladimirsky (KV) distribution which can be matched exactly or numerically to the focusing channel is first studied. Then the matched KV beam is used to approximately match arbitrary beams. Moreover, Waterbag and Maxwell–Boltzmann beams are studied to give analytical solution for code validation. Finally, numerical simulations are presented in different configurations.

Keywords: Kinetic equations; Vlasov equation; paraxial approximation; charged particle beam.

AMS Subject Classification: 65M12, 82D10

1. Introduction

A model which can be used in many cases for the study of plasma as well as beam propagation is the Vlasov equation coupled with the Maxwell or Poisson equations to compute the self-consistent fields. It describes the evolution of a system of particles under the effects of external and self-consistent fields. The unknown $f(t, x, v)$, depending on the time t, the position x, and the velocity v, represents the distribution of particles in phase space for each species. However, the numerical solution of the Vlasov–Maxwell system in phase space requires an important computational effort. In such a situation, we have to take into account the particularities of the physical problem (characteristic length, geometric and physical characteristics) to derive approximate models leading to cheaper computations.

This paper is devoted to the analysis and numerical study of the paraxial model, which is often used in accelerator physics for analyzing the propagation of beams possessing an optical axis, which is assumed to be a straight line. For a physicist's derivation of this model one can refer to the recent book by Davidson and Qin.[1]

In a previous study Degond and Raviart[2,12] presented a rigorous study of the paraxial model as an approximation of the stationary Vlasov–Maxwell equations. The particles of the beam remain close to its optical axis, so that the transverse width l of the beam is very small compared to a characteristic length L, and have about the same kinetic energy. A scaling of the Vlasov–Maxwell equations is given which reflects the geometrical and physical characteristics of a paraxial beam. In the scaled equation $\eta = l/L$ plays the role of a small parameter and the asymptotic expansions of various physical quantities in powers of η are derived. Finally, we give a complete analysis of the linear model: Existence of invariants, presentation of KV and Waterbag distributions, which are exact solutions of the paraxial model.

In Ref. 9, the authors use an asymptotic expansion technique to treat the case of high energy short beams, considering a bunch of highly relativistic charged particles in the interior of a perfectly conducting hollow tube. A new paraxial model is derived using a frame which moves along the optical axis with the light velocity and the bunch is evolving slowly. We also mention the work of Nouri[10] on axisymmetric laminar beams.

These works are really interesting from a mathematical point of view since small parameters are clearly identified and derivations of models are rigorous. However, they do not give any information about the numerical simulation of beams obeying these models.

The aim of the present work is to do the link between the mathematical models and the numerical simulation of actual beams. In particular we describe how a specific beam called a KV beam can be matched to the external focusing fields and how the matched KV beam can be used to approximately match an arbitrary beam.

In Sec. 2, we give a review of the paraxial model obtained from the Vlasov–Maxwell equations. Then, we present an intuitive existence proof of a KV solution for the paraxial model and give some definitions useful for beam focusing in a periodic framework. We next focus our attention to the axisymmetric case, for which we give a new formulation using the invariance of the canonical momentum. This formulation is particularly well adapted for numerical computations. For this model, we give a rigorous presentation of stationary solutions in the nonlinear case (KV and waterbag solutions). Finally, a simple method based on a time splitting scheme for the numerical resolution of the axisymmetric case is presented with different numerical tests illustrating the accuracy of the method.

2. The Paraxial Model

The stationary relativistic Vlasov equation reads

$$\mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{f} + q(E + \mathbf{v} \times B) \cdot \nabla_{\mathbf{p}} \tilde{f} = 0, \tag{1}$$

where q is the charge and m is the mass of one particle, c the velocity of light in free space, $\mathbf{v} = \frac{\mathbf{p}}{\gamma m}$, $\beta = |\mathbf{v}|/c$, $\gamma = (1 + \mathbf{p}^2/(mc)^2)^{1/2}$ or equivalently $\gamma = (1-\beta^2)^{-1/2}$, $\tilde{f}(\mathbf{x}, \mathbf{p})$ represents the distribution function of one species of particles (ions, electrons), depending on the position $\mathbf{x} \in \mathbb{R}^3$ and momentum $\mathbf{p} \in \mathbb{R}^3$. From the distribution function $\tilde{f}$, we compute the charge and current densities

$$\rho(\mathbf{x}) = q \int_{\mathbb{R}^3} \tilde{f}(\mathbf{x}, \mathbf{p}) d\mathbf{p}, \quad \mathbf{j}(\mathbf{x}) = q \int_{\mathbb{R}^3} \mathbf{v} \tilde{f}(\mathbf{x}, \mathbf{p}) d\mathbf{p} \tag{2}$$

and the electromagnetic fields $\mathbf{E}$ (electric field) and $\mathbf{B}$ (magnetic field) are given by the steady-state Maxwell equations

$$\begin{cases} \nabla \times \mathbf{B} = -\mu_0 \mathbf{j}, \\ \nabla \times \mathbf{E} = 0, \\ \nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}, \quad \nabla \cdot \mathbf{B} = 0. \end{cases} \tag{3}$$

The numerical solution of the full Vlasov–Maxwell system can be extremely expensive in computer time. Therefore, whenever possible, it is essential to find simplified models which approximate the Maxwell equations in some sense.

In the case of particle beams, we can derive a simplified model based on the following assumption which are often satisfied in physical problems involving particle beams.

- The beam is steady-state: All partial derivatives with respect to time vanish.
- The beam is sufficiently long so that longitudinal self-consistent forces can be neglected.
- The beam is propagating at constant velocity v_b along the propagation axis z.
- Electromagnetic self-forces are included.
- $\mathbf{p} = (p_x, p_y, p_z)$, $p_z \sim p_b$ and $p_x, p_y \ll p_b$ where $p_b = \gamma m v_b$ is the beam momentum. It follows in particular that

$$\beta \approx \beta_b = (v_b/c)^2, \quad \gamma \approx \gamma_b = (1 - \beta_b^2)^{-1/2}$$

- the beam is thin: the transverse dimensions of the beam are small compared to the characteristic longitudinal dimension.

The paraxial model of approximation of Vlasov–Maxwell's equations is obtained by retaining only the first terms in the asymptotic expansion of the distribution function and the electromagnetic fields with respect to $\eta = l/L$, where l denotes the transverse characteristic length and L is the longitudinal characteristic length,[2] i.e. we assume that

$$\eta = l/L \ll 1.$$

Moreover, for simplicity we will neglect the variation with respect to the longitudinal mean velocity v_b.

Let us now introduce $\mathbf{x} = (x, y)$, $\mathbf{v} = (v_x, v_y)$ and

$$f = f(z, \mathbf{x}, \mathbf{v}), \quad \Phi = \Phi(\mathbf{x}, z), \quad \mathbf{B} = (B_x(z, \mathbf{x}), B_y(z, \mathbf{x})),$$

where the new distribution function f is linked to the solution $\tilde{f}$ of the original Vlasov equation (1) by $\tilde{f}(x, y, z, p_x, p_y, p_z) = f(z, \mathbf{x}, \mathbf{v})\delta(p_z - p_b)$. Then, making the assumptions above, f is a solution (in the sense of distributions) of

$$v_b \frac{\partial f}{\partial z} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \frac{q}{\gamma_b m} \mathbf{F} \cdot \nabla_{\mathbf{v}} f = 0, \tag{4}$$

with $v_\lambda = p_\lambda/(\gamma_b\, m)$, $\lambda \in \{x, y\}$ and

$$\mathbf{E} = -\nabla_{\mathbf{x}} \Phi, \quad -\Delta_{\mathbf{x}} \Phi = qn/\varepsilon_0, \quad n = \int_{\mathbb{R}^2} f d\mathbf{v}, \tag{5}$$

$$\begin{cases} \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} = \mu_0 q v_b n, \\ \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} = -\dfrac{dB_z}{dz} \end{cases} \tag{6}$$

and $\mathbf{F} = (F_x, F_y)$ is given by

$$F_x = -\frac{\partial \Phi}{\partial x} - v_b B_y + v_y B_z, \quad F_y = -\frac{\partial \Phi}{\partial y} + v_b B_x - v_x B_z.$$

Since Maxwell equations are linear, we can split the transverse fields into their external and self-consistent parts:

$$\mathbf{E} = \mathbf{E}^{\mathrm{e}} + \mathbf{E}^{\mathrm{s}}, \quad \mathbf{B} = \mathbf{B}^{\mathrm{e}} + \mathbf{B}^{\mathrm{s}},$$

which are respectively of the forms

$$\mathbf{E}^{\mathrm{e}} = -\nabla_{\mathbf{x}} \Phi^{\mathrm{e}}, \quad \mathbf{B}^{\mathrm{e}} = -\nabla_{\mathbf{x}} \chi^{\mathrm{e}} \tag{7}$$

and

$$\mathbf{E}^{\mathrm{s}} = -\nabla_{\mathbf{x}} \Phi^{\mathrm{s}}, \quad \mathbf{B}^{\mathrm{s}} = \mathbf{curl}_{\mathbf{x}} \psi^{\mathrm{s}} = (\partial_y \psi^{\mathrm{s}}, -\partial_x \psi^{\mathrm{s}}). \tag{8}$$

On the one hand, from Eq. (6), we check that the functions ϕ^{e} and χ^{e} satisfy

$$-\Delta_{\mathbf{x}} \Phi^{\mathrm{e}} = 0, \quad -\Delta_{\mathbf{x}} \chi^{\mathrm{e}} = -\frac{dB_z}{dz}, \tag{9}$$

where B_z is an external magnetic field.

On the other hand, from Eqs. (5) and (6) the self-consistent forces satisfy

$$-\Delta_{\mathbf{x}} \Phi^{\mathrm{s}} = qn/\varepsilon_0, \quad -\Delta_{\mathbf{x}} \psi^{\mathrm{s}} = \mu_0 v_b q n = q v_b n/(\varepsilon_0 c^2). \tag{10}$$

Then, we have

$$\psi^{\mathrm{s}} = \frac{v_b}{c^2} \Phi^{\mathrm{s}}$$

and the self-consistent force field is given by

$$\frac{q}{\gamma_b m} F_x^{\mathrm{s}} = \frac{q}{\gamma_b m}\left(-\frac{\partial \Phi^{\mathrm{s}}}{\partial x} - v_b B_y^{\mathrm{s}}\right) = -\frac{q}{\gamma_b m}(1-\beta_b^2)\frac{\partial \Phi^{\mathrm{s}}}{\partial x} = -\frac{q}{\gamma_b^3 m}\frac{\partial \Phi^{\mathrm{s}}}{\partial x}, \tag{11}$$

$$\frac{q}{\gamma_b m} F_y^{\mathrm{s}} = \frac{q}{\gamma_b m}\left(-\frac{\partial \Phi^{\mathrm{s}}}{\partial y} + v_b B_x^{\mathrm{s}}\right) = -\frac{q}{\gamma_b m}(1-\beta_b^2)\frac{\partial \Phi^{\mathrm{s}}}{\partial y} = -\frac{q}{\gamma_b^3 m}\frac{\partial \Phi^{\mathrm{s}}}{\partial y}, \tag{12}$$

where $\beta = v_b/c$.

For the external forces, we will consider three different types of external focusing forces which are mostly used in accelerators for modeling purposes:

1. Uniform focusing by a continuous electric field of the form

$$\mathbf{E}(\mathbf{x}) = -\frac{\gamma_b\, m}{q}\omega_0^2\,(x\mathbf{e}_x + y\mathbf{e}_y).$$

2. Periodic focusing by a magnetic field of the form

$$\mathbf{B}(\mathbf{x}) = B(z)\,\mathbf{e}_z - \frac{1}{2}B'(z)\,(x\mathbf{e}_x + y\mathbf{e}_y),$$

where the longitudinal component of the magnetic field $B(z)$ is given and satisfies the periodicity condition $B(z+S) = B(z)$.

3. Alternating gradient focusing:

- either by a magnetic field of the form

$$\mathbf{B}(\mathbf{x}) = B'(z)\,(y\mathbf{e}_x + x\mathbf{e}_y),$$

which corresponds to a potential

$$\psi^{\mathrm{e}}(\mathbf{x}) = -\frac{1}{2}B'(z)(x^2 - y^2),$$

- or by an electric field of the form

$$\mathbf{E}(\mathbf{x}) = E'(z)\,(x\mathbf{e}_x - y\mathbf{e}_y),$$

which corresponds to a potential

$$\Phi^{\mathrm{e}}(\mathbf{x}) = -\frac{1}{2}E'(z)(x^2 - y^2).$$

The Vlasov equation (4) we consider can be interpreted as dividing all the terms by the strictly positive velocity v_b, as a transverse Vlasov equation where z plays the role of time. We then get the paraxial model

$$\frac{\partial f}{\partial z} + \frac{\mathbf{v}}{v_b}\cdot\nabla_{\mathbf{x}} f + \frac{q}{\gamma_b m v_b}\left(-\frac{1}{\gamma_b^2}\nabla\Phi^{\mathrm{s}} + \mathbf{E}^{\mathrm{e}} + (\mathbf{v}, v_b)^{\mathrm{T}} \times \mathbf{B}^{\mathrm{e}}\right)\cdot\nabla_{\mathbf{v}} f = 0, \tag{13}$$

coupled with the Poisson equation

$$-\Delta_{\mathbf{x}}\Phi^{\mathrm{s}} = \frac{q}{\varepsilon_0}\int_{\mathbb{R}^2} f(z,\mathbf{x},\mathbf{v})\,d\mathbf{v}. \tag{14}$$

The characteristic curves associated to this Vlasov equation are given by

$$\begin{cases} x' = \dfrac{v_x}{v_b}, \\ y' = \dfrac{v_y}{v_b}, \\ v_x' = -\dfrac{q}{\gamma_b^3 m v_b}\dfrac{\partial\Phi^{\mathrm{s}}}{\partial x} + \dfrac{q}{\gamma_b m v_b}\left(E_x^{\mathrm{e}} + v_y B_z^{\mathrm{e}} - v_b B_y^{\mathrm{e}}\right), \\ v_y' = -\dfrac{q}{\gamma_b^3 m v_b}\dfrac{\partial\Phi^{\mathrm{s}}}{\partial y} + \dfrac{q}{\gamma_b m v_b}\left(E_y^{\mathrm{e}} - v_x B_z^{\mathrm{e}} + v_b B_x^{\mathrm{e}}\right), \end{cases} \tag{15}$$

where the notation $'$ corresponds to the derivative with respect to the longitudinal variable z.

The paraxial model is much simpler than the full Vlasov–Maxwell model. On the one hand, one replaces the stationary Vlasov equation by the paraxial Vlasov equation (4) where the longitudinal coordinates z play the role of a time variable and can be solved numerically by a marching procedure with an initial datum at $z = 0$. On the other hand, the stationary Maxwell's equations are replaced by the two-dimensional Poisson equations (10) where z only acts as a parameter.

3. The Kapchinky–Vladimirsky (KV) Distribution

We introduce two S-periodic functions of z, κ_x and κ_y which are given by the focusing forces and $N_0 = \int f\,d\mathbf{x}\,d\mathbf{v}$ the total number of particles and the parameter K which is a dimensionless parameter linked to the self-consistent force called *perveance*

$$K = \frac{q^2 N_0}{2\pi\varepsilon_0\gamma_b^3 m v_b^2}.$$

The perveance is linked to the plasma frequency ω_p by the relation

$$K = \frac{\omega_p^2 a^2}{2\gamma_b^2 v_b^2},$$

and ε_x, ε_y are parameters, called *emittance*, linked to the constant areas of ellipses in the planes $x - x'$ and $y - y'$.

Theorem 1. *We assume that $\kappa_x(z)$ and $\kappa_y(z)$ satisfy*

$$\frac{d}{dz}\kappa_x(z) \le C\kappa_x(z), \quad \frac{d}{dz}\kappa_y(z) \le C\kappa_y(z), \qquad z \in [0,S]. \tag{16}$$

Let us define the KV distribution[7]

$$f(z,x,y,v_x,v_y) = \frac{N_0}{\pi^2\,\varepsilon_x\,\varepsilon_y}\delta_0\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{(a\frac{v_x}{v_b}-a'x)^2}{\varepsilon_x^2}+\frac{(b\frac{v_y}{v_b}-b'y)^2}{\varepsilon_y^2}-1\right), \tag{17}$$

where a, b *are the solutions of the so-called envelope equations* (*see Refs.* 1 *and* 13)

$$a''+\kappa_x(z)\,a-\frac{2K}{a+b}-\frac{\varepsilon_x^2}{a^3}=0,\quad b''+\kappa_y(z)\,b-\frac{2K}{a+b}-\frac{\varepsilon_y^2}{b^3}=0, \tag{18}$$

where a' *denotes the derivative with respect to* z. *Then,* f *is a measure solution of the paraxial model* (13)–(14) *and the self-consistent electric field solution of the Poisson equation satisfies*

$$E^s(x,y) = \begin{cases} \dfrac{qN_0}{\pi\varepsilon_0(a+b)}\begin{pmatrix} x/a \\ y/b \end{pmatrix} & \textit{if } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\le 1, \\ \to 0 & \textit{as } x,y\to\infty. \end{cases} \tag{19}$$

Let us first give a sense to the distribution f, which can be understood, thanks to the co-area formula (see Ref. 3). Indeed, define $H_z = H_z(x,y,v_x,v_y)$ by

$$H_z(x,y,v_x,v_y) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{(a\frac{v_x}{v_b}-a'x)^2}{\varepsilon_x^2}+\frac{(b\frac{v_y}{v_b}-b'y)^2}{\varepsilon_y^2}.$$

Then for any $e\in H_z(\mathbb{R}^4)$, the manifold

$$H_z^{-1}(e) = \left\{(x,y,v_x,v_y)\in\mathbb{R}^4;\ e = H_z(x,y,v_x,v_y)\right\}$$

is a hypersurface of $\mathbb{R}^4$. Denote by $dS(x,y,v_x,v_y)$ its Euclidean surface element and by $N_z(e)$ the density of states e:

$$N_z(e) = \int_{k\in H_z^{-1}(e)} \frac{dS(k)}{\|\nabla_k H_z(k)\|},$$

where $\|\nabla_k H_z(k)\|$ denotes the Jacobian of $H_z(\cdot)$ with respect to $k\in\mathbb{R}^4$. The distribution f given by (17) is then a measure of $\mathbb{R}^4$ defined as follows: for any Borel set B of $\mathbb{R}^4$

$$f(z)(B) = \int_{k\in B}\frac{dS(k)}{\|\nabla_k H_z(k)\|}. \tag{20}$$

Before proving this theorem, let us give the main arguments. We will first compute some invariants of the Vlasov equation (13) when the total electromagnetic force is linear with respect to (x,y). Then, we will prove that the KV distribution is a measure solution of the Vlasov equation. Finally, we will consider the coupling of the Vlasov and Poisson equations to conclude the proof.

Let us first consider the Vlasov equation (13) with a total force field linear with respect to (x, y), the characteristic curves are then given by the Mathieu–Hill differential equations

$$X' = \frac{V_x}{v_b}, \quad V_x' = -\overline{\kappa}_x(z)\, v_b X, \quad Y' = \frac{V_y}{v_b}, \quad V_y' = -\overline{\kappa}_y(z)\, v_b Y, \tag{21}$$

where $\overline{\kappa}_x(z)$ and $\overline{\kappa}_y(z)$ characterize the total (self-consistent + applied) linear force field. Many systems, such as quadrupole channels, have two planes of symmetry, where the forces may differ. However, as long as there is no coupling between these two forces, the theory is the same, and in the following, we will only consider the motion in the x-direction.

Proposition 1. *(X, V_x) is a solution of* (21) *if and only if it satisfies*

$$\frac{X^2}{a^2} + \frac{\left(a\frac{V_x}{v_b} - a'X\right)^2}{\varepsilon_x^2} = 1, \tag{22}$$

where $a = \sqrt{\varepsilon_x}\, w_x$ and

$$w_x'' + \overline{\kappa}_x(z)\, w_x - \frac{1}{w_x^3} = 0, \tag{23}$$

where $w_x(z)$ is a function of z.

Proof. Let us consider the equation of motion for one particle in the linear force field:

$$X' = \frac{V_x}{v_b}, \quad V_x' = -\overline{\kappa}_x(z)\, v_b X. \tag{24}$$

The generalized solution of second order ODE, $X'' + \overline{\kappa}_x(z)X = 0$, which comes from (24) and can be expressed as

$$u(z) = w_x(z)\, \exp(+i\, \Psi_x(z)), \quad v(z) = w_x(z)\, \exp(-i\, \Psi_x(z)). \tag{25}$$

Then, substituting the solution (25) into Eq. (24)

$$w_x'' + 2\, i\, w_x'\, \Psi_x' + \left(\overline{\kappa}_x(z) + i\, \Psi_x'' - (\Psi_x')^2\right) w_x = 0,$$

and the real and imaginary parts of this last term respectively satisfy the following relations

$$w_x'' + \left(\overline{\kappa}_x(z) - (\Psi_x')^2\right) w_x = 0, \tag{26}$$

$$2\, w_x'\, \Psi_x' + w_x\, \Psi_x'' = 0. \tag{27}$$

From relation (27), we get an expression for the function Ψ_x

$$\Psi_x' = C/w_x^2, \tag{28}$$

where the constant C only depends on the initial data. Let us set C to one to define w_x and reintroduce the dependency on the initial data through a constant ε_x.

Although w_x is not explicit known, it satisfies the following differential equation, which is obtained from (26) and (28)

$$w_x'' + \overline{\kappa}_x(z)\, w_x - \frac{1}{w_x^3} = 0.$$

Any real solution of (24) can be written as a linear combination of u and v in the following form

$$X(z) = \sqrt{\varepsilon_x}\, w_x(z)\, \cos(\Psi_x(z) + \phi_0),$$

where ε_x and ϕ_0 only depend on the initial data. The derivative is then given by

$$X'(z) = \sqrt{\varepsilon_x} \left(w_x'(z)\, \cos(\Psi_x(z) + \phi_0) - \frac{1}{w_x(z)}\, \sin(\Psi_x(z) + \phi_0) \right).$$

Moreover,

$$\begin{aligned}(w_x\, X')^2 &= (w_x')^2 X^2 + \varepsilon_x\, \sin^2(\Psi_x + \phi_0) - 2\, w_x'\, X\, \sqrt{\varepsilon_x} \sin(\Psi_x + \phi_0) \\ &= (w_x')^2 X^2 + \varepsilon_x - \frac{X^2}{w_x^2} - 2\, w_x'\, X\, (w_x'\, X - w_x\, X')\,.\end{aligned}$$

Then, introducing the so-called Twiss parameters (see Refs. 1 and 13).

$$\beta_x = w_x^2, \quad \alpha_x = -w_x\, w_x', \quad \gamma_x = (1 + \alpha_x^2)/\beta_x = 1/w_x^2 + (w_x')^2,$$

we get all particles with the same initial ε_x, but a different ψ_0 will lie on the same phase space ellipse determined by α_x, β_x and γ_x

$$\gamma_x\, X^2 + 2\, \alpha_x\, X\, \frac{V_x}{v_b} + \beta_x\, \frac{V_x^2}{v_b^2} = \varepsilon_x. \tag{29}$$

Finally, we set $a = \sqrt{\varepsilon_x}\, w_x$. Then, we get an invariant of the Vlasov equation in the following form

$$\frac{X^2}{a^2} + \frac{\left(a \frac{V_x}{v_b} - a' X\right)^2}{\varepsilon_x^2} = 1.$$

The existence and uniqueness of a smooth solution local in time come from the smoothness of the focusing function. Moreover, from the following estimate

$$\frac{d}{dz}\left((w_x')^2 + \overline{\kappa}_x(z)\, w_x^2 + \frac{1}{w_x^2} \right) = \frac{d}{dz}(\overline{\kappa}_x(z))\; w_x^2 \le C\, \kappa_x(z)\, w_x^2$$

we conclude that the solution is global in time (see Ref. 2). □

Proposition 1 implies that (22) is an invariant of the Vlasov equation for which the characteristic curves yield the Mathieu–Hill equations (21). Then, $N_0, \varepsilon_x, \varepsilon_y$ being given, let us consider the KV distribution (17)

$$f(z, x, y, v_x, v_y) = \frac{N_0}{\pi^2\, \varepsilon_x\, \varepsilon_y} \delta_0 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(a \frac{v_x}{v_b} - a' x)^2}{\varepsilon_x^2} + \frac{(b \frac{v_y}{v_b} - b' y)^2}{\varepsilon_y^2} - 1 \right), \tag{30}$$

where $a = \sqrt{\varepsilon_x}\, w_x$, $b = \sqrt{\varepsilon_y}\, w_y$ and w_x and w_y are solutions of an equation of type (23).

Proposition 2. *The KV distribution given by* (17) *is a measure solution of the Vlasov equation defined by the characteristic curves* (21)*.*

Before giving the proof of Proposition 2, let us recall the definition of a measure solution of the Vlasov equation.[11]

Consider the transport equation

$$\frac{\partial u}{\partial t} + \mathrm{div}_x(a(t,x)u(t,x)) = 0, \tag{31}$$

where $a \in L^1_{\mathrm{loc}}(\mathbb{R}^+; C^1(\mathbb{R}^n))^n$. For every $u_0 \in \mathcal{M}^1(\mathbb{R}^n)$, the solution of (31) is given by : for any Borel set B,

$$u(t)(B) = u_0(x^{-1}(t)(B)),$$

where the continuous map $x(t)$ is a solution of

$$\frac{dx}{dt} = a(t, x(t)), \quad x(0) = x.$$

Proof of Proposition 2. Let us denote by $\mathbf{x}(z) = (X(z), Y(z), V_x(z), V_y(z))$ the characteristic curves solution of

$$X' = \frac{V_x}{v_b}, \quad V_x' = -\overline{\kappa}_x(z) v_b\, X,$$
$$Y' = \frac{V_y}{v_b}, \quad V_y' = -\overline{\kappa}_y(z) v_b\, Y.$$

From the definition of the KV distribution given by (20), the result becomes straightforward. Indeed, for any Borel set B of $\mathbb{R}^4$

$$\begin{aligned} f(z)(B) &= \int_{k \in B} \frac{dS(k)}{\|\nabla_k H_z(k)\|} = \int_{k\in B} \frac{dS(k)}{\|\nabla_k H_0(\mathbf{x}^{-1}(z,k)\|} \\ &= \int_{k' \in \mathbf{x}^{-1}(z)(B)} \frac{dS(k')}{\|\nabla_{k'} H(0,k')\|} \\ &= f_0\left(\mathbf{x}^{-1}(z)\,(B)\right). \end{aligned}$$

□

Now, we will consider the coupling of the Vlasov equation with the Poisson equation, in order to take into account self-consistent electromagnetic fields.

Proposition 3. *Let us consider the KV distribution* (17)*. Then, the space charge density is given by*

$$qn(x,y) = \begin{cases} \dfrac{qN_0}{\pi ab}, & \textit{if } \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1, \\ 0, & \textit{otherwise.} \end{cases} \tag{32}$$

and the self-consistent electric field solution of the Poisson equation is

$$E^{\mathrm{s}}(x,y) = \begin{cases} \dfrac{qN_0}{\pi\varepsilon_0\,(a+b)}\begin{pmatrix} x/a \\ y/b \end{pmatrix} & \text{if } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} \leq 1, \\ \to 0 & \text{as } x,y \to \infty. \end{cases}$$

Proof. Let us first compute the space charge density, which is obtained by integrating the KV distribution with respect to (v_x, v_y)

$$qn(x,y) = \frac{qN_0}{\pi^2\varepsilon_x\varepsilon_y}\int_{\mathbb{R}^2}\delta_0\Bigg(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{(a\frac{v_x}{v_b}-a'x)^2}{\varepsilon_x^2}+\frac{(b\frac{v_y}{v_b}-b'y)^2}{\varepsilon_y^2}-1\Bigg)dv_x\,dv_y.$$

In fact this "integral" can be restricted to the following domain:

$$S(x,y,z) = \left\{(v_x,v_y)\in\mathbb{R}^2;\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{(a\frac{v_x}{v_b}-a'x)^2}{\varepsilon_x^2}+\frac{(b\frac{v_y}{v_b}-b'y)^2}{\varepsilon_y^2}\leq 1\right\}.$$

Then, setting

$$\overline{v}_x = \frac{a\frac{v_x}{v_b}-a'x}{\varepsilon_x}, \quad \overline{v}_y = \frac{b\frac{v_y}{v_b}-b'y}{\varepsilon_y},$$

we get

$$qn(x,y) = \frac{qN_0}{\pi^2 ab}\int_{\overline{S}}\delta_0\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\overline{v}_x^2+\overline{v}_y^2-1\right)d\overline{v}_x\,d\overline{v}_y,$$

where

$$\overline{S} = \left\{(\overline{v}_x,\overline{v}_y)\in\mathbb{R}^2;\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\overline{v}_x^2+\overline{v}_y^2\leq 1\right\}.$$

Making the change of variables $\overline{v}_x = \sqrt{R}\cos\theta$, $\overline{v}_y = \sqrt{R}\sin\theta$ of Jacobian $\frac{1}{2}$ and using the invariance by rotation

$$\begin{aligned} qn(x,y) &= \frac{qN_0}{\pi ab}\int_{\mathbb{R}^+}\delta_0\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+R-1\right)dR, \\ &= \begin{cases} \dfrac{qN_0}{\pi ab} & \text{if } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\leq 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Finally, since the space charge is uniform inside the ellipse, we can explicitly compute the electric field. Indeed the electric field $\mathbf{E}^{\mathrm{s}}$ satisfies $\operatorname{curl}\mathbf{E}^{\mathrm{s}} = 0$ and

$$\operatorname{div}\mathbf{E}^{\mathrm{s}} = \frac{qN_0}{\pi\varepsilon_0(a+b)}\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{qN_0}{\varepsilon_0\pi ab},$$

which yields the electric field

$$E^s(x,y) = \begin{cases} \dfrac{qN_0}{\pi\varepsilon_0\,(a+b)} \begin{pmatrix} x/a \\ y/b \end{pmatrix} & \text{if } \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1, \\ \to 0 & \text{as } x, y \to \infty. \end{cases}$$ □

From the above expression of the electric field, it follows that the self-consistent force F_x^{s} given by (11) becomes

$$\frac{q}{\gamma_b m} F_x^{\mathrm{s}} = \frac{q}{\gamma_b^3 m} \frac{qN_0}{\pi\varepsilon_0(a+b)} \frac{x}{a} = \frac{2\,K v_b^2}{(a+b)\,a}\, x,$$

and in the same way F_y^{s} given by (12) becomes

$$\frac{q}{\gamma_b m} F_y^{\mathrm{s}} = \frac{q}{\gamma_b^3 m} \frac{qN_0}{\pi\varepsilon_0(a+b)} \frac{y}{b} = \frac{2\,K v_b^2}{(a+b)\,b}\, y.$$

Now, we can easily prove Theorem 1.

Proof of Theorem 1. Consider a linear force field of the following form

$$\overline{\kappa}_x(z) = \kappa_x(z) - \frac{2K}{a\,(a+b)}, \quad \overline{\kappa}_y(z) = \kappa_y(z) - \frac{2K}{b\,(a+b)},$$

where κ_x and κ_y determine the external focusing force and $K = q^2 N_0/(2\,m\,\pi\,\varepsilon_0\,v_b^2)$ the self-consistent force. Since the force field is linear we can apply Proposition 1, which gives an invariant of the Vlasov equation under the condition that a, b are solution of (18)

$$a'' + \kappa_x(z)\,a - \frac{2K}{a+b} - \frac{\varepsilon_x^2}{a^3} = 0, \quad b'' + \kappa_y(z)\,b - \frac{2K}{a+b} - \frac{\varepsilon_y^2}{b^3} = 0.$$

This system of ODEs has a unique local solution since $\kappa_x(z)$ and $\kappa_y(z)$ are smooth enough. Moreover, from the following estimates:

$$\frac{d}{dz}\left((a')^2 + (b')^2 + \kappa_x(z)\,a^2 + \kappa_y(z)\,b^2) + 2K\log(a+b)\; + \frac{\varepsilon_x^2}{a^2} + \frac{\varepsilon_y^2}{b^2} \right) \\ \leq C\,(\kappa_x(z)\,a^2 + \kappa_y(z)\,b^2),$$

we can deduce that the functions a and b are non-negative and bounded. Thus, the solution is global in time.

Applying Proposition 2, we prove that the KV distribution function f given by (17) is a solution of the Vlasov equation (15). Finally, from Proposition 3, we can check that the space charge density is uniform inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

and the self-consistent electric field solution of the Poisson equation is linear with respect to (x,y) inside this ellipse, which concludes the proof. □

4. Envelope Equations

Let us now derive explicitly the envelope equations in the three focusing configurations we are interested in.

1. Uniform focusing. The focusing field is given by

$$\mathbf{E}^{\rm e}(\mathbf{x}) = -\frac{\gamma_b m}{q}\omega_0^2\,(x\mathbf{e}_x + y\mathbf{e}_y).$$

   Using the characteristics equations and the expression of the self-consistent field we get

$$X'' = \frac{V_x'}{v_b} = \frac{q}{\gamma_b^3 m v_b^2}E_x^{\rm s} + \frac{q}{\gamma_b m v_b^2}E_x^{\rm e} = \frac{2K}{(a+b)a}X - \frac{\omega_0^2}{v_b^2}X$$

   and

$$Y'' = \frac{2K}{(a+b)a}Y - \frac{\omega_0^2}{v_b^2}Y.$$

   We thus obtain a Matthieu–Hill equation from which we deduce the envelope equations

$$\begin{cases} a'' + k_0^2\,a - \dfrac{2K}{a+b} = \dfrac{\varepsilon_x}{a^3}, \\ b'' + k_0^2\,b - \dfrac{2K}{a+b} = \dfrac{\varepsilon_y}{b^3}, \end{cases}$$

   where we denote by $k_0 = \omega_0/v_b$. As k_0 and K do not depend on z, these equations admit solutions such that $a' = a'' = b' = b'' = 0$ and $a = b$ if $\varepsilon_x = \varepsilon_y$. Looking for such a solution, we get

$$k_0^2 a^4 - K a^2 - \varepsilon_x^2 = 0.$$

   Taking the positive root of this equation, we obtain

$$a = \sqrt{\frac{K + \sqrt{K^2 + 4\,k_0^2\,\varepsilon_x^2}}{2\,k_0^2}}. \tag{33}$$

2. Periodic focusing by a magnetic field of the form

$$\mathbf{B}^{\rm e}(\mathbf{x}) = B(z)\,\mathbf{e}_z - \frac{1}{2}\,B'(z)\,(x\mathbf{e}_x + y\mathbf{e}_y),$$

   where B is S-periodic. Proceeding as in the previous case and plugging in the expression of $\mathbf{B}^{\rm e}$, we get for an axisymmetric beam such that $a = b$

$$X'' = \frac{K}{a^2}\,X + \frac{q}{\gamma_b\,m\,v_b^2}\left(V_y\,B + v_b\,\frac{B'}{2}\,Y\right),$$
$$Y'' = \frac{K}{a^2}\,Y - \frac{q}{\gamma_b\,m\,v_b^2}\left(V_x\,B + v_b\frac{B'}{2}\,X\right).$$

We then introduce the relativistic Larmor frequency divided by the beam velocity

$$\omega_L(z) = \frac{qB(z)}{2\gamma_b m v_b}.$$

It follows that

$$X'' = \frac{K}{a^2}\,X + 2\,\omega_L Y' + \omega_L' Y, \quad Y'' = \frac{K}{a^2}\,Y - 2\,\omega_L X' - \omega_L' X. \tag{34}$$

These equations take a simpler form in the Larmor frame which is rotating at the Larmor frequency. Taking for θ_L a primitive of $-\omega_L$, the coordinates (X_L, Y_L) in the Larmor frame are obtained by a rotation of angle θ_L from coordinates (X, Y) in the laboratory frame. We then have

$$X(z) = X_L(z)\cos\theta_L(z) - Y_L(z)\sin\theta_L(z),$$
$$Y(z) = X_L(z)\sin\theta_L(z) + Y_L(z)\cos\theta_L(z).$$

Taking the derivative with respect to z, we get

$$X' = (X_L' + \omega_L Y_L)\,\cos\theta_L(z) + (Y_L' - \omega_L X_L)\,\sin\theta_L(z),$$
$$Y' = (X_L' + \omega_L Y_L)\,\sin\theta_L(z) + (Y_L' - \omega_L X_L)\,\cos\theta_L(z)$$

and

$$\begin{aligned} X'' &= \left(X_L'' + 2\omega_L Y_L' + \omega_L' Y_L - \omega_L^2 X_L\right)\,\cos\theta_L(z) \\ &\quad + \left(-Y_L'' + 2\omega_L X_L' + \omega_L' X_L + \omega_L^2 X_L\right)\,\sin\theta_L(z), \\ Y'' &= \left(X_L'' + 2\omega_L Y_L' + \omega_L' Y_L - \omega_L^2 X_L\right)\,\sin\theta_L(z) \\ &\quad - \left(-Y_L'' + 2\omega_L X_L' + \omega_L' X_L + \omega_L^2 Y_L\right)\,\cos\theta_L(z). \end{aligned}$$

Plugging these expressions into (34), we obtain the particle motion equations in the Larmor frame

$$X_L'' - \frac{K}{a^2}\,X_L + \omega_L^2 X_L = 0, \quad Y_L'' - \frac{K}{a^2}\,Y_L + \omega_L^2 Y_L = 0.$$

The equations for X_L and Y_L are both Matthieu–Hill equations

$$\bar\kappa_x(z) = \bar\kappa_y(z) = \omega_L(z)^2 - \frac{K}{a^2}.$$

The envelope equation thus reads

$$a'' + \omega_L^2 a - \frac{K}{a} = \frac{\varepsilon^2}{a^3}. \tag{35}$$

The radius a of a matched beam for given K, ε and ω_L given can be computed by numerically solving for a periodic solution of this equation.

3. Alternating gradient focusing: in the case of a magnetic field of the form

$$\mathbf{B}^{\mathrm{e}}(\mathbf{x}) = B'(z)\,(y\mathbf{e}_x + x\mathbf{e}_y),$$

the particles motion equations read:

$$X'' = \frac{2K}{(a+b)a}X - \frac{qB'}{\gamma_b m v_b^2} v_b X, \quad Y'' = \frac{2K}{(a+b)a}Y + \frac{qB'}{\gamma_b m v_b^2} v_b Y.$$

We thus have Matthieu–Hill equations with

$$\bar{\kappa}_x(z) = -\frac{qB'}{\gamma_b m v_b} - \frac{K}{a^2}$$

and

$$\bar{\kappa}_y(z) = \frac{qB'}{\gamma_b m v_b} - \frac{K}{a^2}.$$

Whence the envelope equations

$$\begin{cases} a'' - \dfrac{qB'}{\gamma_b m v_b} a - \dfrac{2K}{a+b} = \dfrac{\varepsilon_x^2}{a^3}, \\ b'' + \dfrac{qB'}{\gamma_b m v_b} b - \dfrac{2K}{a+b} = \dfrac{\varepsilon_x^2}{b^3}. \end{cases} \tag{36}$$

Similarly for an electric field of the form

$$\mathbf{E}(\mathbf{x}) = E'(z)\,(x\mathbf{e}_x - y\mathbf{e}_y),$$

the particles motion equations read:

$$X'' = \frac{2K}{(a+b)a}X + \frac{qE'}{\gamma_b m v_b^2} X, \quad Y'' = \frac{2K}{(a+b)a}Y - \frac{qE'}{\gamma_b m v_b^2} Y \tag{37}$$

we thus have Matthieu–Hill equations with

$$\bar{\kappa}_x(z) = \frac{qE'}{\gamma_b m v_b^2} - \frac{K}{a^2}$$

and

$$\bar{\kappa}_y(z) = -\frac{qE'}{\gamma_b m v_b^2} - \frac{K}{a^2}.$$

Whence the envelope equations

$$\begin{cases} a'' + \dfrac{qE'}{\gamma_b m v_b^2} a - \dfrac{2K}{a+b} = \dfrac{\varepsilon_x^2}{a^3}, \\ b'' - \dfrac{qE'}{\gamma_b m v_b^2} b - \dfrac{2K}{a+b} = \dfrac{\varepsilon_x^2}{b^3}. \end{cases}$$

5. Focusing a Beam Given by Its Initial Distribution Function

A general beam given by an analytical expression can be focused, thanks to the concept of equivalent beams introduced by Sacherer[14] and Lapostolle.[8]

We shall call two beams of identical particles defined by their distribution function f equivalent if they have the same energy, the same total number of particles and the same order two moments.

We shall use this concept to identify a general beam with a matched KV beam. Hence a general beam will be focused by focusing its equivalent KV beam. This is performed through the following steps: we first compute the parameters to match the KV beam:

- In our model the beam velocity v_b and γ_b are uniquely determined by its energy.
- The beam current determines the total number of particles by the relation $N_0 = \frac{I}{qv_b}$.
- We can then compute the beam perveance

$$K = \frac{q^2 N_0}{2\pi\varepsilon_0 \gamma_b^3 m v_b^2}.$$

- We set the focusing field from the three types we consider (uniform, periodic, alternating gradient).
- These parameters being set we look numerically for an S-periodic solution of the corresponding envelope equation (18) where S is the period of the focusing lattice.

Thus, we have found the initial conditions $a(0)$ and $b(0)$ (with $a'(0) = b'(0) = 0$) of (18) for which the S-periodic KV beam is determined.

We now consider a beam whose shape is given by a known analytical expression f_0. The matched distribution function can be computed by a scaling procedure as follows. Set $x' = v_x/v_b$, $y' = v_y/v_b$ and

$$f(x, y, x', y') = N_0 f_0 \left(\frac{x}{a}, \frac{y}{b}, \frac{x'}{c}, \frac{y'}{d} \right). \tag{38}$$

The beam velocity is chosen identical to the matched KV beam and so is the total number of particles N_0. So it remains to set the parameters a, b, c and d that are chosen such that x_{rms}, y_{rms}, x'_{rms} and y'_{rms} associated to the density f are the same as those of the equivalent matched KV beam. Where we define for a function $\chi(x, y, x', y')$

$$\chi_{\mathrm{rms}}(f) = \sqrt{\frac{\int \chi(x, y, x', y')^2 f(x, y, x', y') dx\, dy\, dx'\, dy'}{\int f(x, y, x', y') dx\, dy\, dx'\, dy'}}.$$

A change of variables in the above integrals gives the relations between the rms quantities associated to f and to f_0 verifying (38):

$$x_{\mathrm{rms}}(f) = a x_{\mathrm{rms}}(f_0), \quad y_{\mathrm{rms}}(f) = b y_{\mathrm{rms}}(f_0),$$

and

$$x'_{\mathrm{rms}}(f) = c x'_{\mathrm{rms}}(f_0), \quad y'_{\mathrm{rms}}(f) = d y'_{\mathrm{rms}}(f_0).$$

Using the expression of the KV distribution given by (17), we obtain

$$x_{\text{rms}}(f_{\text{KV}}) = \frac{a_0}{2}, \qquad y_{\text{rms}}(f_{\text{KV}}) = \frac{b_0}{2},$$
$$x'_{\text{rms}}(f_{\text{KV}}) = \frac{\varepsilon_x}{2a_0}, \quad y'_{\text{rms}}(f_{\text{KV}}) = \frac{\varepsilon_y}{2b_0},$$

where $(a_0, 0)$ and $(b_0, 0)$ are the initial conditions corresponding to a periodic solution (of period S) of the envelope equation. These values a_0 and b_0 can be computed analytically using (33) in the case of uniform focusing and numerically in the other cases.

It now remains to identify the rms quantities associated to f and f_{KV} to determine the parameters a, b, c and d. It follows that

$$a = \frac{a_0}{2x_{\text{rms}}(f_0)}, \quad b = \frac{b_0}{2y_{\text{rms}}(f_0)}, \quad c = \frac{\varepsilon_x}{2a_0 x'_{\text{rms}}(f_0)}, \quad d = \frac{\varepsilon_y}{2b_0 y'_{\text{rms}}(f_0)}.$$

Example. Let us explicit this procedure in the case when f_0 is a normalized semi-Gaussian beam of the form

$$f_0(x, y, x', y') = \frac{1}{2\pi} e^{-\frac{x'^2 + y'^2}{2}}, \qquad \text{if } x^2 + y^2 < 1$$

and 0 otherwise. In this case we have

$$x_{\text{rms}}(f_0) = y_{\text{rms}}(f_0) = \frac{1}{2}, \quad x'_{\text{rms}}(f_0) = y'_{\text{rms}}(f_0) = 1,$$

so that

$$a = a_0, \quad b = b_0, \quad c = \frac{\varepsilon_x}{2a_0}, \quad d = \frac{\varepsilon_y}{2b_0}.$$

Remark. In the case of an axisymmetric beam, we have $x = r\cos\theta$, $y = r\sin\theta$, the computation of x_{rms} and y_{rms} yields

$$x_{\text{rms}} = y_{\text{rms}} = \frac{r_{\text{rms}}}{\sqrt{2}}.$$

6. The Axisymmetric Vlasov Equation

Now, we will restrict ourselves to the axisymmetric case, which plays an important role in applications. Assuming that the distribution function f and the external and self-consistent fields are invariant by rotation, i.e. independent of θ.

Another invariant of motion in the Larmor frame for an axisymmetric beam is

$$\Omega = xv_y - yv_x.$$

Let us pass to cylindrical coordinate,

$$r^2 = x^2 + y^2,$$

and (v_r, v_θ) will represent the cylindrical coordinates of the canonical momentum in the Larmor frame

$$\begin{pmatrix} v_r \\ v_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}.$$

Moreover, the external electromagnetic fields solutions of (9) are given by

$$\phi^{\mathrm{e}} = 0, \quad \chi^{\mathrm{e}} = \frac{dB}{dz} r^2/2,$$

so that the transverse external fields are

$$\mathbf{E}^{\mathrm{e}} = 0, \quad \mathbf{B}^{\mathrm{e}} = -\frac{1}{2}\frac{dB}{dz}\, r \tag{39}$$

Finally, the self-consistent fields in cylindrical coordinates satisfy

$$E_r^{\mathrm{s}} = -\frac{d\phi^{\mathrm{s}}}{dr} = \frac{q}{\varepsilon_0}\left(\frac{1}{r}\int_0^r n(z,s)s\,ds\right). \tag{40}$$

Finally, if we pass in cylindrical coordinates (r, v_r, v_θ), the paraxial Vlasov equation becomes

$$v_b \frac{\partial f}{\partial z} + v_r \frac{\partial f}{\partial r} + \left(\frac{q}{\gamma_b m} F_r + \frac{v_\theta^2}{r}\right)\frac{\partial f}{\partial v_r} + \left(\frac{q}{\gamma_b m} F_\theta + \frac{v_\theta v_r}{r}\right)\frac{\partial f}{\partial v_\theta} = 0, \tag{41}$$

where $F_k = F_k^{\mathrm{e}} + F_k^{\mathrm{s}}$, $k \in \{r, \theta\}$, the external electromagnetic fields are given by

$$F_r^{\mathrm{e}} = -\kappa_0(z) r, \quad F_\theta^{\mathrm{e}} = 0, \tag{42}$$

and the self-consistent forces satisfy

$$F_r^{\mathrm{s}} = \frac{q}{\varepsilon_0}(1-\beta^2)\frac{1}{r}\int_0^r n(z,s)s\,ds, \quad F_\theta^{\mathrm{s}} = 0. \tag{43}$$

6.1. *Construction of steady-state solutions*

This section is devoted to the construction of solutions of the paraxial model (41) independent of z taking into account self-consistent forces. To this aim, we consider a uniform applied potential

$$\Phi^{\mathrm{e}}(r) = k_0^2 r^2/2.$$

We use the invariance of the transverse energy

$$H_\perp(r, v_r, v_\theta) = \frac{1}{2v_b^2}\left(v_r^2 + v_\theta^2\right) + \frac{1}{2}\,k_0^2 r^2 + \frac{q}{\gamma_b^3 m v_b^2}\,\Phi^{\mathrm{s}}(r).$$

Considering a distribution function of the form $f(H_\perp)$, it is easy to check that it is a solution of the paraxial model (41), but the main difficulty is the determination of the charge density $n(r)$ and the self-consistent potential $\Phi^{\mathrm{s}}(r)$. Assume f is compactly supported, then the space charge density n is given by

$$n(r) = \int_{\mathbb{R}^2} f(H_\perp(r, v_r, v_\theta))dv_r\, dv_\theta = \int_0^{2\,\pi}\int_0^{+\infty} f(\overline{H}_\perp(r, v, \psi))v\, dv\, d\psi,$$
$$= \pi \int_0^{+\infty} f(\overline{H}_\perp(r, v^2))dv^2,$$

where $\overline{H}_\perp(r, v, \psi) = \overline{H}_\perp(r, v)$ is the expression of $H_\perp$ in polar velocity coordinates.

The space charge potential Φ^{s} is then found by solving the Poisson equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi^{\mathrm{s}}}{dr}\right) = -\frac{q\pi}{\varepsilon_0}\int_0^{+\infty} f(\overline{H}_\perp(r, v^2))dv^2.$$

Now let us denote by $W(r)$ the potential energy, which becomes using that $1 - \beta^2 = \frac{1}{\gamma_b^2}$,

$$W(r) = \frac{1}{2}k_0^2\, r^2 + \frac{q}{\gamma_b^3 m v_b^2}\Phi^{\mathrm{s}}(r). \tag{44}$$

We first observe that the potential energy $W(r)$ is positive. Indeed, from the Poisson equation, we check that $d\Phi^{\mathrm{s}}/dr$ is negative and the potential vanishes when r goes to infinity, which means that the self-consistent potential is positive decreasing.

Let us denote by H_0 the maximum transverse energy, which might be infinite if f does not have compact support. We introduce the radius $a \in [0, +\infty]$ for which the potential energy $W(r)$ is maximal and the kinetic energy vanishes

$$H_0 = W(a).$$

Note that since the applied potential goes to infinity for large r, when the distribution function f is compactly supported, i.e. H_0 is finite, then the radius a is also finite. With these notations, we have that the maximal kinetic energy $b^2(r)/2$ for any $r \geq 0$, which is given by

$$W(a) = H_0 = \frac{1}{2}b^2(r) + W(r),$$

is such that

$$\frac{v^2}{2v_b^2} \leq \frac{1}{2}b^2(r) = W(a) - W(r) \geq 0.$$

Hence, making the change of variables $H_\perp = \frac{v^2}{2v_b^2} + W(r)$, we get

$$n(r) = \pi \int_0^{+\infty} f(\overline{H_\perp}(r, v^2))dv^2 = \pi \int_0^{2v_b^2(W(a)-W(r))} f(v^2(r))dv^2(r)$$
$$= 2\pi v_b^2 \int_{W(r)}^{W(a)} f(H_\perp)dH_\perp.$$

The KV distribution. If we consider the following distribution

$$f(s) = \delta_0(s - H_0),$$

we recover a simple KV distribution for the axisymmetric case.

The Waterbag distribution. Let us treat the case where the distribution function is uniform

$$f(s) = \begin{cases} f_0/v_b^2 & \text{if } 0 \leq s \leq H_0, \\ 0 & \text{otherwise.} \end{cases}$$

The space charge density is

$$qn(r) = 2\pi q f_0 (W(a) - W(r))$$

and the Poisson equation becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi^s}{dr}\right) = -\frac{q2\pi f_0}{\varepsilon_0}\left(W(a) - W(r)\right). \tag{45}$$

If we set

$$k_1^2 = \frac{2\pi q f_0}{\varepsilon_0}\frac{q}{m\gamma_b^3 v_b^2},$$

and

$$U(r) = W(r) - W(a) + 2\frac{k_0^2}{k_1^2} = \frac{1}{2}k_0^2 r^2 + \frac{q\phi}{m\gamma_b^3 v_b^2} + \frac{2k_0^2}{k_1^2},$$

then, we can replace (45) by the following equation for $U(r)$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right) - k_1^2 U(r) = 0.$$

As we will see later the parameter k_1 plays an important role in the definition of a waterbag distribution, since it allows to determine the shape of the beam. Thus, setting $y(x) = U(r)$ where $x = k_1 r$, the function $y(x)$ satisfies the first Bessel differential equation

$$xy'' + y' = xy \tag{46}$$

and $y = I_0(x)$, where I_0 is the modified Bessel function of the first kind. Finally,

$$U(r) = CI_0(k_1 r).$$

Indeed, the solution of (46) is defined up to a multiplicative constant. However, we know that $U(a) = 2k_0^2/k_1^2$, then

$$U(r) = \frac{2k_0^2}{k_1^2}\frac{I_0(k_1 r)}{I_0(k_1 a)}.$$

Moreover, $U(r)$ is an increasing function of r, which means that the potential energy $W(r)$ is also increasing and positive. Then, the maximal potential energy is obtained at the maximal radius of the beam $r_{\max}$ and we have $a = r_{\max}$. On the other hand, for any $0 < r \leq a$, the maximal kinetic energy is given by

$$\frac{1}{2}\,b^2(r) = W(a) - W(r),$$

and since $W(r)$ is increasing, the velocity $a' = b(0)$ corresponds to the velocity of the maximal kinetic energy. Since the kinetic energy is an increasing function with respect to $v = (v_r^2 + v_\theta^2)^{1/2}$, the point a' corresponds to the maximal modulus of the distribution function f. Finally the boundary of the particle beam in phase space is given by

$$H_0 = \frac{1}{2}\big(v_r^2 + v_\theta^2\big) + W(r) = W(a),$$

or

$$H_0 = \frac{a'^2}{2} + W(0) = W(a).$$

Finally, we get the following relation

$$\frac{v_r^2 + v_\theta^2}{a'^2} + \frac{W(r) - W(0)}{W(a) - W(0)} = 1,$$

where $W(r)$ satisfies

$$W(r) = W(a) - 2\frac{k_0^2}{k_1^2}\left(1 - \frac{I_0(k_1 r)}{I_0(k_1 a)}\right).$$

Then, the phase space contours of the distribution function are described by the following relation:

$$\frac{v_r^2 + v_\theta^2}{a'^2} + \frac{I_0(k_1 r) - 1}{I_0(k_1 a) - 1} = 1, \tag{47}$$

and the charge density is given by

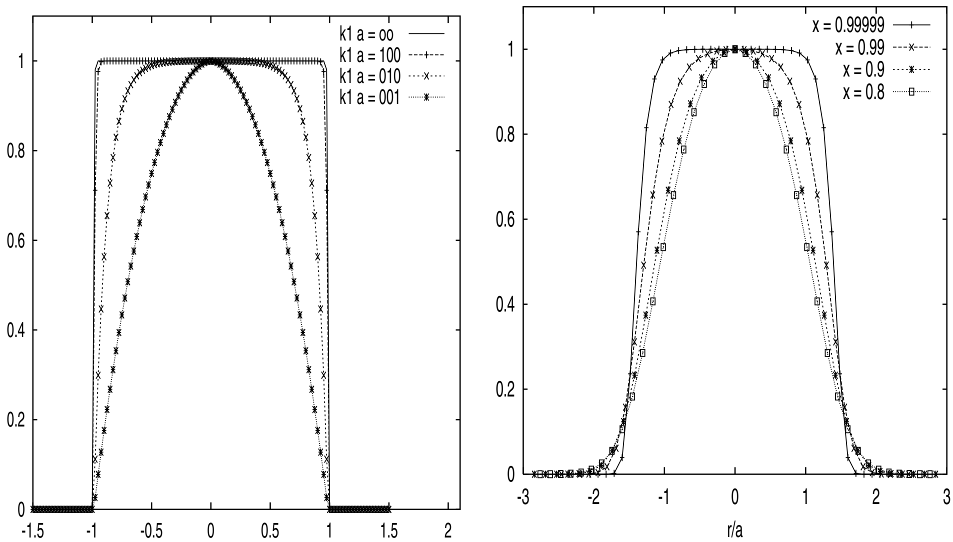
$$n(r) = 4\pi f_0 \frac{k_0^2}{k_1^2}\left(1 - \frac{I_0(k_1 r)}{I_0(k_1 a)}\right).$$

The charge densities obtained for different values of $k_1 a$ are shown in Fig. 1. Usually physicists know the macroscopic values of the beam, but not the value of f_0. For instance, f_0 can be expressed in terms of the total current of the beam I. Indeed,

$$\begin{aligned} I = 2\pi q v_b \int_0^a n(r) r\,dr &= 8\pi^2 q v_b f_0 \frac{k_0^2}{k_1^2} \int_0^a \left(1 - \frac{I_0(k_1 r)}{I_0(k_1 a)}\right) r\,dr \\ &= 4\pi^2 q v_b f_0 \frac{k_0^2}{k_1^2} a^2 \left(1 - \frac{2}{k_1 a}\frac{I_1(k_1 a)}{I_0(k_1 a)}\right), \end{aligned}$$

using that $I_0'(x) = I_1(x)$. Then, the value of the distribution inside the contours defined by (47) is

$$f_0 = \frac{I}{q v_b}\,\frac{k_1^2}{4\pi^2 k_0^2 a^2}\,\frac{k_1 a I_0(k_1 a)}{k_1 a I_0(k_1 a) - 2 I_1(k_1 a)}$$



(a) (b)

Fig. 1. The space charge density of (a) waterbag distribution with respect to $k_1 a = 1, 10, 100$ and ∞, (b) Maxwell–Boltzmann distribution with respect to $\alpha_1 = 0.8$, 0.9, 0.99 and 0.99999.

and ${a'}^2 = b^2(0)$, i.e.

$$ {a'}^2 = 4\frac{k_0^2}{k_1^2}\left(1 - \frac{1}{I_0(k_1 a)}\right). $$

The Maxwell–Boltzmann distribution. Let us consider the following distribution, which is not compactly supported

$$ f(s) = \frac{f_1}{v_b^2}\, \exp(-s/T_\perp). $$

In this case the previous study cannot be directly applied, since the radius a is equal to $+\infty$. Then, the space charge density is

$$ \begin{aligned} qn(r) &= q\frac{f_1}{v_b^2}\int_{\mathbb{R}^2} \exp(-H_\perp(r, v^2)/T_\perp) dv_r\, dv_\theta \\ &= 2\pi T_\perp q f_1\, \exp(-W(r)/T_\perp), \end{aligned} $$

where the potential energy $W(r)$ is given by (44), and the space charge potential Φ is a solution of the nonlinear Poisson equation

$$ \frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = -2\pi\, \frac{q f_1 T_\perp}{\varepsilon_0}\, \exp\left(-\left(\frac{k_0^2}{2}r^2 + \frac{q}{m\gamma_b^3 v_b^2}\Phi^r\right)\Big/ T_\perp\right). $$

where parameters k_0, f_1 and $T_\perp$ have to be defined. Let us rescale the Poisson equation

$$\overline{r} = \sqrt{\frac{2k_0^2}{T_\perp}}\, r, \quad \overline{\Phi}(\overline{r}) = \frac{q}{m\gamma_b^3 v_b^2 T_\perp} \Phi(r)$$

and we get

$$\frac{1}{\overline{r}} \frac{d}{d\overline{r}} \left(\overline{r} \frac{d\overline{\Phi}}{d\overline{r}} \right) = -\alpha_1 \exp\bigl(-\bigl(\overline{r}^2/4 + \overline{\Phi}(\overline{r})\bigr)\bigr)\,, \tag{48}$$

where α_1 is

$$\alpha_1 = \frac{\pi q^2 f_1}{\varepsilon_0\, m\gamma_b^3 v_b^2} \frac{T_\perp}{k_0^2}.$$

If we assume that $\alpha_1 < 1$, it is then easy to prove from a Banach fixed point argument that the nonlinear Poisson equation (48) has a solution. Finally, let us note that Eq. (48) cannot be explicitly solved, but numerical approximations can be computed. See Fig. 1 for an illustration.

6.2. *Invariance of the canonical momentum*

In order to reduce the dimension of the problem, we use the invariance of the canonical angular momentum

$$P(r, v_\theta) = mrv_\theta.$$

Denoting by $I = \frac{P}{m}$ and making the change of variable $(r, v_r, v_\theta) \to (r, v_r, I)$ with $v_\theta = \frac{I}{r}$, we get

$$v_b \frac{\partial f}{\partial z} + v_r \frac{\partial f}{\partial r} + \left(\frac{q}{m\gamma_b^3} E^s(t,r) + \frac{I^2}{r^3} - \frac{1}{4} \left(\frac{qB(z)}{\gamma_b m} \right)^2 r \right) \frac{\partial f}{\partial v_r} = 0.$$

Here the invariant I only appears as a parameter in the equation. It still needs to be discretized but this formulation allows a straightforward parallelization distributing the different values of the invariant over the processors. Then the Vlasov equations corresponding to the values of the invariant are independent. The coupling only occurs through the computation of the charge density. This yields a very efficient parallelization strategy.[5]

7. Numerical Validation

In order to apply the techniques developed in this paper, we present numerical results in four-dimensional phase space obtained using a semi-Lagrangian method.[4,6]

Test 1 : Beam focusing via an applied uniform electric field

The investigation described here is restricted to an axisymmetric space-charge dominated beam in a focusing system which is linear and uniform in the longitudinal direction, but we use a four-dimensional phase space solver. We choose an applied uniform electric field, which can be considered the smooth approximation of a periodic focusing lattice. The beam particles have been given an initially semi-Gaussian distribution with a transverse emittance of $\varepsilon_x = \varepsilon_y = 0.5 \times 10^{-4}$ m.rad

$$f_0(x, y, v_x, v_y) = \frac{n_0}{(2\pi v_{\mathrm{th}}^2)\,(\pi a^2)} \exp\left(-\frac{v_x^2 + v_y^2}{2\,v_{\mathrm{th}}^2}\right), \quad \text{if } x^2 + y^2 \le a^2,$$

and $f_0(x, y, v_x, v_y) = 0$, if $x^2 + y^2 > a^2$ with $a = 8.9 \times 10^{-2}$ m, $v_{\mathrm{th}} = 6.94 \times 10^4$ ms^{-1} and $n_0 = 1.02 \times 10^{11}$. The applied electric field is uniform with respect to z (for uniform focusing see Sec. 4) and given by

$$\mathbf{E}(x, y) = -\begin{pmatrix} x \\ y \end{pmatrix}.$$

We have chosen the rms thermal velocity v_{th} such that the equivalent KV beam is matched using the method described in Sec. 5. The numerical solution is computed using 64×64 points in space and 128×128 points in velocity.

On the one hand, the evolution of the rms quantities of the semi-Gaussian beam and the equivalent KV beam is shown in Fig. 2. In this case, the KV distribution is stationary since the focusing force is uniform, whereas the rms quantities corresponding to the semi-Gaussian beam are oscillating around a value larger than the one corresponding to the KV beam. Indeed, a beam with an initial semi-Gaussian transverse distribution is not exactly a steady state distribution, then the distribution is not in local force balance with the external focusing force. An initial transient

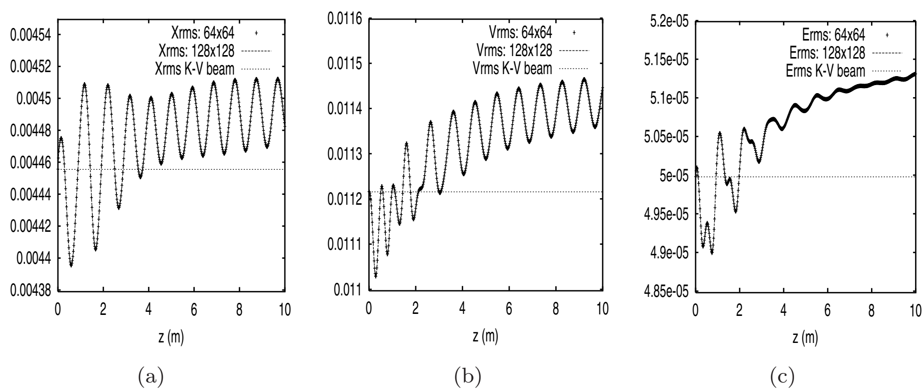


Fig. 2. Test 1 : Comparison of rms quantities obtained from an Eulerian Vlasov solver for a semi-Gaussian beam and from a KV beam: (a) X_{rms}, (b) X'_{rms} and (c) $\varepsilon_{\mathrm{rms}}$.

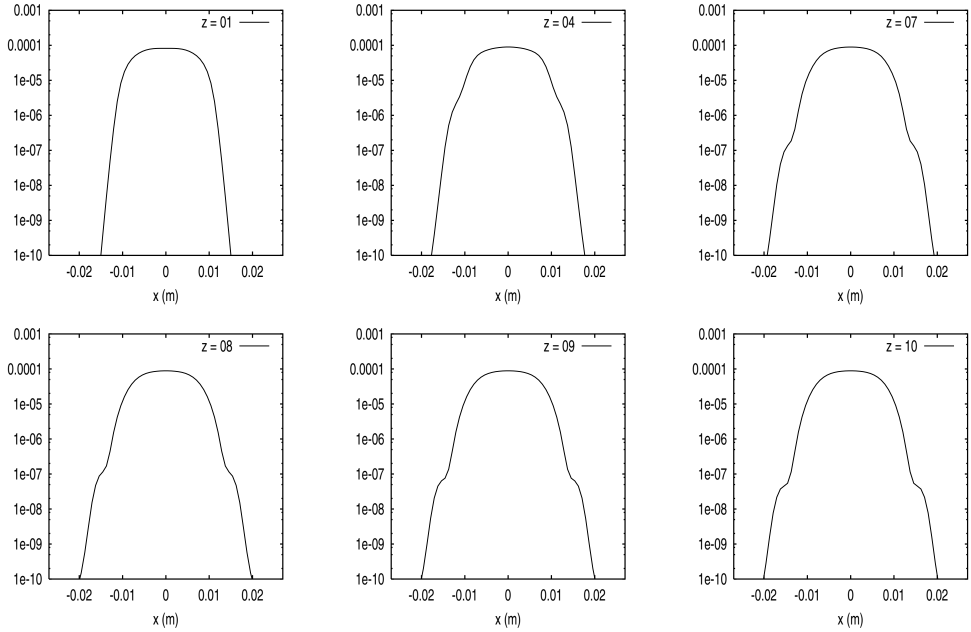


Fig. 3. Test 1 : Evolution of the charge density obtained from an Eulerian Vlasov solver for a semi-Gaussian beam.

therefore will occur until the beam is focused and then oscillates periodically around an equilibrium.

On the other hand, snapshots of the charge density are displayed in Fig. 3. We first observe that the numerical solution computed using a $2D_x \times 2D_v$ solver, remains close to an axisymmetric distribution and these results agree well with previous results presented in the literature[5,4] using an axisymmetric Vlasov solver. We notice that the beam at first becomes hollow, then regions of high density propagate to the core of the beam and out again, creating space charge waves. These waves are damped by phase mixing after a few lattice periods since the beam oscillates around an equilibrium (see also the evolution of rms quantities). After the transient regime, we observe that particles are trapped outside the core of the beam like a "halo". This phenomenon has also been observed on numerical simulations performed using an axisymmetric solver.

Test 2 : Alternating gradient focusing

We now consider a beam of protons focused with an alternating gradient method: the applied electric field is given by

$$\mathbf{E}(x, y, z) = \begin{pmatrix} +k_0(z)x \\ -k_0(z)y \end{pmatrix},$$

where for $z \in (0,1)$

$$k_0(z) = \begin{cases} +1 & \text{if } 0 < z < 1/8, \quad \text{or} \quad 7/8 < z < 1 \\ 0 & \text{if } 1/8 < z < 3/8, \quad \text{or} \quad 5/8 < z < 7/8 \\ -1 & \text{if } 3/8 < z < 5/8. \end{cases}$$

The emittance of the beam is $\varepsilon = 2 \times 10^{-4}\,\pi$mrad. The initial value of the distribution function is a Gaussian distribution with $n_0 = \frac{I}{q\, v_b}$, the current $I = 0.1\,A$ and the beam energy is $W = 0.2\,\mathrm{MeV}$, so that $v_b = 6.19 \times 10^6\,\mathrm{ms}^{-1}$

$$f_0(x, y, v_x, v_y) = \frac{n_0}{4\pi^2 dcba} \exp\left(-\frac{r^2}{2}\right), \quad r^2 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{v_x}{c}\right)^2 + \left(\frac{v_y}{d}\right)^2.$$

In this case, the KV distribution is not exactly known, but we numerically solved the envelope equations (18) to get a periodic solution, which is necessary to compute the KV beam (17). Therefore, the parameters (a, b, c, d) associated to the Gaussian beam are computed using rms quantities and the notion of equivalent beams presented above.

The evolution of rms quantities of the Gaussian beam and the equivalent KV beam is shown in Fig. 4. As expected the rms quantities associated to the Gaussian beam are not exactly periodic, but they remain very close to the ones corresponding to the periodic KV distribution. More precisely, we observe that the oscillation frequency of the rms quantities y_{rms} and y'_{rms} of both beams are very close together,

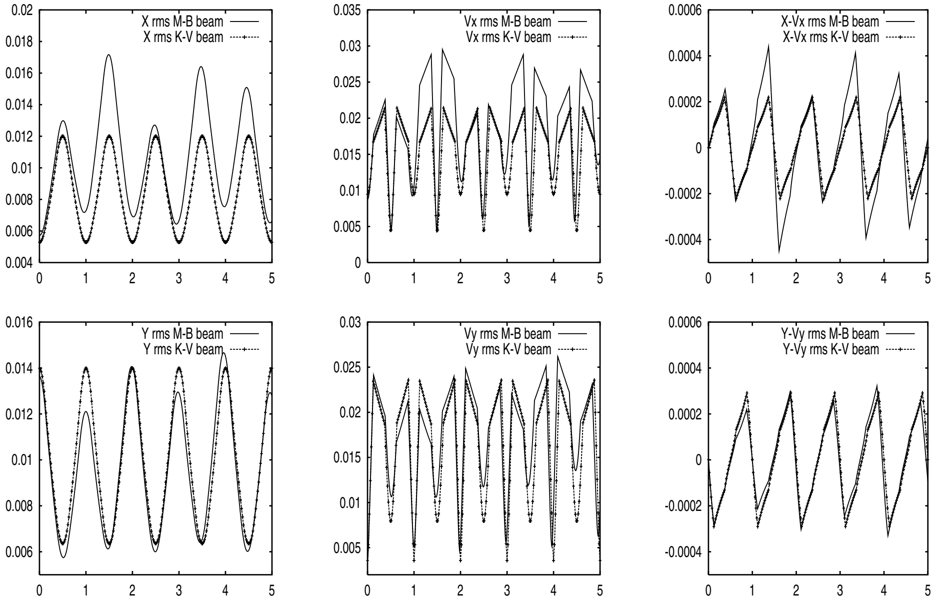


Fig. 4. Test 2: Evolution of rms quantities obtained from an Eulerian Vlasov solver for a Gaussian beam and an equivalent KV beam.

whereas the amplitude of $x_{\rm rms}$ and $x'_{\rm rms}$ of the Gaussian distribution are larger than those of the KV beam.

This phenomenon can be observed more precisely on the snapshots of the projection on the $x-y$, $x-v_x$ and $y-v_y$ planes of the distribution function displayed in Fig. 5. The contours represent densities of 0.5, 0.1, 0.01 and 0.001 in units of the

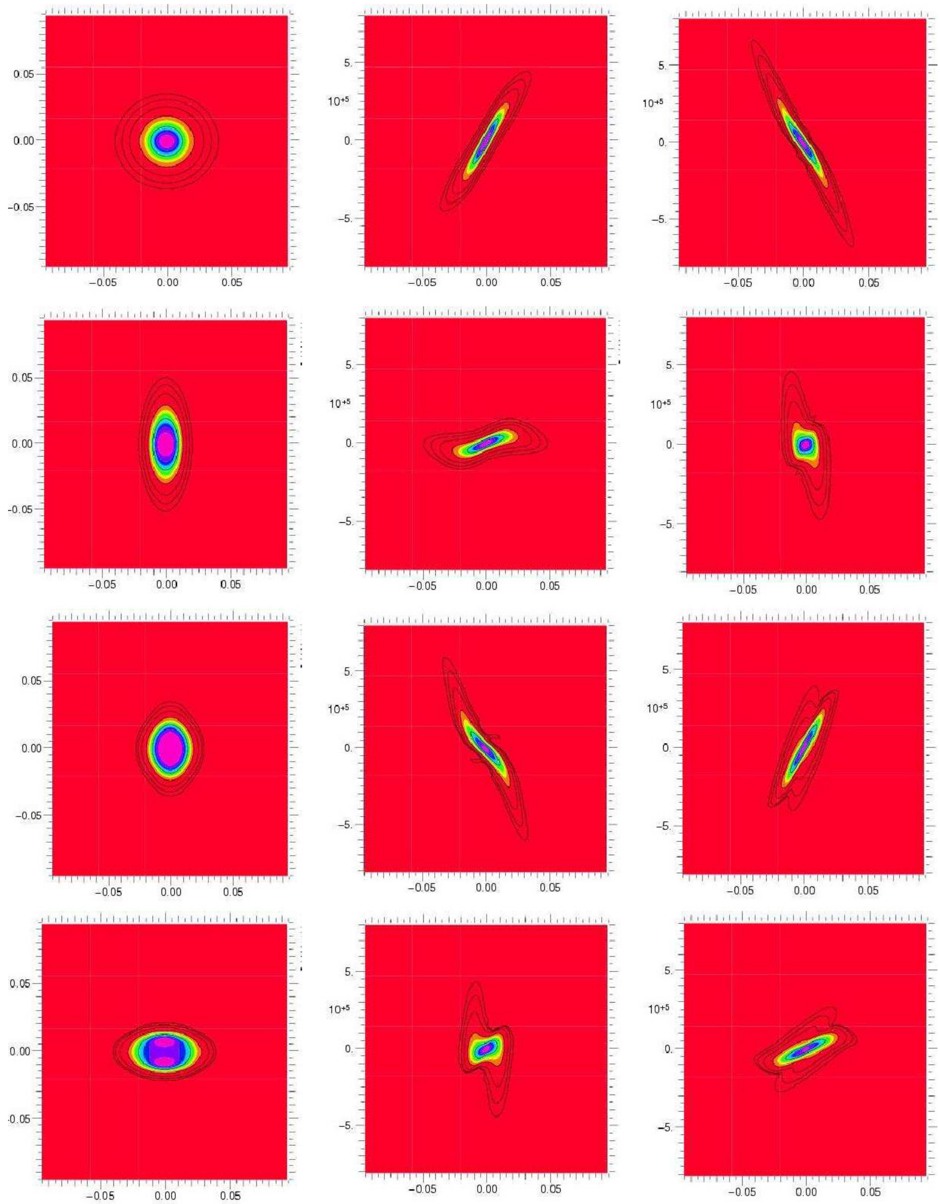


Fig. 5. z evolution of the $x-y$, $x-v_x$, $y-v_y$ and v_x-v_y projections of the distribution function obtained from an Eulerian Vlasov solver for a Gaussian beam z=0, 1, 2, 4.

maximum phase space density. The axial phase-space is not elliptical (as it is for the KV beam) because the beam is matched to the very nonlinear forces, but the beam is alternatively focused in one direction and defocused in the other, then after several periods (around 100) the beam stabilizes and becomes quasi-periodic. The simulations performed using a Vlasov solver are particularly well suited to study this transient regime, where nonlinear forces are dominant. Moreover, it allows to justify numerically the concept of equivalent beams introduced by Sacherer[14] and Lapostolle[8] (see Sec. 5).

8. Conclusion

In this paper we introduced mathematical tools for the numerical simulation of charged particle beams in the paraxial approximation. After recalling the paraxial Vlasov–Poisson system (13)–(14). We introduced three analytical or quasi-analytical steady-state solutions of the model useful for simulation and code validation purposes. One of them, the KV distribution is even more essential, as we were able to show how it can be used to approximately match an arbitrary beam to the external focusing forces. We have thus defined a solid mathematical framework for numerical beam simulations. This framework was applied on some realistic examples using a Eulerian solver based on the PFC method on a uniform grid.[4–6] For larger problems needing more efficiency, we investigate using unstructured grids as well as adaptive mesh refinement.

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