SIAM J. NUMER. ANAL.
Vol. 41, No. 2, pp. 563–588

NUMERICAL APPROXIMATION OF THE LIFSHITZ–SLYOZOV–WAGNER EQUATION*

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Abstract. The Lifshitz–Slyozov–Wagner theory of coarsening (Ostwald ripening) describes the late stages of the growth by diffusional mass transfer of the grains of a new phase from a supersaturated solution. It results in a nonlinear transport equation with a nonlocal nonlinearity for the volume distribution function of the grains. A time explicit finite volume numerical scheme is proposed to solve this equation in self-similar variables and is shown to converge under a CFL condition. Numerical simulations are also presented.

Key words. Lifshitz–Slyozov–Wagner model, Ostwald ripening, finite volume method, convergence

AMS subject classifications. 35L60, 65M60, 82C21

PII. S0036142902407599

1. Introduction. The theory of coarsening (Ostwald ripening) in alloys describes the late stages of the formation and growth of grains of a new phase from a supersaturated solution. During these stages, no new grains can form, and the determining process is the growth of the grains by diffusional mass exchange [13, 25]. More precisely, the grains of the new phase that are larger than some critical size grow at the expense of smaller ones, the critical size varying in time as a function of the degree of supersaturation. A mean-field approach for this process has been formulated by Lifshitz and Slyozov [13] and Wagner [25]. For very dilute solutions at large times, the variation in the degree of supersaturation may be neglected, and the time evolution of the volume distribution function f of the grains is given by

$$\partial_t f + \partial_x (\mathcal{V}\ f) = 0, \quad (t,x) \in \mathbb{R}_+^2, \tag{1.1}$$

with the constraint (total volume conservation)

$$\int_0^\infty x\ f(t,x)\ dx = \text{ const.}, \quad t \in \mathbb{R}_+. \tag{1.2}$$

Here $x \in \mathbb{R}_+ := (0,+\infty)$ is the volume of the grains, $t \in \mathbb{R}_+$ is the time variable, and $\mathcal{V} = \mathcal{V}(t,x)$ denotes the rate of growth of the grains, which is determined by the mechanism of mass transfer between the grains, e.g., volume diffusion [13, 25] or grain-boundary diffusion [22]. In general, one has $\mathcal{V}(t,x) = k(x)u(t) - q(x)$, where k and q are computed from the modeling of the mechanism of mass transfer between the grains [13, 22, 25]. For instance, in the model considered in [13], where the mass transfer is driven by diffusion, $\mathcal{V}$ is explicitly computable and $k(x) = 3\ x^{1/3}$, $q(x) = 3$,

*Received by the editors May 14, 2002; accepted for publication (in revised form) November 13, 2002; published electronically May 6, 2003. This research was partially supported by the ACI "Jeunes Chercheurs" *Analyse mathématique et simulation numérique de particules chargées*, grant 2000–2526.

http://www.siam.org/journals/sinum/41-2/40759.html

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$x \in \mathbb{R}_+$. The function u is then determined by requiring that the solution f to (1.1) comply with (1.2), that is,

$$u(t) \int_0^\infty k(x)\ f(t,x)\ dx = \int_0^\infty q(x)\ f(t,x)\ dx, \quad t \in \mathbb{R}_+. \tag{1.3}$$

The main purpose of this work is to present a numerical scheme for solving (1.1)–(1.2) and to study the properties and the convergence of this scheme when the functions k and q determining the rate of growth of the grains $\mathcal{V}$ are given by

$$k(x) = x^\alpha \quad \text{and} \quad q(x) = 1, \quad x \in \mathbb{R}_+, \tag{1.4}$$

for some $\alpha \in (0,1)$. (Recall that $\alpha = 1/3$ is the case considered in [13].) We will not, however, study (1.1)–(1.2) directly but will first perform a couple of transformations to obtain an equivalent formulation more suitable for our purposes. As in [17], we first introduce the number $F(t,x)$ of grains of size larger than x at time t, that is,

$$F(t,x) = \int_x^\infty f(t,x')\ dx', \quad (t,x) \in \mathbb{R}_+^2. \tag{1.5}$$

The constraint (1.2) then straightforwardly translates to the conservation of the L^1-norm of $F(t)$ throughout time evolution, and F solves

$$\partial_t F + \mathcal{V}\ \partial_x F = 0, \qquad \int_0^\infty F(t,x)\ dx = \text{ const.}, \quad (t,x) \in \mathbb{R}_+^2. \tag{1.6}$$

The second transformation we shall perform is related to the large time behavior of solutions to (1.1)–(1.2) and is motivated by the following fact: formal asymptotic expansions performed in [13] for $\alpha = 1/3$ indicate that the pair (f,u) approaches a self-similar form as time increases to infinity with the following scaling:

$$f(t,x) \sim t^{-2}\ f_\infty\left(\frac{x}{t}\right) \quad \text{and} \quad u(t) \sim u_\infty\ t^{-\alpha}.$$

Observing that convergence to a self-similar profile translates to convergence to a steady state in self-similar variables, we introduce

$$\begin{cases} F(t,x) = (1+t)^{-1}\ G\left(\ln(1+t), x/(1+t)\right), \\ \\ u(t) = (1+t)^{-\alpha}\ v\left(\ln(1+t)\right), \end{cases} \qquad (t,x) \in \mathbb{R}_+^2. \tag{1.7}$$

It then follows from (1.6) that (G,v) satisfies

$$\partial_t G + \mathcal{W}\ \partial_x G = G, \qquad \int_0^\infty G(t,x) = \text{ const.}, \tag{1.8}$$

where $\mathcal{W}(t,x) = x^\alpha\ v(t) - 1 - x$, $(t,x) \in \mathbb{R}_+^2$. In this paper, we will focus on this alternative formulation of the Lifshitz–Slyozov–Wagner (LSW) equation (1.1)–(1.2). We investigate the properties and the convergence of a numerical scheme for (1.8) built upon an explicit Euler discretization with respect to the time variable t and a finite volume discretization with respect to the volume variable x. Finally, numerical simulations will be presented, allowing us to check the numerical convergence of the scheme and to compare the large time behavior of our approximation with that expected for the solution to (1.8).

Let us provide some comments about the behavior of our numerical scheme, referring to section 5 for a more complete discussion. Unlike what was conjectured by Lifshitz and Slyozov [13], the large time behavior of solutions to (1.8) is complex and very sensitive to perturbations. In particular, according to the analysis in [14, 20], the behavior for large times of solutions to (1.8) with compactly supported initial data changes drastically in the presence of a small diffusion (say, an additional term $\eta\, \partial_x^2 G$ on the right-hand side of (1.8) with $\eta > 0$). Since our numerical scheme is a classical upwind method, a small numerical diffusion comes into play during the simulations. It is then unlikely that our scheme reproduces the correct large time behavior for compactly supported initial data, and this is exactly what we observe in the numerical simulations. On the other hand, our scheme gives the correct limit for noncompactly supported initial data. In order to capture the expected behavior for large times for arbitrary initial data, it seems that a less diffusive numerical scheme is needed. One possibility is to use a higher-order scheme, and this is the approach developed by Carrillo and Goudon in [2], where a WENO (weight essentially nonoscillatory)-type scheme is used to numerically compute the solutions to (1.8). (The main focus of [2] is actually the variant of (1.8) described in Remark 1.1 below.) The numerical simulations reported in [2] show that such a scheme gives the expected behavior for intermediate times, providing better results than our scheme. Still, for larger times, some numerical diffusion effects also come into play and drive the numerical solution away from the theoretical predictions. Another approach relies on a nonlinear and antidissipative scheme [5]. It has been recently considered by Lagoutière and seems to successfully compute the correct behavior, even for large times [8]. Let us point out, however, that no convergence proof seems to be available for these schemes.

Before describing our results more precisely, let us recall that the LSW equation (1.1)–(1.2) has been the object of several studies recently; existence and uniqueness of weak solutions have been proved in [10, 17, 19] for the initial value problem (1.1)–(1.2) under various assumptions on the functions k and q determining the growth rate of the grains $\mathcal{V}$ and the initial data. Also, the large time asymptotics have been investigated in [1, 16] by analytical means and in [2, 6] by numerical simulations.

Remark 1.1. A different version of the LSW equation (originally introduced in [13]), in which the constraint (1.2) is replaced by

$$u(t) + A \int_0^\infty x\ f(t,x)\ dx = Q, \quad t \in \mathbb{R}_+, \tag{1.9}$$

is actually the main concern of [2, 18]. In (1.9), Q is the total initial supersaturation and A is a physical constant [13]. Still, the large time behavior of solutions to (1.1), (1.9) is expected to be the same as that of (1.1), (1.2), provided that $u(t)$ defined by (1.9) converges to zero, which is true for initial data with a sufficiently wide support [2, 18]. Let us also mention that the well-posedness of the initial value problem (1.1), (1.9) has been studied in [3, 9, 17].

We now briefly outline the contents of the paper. In the next section, we introduce the numerical approximation of (1.8) and state the convergence result, which we prove in sections 3 and 4. Two points are worth mentioning here. First, it readily follows from (1.5) and the nonnegativity of f that $x \mapsto F(t,x)$ is nonincreasing and so is $x \mapsto G(t,x)$ by (1.8). At the discrete level, our approximation of G also enjoys this property. Secondly, since the definition of v involves the inverse of a moment of G (recall the definition (1.3) of u), an important step of the convergence proof is the derivation of a uniform L^∞-estimate on the approximations of v. For compactly supported initial data, such a bound has been obtained by estimating the time evolution of the support of the solution [10, 17], but this method does not seem to work here because of the (small) viscosity induced by the numerical approximation. We therefore use a different approach and obtain a new L^∞-bound in terms of the first moment of G. Since the proof of this estimate is quite technical at the discrete level, we also provide a (formal) proof for (1.1)–(1.2) in the appendix, hoping to clarify the underlying idea. The final section (section 5) is devoted to some numerical simulations performed with the numerical scheme presented in section 2.

2. Main results. Before describing our numerical scheme and stating a convergence result, we first introduce some notation and assumptions and recall previous results on (1.8). As already mentioned, we focus on the approximation of the initial value problem

$$\partial_t G + \partial_x (\mathcal{W}\ G) = S, \quad (t,x) \in \mathbb{R}_+^2, \tag{2.1}$$

$$\alpha\ v(t) \int_0^\infty x^{\alpha-1}\ G(t,x)\ dx = G(t,0), \quad t \in \mathbb{R}_+, \tag{2.2}$$

$$G(0,x) = G_0(x), \quad x \in \mathbb{R}_+, \tag{2.3}$$

where $\alpha \in (0,1)$ is fixed,

$$\begin{aligned} \mathcal{W}(t,x) &= x^\alpha\ v(t) - 1 - x, \quad (t,x) \in \mathbb{R}_+^2, \\ S(t,x) &= \alpha\ x^{\alpha-1}\ v(t)\ G(t,x), \quad (t,x) \in \mathbb{R}_+^2, \end{aligned} \tag{2.4}$$

and we assume that the initial datum G_0 satisfies

$$\begin{aligned} &G_0 \in W^{1,1}(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, xdx) \quad \text{is a nonnegative and nonincreasing function} \\ &\text{and } G_0 \not\equiv 0. \end{aligned} \tag{2.5}$$

Here and below, the notation $L^1(\mathbb{R}_+, xdx)$ stands for the space of the Lebesgue measurable real-valued functions on $\mathbb{R}_+$ which are integrable with respect to the measure xdx.

Observe that (2.1) is nothing but $\partial_t G + \mathcal{W}\ \partial_x G = G$ written in conservative form. Next, as a consequence of [10, Theorem 2] and Proposition A.1, there are at least a pair of nonnegative functions (G, v) satisfying

$$G \in \mathcal{C}([0,T]; L^1(\mathbb{R}_+)) \cap \mathcal{C}^1(0,T; L^1(\mathbb{R}_+, \min\{x,1\}dx)), \qquad v \in L^\infty(0,T)$$

for each $T \in \mathbb{R}_+$ and (2.1), (2.2), (2.3) with $\mathcal{W}$ given by (2.4). In addition, $x \mapsto G(t,x)$ is a nonincreasing function for each $t \geq 0$ and

$$\int_0^\infty G(t,x)\ dx = \int_0^\infty G_0(x)\ dx, \quad t \geq 0, \tag{2.6}$$

the identity (2.6) being actually equivalent to (2.2). Furthermore, the uniqueness of the pair (G, v) follows from [18] if G_0 is compactly supported, and from [19] in the general case. (Only the case $\alpha = 1/3$ is actually considered in [18, 19], but their proofs extend to $\alpha \in (0,1)$.) Let us finally point out that the integrability assumption $G_0 \in L^1(\mathbb{R}_+; xdx)$ is not needed for the existence of a solution to (2.1), (2.2), (2.3). It can probably be dispensed with herein also but allows us to avoid many technicalities in the proof of the L^∞-bound for v and its approximations.

Next, let $h \in (0,1)$ denote the mesh size and set

$$x_{-1/2} = 0, \quad x_i = x_{i-1/2} + \frac{h}{2}, \quad x_{i+1/2} = x_{i-1/2} + h, \tag{2.7}$$

and $\Lambda_i^h = [x_{i-1/2}, x_{i+1/2})$ for $i \geq 0$. Since x ranges in the unbounded domain $\mathbb{R}_+$, the numerical solution will actually be computed on the bounded domain $[0, x_{I^h+1/2})$, where I^h is a large integer depending on h. We shall of course require that $h\ I^h \to +\infty$ as $h \to 0$. We then define the approximation $G^{0,h}$ of the initial datum G_0 as usual by

$$G^{0,h} = \sum_{i=0}^{I^h} G_i^{0,h}\ \mathbf{1}_{\Lambda_i^h} \quad \text{with} \quad G_i^{0,h} = \frac{1}{h} \int_{\Lambda_i^h} G_0(x)\ dx, \tag{2.8}$$

and recall that

$$\|G^{0,h}\|_{L^1} \leq \|G_0\|_{L^1} \quad \text{and} \quad \lim_{h\to 0} \|G^{0,h} - G_0\|_{L^1} = 0. \tag{2.9}$$

Here and below, $\mathbf{1}_E$ denotes the characteristic function of the subset E of $\mathbb{R}_+$.

Finally, let $T \in \mathbb{R}_+$ be some final time and N the number of time iterations, and set

$$\Delta t = \frac{T}{N}, \qquad t^n = n\ \Delta t, \quad 0 \leq n \leq N.$$

The data h, Δt, and I^h have to fulfill the following conditions: we first require that the domain of computation approach $[0, +\infty)$ and the discrete initial data be close enough to G_0, that is,

$$\lim_{h\to 0} h\ I^h = +\infty, \qquad \|G^{0,h}\|_{L^1} \geq \frac{1}{2}\ \|G_0\|_{L^1}. \tag{2.10}$$

We also impose the following CFL condition:

$$10\ \frac{\Delta t}{h}\ \left(h\ I^h\right) \leq 1. \tag{2.11}$$

Observe that the above constraints are satisfied when $I^h \sim h^{-1-\vartheta}$ for some $\vartheta > 0$ and when $\Delta t\ h^{-1-\vartheta}$ is sufficiently small.

Denoting by $G_i^{n,h}$ an approximation of the mean value of $G(t^n)$ on Λ_i^h for $i \in \{0, \ldots, I^h\}$, and by v^n an approximation of $v(t^n)$, the numerical scheme to be studied in this paper reads

$$G_i^{n+1,h} = G_i^{n,h} - \frac{\Delta t}{h}\ \left(F_{i+1/2}^{n,h} - F_{i-1/2}^{n,h}\right) + \Delta t\ S_i^{n,h}, \quad 0 \leq i \leq I^h, \tag{2.12}$$

$$G_{-1}^{n,h} = G_{I^h+1}^{n,h} = 0, \tag{2.13}$$

$$h\ v^{n+1}\ \left(\sum_{i=0}^{I^h} a_i^h\ G_i^{n+1,h}\right) = G_0^{n+1,h}, \tag{2.14}$$

for $n \in \{0, \ldots, N-1\}$, with initial data $(G_i^{0,h})_{0\leq i\leq I^h}$ defined in (2.8) and v^0 given by (2.14) with $n = -1$. In (2.12) the approximate flux $F_{i+1/2}^{n,h}$ is given by

$$F_{i+1/2}^{n,h} = \nu_+^n(x_{i+1/2})\ G_i^{n,h} - \nu_-^n(x_{i+1/2})\ G_{i+1}^{n,h}, \quad -1 \leq i \leq I^h, \tag{2.15}$$

with

$$\nu^n(x) = x^\alpha\ v^n - 1 - x, \quad x \in \mathbb{R}_+, \tag{2.16}$$

$\nu^n_+(x) = \max\{0, \nu^n(x)\}$, $\nu^n_-(x) = \max\{0, -\nu^n(x)\}$, and the source term $S_i^{n,h}$ is given by

$$S_i^{n,h} = a_i^h\ v^n\ G_i^{n,h}, \quad \text{with} \quad a_i^h = \frac{\alpha}{h} \int_{\Lambda_i^h} x^{\alpha-1}\ dx, \quad 0 \le i \le I^h. \tag{2.17}$$

Remark 2.1. Note that the boundary condition $G^{n,h}_{I^h+1} = 0$ in (2.13) is needed only when $\nu^n(x_{I^h+1/2}) < 0$.

Before stating some properties on the scheme (2.12)–(2.17), let us briefly comment on its derivation, which relies obviously on an explicit Euler scheme for the time variable and on a finite volume approach for the volume variable (see, e.g., [7, 12]). Concerning the latter, the formula (2.15) comes from the approximation by a classical upwind scheme of the fluxes $\mathcal{W}(t^n, x_{i+1/2})$ and $\mathcal{W}(t^n, x_{i-1/2})$ arising from the integration of (2.1) over the cell Λ_i^h. As for (2.14), it is a discrete version of (2.2), which guarantees the conservation of the L^1-norm of G^h; see (2.20) below.

Under the conditions (2.10), (2.11) and if hI^h is large enough, the solution $(G_i^{n,h})$ to the scheme (2.12)–(2.17) enjoys properties similar to those of G, which we gather in Proposition 2.2 below.

PROPOSITION 2.2. *There is a positive constant $x_\star$ depending only on α, G_0, and T such that, if*

$$h\ I^h \ge x_\star \tag{2.18}$$

and the conditions (2.10), (2.11) *are fulfilled, the solution* $(G_i^{n,h})$ *to the scheme* (2.12)–(2.17) *satisfies the following:*

- *nonnegativity and monotonicity:*

$$0 \le G_{i+1}^{n,h} \le G_i^{n,h} \le G_0^{n,h}, \quad 0 \le i \le I^h - 1, \tag{2.19}$$

- *conservation of the total volume:*

$$\sum_{i=0}^{I^h} h\ G_i^{n,h} = \sum_{i=0}^{I^h} h\ G_i^{0,h} \tag{2.20}$$

for $n \in \{0, \dots, N\}$.

We next define the numerical approximation (G^h, v^h) of (G, v) by

$$G^h(t,x) = \sum_{i=0}^{I^h} G_i^{n,h}\ \mathbf{1}_{\Lambda_i^h}(x), \quad v^h(t) = v^n, \quad x \in \mathbb{R}_+, \tag{2.21}$$

for $t \in [t^n, t^{n+1})$ and $n \in \{0, \dots, N-1\}$, and

$$G^h(T,x) = \sum_{i=0}^{I^h} G_i^{N,h}\ \mathbf{1}_{\Lambda_i^h}(x), \quad v^h(T) = v^N, \quad x \in \mathbb{R}_+. \tag{2.22}$$

We may now state our main result.

THEOREM 2.3. *Assume that the conditions* (2.10), (2.11), *and* (2.18) *are fulfilled and that G_0 satisfies* (2.5). *Then*

$$G^h \longrightarrow G \quad in \quad L^\infty(0,T; L^1(\mathbb{R}_+)), \tag{2.23}$$

$$v^h \overset{*}{\rightharpoonup} v \quad in \quad L^\infty(0,T), \tag{2.24}$$

where (G, v) is the weak solution to (2.1), (2.2), (2.3) *on* $[0, T]$ *with initial datum* G_0. *More precisely,* (G, v) *is a pair of nonnegative functions satisfying*

$$\left\{ \begin{array}{l} G \in \mathcal{C}([0,T]; L^1(\mathbb{R}_+)) \cap L^\infty\big(0,T; W^{1,1}(\mathbb{R}_+; (1+x)dx)\big), \\ \\ v \in L^\infty(0,T) \end{array} \right. \tag{2.25}$$

and

$$\int_0^\infty (G(t) - G_0)\ \varphi\ dx = \int_0^t \int_0^\infty (G(s) - \mathcal{W}(s)\ \partial_x G(s))\ \varphi\ dxds \tag{2.26}$$

for each $t \in [0, T]$ *and* $\varphi \in L^\infty(\mathbb{R}_+)$*, where* v *and* $\mathcal{W}$ *are given by* (2.2) *and* (2.4)*, respectively. Equivalently,* G *satisfies* (2.6)*. In addition,* $x \mapsto G(t, x)$ *is nonincreasing for each* $t \in [0, T]$.

Observe that each term in (2.26) makes sense since, by (2.25), $\mathcal{W}$ and $\partial_x G$ belong to $L^\infty((0,T) \times \mathbb{R}_+, (1+x)^{-1}dtdx)$ and $L^\infty(0,T; L^1(\mathbb{R}_+, (1+x)dx))$, respectively.

Let us finally explain the main steps of the proof of Theorem 2.3. Similarly to the existence proof in [10], the proof of Theorem 2.3 relies on estimates of G^h and its discrete gradient in $L^\infty(0,T; L^1(\mathbb{R}_+, (1+x)dx))$ and on an $L^\infty(0,T)$-estimate on v^h. On the continuous equation (2.1), (2.2), (2.3), these bounds are obtained as follows: uniform estimates for G in $L^1(\mathbb{R}_+)$ and $L^\infty(\mathbb{R}_+)$ are straightforward consequences of (2.6) and (2.1), respectively. The main new observation then is that an upper bound on v can be obtained from (2.2) and the previous estimates in terms of the $L^1(\mathbb{R}_+; xdx)$-norm of G (Lemma 3.1). Inserting this estimate into (2.1) yields a uniform estimate for G in $L^1(\mathbb{R}_+; xdx)$, and thus an upper bound for v in $L^\infty(0,T)$ (Lemma 3.2). The equation satisfied by $\partial_x G$ then reveals an $L^1(\mathbb{R}_+)$-weak compactness estimate on $\partial_x G$, which, in turn, implies some time equicontinuity on G (see Lemmas 3.5 and 3.7 and (4.12) below). From these estimates, one deduces that G lies in compact subsets of $\mathcal{C}([0,T]; L^1(\mathbb{R}_+))$ and $L^1(0,T; W^{1,1}(\mathbb{R}_+))$. At the discrete level, we perform the same steps for (G^h, v^h), which is possible thanks to the conditions (2.10), (2.11), and (2.18).

3. Properties of $(G_i^{n,h})$. This section is devoted to the proof of Proposition 2.2 and the uniform bounds satisfied by $(G_i^{n,h})$. The parameters h, Δt, and I^h being fixed such that (2.10), (2.11), and (2.18) are fulfilled, we omit the superscript h throughout this section. Also, owing to (2.10), we may assume without loss of generality that $h \in (0, 1)$ and $x_{I+1/2} \geq 2$.

Lemma 3.1. *Let* $n \in \{0, \ldots, N\}$ *be such that*

$$\sum_{i=0}^{I} h\ G_i^n = \sum_{i=0}^{I} h\ G_i^0 \quad \textit{and} \quad G_i^n \geq 0 \quad \textit{for} \quad i \in \{0, \ldots, I\}. \tag{3.1}$$

Then there is a positive constant C_1 *depending only on* α *and* $\|G_0\|_{L^1}$ *such that*

$$0 \leq v^n \leq C_1\ G_0^n\ \left(1 + \left(\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n \right)^{1-\alpha} \right), \tag{3.2}$$

where $\ell \in \{0, \ldots, I/2\}$ *denotes the integer such that* $1 \in \Lambda_\ell$.

Proof. We infer from (2.10), (3.1), and the Hölder inequality that

$$\begin{aligned}
\frac{\|G_0\|_{L^1}}{2} &\le \sum_{i=0}^{I} h\ G_i^0 = \sum_{i=0}^{I} h\ G_i^n \\
&\le \left(\sum_{i=0}^{I} h\ x_{i+1/2}^{\alpha-1}\ G_i^n \right)^{1/(2-\alpha)} \quad \left(\sum_{i=0}^{I} h\ x_{i+1/2}\ G_i^n \right)^{(1-\alpha)/(2-\alpha)}, \\
\left(\frac{\|G_0\|_{L^1}}{2} \right)^{2-\alpha} &\le \left(\frac{1}{\alpha} \sum_{i=0}^{I} h\ a_i\ G_i^n \right) \left(\sum_{i=0}^{I} h\ x_{i+1/2}\ G_i^n \right)^{1-\alpha}.
\end{aligned}$$

Multiplying both sides of the above inequality by v^n and using (2.14) yields

$$\left(\frac{\|G_0\|_{L^1}}{2} \right)^{2-\alpha} \quad v^n \le \frac{G_0^n}{\alpha} \quad \left(\sum_{i=0}^{I} h\ x_{i+1/2}\ G_i^n \right)^{1-\alpha}.$$

Since $x_{i+1/2} \le x_{\ell+1/2} \le 2$ for $0 \le i \le \ell$, and $x_{i+1/2} \le 2\ x_{i-1/2}$ for $i \ge \ell+1$, we deduce from (2.9) and (3.1) that

$$\begin{aligned}
\sum_{i=0}^{I} h\ x_{i+1/2}\ G_i^n &\le 2 \sum_{i=0}^{\ell} h\ G_i^n + 2 \sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n \\
&\le 2\ \|G_0\|_{L^1} + 2 \sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n.
\end{aligned}$$

Combining the previous two inequalities, we end up with

$$\begin{aligned}
v^n &\le \left(\frac{2}{\|G_0\|_{L^1}} \right)^{2-\alpha} \frac{2^{1-\alpha}\ G_0^n}{\alpha} \left(\|G_0\|_{L^1}^{1-\alpha} + \left(\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n \right)^{1-\alpha} \right) \\
&\le C_1\ G_0^n \left(1 + \left(\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n \right)^{1-\alpha} \right),
\end{aligned}$$

whence (3.2). ∎

Lemma 3.2. *Let $n \in \{0, \dots, N-1\}$ be such that* (3.1) *holds true and*

$$\nu_+^n(x_{I+1/2}) = 0. \tag{3.3}$$

Then

$$\sum_{i=0}^{I} h\ G_i^{n+1} = \sum_{i=0}^{I} h\ G_i^0, \tag{3.4}$$

$$0 \le G_i^{n+1} \le (1 + \Delta t)\ \sup_j \{G_j^n\}, \quad 0 \le i \le I, \tag{3.5}$$

$$\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^{n+1} \le \left(1 + C_2\ \left(1 + \sup_i \{G_i^n\}\right)\ \Delta t\right)\ \sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n$$
$$+\ C_2\ \left(1 + \left(\sup_i \{G_i^n\}\right)^{(1+\alpha)/\alpha}\right)\ \Delta t, \tag{3.6}$$

where C_2 is a positive constant depending only on α and $\|G_0\|_{L^1}$.

Proof. We first check (3.4). For $n \in \{0, \dots, N-1\}$, it follows from (2.12), (2.13), (2.14), and (2.15) that

$$\begin{aligned}\sum_{i=0}^{I} h\, G_i^{n+1} &= \sum_{i=0}^{I} h\, G_i^n - \Delta t \sum_{i=1}^{I+1} F_{i-1/2}^n + \Delta t \sum_{i=0}^{I} F_{i-1/2}^n + h\, \Delta t\, v^n \sum_{i=0}^{I} a_i\ G_i^n \\ &= \sum_{i=0}^{I} h\, G_i^n - \Delta t\, G_0^n - \Delta t\, \nu_+^n(x_{I+1/2})\ G_I^n + h\, \Delta t\, v^n \sum_{i=0}^{I} a_i\ G_i^n \\ &= \sum_{i=0}^{I} h\, G_i^n - \Delta t\, \nu_+^n(x_{I+1/2})\ G_I^n,\end{aligned}$$

whence (3.4) by (3.3). Before completing the proof of Lemma 3.2, we provide an alternative formulation of (2.12). By (2.15) we have

$$\begin{aligned}F_{i+1/2}^n - F_{i-1/2}^n &= \nu_+^n(x_{i+1/2})\ G_i^n - \nu_-^n(x_{i+1/2})\ G_{i+1}^n \\ &\quad - \nu_+^n(x_{i-1/2})\ G_{i-1}^n + \nu_-^n(x_{i-1/2})\ G_i^n \\ &= (\nu^n(x_{i+1/2}) - \nu^n(x_{i-1/2}))\ G_i^n \\ &\quad + \nu_-^n(x_{i+1/2})\ (G_i^n - G_{i+1}^n) + \nu_+^n(x_{i-1/2})\ (G_i^n - G_{i-1}^n).\end{aligned}$$

Since

$$\nu^n(x_{i-1/2}) - \nu^n(x_{i+1/2}) + h\ a_i\ v^n = h, \tag{3.7}$$

we insert the above formula for $F_{i+1/2}^n - F_{i-1/2}^n$ into (2.12) and obtain

$$G_i^{n+1} = (1+\Delta t)\ G_i^n + \frac{\Delta t}{h}\ \nu_-^n(x_{i+1/2})\ (G_{i+1}^n - G_i^n)$$
$$+\ \frac{\Delta t}{h}\ \nu_+^n(x_{i-1/2})(G_{i-1}^n - G_i^n). \tag{3.8}$$

Now, since $\nu_+^n(x_{I+1/2}) = 0$ by (3.3) and $x_{I+1/2} \ge 1$, we have $v^n \le 2\ x_{I+1/2}^{1-\alpha}$, and (2.11) ensures that

$$\left|\nu^n(x_{i+1/2})\right| \le x_{i+1/2}^{\alpha}\ v^n + 1 + x_{i+1/2} \le 2\ x_{I+1/2} + 2\ x_{I+1/2} \le \frac{h}{2\ \Delta t} \tag{3.9}$$

for $0 \le i \le I$. Owing to (3.9) and the nonnegativity (3.1) of $(\Delta t\ G_i^n)$, it follows from (3.8) that G_i^{n+1} lies above a convex combination of G_{i-1}^n, G_i^n, and G_{i+1}^n, which are nonnegative by (3.1), whence the nonnegativity of G_i^{n+1} for $0 \le i \le I$. Similarly, we infer from (3.8) and (3.9) that $G_i^{n+1}/(1+\Delta t)$ is a convex combination of G_{i-1}^n, G_i^n, and G_{i+1}^n, from which we deduce (3.5).

We next turn to (3.6). We infer from (2.13), (3.1), and (3.8) that

$$
\begin{aligned}
&\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ \left(G_i^{n+1} - (1+\Delta t)\ G_i^n\right) \\
&= \Delta t \sum_{i=\ell+2}^{I+1} x_{i-3/2}\ \nu_-^n(x_{i-1/2})\ G_i^n + \Delta t \sum_{i=\ell}^{I-1} x_{i+1/2}\ \nu_+^n(x_{i+1/2})\ G_i^n \\
&\quad - \Delta t \sum_{i=\ell+1}^{I} x_{i-1/2}\ \left(\nu_-^n(x_{i+1/2}) + \nu_+^n(x_{i-1/2})\right)\ G_i^n \\
&\le \Delta t \sum_{i=\ell+1}^{I} (x_{i-3/2} - x_{i-1/2})\ \nu_-^n(x_{i-1/2})\ G_i^n \\
&\quad + \Delta t \sum_{i=\ell+1}^{I} x_{i-1/2}\ \left(\nu_-^n(x_{i-1/2}) - \nu_-^n(x_{i+1/2})\right)\ G_i^n \\
&\quad + \Delta t \sum_{i=\ell+1}^{I} (x_{i+1/2} - x_{i-1/2})\ \nu_+^n(x_{i-1/2})\ G_i^n \\
&\quad + \Delta t \sum_{i=\ell+1}^{I} x_{i+1/2}\ \left(\nu_+^n(x_{i+1/2}) - \nu_+^n(x_{i-1/2})\right)\ G_i^n \\
&\quad + \Delta t\ x_{\ell+1/2}\ \nu_+^n(x_{\ell+1/2})\ G_\ell^n \\
&\le \Delta t \sum_{i=\ell+1}^{I} h\ \nu^n(x_{i-1/2})\ G_i^n \\
&\quad + \Delta t \sum_{i=\ell+1}^{I} x_{i-1/2}\ \left(\nu^n(x_{i-1/2}) - \nu^n(x_{i+1/2})\right)_-\ G_i^n \\
&\quad + \Delta t \sum_{i=\ell+1}^{I} x_{i+1/2}\ \left(\nu^n(x_{i+1/2}) - \nu^n(x_{i-1/2})\right)_+\ G_i^n \\
&\quad + \Delta t\ x_{\ell+1/2}\ \nu_+^n(x_{\ell+1/2})\ G_\ell^n,
\end{aligned}
$$

the last inequality being a consequence of the subadditivity of $r \mapsto r_+$ and $r \mapsto r_-$. Since $h \in (0,1)$, the choice of ℓ guarantees that $x_{\ell+1/2} \le 2$ and $x_{i+1/2} \le 2\ x_{i-1/2}$ for $i \ge \ell+1$. Consequently, for $i \ge \ell+1$,

$$
\begin{aligned}
x_{i-1/2}\ \left(\nu^n(x_{i-1/2}) - \nu^n(x_{i+1/2})\right)_- &\le v^n\ x_{i-1/2}\ \left(x_{i-1/2}^\alpha - x_{i+1/2}^\alpha\right)_- \\
&\le v^n\ x_{i-1/2}\ \alpha\ h\ x_{i-1/2}^{\alpha-1} \\
&\le h\ v^n\ x_{i-1/2}^\alpha,
\end{aligned}
$$

$$
x_{i+1/2}\ \left(\nu^n(x_{i+1/2}) - \nu^n(x_{i-1/2})\right)_+ \le 2\ h\ v^n\ x_{i-1/2}^\alpha,
$$

and $\nu^n(x_{i-1/2}) \le v^n\ x_{i-1/2}^\alpha$, while

$$
x_{\ell+1/2}\ \nu_+^n(x_{\ell+1/2})\ \le v^n\ x_{\ell+1/2}^{1+\alpha} \le 4\ v^n.
$$

Therefore, since $G_i^n \geq 0$ by (3.1),

$$\begin{aligned}
&\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ \left(G_i^{n+1} - (1+\Delta t)\ G_i^n\right) \\
&\leq \Delta t\ v^n \sum_{i=\ell+1}^{I} h\ x_{i-1/2}^{\alpha}\ G_i^n + 3\ \Delta t\ v^n \sum_{i=\ell+1}^{I} h\ x_{i-1/2}^{\alpha}\ G_i^n + 4\ \Delta t\ v^n\ G_\ell^n \\
&\leq 4\ \Delta t\ v^n \left(\sum_{i=\ell+1}^{I} h\ G_i^n\right)^{1-\alpha} \left(\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n\right)^{\alpha} + 4\ \Delta t\ v^n\, G_\ell^n
\end{aligned}$$

by the Hölder inequality. Using (3.1) once more yields

$$\begin{aligned}
&\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ \left(G_i^{n+1} - (1+\Delta t)\ G_i^n\right) \\
&\leq 4\ \Delta t\ \|G_0\|_{L^1}^{1-\alpha}\ v^n \left(\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n\right)^{\alpha} + 4\ \Delta t\ v^n\ G_\ell^n. \qquad (3.10)
\end{aligned}$$

Owing to (3.1), we may use Lemma 3.1 and insert (3.2) into (3.10) to obtain, with the help of the Young inequality,

$$\begin{aligned}
&\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ \left(G_i^{n+1} - (1+\Delta t)\ G_i^n\right) \\
&\leq C_1'\ \Delta t\ G_0^n \left(\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n\right)^{\alpha} + C_1'\ \Delta t\ G_0^n \sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n \\
&\quad + C_1'\ \Delta t\ G_0^n\ \sup_i\ \{G_i^n\} + C_1'\ \Delta t\ G_0^n\ \sup_i\ \{G_i^n\} \left(\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n\right)^{1-\alpha} \\
&\leq 2\ C_1'\ \Delta t\ \sup_i\ \{G_i^n\} \left(\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n + 1 + \sup_i\ \{G_i^n\}^{1/\alpha}\right),
\end{aligned}$$

with $C_1' = 4C_1 \max\{1, \|G_0\|_{L^1}^{1-\alpha}\}$, whence (3.6), and the proof of Lemma 3.2 is complete. $\square$

We now introduce

$$\begin{aligned}
&K_1 := 2\ C_1\ \|G_0\|_{L^\infty}\ e^T, \qquad K_2 := C_2\ \left(1 + \|G_0\|_{L^\infty}\ e^T\right), \\
&K_3 := C_2\ \left(1 + \|G_0\|_{L^\infty}^{(1+\alpha)/\alpha}\ e^{(1+\alpha)T/\alpha}\right), \\
&x_\star := \left\{K_1\ e^{K_2 T}\ \left(1 + \int_0^\infty x\ G_0(x)\ dx + K_3\ T\right)\right\}^{1/(1-\alpha)}.
\end{aligned}$$

Proposition 3.3. *Assume that* (2.18) *holds true. For* $n \in \{0, \dots, N\}$, *we have*

$$\sum_{i=0}^{I} h\ G_i^n = \sum_{i=0}^{I} h\ G_i^0, \qquad (3.11)$$

$$0 \le G_i^n \le (1+\Delta t)^n \ \|G_0\|_{L^\infty}, \quad 0 \le i \le I, \tag{3.12}$$

$$\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n \le (1+K_2\ \Delta t)^n \int_0^\infty x\ G_0(x)\ dx$$

$$+ K_3\ \Delta t \sum_{j=0}^{n-1} (1+K_2\ \Delta t)^j\ , \tag{3.13}$$

$$0 \le v^n \le K_1 \left(1 + \sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n\right) \le x_\star^{1-\alpha}. \tag{3.14}$$

Proof. We proceed by induction on $n \in \{0, \dots, N\}$ and first consider the case $n = 0$. The assertion (3.11) is obvious in that case, while (3.12) and (3.13) readily follow from (2.5) and (2.8). We then infer from Lemma 3.1, (3.12), (3.13), and the Young inequality that

$$\begin{aligned} v^0 &\le 2\ C_1\ G_0^0 \left(1 + \sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^0\right) \\ &\le K_1 \left(1 + \sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^0\right) \\ &\le K_1\ e^{K_2 T} \left(1 + \int_0^\infty x\ G_0(x)\ dx + K_3\ T\right) = x_\star^{1-\alpha}, \end{aligned}$$

and we have checked that Proposition 3.3 is valid for $n = 0$. Consider now $n \in \{0, \dots, N-1\}$ such that the assertions (3.11)–(3.14) hold true. By (2.18) and (3.14), we have

$$\nu^n(x_{I+1/2}) \le (h\ I)^\alpha\ v^n - (h\ I) \le (h\ I)^\alpha \ \left(x_\star^{1-\alpha} - (h\ I)^{1-\alpha}\right) \le 0,$$

and thus $\nu_+^n(x_{I+1/2}) = 0$. This fact, together with (3.11) and (3.12), allows us to use Lemma 3.2 to conclude that (3.4)–(3.6) hold true. Then, (3.11) for $n+1$ follows at once from (3.4), while (3.12) for $n+1$ is a consequence of (3.5) and (3.12) for n. In addition, inserting (3.12) into (3.6) yields

$$\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^{n+1} \le (1+K_2\ \Delta t) \sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^n + K_3\ \Delta t.$$

Taking into account (3.13) for n, we deduce (3.13) for $n+1$. A straightforward consequence of (3.13) for $n+1$ is that

$$\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^{n+1} \le e^{K_2\ T} \left(\int_0^\infty x\ G_0(x)\ dx + K_3\ T\right). \tag{3.15}$$

Since (3.11) and (3.12) hold true for $n+1$ by the previous analysis, we are in a position to apply Lemma 3.1 and conclude that

$$v^{n+1} \le C_1\ G_0^{n+1} \left(1 + \left(\sum_{i=\ell+1}^{I} h\ x_{i-1/2}\ G_i^{n+1}\right)^{1-\alpha}\right).$$

Using the Young inequality and (3.12) for $n+1$, we are led to

$$\begin{aligned} v^{n+1} &\le C_1 \ \|G_0\|_{L^\infty} \ e^T \ \left(2 + \sum_{i=\ell+1}^{I} h \ x_{i-1/2} \ G_i^{n+1}\right) \\ &\le K_1 \ \left(1 + \sum_{i=\ell+1}^{I} h \ x_{i-1/2} \ G_i^{n+1}\right), \end{aligned}$$

whence the first inequality in (3.14) for $n+1$. Combining this last inequality with (3.15) finally entails that $v^{n+1} \le x_\star^{1-\alpha}$, and the proof of Proposition 3.3 is complete. ∎

Summarizing the outcome of Proposition 3.3, we have proved that (G_i^n) satisfies the following estimate.

COROLLARY 3.4. *There is a positive constant C_3 depending only on α, G_0, and T such that*

$$\sup_i \ \{G_i^n\} + v^n + \sum_{i=0}^{I} h \ (1 + x_{i-1/2}) \ G_i^n \le C_3$$

for $n \in \{0, \dots, N\}$.

Recalling that the solution $G(t)$ to (2.1), (2.2), (2.3) is nonincreasing with respect to the variable x for each $t \ge 0$, we now show that this property is also enjoyed by (G_i^n).

LEMMA 3.5. *For $n \in \{0, \dots, N\}$,*

$$G_{i+1}^n \le G_i^n, \quad 0 \le i \le I, \tag{3.16}$$

$$\sum_{i=0}^{I} \left|G_{i+1}^n - G_i^n\right| \le \|\partial_x G_0\|_{L^1} \ e^T. \tag{3.17}$$

Proof. For $n \in \{0, \dots, N\}$ and $i \in \{0, \dots, I\}$, we set $g_{i+1/2}^n = (G_{i+1}^n - G_i^n)/h$ and use (3.8) to compute $g_{i+1/2}^n$:

$$\begin{aligned} h \ g_{i+1/2}^{n+1} &= (1+\Delta t) \ G_{i+1}^n + \Delta t \ \nu_-^n(x_{i+3/2}) \ g_{i+3/2}^n - \Delta t \ \nu_+^n(x_{i+1/2}) \ g_{i+1/2}^n \\ &\quad - (1+\Delta t) \ G_i^n - \Delta t \ \nu_-^n(x_{i+1/2}) \ g_{i+1/2}^n + \Delta t \ \nu_+^n(x_{i-1/2}) \ g_{i-1/2}^n, \end{aligned}$$

$$\begin{aligned} g_{i+1/2}^{n+1} &= \left(1 + \Delta t - \frac{\Delta t}{h} \ \left|\nu^n(x_{i+1/2})\right|\right) \ g_{i+1/2}^n \\ &\quad + \frac{\Delta t}{h} \ \nu_-^n(x_{i+3/2}) \ g_{i+3/2}^n + \frac{\Delta t}{h} \ \nu_+^n(x_{i-1/2}) \ g_{i-1/2}^n. \end{aligned} \tag{3.18}$$

Since $r \mapsto r_+$ and $r \mapsto r_-$ are subadditive, we realize that

$$\begin{aligned} &\nu_-^n(x_{i+3/2}) + \nu_+^n(x_{i-1/2}) - |\nu^n(x_i + 1/2)| \\ &\quad = \nu_-^n(x_{i+3/2}) - \nu_-^n(x_{i+1/2}) + \nu_+^n(x_{i-1/2}) - \nu_+^n(x_{i+1/2}) \\ &\quad \le \left(\nu^n(x_{i+3/2}) - \nu^n(x_{i+1/2})\right)_- + \left(\nu^n(x_{i-1/2}) - \nu^n(x_{i+1/2})\right)_+ \\ &\quad \le \left((x_{i+3/2}^\alpha - x_{i+1/2}^\alpha) \ v^n - h\right)_- + \left((x_{i-1/2}^\alpha - x_{i+1/2}^\alpha) \ v^n + h\right)_+. \end{aligned}$$

Then,

$$\nu_-^n(x_{i+3/2}) + \nu_+^n(x_{i-1/2}) - |\nu^n(x_{i+1/2})| \le 2h. \tag{3.19}$$

Introducing

$$\lambda_{2,i}^n = \frac{\Delta t}{(1+3\ \Delta t)h}\ \nu_-^n(x_{i+3/2}), \qquad \lambda_{3,i}^n = \frac{\Delta t}{(1+3\ \Delta t)h}\ \nu_+^n(x_{i-1/2}),$$

$$\lambda_{1,i}^n = \frac{1}{(1+3\ \Delta t)}\ \left(1+\Delta t - \frac{\Delta t}{h}\ |\nu^n(x_{i+1/2})|\right), \qquad \lambda_{4,i}^n = 1 - \sum_{j=1}^{3} \lambda_{j,i}^n,$$

we clearly have $\lambda_{2,i}^n \geq 0$, $\lambda_{3,i}^n \geq 0$, while (3.9) ensures that $\lambda_{1,i}^n \geq 0$. In addition, it follows from (3.19) that

$$1 - \lambda_{4,i}^n \leq \frac{1}{(1+3\ \Delta t)}\ \left(1 + \Delta t + \frac{\Delta t}{h}\ 2\ h\right) \leq 1,$$

whence $\lambda_{4,i}^n \geq 0$. Consequently, $\lambda_{j,i}^n \in [0,1]$ for $1 \leq j \leq 4$, and $g_{i+1/2}^{n+1}/(1+3\Delta t)$ is a convex combination of $g_{i+1/2}^n$, $g_{i+3/2}^n$, $g_{i-1/2}^n$, and 0.

Now, let $\Psi : \mathbb{R} \to [0,+\infty)$ be a nonnegative and convex function with $\Psi(0)=0$ and such that

$$\Psi(\lambda\ r) \leq \lambda^\gamma\ \Psi(r), \quad (r,\lambda) \in [0,+\infty) \times [1,+\infty), \tag{3.20}$$

for some $\gamma \geq 1$. The convexity of Ψ then entails that

$$\sum_{i=0}^{I} \Psi\left(\frac{g_{i+1/2}^{n+1}}{(1+3\ \Delta t)}\right) \leq \sum_{i=0}^{I} \left(\lambda_{1,i}^n\ \Psi(g_{i+1/2}^n) + \lambda_{2,i}^n\ \Psi(g_{i+3/2}^n) + \lambda_{3,i}^n\ \Psi(g_{i-1/2}^n)\right).$$

Since $\nu_+^n(x_{-1/2}) = 0$, we have $\lambda_{3,0}^n = 0$ and

$$\begin{aligned}
(1+3\ \Delta t)\ \sum_{i=0}^{I} \Psi\left(\frac{g_{i+1/2}^{n+1}}{(1+3\ \Delta t)}\right) &\leq \sum_{i=0}^{I} \left(1+\Delta t - \frac{\Delta t}{h}\ |\nu^n(x_{i+1/2})|\right)\ \Psi(g_{i+1/2}^n) \\
&\quad + \sum_{i=1}^{I} \frac{\Delta t}{h}\ \nu_-^n(x_{i+1/2})\ \Psi(g_{i+1/2}^n) \\
&\quad + \sum_{i=0}^{I-1} \frac{\Delta t}{h}\ \nu_+^n(x_{i+1/2})\ \Psi(g_{i+1/2}^n) \\
&\leq (1+\Delta t)\ \sum_{i=0}^{I} \Psi(g_{i+1/2}^n).
\end{aligned}$$

Owing to (3.20), we end up with

$$\begin{aligned}
\sum_{i=0}^{I} \Psi(g_{i+1/2}^{n+1}) &= \sum_{i=0}^{I} \Psi\left(\frac{(1+3\ \Delta t)\ g_{i+1/2}^{n+1}}{(1+3\ \Delta t)}\right) \\
&\leq (1+3\ \Delta t)^{\gamma-1}\ (1+\Delta t)\ \sum_{i=0}^{I} \Psi(g_{i+1/2}^n).
\end{aligned}$$

The discrete Gronwall lemma yields

$$\sum_{i=0}^{I} \Psi(g_{i+1/2}^n) \leq e^{(3(\gamma-1)+1)T}\ \sum_{i=0}^{I} \Psi(g_{i+1/2}^0). \tag{3.21}$$

We first take $\Psi(r) = r_+$, which obviously satisfies (3.20) with $\gamma = 1$. The assertion (3.16) then readily follows from (3.21) since $g^0_{i+1/2} \le 0$ for $i \in \{0, \ldots, I\}$ by (2.5) and (2.8). Similarly, $\Psi(r) = |r|$ satisfies (3.20) with $\gamma = 1$, and (3.17) is a straightforward consequence of (3.21), taking into account that

$$\sum_{i=0}^{I} h \ \left| g^0_{i+1/2} \right| \le \|\partial_x G_0\|_{L^1},$$

and the proof of Lemma 3.5 is complete. $\square$

Remark 3.6. Note that $\Psi(r) = r^p$ satisfies (3.20) for $p \in [1, \infty)$ with $\gamma = p$. In that case, (3.21) simply means that the discrete L^p-norm of $(g^n_{i+1/2})$ remains finite if it is initially finite.

At this point, note that Proposition 2.2 is a consequence of Proposition 3.3 and Lemma 3.5.

We end this section with the time equicontinuity of (G^n_i).

LEMMA 3.7. *For each $R \ge 1$ there is a constant $C_4(R)$ depending only on α, G_0, T, and R such that, for $n \in \{0, \ldots, N-1\}$,*

$$\sum_{i=0}^{\ell_R} h \ \left| G^{n+1}_i - G^n_i \right| \le C_4(R) \ \Delta t, \tag{3.22}$$

where ℓ_R denotes the integer such that $R \in \Lambda_{\ell_R}$.

Proof. By Corollary 3.4, Lemma 3.5, and (3.8), we have for $n \in \{0, \ldots, N-1\}$ and $i \in \{0, \ldots, \ell_R\}$

$$\begin{aligned} \left| G^{n+1}_i - G^n_i \right| &\le \Delta t \ \left\{ C_3 + x^\alpha_{i+1/2} \ v^n \ \frac{G^n_i - G^n_{i+1}}{h} + x^\alpha_{i-1/2} \ v^n \ \frac{G^n_{i-1} - G^n_i}{h} \right\} \\ &\le \Delta t \ \left\{ C_3 + (1+R)^\alpha \ C_1 \ \left(\frac{G^n_i - G^n_{i+1}}{h} + \frac{G^n_{i-1} - G^n_i}{h} \right) \right\}. \end{aligned}$$

We sum up the above inequalities for $i \in \{0, \ldots, \ell_R\}$ and use (3.17) to conclude that (3.22) holds true. $\square$

4. Convergence. As a consequence of the analysis of the previous section, the sets $\{G^h\}_h$ and $\{v^h\}_h$ enjoy the following compactness properties.

LEMMA 4.1. *There are a subsequence of (G^h, v^h) (not relabeled) and a pair of nonnegative functions $G \in \mathcal{C}([0,T]; L^1(\mathbb{R}_+))$ and $v \in L^\infty(0,T)$ such that*

$$G^h \longrightarrow G \quad in \quad L^\infty(0,T; L^1(\mathbb{R}_+)), \tag{4.1}$$

$$v^h \ \overset{*}{\rightharpoonup} \ v \quad in \quad L^\infty(0,T), \tag{4.2}$$

and $G \in L^\infty(0,T; L^1(\mathbb{R}_+, x dx))$ satisfies (2.6).

Proof. We introduce the auxiliary function

$$\mathcal{G}^h(t,x) = \sum_{i=0}^{I^h} \left\{ G^{n,h}_i + \frac{(t-t^n)}{\Delta t} \ \left(G^{n+1,h}_i - G^{n,h}_i \right) \right\} \ \mathbf{1}_{\Lambda^h_i}(x), \quad (t,x) \in [t^n, t^{n+1}] \times \mathbb{R}_+,$$

for $n \in \{0, \ldots, N-1\}$. Clearly, $\mathcal{G}^h \in \mathcal{C}([0,T]; L^1(\mathbb{R}_+))$, and it readily follows from Lemma 3.7 that

$$\sup_{[0,T]} \left\| G^h(t) - \mathcal{G}^h(t) \right\|_{L^1(0,R)} \le C_4(R) \ \Delta t \tag{4.3}$$

for any $R \ge 1$.

We next fix $R \geq 1$. On the one hand, it follows from Corollary 3.4 and (3.17) that $(\mathcal{G}^h)$ is bounded in $L^\infty(0,T;BV(0,R))$. On the other hand, an easy computation shows that (3.22) implies

$$\left\|\mathcal{G}^h(t) - \mathcal{G}^h(s)\right\|_{L^1(0,R)} \leq C_4(R)\ |t-s|$$

for $(s,t) \in [0,T] \times [0,T]$. Since $BV(0,R)$ is compactly embedded in $L^1(0,R)$, a classical compactness result [21, Theorem 5] entails that

$$(\mathcal{G}^h) \quad \text{is relatively compact in} \quad \mathcal{C}([0,T];L^1(0,R)), \tag{4.4}$$

and (4.4) is valid for every $R \geq 1$. Also, by Corollary 3.4, there is a constant C depending only on α, G_0, and T such that

$$\int_0^\infty \left(G^h(t,x) + \mathcal{G}^h(t,x)\right)\ x\ dx \leq C, \quad t \in [0,T]. \tag{4.5}$$

Thanks to (4.5), we may improve the compactness (4.4) of $(\mathcal{G}^h)$ and conclude that $(\mathcal{G}^h)$ is relatively compact in $\mathcal{C}([0,T];L^1(\mathbb{R}_+))$. Consequently, there are a subsequence of $(\mathcal{G}^h)$ (not relabeled) and a function G in $\mathcal{C}([0,T];L^1(\mathbb{R}_+))$ such that

$$\mathcal{G}^h \longrightarrow G \quad \text{in} \quad \mathcal{C}([0,T];L^1(\mathbb{R}_+)).$$

Recalling (4.3) and (4.5), we readily conclude that (4.1) holds true and that G is a nonnegative function in $L^\infty(0,T;L^1(\mathbb{R}_+,xdx))$. Moreover, the convergence (4.1) of (G^h), (2.9), and (3.11) imply that G satisfies (2.6). The convergence (4.2) and the nonnegativity of v are then straightforward consequences of Corollary 3.4 and the nonnegativity of v^h. $\square$

We next show that (3.11), (3.21), and the integrability of $\partial_x G_0$ guarantee that G enjoys the regularity properties claimed in Theorem 2.3.

LEMMA 4.2. *We have $\partial_x G \in L^\infty(0,T;L^1(\mathbb{R}_+,(1+x)dx))$, and $t \mapsto G(t,x)$ is nonincreasing for each $t \geq 0$.*

Proof. For $\varphi \in L^\infty(\mathbb{R}_+)$, we define the discrete gradient $D_h\varphi$ by

$$D_h\varphi(x) := \frac{\varphi(x+h) - \varphi(x)}{h}, \quad x \in \mathbb{R}_+.$$

We first observe that

$$D_h G^h(t,x) = \sum_{i=0}^{I^h} \left(\frac{G_{i+1}^{n,h} - G_i^{n,h}}{h}\right)\ \mathbf{1}_{\Lambda_i^h}(x), \quad x \in \mathbb{R}_+, \tag{4.6}$$

for $t \in [t^n, t^{n+1})$ and $n \in \{1,\dots,N\}$ and recall that Lemma 3.5 and in particular (3.21) provide some information on $((G_{i+1}^{n,h} - G_i^{n,h})/h)$. It turns out that the available information allows us to show the weak compactness of (D_hG^h) in $L^1((0,T)\times\mathbb{R}_+)$. Indeed, we first notice that

$$\sup_{t\in[0,T]} \int_0^\infty (1+x)\ \left|D_hG^h(t,x)\right|\ dx \leq C \tag{4.7}$$

for some constant C depending only on α, G_0, and T. Indeed, (4.7) readily follows from (3.11), (3.16), and (3.17), thanks to the identity

$$\sum_{i=0}^{I^h} x_{i+1/2}\ \left|G_{i+1}^{n,h} - G_i^{n,h}\right| = \sum_{i=0}^{I^h} h\ G_i^{n,h}.$$

We next recall that, since $\partial_x G_0 \in L^1(\mathbb{R}_+)$, a refined version of the de la Vallée–Poussin theorem [11, Proposition I.1.1] ensures that there is a nonnegative and convex function $\Psi_0 \in \mathcal{C}^1([0,+\infty)) \cap W^{2,\infty}_{\text{loc}}(\mathbb{R}_+)$ satisfying

$$\lim_{r\to+\infty} \frac{\Psi_0(r)}{r} = +\infty, \tag{4.8}$$

with $\Psi_0(0) = 0$, $\Psi_0'(0) \geq 0$, Ψ_0' a concave function on $[0,+\infty)$, and such that $\Psi_0(|\partial_x G_0|) \in L^1(\mathbb{R}_+)$. (See also, e.g., [4, p. 38] for the construction of such a function Ψ_0 without the requirement that Ψ_0' be concave.) Thanks to the concavity of Ψ_0' and the nonnegativity of $\Psi_0'(0)$, we have $\Psi_0'(\lambda r) \leq \lambda\ \Psi_0'(r)$ for $r \geq 0$ and $\lambda \geq 1$. Integrating this inequality yields

$$\Psi_0(\lambda r) \leq \lambda^2\ \Psi_0(r), \quad (r,\lambda) \in [0,+\infty)\times[1,+\infty). \tag{4.9}$$

Since Ψ_0 is nondecreasing, the function Ψ defined by $\Psi(r) = \Psi_0(|r|)$ is a nonnegative and convex function with $\Psi(0) = 0$, which satisfies (3.20) with $\gamma = 2$ by (4.9). Consequently, we infer from (3.21) and (4.6) that

$$\sup_{t\in[0,T]} \int_0^\infty \Psi_0\left(\left|D_h G^h(t,x)\right|\right)\ dx \leq e^{4T} \int_0^\infty \Psi_0\left(\left|D_h G^h(0,x)\right|\right)\ dx. \tag{4.10}$$

However,

$$\int_0^\infty \Psi_0\left(\left|D_h G^h(0,x)\right|\right)\ dx = h\ \sum_{i=0}^{I^h} \Psi_0\left(\left|\frac{G^{0,h}_{i+1} - G^{0,h}_i}{h}\right|\right), \tag{4.11}$$

and it follows from (2.5), (2.8), (4.9), the convexity of Ψ_0, and the Jensen inequality that

$$\begin{aligned}
&\Psi_0\left(\left|\frac{G^{0,h}_{i+1} - G^{0,h}_i}{h}\right|\right)\\
&\qquad \leq \Psi_0\left(\frac{1}{h}\ \int_{\Lambda_i^h} |\partial_x G_0|\ dx + \frac{1}{h}\ \int_{\Lambda_{i+1}^h} |\partial_x G_0|\ dx\right)\\
&\qquad \leq 2\ \left\{\Psi_0\left(\frac{1}{h}\ \int_{\Lambda_i^h} |\partial_x G_0|\ dx\right) + \Psi_0\left(\frac{1}{h}\ \int_{\Lambda_{i+1}^h} |\partial_x G_0|\ dx\right)\right\}\\
&\qquad \leq \frac{2}{h}\ \left\{\int_{\Lambda_i^h} \Psi_0\left(|\partial_x G_0|\right)\ dx + \int_{\Lambda_{i+1}^h} \Psi_0\left(|\partial_x G_0|\right)\ dx\right\},
\end{aligned}$$

whence

$$h\ \sum_{i=0}^{I^h} \Psi_0\left(\left|\frac{G^{0,h}_{i+1} - G^{0,h}_i}{h}\right|\right) \leq 4\ \|\partial_x G_0\|_{L^1}.$$

Inserting this estimate into (4.11), we deduce from (4.10) that

$$\sup_{t\in[0,T]} \int_0^\infty \Psi_0\left(\left|D_h G^h(t,x)\right|\right)\ dx \leq C \tag{4.12}$$

for some constant C depending on α, G_0, and T. Owing to (4.7), (4.8), and (4.12), we are in a position to apply the Dunford–Pettis theorem and conclude that $(D_h G^h)$ is relatively weakly sequentially compact in $L^1((0,T)\times\mathbb{R}_+)$. We may thus extract a subsequence of $(D_h G^h)$ (not relabeled) such that

$$D_h G_h \rightharpoonup g \quad \text{in} \quad L^1((0,T)\times\mathbb{R}_+) \tag{4.13}$$

for some function $g\in L^1((0,T)\times\mathbb{R}_+)$. Now, it follows from (4.7) and (4.13) that g belongs to $L^\infty(0,T;L^1(\mathbb{R}_+,(1+x)dx))$, while a classical computation entails that $g=\partial_x G$ in the sense of distributions. In addition, $D_h G^h\le 0$ by (3.16), whence $\partial_x G\le 0$, and the proof of Lemma 4.2 is complete. ∎

We are now in a position to complete the proof of Theorem 2.3.

Proof of Theorem 2.3. We first observe that Lemmas 4.1 and 4.2 ensure that (G,v) enjoy the regularity properties (2.25).

We next consider $\varphi\in\mathcal{C}_0^\infty([0,T)\times[0,+\infty))$ with $\operatorname{supp}\varphi\subset[0,\tau)\times[0,R)$ for some $\tau\in[0,T)$ and $R\in\mathbb{R}_+$ and set

$$\varphi_i^{n,h}=\frac{1}{h\ \Delta t}\int_{t^n}^{t^{n+1}}\int_{\Lambda_i^h}\varphi(t,x)\ dxdt$$

for $i\ge 0$ and $n\in\{0,\dots,N-1\}$. We also assume that h and Δt are sufficiently small so that $\tau\le t^{N-2}$ and $R\le x_{I^h-1/2}$, and we denote by ℓ_R^h the integer such that $R\in\Lambda_{\ell_R^h}^h$. In the following we denote by C_φ any nonnegative constant depending only on α, G_0, T, and φ.

We multiply (3.8) by $\varphi_i^{n,h}$ and sum up the resulting identities to obtain

$$Y_1^h=Y_2^h,$$

where

$$Y_1^h:=\sum_{n=0}^{N-1}\sum_{i=0}^{I^h}h\ (G_i^{n+1,h}-G_i^{n,h})\ \varphi_i^{n,h}$$

and

$$\begin{aligned}Y_2^h:=&\ h\ \Delta t\ \sum_{n=0}^{N-1}\sum_{i=0}^{I^h}G_i^{n,h}\ \varphi_i^{n,h}+\Delta t\ \sum_{n=0}^{N-1}\sum_{i=0}^{I^h}\nu_-^n(x_{i+1/2})\ \left(G_{i+1}^{n,h}-G_i^{n,h}\right)\ \varphi_i^{n,h}\\&+\Delta t\ \sum_{n=0}^{N-1}\sum_{i=0}^{I^h}\nu_+^n(x_{i-1/2})\ \left(G_{i-1}^{n,h}-G_i^{n,h}\right)\ \varphi_i^{n,h}.\end{aligned}$$

We next introduce

$$Z_1^h:=-\int_0^T\int_0^\infty G^h(t,x)\ \partial_t\varphi(t,x)\ dtdx-\int_0^\infty G_0(x)\ \varphi(0,x)\ dx,$$

$$\begin{aligned}Z_2^h:=&\int_0^T\int_0^\infty G^h(t,x)\ \varphi(t,x)\ dtdx-\int_0^T G^h(t,0)\ \varphi(t,0)\ dt\\&+\int_0^T\int_0^\infty G^h(t,x)\ \partial_x\left(\mathcal{W}^h\ \varphi\right)(t,x)\ dxdt,\end{aligned}$$

where

$$\mathcal{W}^h(t,x) = x^\alpha\ v^h(t) - 1 - x, \quad (t,x) \in [0,T] \times \mathbb{R}_+.$$

On the one hand, since φ is compactly supported, it follows at once from (4.1) and (4.2) that

$$\lim_{h,\Delta t \to 0} Z_1^h = -\int_0^T \int_0^\infty G(t,x)\ \partial_t \varphi(t,x)\ dtdx - \int_0^\infty G_0(x)\ \varphi(0,x)\ dx, \tag{4.14}$$

and

$$\begin{aligned} &\lim_{h,\Delta t \to 0} \left(Z_2^h + \int_0^T G^h(t,0)\ \varphi(t,0)\ dt \right) \\ &\qquad = \int_0^T \int_0^\infty G(t,x)\ \left(\varphi + \partial_x \left(\mathcal{W}\ \varphi\right)\right)(t,x)\ dxdt. \end{aligned} \tag{4.15}$$

On the other hand, we have

$$\begin{aligned} &\int_0^T \big(G^h(t,0) - G(t,0)\big)\ \varphi(t,0)\ dt \\ &\quad = \int_0^T \sum_{i=0}^{I^h} \big(G^h(t,x_{i-1/2}) - G^h(t,x_{i+1/2})\big)\ \varphi(t,0)\ dt + \int_0^\infty \partial_x G(t,x)\ \varphi(t,0)\ dxdt \\ &\quad = \int_0^T \int_0^\infty \big(\partial_x G(t,x) - D_h G^h(t,x)\big)\ \varphi(t,0)\ dxdt. \end{aligned}$$

We then infer from (4.13) that the right-hand side of the above identity converges to zero as $h \to 0$. Inserting this result into (4.15), we end up with

$$\begin{aligned} \lim_{h,\Delta t \to 0} Z_2^h = &- \int_0^T G(t,0)\ \varphi(t,0)\ dt \\ &+ \int_0^T \int_0^\infty G(t,x)\ \left(\varphi + \partial_x \left(\mathcal{W}\ \varphi\right)\right)(t,x)\ dxdt. \end{aligned} \tag{4.16}$$

Having identified the limits of (Z_1^h) and (Z_2^h) as $h \to 0$, we next aim at comparing the terms Y_k^h and Z_k^h, $k = 1,2$, in order to show that $Z_1^h - Z_2^h$ converges to zero as $(h, \Delta t) \to 0$.

We first compute $(Z_1^h - Y_1^h)$. Since G^h is constant on $[t^n, t^{n+1}) \times \Lambda_i^h$ for $i \geq 0$ and $n \in \{0, \ldots, N-1\}$, we have

$$\begin{aligned} Z_1^h &= -\sum_{n=0}^{N-1} \int_0^\infty G^h(t^n,x)\ \big(\varphi(t^{n+1},x) - \varphi(t^n,x)\big)\ dx - \int_0^\infty G_0(x)\ \varphi(0,x)\ dx \\ &= \sum_{n=0}^{N-1} \int_0^\infty \big(G^h(t^{n+1},x) - G^h(t^n,x)\big)\ \varphi(t^{n+1},x)\ dx \\ &\quad + \int_0^\infty \big(G^h(0,x) - G_0(x)\big)\ \varphi(0,x)\ dx, \end{aligned}$$

from which we deduce that

$$\begin{aligned}\left|Z_1^h - Y_1^h\right| &\leq \sum_{n=0}^{N-1} \sum_{i=0}^{\ell_R^h} \left|G_i^{n+1,h} - G_i^{n,h}\right| \int_{t^n}^{t^{n+1}} \int_{\Lambda_i^h} |\partial_t \varphi| \ dxdt \\ &\quad + \|G^h(0,.) - G_0(.)\|_{L^1} \ \|\varphi\|_{L^\infty} \\ &\leq T \ \|\partial_t \varphi\|_{L^\infty} \sup_{0\leq n\leq N-1} \sum_{i=0}^{\ell_R^h} h \ \left|G_i^{n+1,h} - G_i^{n,h}\right| \\ &\quad + \|G^h(0,.) - G_0(.)\|_{L^1} \ \|\varphi\|_{L^\infty}.\end{aligned}$$

We now use (2.9) and Lemma 3.7 to conclude that

$$\left|Z_1^h - Y_1^h\right| \leq C_\varphi \ \Delta t. \tag{4.17}$$

We next turn to $(Z_2^h - Y_2^h)$. Since $\mathcal{W}^h(t^n, x_{i+1/2}) = \nu^n(x_{i+1/2})$, G^h is constant on $[t^n, t^{n+1}) \times \Lambda_i^h$ for $i \geq 0$, and $n \in \{0, \dots, N-1\}$, we have

$$\begin{aligned}Z_2^h &= \sum_{n=0}^{N-1} \sum_{i=0}^{I^h} G_i^{n,h} \int_{t^n}^{t^{n+1}} \int_{\Lambda_i^h} \varphi(t,x) \ dxdt - \sum_{n=0}^{N-1} G_0^n \int_{t^n}^{t^{n+1}} \varphi(t,0) \ dt \\ &\quad + \sum_{n=0}^{N-1} \sum_{i=0}^{I^h} G_i^{n,h} \int_{t^n}^{t^{n+1}} \left(\nu^n(x_{i+1/2}) \ \varphi(t, x_{i+1/2}) - \nu^n(x_{i-1/2}) \ \varphi(t, x_{i-1/2})\right) \ dt.\end{aligned}$$

Since $\nu^n(x) = \nu_+^n(x) - \nu_-^n(x)$ and $\nu^n(x_{-1/2}) = -1$, a discrete integration by parts yields

$$\begin{aligned}Z_2^h &= \sum_{n=0}^{N-1} \sum_{i=0}^{I^h} G_i^{n,h} \int_{t^n}^{t^{n+1}} \int_{\Lambda_i^h} \varphi(t,x) \ dxdt \\ &\quad + \sum_{n=0}^{N-1} \sum_{i=0}^{I^h} \nu_+^n(x_{i-1/2}) \left(G_{i-1}^{n,h} - G_i^{n,h}\right) \int_{t^n}^{t^{n+1}} \varphi(t, x_{i-1/2}) \ dt \\ &\quad + \sum_{n=0}^{N-1} \sum_{i=0}^{I^h} \nu_-^n(x_{i+1/2}) \left(G_{i+1}^{n,h} - G_i^{n,h}\right) \int_{t^n}^{t^{n+1}} \varphi(t, x_{i+1/2}) \ dt.\end{aligned}$$

It is then easy to compute $(Z_2^h - Y_2^h)$ and deduce from Corollary 3.4 that

$$\begin{aligned}\left|Z_2^h - Y_2^h\right| &\leq h \ \Delta t \sum_{n=0}^{N-1} \sum_{i=0}^{\ell_R^h} \nu_+^n(x_{i-1/2}) \left|G_{i-1}^{n,h} - G_i^{n,h}\right| \|\partial_x \varphi\|_{L^\infty} \\ &\quad + h \ \Delta t \sum_{n=0}^{N-1} \sum_{i=0}^{\ell_R^h} \nu_-^n(x_{i+1/2}) \left|G_{i+1}^{n,h} - G_i^{n,h}\right| \|\partial_x \varphi\|_{L^\infty} \\ &\leq C_\varphi \ h \sup_{0\leq n\leq N-1} \sum_{i=0}^{I^h} \left|G_{i+1}^{n,h} - G_i^{n,h}\right|,\end{aligned}$$

whence

$$\left|Z_2^h - Y_2^h\right| \leq C_\varphi \ h \tag{4.18}$$

by (3.17). Since $Y_1^h = Y_2^h$, we infer from (4.17) and (4.18) that $(Z_1^h - Z_2^h)$ converges to zero as $(h, \Delta t) \to 0$. This fact, together with (4.14) and (4.16), immediately ensures that G satisfies

$$\int_0^T \int_0^\infty G(t,x)\ \partial_t \varphi(t,x)\ dtdx + \int_0^\infty G_0(x)\ \varphi(0,x)\ dx$$
$$= -\int_0^T G(t,0)\ \varphi(t,0)\ dt + \int_0^T \int_0^\infty G(t,x)\ \left(\varphi + \partial_x \left(\mathcal{W}\ \varphi\right)\right)(t,x)\ dxdt.$$

Owing to the regularity of G, standard approximation arguments allow us to conclude from the previous identity that G actually satisfies (2.26).

To conclude the proof, it remains to show that v is given by (2.2). The easiest way to see it is to take $\varphi \equiv 1$ in (2.26), from which (2.2) readily follows, since we already know that G satisfies (2.6). We may, however, prove it directly by passing to the limit in (2.14). Indeed, consider $\varphi \in \mathcal{C}(0,T)$. Arguing as in the proof of (4.16), we realize that

$$\lim_{(h,\Delta t)\to 0} \int_0^T \left(G^h(t,0) - G(t,0)\right)\ \varphi(t)\ dt = 0. \tag{4.19}$$

We next claim that

$$\lim_{(h,\Delta t)\to 0} \sup_{t\in[0,T]} \int_0^\infty x^{\alpha-1}\ \left|G^h(t,x) - G(t,x)\right|\ dx = 0. \tag{4.20}$$

Indeed, it follows from Corollary 3.4 and (2.25) that, for $\delta \in (0,1)$ and $t \in [0,T]$,

$$\begin{aligned}\int_0^\infty x^{\alpha-1}\ \left|G^h(t,x) - G(t,x)\right|\ dx &\le \frac{\delta^\alpha}{\alpha}\ \left(\|G^h(t)\|_{L^\infty} + \|G(t)\|_{L^\infty}\right)\\ &\quad + \delta^{\alpha-1} \int_\delta^\infty \left|G^h(t,x) - G(t,x)\right| dx\\ &\le C\ \delta^\alpha + \delta^{\alpha-1} \sup_{s\in[0,T]} \left\|G^h(s) - G(s)\right\|_{L^1}\end{aligned}$$

for some constant C depending only on α, G_0, T, and G. Thanks to (4.1), we may pass to the limit as $(h,\Delta t) \to 0$ and obtain

$$\limsup_{(h,\Delta t)\to 0} \sup_{t\in[0,T]} \int_0^\infty x^{\alpha-1}\ \left|G^h(t,x) - G(t,x)\right|\ dx \le C\ \delta^\alpha.$$

As $\delta \in (0,1)$ is arbitrary, we let $\delta \to 0$ to obtain the claim (4.20).

Now, owing to (4.2) and (4.20), it is straightforward to check that

$$\lim_{(h,\Delta t)\to 0} \int_0^T \int_0^\infty x^{\alpha-1}\ \left(v^h(t)\ G^h(t,x) - v(t)\ G(t,x)\right)\ \varphi(t)\ dxdt = 0. \tag{4.21}$$

Thanks to (4.19) and (4.21), we may pass to the limit in (2.14) and conclude that v is given by (2.2). $\square$

5. Numerical simulations. In this section, we perform numerical experiments with $\alpha = 1/3$, which corresponds to the original model of Lifshitz and Slyozov [13]. Our aim is twofold: first, to study the numerical accuracy of the scheme analyzed in the previous sections and second, to see its behavior for large times.

We first check the order of the scheme with the following explicit stationary solution to (2.1)–(2.2):

$$G_{LS}(x) := \frac{6}{\left(1-(2x)^{1/3}\right)^{5/3}\ \left(1+(x/4)^{1/3}\right)^{4/3}}\ \exp\left(-\frac{(2x)^{1/3}}{1-(2x)^{1/3}}\right) \tag{5.1}$$

Table 1
Relative errors in the L^1- and L^∞-norms with respect to the number of grid points I^h.

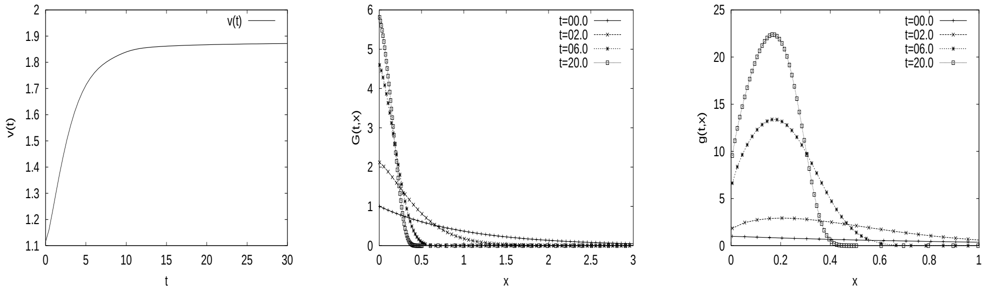
Number of points	Number of iterations	L^1 Error	L^∞ Error
200	221	$1.4\ 10^{-3}$	$1.5\ 10^{-3}$
400	443	$7.5\ 10^{-4}$	$7.5\ 10^{-4}$
800	887	$4.0\ 10^{-4}$	$3.8\ 10^{-4}$
1600	1776	$2.0\ 10^{-4}$	$2.0\ 10^{-4}$

for $x \in [0,1/2]$ and $G_{LS}(x) = 0$ if $x \geq 1/2$. Since G_{LS} is compactly supported in $[0,1/2]$, we take $hI^h = 1$ and $T = 1$. We compute the relative errors at $T = 1$ in the L^1- and L^∞-norms for different values of I^h, which are reported in Table 1. As expected, the scheme is first-order; that is, the error is proportional to h.

We next turn to the large time behavior and first recall that, for (2.1)–(2.3), it is much more complex that originally conjectured by Lifshitz and Slyozov in [13]. As already mentioned, formal asymptotic expansions performed in [13] indicate that the pair (G, v) converges towards a stationary solution (G_∞, v_∞) to (2.1)–(2.3), and it was further conjectured in [13] (and also in [25], but for a different choice of functions k and q) that the asymptotic profile (G_∞, v_∞) does not depend on the shape of the initial data G_0 but only on $\|G_0\|_{L^1}$. More precisely, the conjecture in [13] states that $G_\infty = a\, G_{LS}$ (defined by (5.1) above), with $a = \|G_0\|_{L^1}\ \|G_{LS}\|_{L^1}^{-1}$, while $v_\infty = V_{LS} := 3/2^{2/3}$. It was, however, noticed in [13] that (2.1)–(2.2) actually has a continuum $(G_V)_{V \geq V_{LS}}$ of stationary solutions (with $G_{V_{LS}} = G_{LS}$) satisfying $\|G_V\|_{L^1} = \|G_{LS}\|_{L^1}$, but it was argued that G_V is "unstable" for $V > V_{LS}$. This conjecture turns out to be false, as noticed on the ground of physical arguments in [6, 15] and confirmed by numerical simulations performed in [6]. More precisely, if the initial datum is compactly supported, the asymptotic profile (G_∞, v_∞) is determined by the way in which the initial datum vanishes at the edge of its support. Mathematical proofs of these facts have subsequently been supplied in [1] for $\alpha = 1$ by means of the Laplace transform. Though a convergence proof is still lacking in the general case $\alpha \in (0,1)$, necessary conditions for convergence are provided in [16] when $\alpha = 1/3$. In addition, it is established in [16] that, if one can prove that $v(t)$ converges to some $V > V_{LS}$ as $t \to +\infty$, then $G(t)$ converges towards G_V as $t \to +\infty$. Analogous results for the variant (1.1), (1.9) have subsequently been obtained in [18], still in the case $\alpha = 1/3$. It is also shown in [1, 16, 18] that there are initial data for which convergence towards a stationary solution does not hold at all. In addition, several numerical simulations have been performed in [2] with an accurate numerical method. The results in [2] provide further numerical evidence that the solutions to (2.1)–(2.3) with compactly supported initial data do not converge to the asymptotic profile (G_{LS}, V_{LS}) conjectured in [13] but to the one determined by the way the initial datum vanishes at the edge of its support; that is, (G_V, V) for some $V > V_{LS}$. Still, it is expected that the Lifshitz–Slyozov conjecture is valid for noncompactly supported initial data (with a "smooth" behavior for large x), and the aim of our first computations is to provide some numerical evidence of this fact. We thus choose

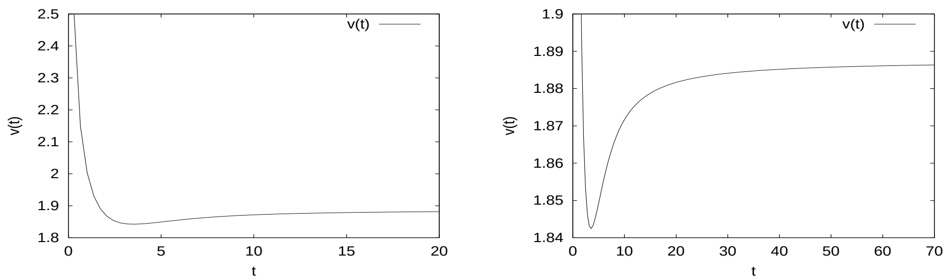
$$G_0(x) = \|G_{LS}\|_{L^1}\ \exp(-x), \quad x \in \mathbb{R}_+, \tag{5.2}$$

and report in Figure 1 the time evolution of v, G, and $g = -\partial_x G$ obtained by the scheme (2.12)–(2.17). For this simulation, we take the number of grid points $I^h = 1000$ with $h\, I^h = 10$, and the final time is $T = 30$. Noticing that $v(0) < V_{LS}$, we see that the function v first increases rapidly towards V_{LS} and then stabilizes to this value



(a) (b) (c)

FIG. 1. *Evolution of* (a) $v(t)$, (b) $G(t,x)$, *and* (c) $g(t,x)$ *corresponding to the initial datum* (5.2).



(a) (b)

FIG. 2. (a) *Evolution of* $v(t)$, (b) *zoom on the small variations of* $v(t)$ *corresponding to the initial datum* (5.3).

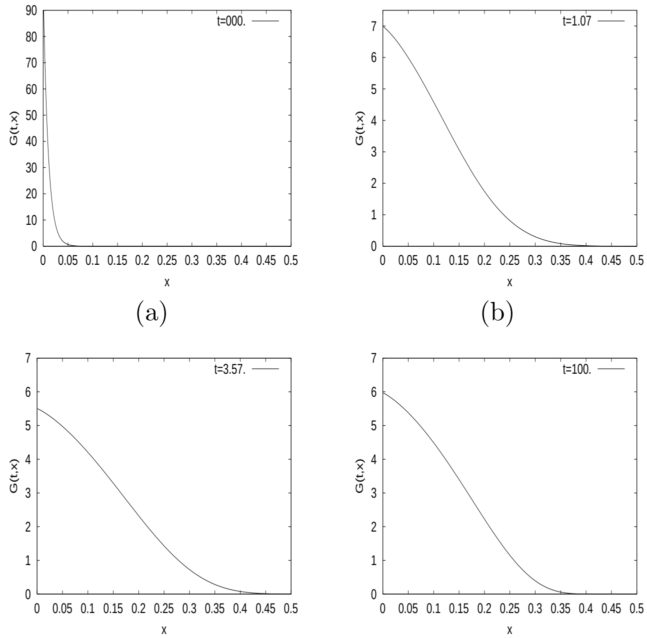
as conjectured. As for G, its support decreases with time towards $[0,1/2]$, and $G(t)$ converges to the stationary solution G_{LS} with a good accuracy.

We next investigate what happens to a similar initial datum but with $v(0) > V_{LS}$. More precisely, we take

$$G_0(x) = 100\,\|G_{LS}\|_{L^1}\,\exp(-100\,x), \quad x \in \mathbb{R}_+, \tag{5.3}$$

and observe that G_0 has the same L^1-norm as (5.2) but decreases faster for large x. In this case, we take $I^h = 1000$ with $h\,I^h = 1$, and the final time is $T = 100$. We observe that the behavior of v differs from that in the previous simulation. Indeed, $v(0)$ being greater than V_{LS}, v first decreases with time but to a smaller value than V_{LS}, as shown in Figure 2. It then increases again towards V_{LS} and finally stabilizes to V_{LS}. The evolution of G in that case is presented in Figure 3, which shows the convergence of G to G_{LS}.

Finally, following the previous discussion on the large time behavior for compactly supported initial data, one may wonder what the behavior of our scheme in that case might be. We have performed numerical simulations with $G_0(x) = (1-x)_+$ and observe that, in this case also, the numerical solution converges to (G_{LS}, V_{LS}), which is definitely not the behavior predicted by the theory [16]. It is, however, not surprising, as the numerical scheme induces some small diffusive effects, and diffusion is known to significantly modify large time behavior. More precisely, it is conjectured that solutions to a diffusive perturbation of (2.1)–(2.3) with a time-dependent diffusion coefficient vanishing for large times should converge to (G_{LS}, V_{LS}) (see [14, 20, 23]



(a) (b) (c) (d)

FIG. 3. *Evolution of $G(t,x)$ at time $t = 0, 1.07, 3.57, 100$ corresponding to the initial datum* (5.3).

and the references therein). The initial datum $G_0(x) = (1-x)_+$ being quite far from its expected limit, the diffusive effects of the scheme become not negligible after some time and thus induce a difference between the behavior of the numerical and the exact solutions. In an attempt to quantify the time of appearance of diffusive effects, it is interesting to look at the behavior of the scheme if the initial datum is one of the stationary solutions (G_V, V) for some $V > V_{LS}$. Given $V > V_{LS}$, this solution G_V is given by

$$G_V(x) := \frac{6\,\left(1-(x/x_0)^{1/3}\right)^{-\lambda_0}}{\left(1-(x/x_-)^{1/3}\right)^{\lambda_-}\,\left(1-(x/x_+)^{1/3}\right)^{\lambda_+}}, \tag{5.4}$$

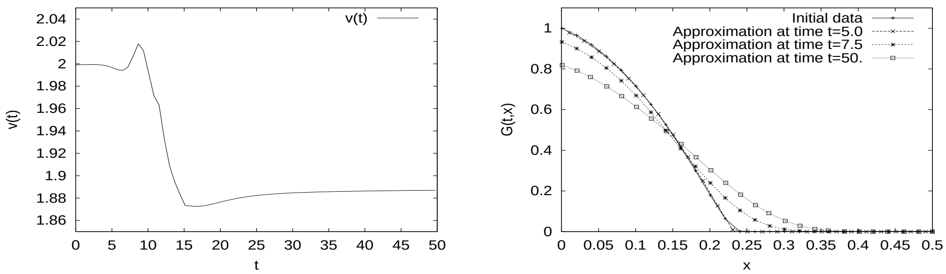
where λ_-, λ_0, λ_+ satisfy

$$\lambda_* = \frac{3\,x_*^{2/3}}{3\,x_*^{2/3} - V} \qquad \text{for} \qquad * \in \{-,0,+\}$$

and $x_-^{1/3}$, $x_0^{1/3}$, $x_+^{1/3}$ are solutions of the following equation:

$$X^3 - V\,X + 1 = 0 \quad \text{with} \quad x_- \leq 0 \leq x_0 \leq x_+.$$

We choose $V = 2$ and $G_0 = G_2$ with $I^h = 1000$, $h\,I^h = 1$, and $T = 50$. The numerical simulations are reported in Figure 4, and we observe that, for $t \in [0,5]$, the computed solution remains close to G_2. After that time, diffusive effects come into play and the numerical solution evolves towards (G_{LS}, V_{LS}). In order to capture the expected behavior for compactly supported initial data for larger times, one thus needs a less



(a) (b)

FIG. 4. *Evolution of* (a) $v(t)$ *and* (b) $G(t,x)$ *corresponding to the initial datum* G_2.

diffusive numerical scheme such as the one used in [2], to which we refer for a more complete discussion on that issue.

Appendix. An L^∞-estimate for u. We consider a nonnegative function $f_0 \in L^1(\mathbb{R}_+, (1+x^2)dx)$, $f_0 \not\equiv 0$, and denote by f a weak solution to (1.1)–(1.2) (in the sense of [10, Theorem 2]) with initial datum f_0, the functions k and q being still given by (1.4) with $\alpha \in (0,1)$, so that $\mathcal{V}(t,x) = x^\alpha\ u(t) - 1$. We have the following result.

PROPOSITION A.1. *There is a constant C depending only on α and f_0 such that, for each $t \geq 0$,*

$$u(t) + \int_0^\infty x^2\ f(t,x)\ dx \leq C\ \exp(Ct). \tag{A.1}$$

Proof. For $\lambda \in [0,2]$ and $t \geq 0$, we set

$$M_\lambda(t) := \int_0^\infty x^\lambda\ f(t,x)\ dx.$$

Consider $t \in \mathbb{R}_+$. Since $\mathcal{V}(t,0) = -1$, we infer from (1.1) that $M_0(t) \leq M_0(0)$ and

$$\frac{dM_2}{dt}(t) \leq 2 \int_0^\infty x\ \mathcal{V}(t,x)\ f(t,x)\ dx \leq 2\ u(t)\ M_{1+\alpha}(t). \tag{A.2}$$

We next infer from the Hölder inequality and (1.2) that

$$M_{1+\alpha}(t) \leq M_1(t)^{1-\alpha}\ M_2(t)^\alpha \leq C\ M_2(t)^\alpha,$$

$$0 < M_1(0) = M_1(t) \leq M_\alpha(t)^{1/(2-\alpha)}\ M_2(t)^{(1-\alpha)/(2-\alpha)}.$$

As a consequence of the first inequality and (A.2), we deduce that

$$\frac{dM_2}{dt}(t) \leq C\ u(t)\ M_2(t)^\alpha, \tag{A.3}$$

while the second inequality and (1.3) yield

$$u(t) = \frac{M_0(t)}{M_\alpha(t)} \leq \frac{M_0(0)}{M_1(0)^{2-\alpha}}\ M_2(t)^{1-\alpha}. \tag{A.4}$$

Combining (A.3) and (A.4), we end up with

$$\frac{dM_2}{dt}(t) \le C\ M_2(t)$$

and use the Gronwall lemma to conclude that $M_2(t) \le C \exp(Ct)$ for $t \ge 0$. A similar bound for u then follows by (A.4). $\square$

Remark A.2. Observe that (A.3) and (A.4) are the continuous analogues of (3.10) and (3.2), respectively.

Acknowledgments. We thank Professor J. A. Carrillo for enlightening discussions on the computational issues of the LSW model. We are also grateful to the referees for their careful reading of the manuscript and their helpful suggestions.

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