Global existence for the Vlasov-Darwin system in $\mathbb{R}^3$ for small initial data.

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Abstract

We prove the global existence of weak solutions to the Vlasov-Darwin system in $\mathbb{R}^3$ for small initial data. The Vlasov-Darwin system is an approximation of the Vlasov-Maxwell model which is valid when the characteristic speed of the particles is smaller than the light velocity, but not too small. In contrast to the Vlasov-Maxwell system, the total energy conservation does not provide a L^2-bound on the transverse part of the electric field. This difficulty may be overcome by exploiting the underlying elliptic structure of the Darwin equations under a smallness assumption on the initial data. We finally investigate the convergence of the Vlasov-Darwin system towards the Vlasov-Poisson system.

1 Introduction.

In many problems encountered in plasma physics or beam propagation, the numerical resolution of the full Vlasov-Maxwell system can be extremely expensive in computer time. However, it is sometimes possible to use simplified models which approximate the Maxwell equations in some sense [25] and are not subjected to the CFL constraint on the time step. Such situations are encountered, for instance, when no high frequency phenomenon occurs. A first approximation in this case would be the Poisson equation, which neglects altogether magnetic effects. The next approximation would be the quasi-static model which adds the magnetostatic equation to the Poisson equation. This model has the drawback, when coupled to the Vlasov equation, not to be consistent with the charge conservation. Actually, the first model including magnetic effects and charge conservation is the Darwin model. This model was first introduced by Darwin [8], and developed from a more mathematical point of view in [11, 25].

The Darwin model differs from the Maxwell equations by neglecting the "transverse part" of the displacement current, which is the time derivative of the electric field. The underlying elliptic structure of the Darwin model induces some nice features for numerical computation: it avoids error propagations inherent to hyperbolic equations as well as the restriction on the time step. The numerical approximation of the Vlasov-Darwin model has been investigated in [28, 17] (see also the references therein), while the well-posedness of the Darwin model is studied in [11]. But, to the best of our knowledge, no existence result seems to be available for the coupled Vlasov-Darwin system and the purpose of this work is to show the existence of a solution to the relativistic Vlasov-Darwin system in $\mathbb{R}^3$ under a smallness assumption on the initial distribution function.

The relativistic Vlasov equation reads

$$\frac{\partial f}{\partial t} + v(\xi) \cdot \nabla_x f + q\,(E + v(\xi) \times B) \cdot \nabla_\xi f = 0, \tag{1}$$

where q is the charge, $f(t, x, \xi)$ represents the distribution function of one species of particles (ions, electrons), depending on time t, position x and impulsion ξ. The relativistic velocity of particles $v(\xi)$ is given by

$$v(\xi) = \frac{\xi/m}{\sqrt{1 + |\xi|^2/m^2\, c^2}},$$

where m is the mass of one particle. From the distribution function f, we compute the charge and current densities

$$\rho(t, x) = q \int_{\mathbb{R}^3} f(t, x, \xi) d\xi, \quad j(t, x) = q \int_{\mathbb{R}^3} v(\xi)\, f(t, x, \xi) d\xi \tag{2}$$

and the electromagnetic fields E_T (transverse component of E), E_L (longitudinal component of E) and B (magnetic field) are given by the Darwin equations

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial E_L}{\partial t} - \nabla \times B = -\mu_0\, j, \\[2ex] \dfrac{\partial B}{\partial t} + \nabla \times E_T = 0, \\[2ex] \nabla \cdot E_L = \dfrac{\rho}{\varepsilon_0}, \quad \nabla \cdot B = 0, \\[2ex] E = E_T + E_L \text{ and } \quad \nabla \cdot E_T = 0, \quad \nabla \times E_L = 0. \end{cases} \tag{3}$$

Recall that the difference between the above Darwin model and the Maxwell equations is that the transverse part of the displacement current is neglected in the former.

Before describing our results, let us briefly mention that the Vlasov-Poisson and Vlasov-Maxwell equations have been studied by several authors: the existence of classical solutions has been obtained in [3, 24, 27] (Vlasov-Poisson) and [2, 10, 13, 14, 15] (Vlasov-Maxwell). The existence of weak solutions is described in [1, 21, 19, 23] (Vlasov-Poisson) and [22] (Vlasov-Maxwell).

Let us point out here the main difficulty encountered in the study of (1)-(3). Similarly to the Vlasov-Poisson and Vlasov-Maxwell systems, *a priori* estimates such as conservation of positivity, L^p norms and total energy are available, but here, the total energy is given by

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} m^2c^2(\gamma(\xi)-1)f(t)\,dx\,d\xi + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^3}\left(|E_L(t)|^2 + c^2|B(t)|^2\right)\,dx,$$

where

$$\gamma(\xi) = \sqrt{1+|\xi|^2/c^2m^2}.$$

Unfortunately it does not give any bound on the transverse component of the electric field E_T. Then, the procedure introduced by DiPerna and Lions [22] cannot be directly applied. Nevertheless, the underlying elliptic structure of the Darwin model allows to establish an estimate on E_T when the initial data is sufficiently small.

For a better understanding of this restriction and in order to study the behavior of the solution of the Vlasov-Darwin model when the ratio between the characteristic velocity of particles and the light velocity becomes small, we start by writing the Vlasov-Darwin system in a dimensionless form using the following characteristic scales:

$$\begin{aligned}
\overline{L} &= \text{characteristic length},\\
\overline{t} &= \text{characteristic time},\\
\overline{\xi} &= \text{characteristic impulsion},\\
\overline{v} &= \text{characteristic velocity},\\
\overline{\rho},\overline{j} &= \text{charge and current densities scaling factors},\\
\overline{E},\overline{B} &= \text{electric and magnetic fields scaling factors},\\
\overline{f} &= \text{distribution function scaling factor}.
\end{aligned}$$

Then, the unknowns are written with respect to the characteristic values, for example

$$f = f'\,\overline{f},\quad E = E'\,\overline{E},\quad B = B'\,\overline{B}.$$

To simplify the notations, we drop the primes and replace f' by f, E' by E, and so on. The dimensionless Vlasov-Darwin equations have the following form

$$\frac{\partial f}{\partial t} + \frac{\overline{v}\,\overline{t}}{\overline{L}}v(\xi)\cdot\nabla_x f + q\,\overline{B}\,\overline{t}\left(\frac{c}{\overline{\xi}}\frac{\overline{E}}{c\,\overline{B}}E + \frac{\overline{v}}{\overline{\xi}}v(\xi)\times B\right)\cdot\nabla_\xi f = 0,$$

with

$$
\begin{cases}
\dfrac{\overline{L}}{c\,\overline{t}}\,\dfrac{\overline{E}}{c\,\overline{B}}\,\dfrac{\partial E_L}{\partial t} - \nabla \times B = -\dfrac{\overline{j}\,\overline{L}}{\overline{B}}\,\mu_0\, j, \\[2ex]
\dfrac{\overline{L}}{c\overline{t}}\,\dfrac{c\overline{B}}{\overline{E}}\,\dfrac{\partial B}{\partial t} + \nabla \times E_T = 0, \\[2ex]
\varepsilon_0 \dfrac{\overline{E}}{\overline{L}\,\overline{\rho}} \nabla \cdot E_L = \rho,
\end{cases}
$$

where

$$
v(\xi) = \frac{\overline{\xi}}{m\,\overline{v}}\,\frac{\xi}{\sqrt{1 + \overline{\xi}^2\,|\xi|^2/m^2c^2}}
$$

and the continuity equation can be written as

$$
\frac{\overline{\rho}\,\overline{L}}{\overline{j}\,\overline{t}}\quad\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0.
$$

Let us denote by ε the ratio between the characteristic velocity and the light velocity. Physical considerations lead to the following scaling

$$
\varepsilon = \frac{\overline{v}}{c},\quad \frac{\overline{L}}{\overline{t}} = \overline{v},\quad \frac{\overline{\xi}}{m\,\overline{v}} = 1,\quad \frac{c\,\overline{B}}{\overline{E}} = 1,\quad \overline{j} = \overline{v}\,\overline{\rho},\quad \varepsilon_0 \frac{\overline{E}}{\overline{L}\,\overline{\rho}} = 1 \quad \text{and} \quad \omega_c = \frac{q}{m}\overline{B}.
$$

The quantity ω_c is the cyclotron frequency which is assumed to be of order of $\varepsilon/\overline{t}$. The Vlasov equation now becomes

$$
\frac{\partial f}{\partial t} + v(\xi) \cdot \nabla_x f + (E + \varepsilon\, v(\xi) \times B) \cdot \nabla_\xi f = 0, \tag{4}
$$

where the relativistic velocity is given by

$$
v(\xi) = \frac{\xi}{\sqrt{1 + \varepsilon^2 |\xi|^2}} \tag{5}
$$

and the electromagnetic fields satisfy the scaled Darwin model

$$
\begin{cases}
\varepsilon \dfrac{\partial E_L}{\partial t} - \nabla \times B = -\varepsilon\, j, \\[2ex]
\varepsilon\, \dfrac{\partial B}{\partial t} + \nabla \times E_T = 0, \\[2ex]
\nabla \cdot E_L = \rho, \quad \nabla \cdot B = 0, \\[2ex]
E = E_T + E_L \text{ and } \quad \nabla \cdot E_T = 0, \quad \nabla \times E_L = 0.
\end{cases} \tag{6}
$$

The charge and current densities ρ, j are computed from the distribution function f

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,\xi) d\xi, \quad j(t,x) = \int_{\mathbb{R}^3} v(\xi)\, f(t,x,\xi) d\xi,$$

while the total energy becomes

$$\mathcal{E}(t) = \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\gamma(\xi)-1}{\varepsilon^2} f(t)\, dx\, d\xi + \frac{1}{2}\int_{\mathbb{R}^3} \left(|E_L(t)|^2 + |B(t)|^2\right)\, dx, \tag{7}$$

where

$$\gamma(\xi) = \sqrt{1+\varepsilon^2|\xi|^2}. \tag{8}$$

Finally, at time $t=0$, the distribution function is given by

$$f(t=0,x,\xi) = f_0(x,\xi), \quad (x,\xi) \in \mathbb{R}^3\times\mathbb{R}^3. \tag{9}$$

2 Main results.

We assume that the initial datum f_0 is a non negative function such that

$$f_0 \in L^1 \cap L^\infty(\mathbb{R}^3\times\mathbb{R}^3), \quad \frac{x}{|x|^3} * \int_{\mathbb{R}^3} f_0(x,\xi)\, d\xi \in \mathbb{L}^2(\mathbb{R}^3_x), \tag{10}$$

and has finite kinetic energy

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\gamma(\xi)-1}{\varepsilon^2} f_0\, dx\, d\xi < \infty. \tag{11}$$

Moreover, we set

$$\mathcal{A}_0 = \mathcal{A}_2 + \varepsilon\, \mathcal{A}_1^{2/3}\mathcal{A}_2^{1/3}, \tag{12}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are given by

$$\mathcal{A}_1 = \|f_0\|_{L^1} \quad \text{and} \quad \mathcal{A}_2 = \|f_0\|_{L^\infty}^{1/2} \left(\|f_0\|_{L^1} + \varepsilon^2\, \mathcal{E}(0)\right)^{1/2} \tag{13}$$

and $\mathcal{E}(0)$ is the total initial energy (7) where $E_L(0)$ and $B(0)$ are given by

$$\nabla\cdot E_L(0) = \rho(0), \quad \Delta B(0) = \nabla\times j(0).$$

Theorem 1 *There exists a positive constant σ such that, if the initial datum f_0 satisfies (10)-(11) and*

$$\varepsilon^{1/2}\, \sigma\, \mathcal{A}_0 < 1, \tag{14}$$

there is a weak solution (f, E, B) to the Vlasov-Darwin system (4)-(6), where $f \in L^\infty(\mathbb{R}^+, L^1 \cap L^\infty(\mathbb{R}^6))$ satisfies the Vlasov equation (4) in the distribution sense, the charge and current densities ρ, j are such that

$$\rho, \quad j \in L^\infty(\mathbb{R}_t^+; L^{4/3}(\mathbb{R}_x^3))$$

and E_T, E_L, B are solutions in the distribution sense of the Darwin equations (6) with

$$(E_T, E_L, B) \in L^\infty(\mathbb{R}_t^+; \mathbb{L}^2 + \mathbb{L}^6(\mathbb{R}_x^3)) \times L^\infty(\mathbb{R}_t^+; \mathbb{L}^2(\mathbb{R}_x^3)) \times L^\infty(\mathbb{R}_t^+; \mathbb{L}^2(\mathbb{R}_x^3)).$$

Observe that since f and the electromagnetic fields belong to L^2_{loc}, the product

$$f\,(E + v(\xi) \times B) \in L^1_{loc}$$

and the Vlasov equation makes sense in $\mathcal{D}'$. Also, it is clear that E_T, E_L, B and f are time continuous with values in $\mathcal{D}'$ and thus take the correct initial data.

To prove this result, we first give *a priori* estimates on the distribution function f and on the electric fields E_T, E_L and magnetic field B. While the bounds on f, E_L and B are obtained in a classical way, that on E_T requires a different approach and is performed by a duality technique exploiting the elliptic structure of the Darwin system. We next give a regularization of the Vlasov-Darwin system for which these estimates hold uniformly. Then, passing to the limit in the regularized problem, we prove the existence of a weak solution to the Vlasov-Darwin system.

The second result concerns the convergence of the sequence $(f^\varepsilon, E_T^\varepsilon, E_L^\varepsilon, B^\varepsilon)$ when the parameter ε goes to zero, which means that the characteristic velocity of particles is much smaller than the light velocity. In that case the Darwin system reduces to the Poisson equation [11, 25]. Then, the following theorem holds.

Theorem 2 *Let us assume the initial datum f_0 satisfies (10) and*

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |\xi|^2 f_0 \, dx \, d\xi < +\infty. \tag{15}$$

Then, for ε sufficiently small, there exists a weak solution $(f^\varepsilon, E_T^\varepsilon, E_L^\varepsilon, B^\varepsilon)$ of the relativistic Vlasov-Darwin system (4)-(6). Moreover, there are a subsequence $(f^{\varepsilon_k}, E_T^{\varepsilon_k}, E_L^{\varepsilon_k}, B^{\varepsilon_k})_{k \in \mathbb{N}}$ and a couple (f, E) such that

$$\begin{aligned} f^{\varepsilon_k} &\rightharpoonup f \text{ weakly in } L^\infty(\mathbb{R}^+ \times \mathbb{R}^6), \text{ when } \varepsilon_k \to 0, \\ (E_L^{\varepsilon_k}, E_T^{\varepsilon_k}, B^{\varepsilon_k}) &\to (E, 0, 0) \text{ strongly in } L^\infty\left(\mathbb{R}^+; L^{5/4}_{loc}(\mathbb{R}^3)\right), \text{ when } \varepsilon_k \to 0, \end{aligned}$$

where (f, E) is a weak solution of the Vlasov-Poisson system, that is

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \quad \nabla \cdot E = \rho, \quad \nabla \times E = 0.$$

Let us point out here that the convergence of the Vlasov-Maxwell system to the Vlasov-Poisson system has been performed in [10] within the framework of classical solutions.

3 *A priori* estimates.

Throughout this section, we consider a non negative function f_0 satisfying (10), (11) and assume that (f, E, B) is a smooth solution of the Vlasov-Darwin system. We denote by C any positive constant which may vary from line to line but does not depend on $\varepsilon \in (0,1)$ and f_0.

The first *a priori* estimate is the non negativity of the distribution function f and the boundedness of its L^p norms, $p \in [1, \infty]$. Indeed, for all functions $\Phi \in W^{1,\infty}_{loc}(\mathbb{R})$, we have

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \Phi(f(t,x,\xi))dx\,d\xi \leq \int_{\mathbb{R}^3\times\mathbb{R}^3} \Phi(f_0(x,\xi))dx\,d\xi,$$

whence

$$\|f(t)\|_{L^1} \leq \|f_0\|_{L^1}, \quad \|f(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty}. \tag{16}$$

In the following result, we establish a bound on the total energy.

Proposition 1 *For $t \geq 0$, there holds*

$$\mathcal{E}(t) = \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\gamma(\xi)-1}{\varepsilon^2} f(t)\,dx\,d\xi + \frac{1}{2}\int_{\mathbb{R}^3} \left(|E_L(t)|^2 + |B(t)|^2\right) dx \leq \mathcal{E}(0), \tag{17}$$

Moreover, we have

$$\int_{\mathbb{R}^3}\int_{|\xi|\leq 1/\varepsilon} f(t)|\xi|^2\,d\xi\,dx + \frac{1}{\varepsilon}\int_{\mathbb{R}^3}\int_{|\xi|>1/\varepsilon} f(t)|\xi|\,d\xi\,dx \leq (1+\sqrt{2})\ \mathcal{E}(t). \tag{18}$$

Proof. The first estimate (17) is classical, see e.g. [6, Proposition 1.6].

We next observe that

$$\frac{1}{1+\sqrt{2}}\ \frac{|\xi|^2}{\max(1, \varepsilon\,|\xi|)} \leq \frac{|\xi|^2}{1+\sqrt{1+\varepsilon^2|\xi|^2}} = \frac{\gamma(\xi)-1}{\varepsilon^2},$$

which proves the inequality (18). □

The estimate on the total energy gives a uniform bound on (E_L,B) in $L^\infty_t(\mathbb{L}^2_x)$ and on the first moment in ξ of the distribution function in $L^\infty_t(L^1_x)$. The latter provides estimates on the densities ρ and j. We first recall a classical interpolation lemma [6, Lemma 1.1].

Lemma 1 *Let f be a non negative function such that*

$$f \in L^1 \cap L^\infty(\mathbb{R}^6), \quad \int_{\mathbb{R}^3} f|\xi|^m d\xi \in L^1(\mathbb{R}^3),$$

for some $m \geq 0$.

Then, there exists a positive constant $C > 0$, depending only on m, such that

$$\|\rho\|_{L^{1+m/3}} \leq C \, \|f\|_{L^\infty}^{m/(m+3)} \left(\int_{\mathbb{R}^6} |\xi|^m f \; dx \, d\xi \right)^{3/(3+m)},$$

$$\varepsilon \|j\|_{\mathbb{L}^{1+m/3}} \leq C \, \|f\|_{L^\infty}^{m/(m+3)} \left(\int_{\mathbb{R}^6} |\xi|^m f \; dx \, d\xi \right)^{3/(3+m)}.$$

We now proceed to show that the total energy estimate (17) provides a bound for the first moment in ξ of f which, in turn, yields the boundedness of ρ and j in $L^{4/3}_{t,x}$.

Proposition 2 *For $t \geq 0$ the charge and current densities satisfy*

$$\|j(t)\|_{\mathbb{L}^1} \leq \int_{\mathbb{R}^6} |\xi| \; f(t,x,\xi) \, dx d\xi \;\; \leq \;\; C \, \left(\|f_0\|_{L^1} + \mathcal{E}(0) \right), \tag{19}$$

$$\|\rho(t)\|_{L^{4/3}} + \varepsilon \|j(t)\|_{\mathbb{L}^{4/3}} \;\; \leq \;\; C \, \|f_0\|_{L^\infty}^{1/4} \; \left(\|f_0\|_{L^1} + \mathcal{E}(0) \right)^{3/4}. \tag{20}$$

Proof. Let us prove that the first moment in ξ of f is bounded in $L^1(\mathbb{R}^3_x)$: indeed

$$\begin{aligned} \int_{\mathbb{R}^3} |\xi| f(t,x,\xi) \, d\xi \;\; &= \;\; \int_{|\xi| \leq 1/\varepsilon} |\xi| f(t,x,\xi) \, d\xi + \int_{|\xi| > 1/\varepsilon} |\xi| f(t,x,\xi) \, d\xi \\ &\leq \;\; \int_{|\xi| \leq 1/\varepsilon} (1 + |\xi|^2) f(t,x,\xi) \, d\xi + \varepsilon \left(\frac{1}{\varepsilon} \int_{|\xi| > 1/\varepsilon} |\xi| f(t,x,\xi) \, d\xi \right). \end{aligned}$$

After integrating with respect to x, we obtain

$$\begin{aligned} \int_{\mathbb{R}^6} |\xi| f(t,x,\xi) \, d\xi \, dx \;\; &\leq \;\; \int_{\mathbb{R}^6} f(t,x,\xi) \, d\xi \, dx + \int_{\mathbb{R}^3} \int_{|\xi| \leq 1/\varepsilon} |\xi|^2 f(t,x,\xi) \, d\xi \, dx \\ &+ \;\; \varepsilon \left(\frac{1}{\varepsilon} \int_{\mathbb{R}^3} \int_{|\xi| > 1/\varepsilon} |\xi| f(t,x,\xi) \, d\xi \, dx \right). \end{aligned}$$

Finally, Proposition 1 and (16) yield

$$\int_{\mathbb{R}^6} |\xi| f(t,x,\xi) \, dx \, d\xi \leq \|f_0\|_{L^1} + C \; \mathcal{E}(0). \tag{21}$$

To conclude the proof of (19), we notice that

$$|j(t,x)| = \left| \int_{\mathbb{R}^3} \frac{\xi}{\sqrt{1+\varepsilon^2|\xi|^2}}\, f(t,x,\xi)\, d\xi \right| \leq \int_{\mathbb{R}^3} |\xi| f(t,x,\xi)\, d\xi.$$

Now, applying the interpolation lemma (Lemma 1) with $m = 1$ and using (16) and (21), we obtain

$$\begin{aligned} \|\rho(t)\|_{L^{4/3}} \ + \ \varepsilon \|j(t)\|_{\mathbb{L}^{4/3}} \ &\leq \ C\, \|f(t)\|_{L^\infty}^{1/4} \left(\int_{\mathbb{R}^6} |\xi| f(t,x,\xi)\, d\xi\, dx \right)^{3/4}, \\ &\leq \ C\, \|f_0\|_{L^\infty}^{1/4} \left(\|f_0\|_{L^1} + \mathcal{E}(0) \right)^{3/4}. \end{aligned}$$

□

Estimates on the transverse electric field E_T.

In contrast to the Vlasov-Maxwell system, the total energy estimate (17) gives no piece of information on the transverse component of the electric field E_T. Nevertheless E_T satisfies the following elliptic equation

$$-\Delta E_T + \varepsilon^2\, \frac{\partial^2 E_L}{\partial t^2} = -\varepsilon^2\, \frac{\partial j}{\partial t}. \tag{22}$$

Furthermore, the Vlasov equation (4) allows us to compute $\partial j/\partial t$ which is given by

$$\begin{aligned} -\frac{\partial j}{\partial t} &= \int_{\mathbb{R}^3} v(\xi) \otimes v(\xi)\, \nabla_x f\, d\xi \\ &\quad - \int_{\mathbb{R}^3} (1+\varepsilon^2|\xi|^2)^{-1/2} (Id - \varepsilon^2 v(\xi) \otimes v(\xi)) (E_L + E_T + \varepsilon\, v(\xi) \times B) f\, d\xi. \end{aligned}$$

and explicitly depends on E_T. Indeed, the derivative of the relativistic velocity is

$$\partial_j v_i(\xi) = \frac{1}{(1\ +\ \varepsilon^2|\xi|^2)^{1/2}} \left(\delta_{i,j}\ -\ \varepsilon^2\, v_i(\xi)\, v_j(\xi) \right)$$

and we have set

$$v(\xi) \otimes v(\xi) = (v_i(\xi)\, v_j(\xi))_{1 \leq i,j \leq 3}.$$

Owing to the poor regularity of f, the source term $\partial j/\partial t$ does not enjoy the smoothness properties required to use a variational method and exploit the non-negativity of the matrix $(Id - \varepsilon^2\ v(\xi) \otimes v(\xi))$ to obtain an $\mathbb{L}^2$-estimate on ∇E_T. We instead use a duality argument which provides an estimate on E_T at the expense of a smallness condition on f_0.

Proposition 3 *There exists a positive constant $\sigma > 0$ such that, if the initial datum f_0 satisfies*

$$\varepsilon^{1/2}\,\sigma\,\mathcal{A}_0 < 1,$$

where $\mathcal{A}_0$ is given by (12), there exists a positive constant $C(f_0)$ depending on f_0 such that

$$\|E_T(t)\|_{\mathbb{L}^2+\mathbb{L}^6} \le C(f_0)\,\varepsilon^{1/2},$$

where the functional space $\mathbb{L}^2+\mathbb{L}^6$ is endowed with the norm $\|.\|_{\mathbb{L}^2+\mathbb{L}^6}$ defined by

$$\|E_T\|_{\mathbb{L}^2+\mathbb{L}^6} = \inf\left\{\,\|a\|_{\mathbb{L}^2} + \|b\|_{\mathbb{L}^6},\quad E_T = a+b,\quad a\in\mathbb{L}^2(\mathbb{R}^3),\quad b\in\mathbb{L}^6(\mathbb{R}^3)\right\}.$$

Before proving this proposition, we recall some results concerning the Poisson equation and refer, for instance, to [16, 18] for a proof.

Lemma 2 *Consider $p\in(1,3)$ and $g\in L^p(\mathbb{R}^3)$.*

- *Let $k\in\{1,2,3\}$. The Poisson equation*

$$-\Delta u = \frac{\partial g}{\partial x_k}, \tag{23}$$

  has a unique solution $u\in L^q(\mathbb{R}^3)$, $q = 3\,p/(3-p)$, $\nabla u\in L^p(\mathbb{R}^3)$, $1<p<3$, and there holds

$$\left\|\frac{\partial u}{\partial x_j}\right\|_{L^p} \le C\,\|g\|_{L^p},\quad 1\le j\le 3.$$

- *The solution $u = G * g$, where $G(x) = \frac{1}{|S^2||x|}$, of the Poisson equation*

$$-\Delta u = g$$

  satisfies the Calderón-Zygmund inequality

$$\left\|\frac{\partial^2 u}{\partial x_i\partial x_j}\right\|_{L^p} \le C\,\|g\|_{L^p},\quad 1\le i,j\le 3.$$

Lemma 3 *Let $g\in\mathbb{L}^2(\mathbb{R}^3)\cap\mathbb{L}^{6/5}(\mathbb{R}^3)$. Then there are two functions g_T and g_L in $\mathbb{L}^2(\mathbb{R}^3)\cap\mathbb{L}^{6/5}(\mathbb{R}^3)$ such that*

$$g = g_T + g_L,\quad \nabla\times g_L = 0,\quad \nabla\cdot g_T = 0.$$

Moreover, there exists a positive constant C, independent of g, such that

$$\|g_T\|_{\mathbb{L}^{6/5}} \le C\,\|g\|_{\mathbb{L}^{6/5}},\quad \|g_T\|_{\mathbb{L}^2} \le C\,\|g\|_{\mathbb{L}^2}.$$

Proof. Since the function g belongs to $\mathbb{L}^2(\mathbb{R}^3)$, the Helmoltz decomposition [5] ensures that there are two functions g_T and g_L in $\mathbb{L}^2(\mathbb{R}^3)$ such that

$$g = g_T + g_L, \quad \nabla \times g_L = 0, \quad \nabla \cdot g_T = 0.$$

In addition, $g_T = \nabla \times \psi_g$ where $\psi_g \in Y$ is the unique solution to

$$-\Delta \psi_g = \nabla \times g,$$

and Y is defined as the completion of the set

$$\left\{ u \in \left(\mathcal{D}(\mathbb{R}^3)\right)^3; \quad \nabla \cdot u = 0 \right\}, \tag{24}$$

for the gradient norm $u \mapsto \|\nabla u\|_{\mathbb{L}^2}$ (we refer, for instance, to [9, Chapitre IX] or [5]). Moreover, since $g \in \mathbb{L}^{6/5}(\mathbb{R}^3)$, Lemma 2 ensures the existence of a positive constant C such that

$$\|\nabla \psi_g\|_{\mathbb{L}^{6/5}} \leq C \ \|g\|_{\mathbb{L}^{6/5}}.$$

Consequently,

$$\|g_T\|_{\mathbb{L}^{6/5}} = \|\nabla \times \psi_g\|_{\mathbb{L}^{6/5}} \leq C \, \|\nabla \psi_g\|_{\mathbb{L}^{6/5}} \leq C \ \|g\|_{\mathbb{L}^{6/5}}.$$

Finally, as g and g_T are in $\mathbb{L}^{6/5}(\mathbb{R}^3)$, the longitudinal component $g_L = g - g_T$ is such that $g_L \in \mathbb{L}^{6/5}(\mathbb{R}^3)$. □

Lemma 4 *Let $g_T \in \mathbb{L}^2(\mathbb{R}^3) \cap \mathbb{L}^{6/5}(\mathbb{R}^3)$ be such that $\nabla \cdot g_T$=0. Then, the following problem*

$$-\Delta \psi = g_T \tag{25}$$

has a unique solution $\psi \in Y$. Moreover, there exists a positive constant C, which does not depend on g_T, such that

$$\|\psi\|_{\mathbb{L}^\infty} + \|\nabla \psi\|_{\mathbb{L}^6} \leq C \ \max\left\{\|g_T\|_{\mathbb{L}^2}, \|g_T\|_{\mathbb{L}^{6/5}}\right\}.$$

Proof. Existence and uniqueness of a solution to (25) follows from a variational method in the Hilbert space Y with the gradient norm $u \mapsto \|\nabla u\|_{\mathbb{L}^2}$. On the one hand, from the Gagliardo-Nirenberg-Sobolev inequality [5], we establish

$$\|\psi\|_{\mathbb{L}^6} \leq C \ \|\nabla \psi\|_{\mathbb{L}^2} \leq C \ \|D^2 \psi\|_{\mathbb{L}^{6/5}}. \tag{26}$$

On the other hand, since $g_T \in \mathbb{L}^{6/5}(\mathbb{R}^3) \cap \mathbb{L}^2(\mathbb{R}^3)$, the Calderón-Zygmund inequality applied to the Poisson equation gives

$$\|D^2 \psi\|_{\mathbb{L}^{6/5}} \leq C \ \|g_T\|_{\mathbb{L}^{6/5}} \tag{27}$$

and

$$\|D^2\psi\|_{\mathbb{L}^2} \leq C\ \|g_T\|_{\mathbb{L}^2}. \tag{28}$$

Since $W^{1,2}(\mathbb{R}^3)$ is continuously embedded in $\mathbb{L}^6(\mathbb{R}^3)$, we obtain

$$\|\nabla\psi\|_{\mathbb{L}^6} \leq C\ \left(\|\nabla\psi\|_{\mathbb{L}^2} + \|D^2\psi\|_{\mathbb{L}^2}\right). \tag{29}$$

Thus, gathering inequalities (26), (27), (28) and (29), we get the first bound on $\nabla\psi$: there exists a positive constant C, which does not depend on g, such that

$$\|\nabla\psi\|_{\mathbb{L}^6} \leq C\ \left(\|\nabla\psi\|_{\mathbb{L}^2} + \|D^2\psi\|_{\mathbb{L}^2}\right) \leq C\ \max\left\{\|g_T\|_{\mathbb{L}^2}, \|g_T\|_{\mathbb{L}^{6/5}}\right\}. \tag{30}$$

Then, inequalities (26) and (30) imply that the function ψ belongs to $W^{1,6}(\mathbb{R}^3)$ which is continuously embedded in $\mathbb{L}^\infty(\mathbb{R}^3)$. Therefore, we have

$$\|\psi\|_{\mathbb{L}^\infty} \leq C\ \left(\|\psi\|_{\mathbb{L}^6} + \|\nabla\psi\|_{\mathbb{L}^6}\right) \leq C\ \max\left\{\|g_T\|_{\mathbb{L}^2}, \|g_T\|_{\mathbb{L}^{6/5}}\right\}, \tag{31}$$

which completes the proof. □

Lemma 5 *Set*

$$\mathcal{M}_{-1}(t,x) = \int_{\mathbb{R}^3} (1\,+\,\varepsilon^2|\xi|^2)^{-1/2}\, f(t,x,\xi)\, d\xi.$$

Then, we have

$$\varepsilon^2\, \|\mathcal{M}_{-1}(t)\|_{L^2} \leq\ \varepsilon^{1/2}\, C\, \|f_0\|^{1/2}_{L^\infty}\ \left(\|f_0\|_{L^1} + \varepsilon^2\, \mathcal{E}(0)\right)^{1/2}. \tag{32}$$

Proof. Let us consider an arbitrary $R > 0$ and split $\mathcal{M}_{-1}(t)$ in the following way

$$\begin{aligned}
&\varepsilon^2 \int_{\mathbb{R}^3} f(t,x,\xi)\,(1+\varepsilon^2|\xi|^2)^{-1/2}\, d\xi \\
&= \ \varepsilon^2 \int_{|\xi|\leq R} f(t,x,\xi)\,(1+\varepsilon^2|\xi|^2)^{-1/2}\, d\xi + \varepsilon^2 \int_{|\xi|>R} f(t,x,\xi)\,(1+\varepsilon^2|\xi|^2)^{-1/2}\, d\xi, \\
&\leq \ \varepsilon^2\, \|f(t)\|_{L^\infty} \int_{|\xi|\leq R} (1+\varepsilon^2|\xi|^2)^{-1/2}\, d\xi \ +\ \frac{1}{R^2} \int_{|\xi|>R} f(t,x,\xi)\,(1+\varepsilon^2|\xi|^2)^{1/2}\, d\xi.
\end{aligned}$$

Since

$$\varepsilon^2 \int_{|\xi|\leq R} (1+\varepsilon^2|\xi|^2)^{-1/2}\, d\xi = |S^2| \int_0^R \frac{\varepsilon^2\, r^2\, dr}{(1+\varepsilon^2\, r^2)^{1/2}} \leq \frac{\varepsilon\, |S^2|}{2}\, R^2,$$

we have

$$\begin{aligned}
&\varepsilon^2 \int_{\mathbb{R}^3} f(t,x,\xi)\,(1+\varepsilon^2|\xi|^2)^{-1/2}\,d\xi \\
&\leq \;\|f(t)\|_{L^\infty}\,\frac{\varepsilon|S^2|}{2}\,R^2 + \frac{1}{R^2}\int_{\mathbb{R}^3} f(t,x,\xi)\,(1+\varepsilon^2|\xi|^2)^{1/2}\,d\xi.
\end{aligned}$$

Optimizing with respect to R we obtain

$$\varepsilon^2\,\mathcal{M}_{-1}(t,x) \leq \varepsilon^{1/2}\,C\,\|f(t)\|_{L^\infty}^{1/2}\,\left(\int_{\mathbb{R}^3} f(t,x,\xi)\,(1+\varepsilon^2|\xi|^2)^{1/2}\,d\xi\right)^{1/2}.$$

Now, we need to give a bound on the right hand side. On the one hand, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} f(t,x,\xi)\,(1+\varepsilon^2|\xi|^2)^{1/2}\,d\xi \\
&= \int_{|\xi|\leq 1/\varepsilon} f(t,x,\xi)\,(1+\varepsilon^2|\xi|^2)^{1/2}\,d\xi + \int_{|\xi|>1/\varepsilon} f(t,x,\xi)\,(1+\varepsilon^2|\xi|^2)^{1/2}\,d\xi \\
&\leq \; C\left(\rho(t,x) \,+\, \varepsilon^2\,\left(\frac{1}{\varepsilon}\int_{|\xi|>1/\varepsilon} |\xi| f(t,x,\xi)\,d\xi\right)\right).
\end{aligned}$$

On the other hand, by (16),

$$\|f(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty}.$$

Taking the L^2_x norm of $\mathcal{M}_{-1}$ and using (17) we finally obtain

$$\begin{aligned}
\varepsilon^2\|\mathcal{M}_{-1}(t)\|_{L^2} &\leq \varepsilon^{1/2}\,C\,\|f(t)\|_{L^\infty}^{1/2}\,\left(\|f(t)\|_{L^1} + \varepsilon^2\,(\mathcal{E}(t))\right)^{1/2} \\
&\leq \varepsilon^{1/2}\,C\,\|f_0\|_{L^\infty}^{1/2}\,\left(\|f_0\|_{L^1} + \varepsilon^2\mathcal{E}(0)\right)^{1/2} = \varepsilon^{1/2}\mathcal{A}_2.
\end{aligned}$$

□

We can now give the proof of Proposition 3.

Proof of Proposition 3. Let $g \in \mathbb{L}^2(\mathbb{R}^3)\cap\mathbb{L}^{6/5}(\mathbb{R}^3)$. On the one hand, thanks to Lemma 3, the function g may be written as

$$g = g_T \,+\, g_L, \quad \nabla\times g_L \,=\, 0, \quad \nabla\cdot g_T \,=\, 0.$$

On the other hand, from Lemma 4, there exists a unique solution $\psi \in Y$ to the following problem

$$-\Delta\psi = g_T. \tag{33}$$

Since $E_L(t)$ and $B(t)$ belong to $\mathbb{L}^2(\mathbb{R}^3_x)$ and $j(t)$ belongs to $\mathbb{L}^{4/3}(\mathbb{R}^3_x)$ it follows from (22) and (33) that

$$< -\Delta E_T + \varepsilon^2 \frac{\partial^2 E_L}{\partial t^2} + \varepsilon^2 \frac{\partial j}{\partial t}, \psi >= 0.$$

We now compute the three terms of the above equality. It first follows from (33) that

$$< -\Delta E_T, \psi >= -\int_{\mathbb{R}^3} E_T \cdot \Delta\psi \, dx = \int_{\mathbb{R}^3} E_T \cdot g_T \, dx = \int_{\mathbb{R}^3} E_T \cdot g \, dx. \tag{34}$$

Since $\nabla \times E_L = 0$ and $\nabla \cdot \psi = 0$, we obtain

$$\varepsilon^2 < \frac{\partial^2 E_L}{\partial t^2}, \psi >= 0. \tag{35}$$

Finally, we need to evaluate the following duality product

$$< \varepsilon^2 \frac{\partial j}{\partial t}, \psi > . \tag{36}$$

From the Vlasov equation, $\partial j/\partial t$ can be easily computed

$$-\frac{\partial j}{\partial t} = \int_{\mathbb{R}^3} v(\xi)\,(v(\xi) \cdot \nabla_x f)\, d\xi \tag{37}$$

$$-\int_{\mathbb{R}^3} (1+\varepsilon^2|\xi|^2)^{-1/2}(Id - \varepsilon^2 v(\xi)\otimes v(\xi))(E_L + \varepsilon\, v(\xi)\times B) f \, d\xi, \tag{38}$$

$$-\int_{\mathbb{R}^3} (1+\varepsilon^2|\xi|^2)^{-1/2}(Id - \varepsilon^2 v(\xi)\otimes v(\xi)) E_T\, f \, d\xi. \tag{39}$$

An estimate on $\partial j/\partial t$ will be obtained in three steps.

First step. Let us begin with the following term

$$\varepsilon^2 < \int_{\mathbb{R}^3} v(\xi)\,(v(\xi) \cdot \nabla_x f)\, d\xi, \psi > .$$

Thanks to the regularity of the charge density $\rho(t) \in \mathbb{L}^1(\mathbb{R}^3) \cap \mathbb{L}^{4/3}(\mathbb{R}^3)$ and of $\psi \in W^{1,6}(\mathbb{R}^3)$ given by Lemma 4, we obtain

$$\varepsilon^2 < \int_{\mathbb{R}^3} v(\xi)\,(v(\xi) \cdot \nabla_x f)\, d\xi, \psi >= -\varepsilon^2 \int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,\xi)\, {}^t v(\xi)\,(\nabla\psi)\, v(\xi)\, dx\, d\xi. \tag{40}$$

Introducing

$$\mathcal{K}(t,x) = \int_{\mathbb{R}^3} f(t,x,\xi)\, |v(\xi)|^2 \, d\xi,$$

it follows from the Hölder inequality that

$$\varepsilon^2 \left| < \int_{\mathbb{R}^3} v(\xi)\,(v(\xi)\,\cdot\,\nabla_x f)\,d\xi, \psi > \right| \leq \varepsilon^2\, C\, \|\mathcal{K}(t)\|_{L^{6/5}}\, \|\nabla\,\psi\|_{\mathbb{L}^6}. \tag{41}$$

It remains to estimate $\mathcal{K}(t,.)$ in $L^{6/5}(\mathbb{R}^3)$. On the one hand, we have

$$\varepsilon^2 \mathcal{K}(t,x) = \varepsilon^2 \int_{\mathbb{R}^3} f(t,x,\xi)\, \frac{|\xi|^2}{(1+\varepsilon^2|\xi|^2)}\, d\xi \leq \int_{\mathbb{R}^3} f(t,x,\xi)\, d\xi,$$

whence,

$$\varepsilon^2\, \|\mathcal{K}(t)\|_{L^{4/3}} \leq \|\rho(t)\|_{L^{4/3}}. \tag{42}$$

On the other hand,

$$\begin{aligned} \varepsilon^2 \mathcal{K}(t,x) &= \varepsilon^2 \int_{|\xi|\leq 1/\varepsilon} f(t,x,\xi)\, |v(\xi)|^2\, d\xi + \varepsilon^2 \int_{|\xi|>1/\varepsilon} f(t,x,\xi)\, |v(\xi)|^2\, d\xi \\ &\leq \varepsilon^2 \int_{|\xi|\leq 1/\varepsilon} f(t,x,\xi)\, |\xi|^2 d\xi + \varepsilon^2 \left(\frac{1}{\varepsilon} \int_{|\xi|>1/\varepsilon} f(t,x,\xi)\, |\xi|\, d\xi \right). \end{aligned}$$

Thanks to (18), we get

$$\varepsilon^2 \int_{\mathbb{R}^3} \mathcal{K}(t,x)\, dx \leq \varepsilon^2\, (1+\sqrt{2})\, \mathcal{E}(t). \tag{43}$$

Finally, we conclude from (42), (43) and the Hölder inequality

$$\begin{aligned} \varepsilon^2\, \|\mathcal{K}(t)\|_{L^{6/5}} &\leq \left\|\varepsilon^2 \mathcal{K}(t)\right\|_{L^1}^{1/3}\, \left\|\varepsilon^2 \mathcal{K}(t)\right\|_{L^{4/3}}^{2/3} \\ &\leq \varepsilon^{2/3} \Big((1+\sqrt{2})\, \mathcal{E}(t) \Big)^{1/3}\, \|\rho(t)\|_{L^{4/3}}^{2/3}. \end{aligned} \tag{44}$$

Gathering (40), (41) and (44), we obtain

$$|\varepsilon^2 < \int_{\mathbb{R}^3} v(\xi)\,(v(\xi)\cdot\,\nabla_x f)\,d\xi, \psi > | \leq \varepsilon^{2/3} C\, (\mathcal{E}(t))^{1/3}\, \|\rho(t)\|_{L^{4/3}}^{2/3}\, \|\nabla\,\psi\|_{\mathbb{L}^6}. \tag{45}$$

Finally, thanks to (45), (17) and (20), we end up with

$$\begin{aligned} &\varepsilon^2 \left| < \int_{\mathbb{R}^3} v(\xi)\,(v(\xi)\cdot\,\nabla_x f)\,d\xi, \psi > \right| \\ &\leq C\varepsilon^{2/3}\, (\mathcal{E}(0))^{1/3}\, \|f_0\|_{L^\infty}^{1/6}\, (\|f_0\|_{L^1} + \mathcal{E}(0))^{1/2}\, \|\nabla\,\psi\|_{\mathbb{L}^6}. \end{aligned} \tag{46}$$

Second step. We now study the term (38) which does not contain the transverse component E_T.

$$< \varepsilon^2 \int_{\mathbb{R}^3} (1+\varepsilon^2|\xi|^2)^{-1/2} \left(Id - \varepsilon^2 v(\xi)\otimes v(\xi)\right) f\,(E_L + \varepsilon v(\xi)\times B) d\xi\,, \psi > . \tag{47}$$

On the one hand, applying the Hölder inequality, we have

$$\begin{aligned} & \varepsilon^2 \left| \int_{\mathbb{R}^6} f\ \left((1+\varepsilon^2|\xi|^2)^{-1/2} \left(Id - \varepsilon^2 v(\xi)\otimes v(\xi)\right)\ E_L\right)\cdot \psi\ dx\,d\xi \right| \\ \leq\ & \varepsilon^2\ C \int_{\mathbb{R}^3} \mathcal{M}_{-1} |E_L||\psi|\,dx. \\ \leq\ & \varepsilon^2\ C\ \|\mathcal{M}_{-1}\|_{L^2}\ \|E_L\|_{\mathbb{L}^2}\ \|\psi\|_{\mathbb{L}^\infty}. \end{aligned} \tag{48}$$

On the other hand, observing that $\varepsilon\,|v(\xi)| \leq 1$, we proceed in the same way to establish

$$\begin{aligned} \varepsilon^2 \left| \int_{\mathbb{R}^6} f(1\,+\,\varepsilon^2|\xi|^2)^{-1/2} \left(\left(Id\,-\,\varepsilon^2 v(\xi)\otimes v(\xi)\right)\,\varepsilon\, v(\xi)\times B\right)\cdot \psi d\xi\,dx \right| \\ \leq \varepsilon^2\ C\,\|\mathcal{M}_{-1}\|_{L^2}\ \|B\|_{\mathbb{L}^2}\,\|\psi\|_{\mathbb{L}^\infty}. \end{aligned} \tag{49}$$

Gathering (48), (49), we use the bound on $\mathcal{M}_{-1}$ given by (32) and (17) to obtain the following estimate

$$\begin{aligned} \left| \varepsilon^2 \int_{\mathbb{R}^6} (1\,+\,\varepsilon^2|\xi|^2)^{-1/2} \left(Id\,-\,\varepsilon^2 v(\xi)\otimes v(\xi)\right)\ f\,(E_L + \varepsilon\, v(\xi)\times B)\,\psi d\xi\,dx \right| \\ \leq \varepsilon^{1/2}\,C\,\|f_0\|_{L^\infty}^{1/2}\ \left(\|f_0\|_{L^1} + \varepsilon^2\mathcal{E}(0)\right)^{1/2}\ \mathcal{E}(0)^{1/2}\,\|\psi\|_{\mathbb{L}^\infty}. \end{aligned} \tag{50}$$

Third step. Finally, we handle the term involving the transverse component E_T given by (39): we have

$$\begin{aligned} \left| < \int_{\mathbb{R}^3} (1\,+\,\varepsilon^2|\xi|^2)^{-1/2} \left(Id\,-\,\varepsilon^2 v(\xi)\otimes v(\xi)\right)\ f\,E_T\,d\xi, \psi > \right| \\ \leq C\,\int_{\mathbb{R}^3} \mathcal{M}_{-1}\,|E_T|\,|\psi|dx. \end{aligned} \tag{51}$$

Thanks to (34), (35) and (36), we get

$$\int_{\mathbb{R}^3} E_T\ g\,dx = -\varepsilon^2 < \frac{\partial j}{\partial t}, \psi >$$

and the previous three steps (46), (50), (51) ensure that there exists a positive constant $C(f_0)$ and a positive constant $C > 0$ independent of f_0, such that for any function $g \in \mathbb{L}^{6/5} \cap \mathbb{L}^2(\mathbb{R}^3)$

$$\begin{aligned}\left|\int_{\mathbb{R}^3} E_T\, g\, dx\right| &\leq C(f_0)\,(\|\psi\|_{\mathbb{L}^\infty}\,\varepsilon^{1/2} + \|\nabla\psi\|_{\mathbb{L}^6}\,\varepsilon^{2/3}) \\ &+ C\,\varepsilon^2 \int_{\mathbb{R}^3} \mathcal{M}_{-1}\,|E_T|\,dx\,\|\psi\|_{\mathbb{L}^\infty}.\end{aligned}$$

Thus, thanks to Lemma 3 and Lemma 4, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3} E_T\, g\, dx\right| &\leq C(f_0)\,\varepsilon^{1/2}\,(1+\varepsilon^{1/6})\,\|g\|_{\mathbb{L}^{6/5}\cap\mathbb{L}^2} \\ &+ C\,\varepsilon^2 \int_{\mathbb{R}^3} \mathcal{M}_{-1}\,|E_T|\,dx\,\|g\|_{\mathbb{L}^{6/5}\cap\mathbb{L}^2}.\end{aligned} \tag{52}$$

Recalling that the dual space of $(\mathbb{L}^2 \cap \mathbb{L}^{6/5})$ is $\mathbb{L}^2 + \mathbb{L}^6$ (see for instance [4, Theorem 2.7.1]), we actually deduce from (52) an estimate on E_T in $\mathbb{L}^2 + \mathbb{L}^6$. Indeed, for all $a \in \mathbb{L}^2(\mathbb{R}^3)$ and $b \in \mathbb{L}^6(\mathbb{R}^3)$ such that $E_T = a + b$,

$$\varepsilon^2 \int_{\mathbb{R}^3} \mathcal{M}_{-1}\,|E_T|\,dx \leq \|\varepsilon^2\mathcal{M}_{-1}\|_{L^2}\,\|a\|_{\mathbb{L}^2} + \|\varepsilon^2\mathcal{M}_{-1}\|_{L^{6/5}}\,\|b\|_{\mathbb{L}^6}. \tag{53}$$

By (32), we have

$$\varepsilon^2\,\|\mathcal{M}_{-1}\|_{L^2} \leq \varepsilon^{1/2}\,C\,\|f_0\|_{L^\infty}^{1/2}\,\left(\|f_0\|_{L^1} + \varepsilon^2\,\mathcal{E}(0)\right)^{1/2} = \varepsilon^{1/2}\,C\,\mathcal{A}_2, \tag{54}$$

where $\mathcal{A}_2$ is defined by (13). Let us also observe that

$$\varepsilon^2\,\|\mathcal{M}_{-1}\|_{L^1} = \varepsilon^2 \int_{\mathbb{R}^6} (1+\varepsilon^2|\xi|^2)^{-1/2}\,f\,dx\,d\xi \leq \varepsilon^2\|\rho(t)\|_{L^1} \leq \varepsilon^2\|f_0\|_{L^1} = \varepsilon^2\,\mathcal{A}_1,$$

where $\mathcal{A}_1$ is given by (13). Thus, using an interpolation inequality

$$\begin{aligned}\varepsilon^2\,\|\mathcal{M}_{-1}\|_{L^{6/5}} &\leq \left(\varepsilon^2\,\|\mathcal{M}_{-1}\|_{L^1}\right)^{2/3}\,\left(\varepsilon^2\,\|\mathcal{M}_{-1}\|_{L^2}\right)^{1/3}, \\ &\leq \left(\varepsilon^2\,\mathcal{A}_1\right)^{2/3}\,\left(\varepsilon^{1/2}\,C\,\mathcal{A}_2\right)^{1/3}, \\ &\leq C\,\varepsilon^{3/2}\,\mathcal{A}_1^{2/3}\,\mathcal{A}_2^{1/3}.\end{aligned} \tag{55}$$

Gathering (52), (53), (54) and (55), there exists a positive constant $\sigma > 0$, such that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} E_T\, g\, dx\right| &\leq C(f_0)\,\varepsilon^{1/2}(1+\varepsilon^{1/6})\,\|g\|_{\mathbb{L}^{6/5}\cap\mathbb{L}^2} \\ &+ \varepsilon^{1/2}\,\sigma\,\mathcal{A}_0\,(\|a\|_{\mathbb{L}^2} + \|b\|_{\mathbb{L}^6})\,\|g\|_{\mathbb{L}^{6/5}\cap\mathbb{L}^2},\end{aligned}$$

for any $a \in \mathbb{L}^2(\mathbb{R}^3)$ and $b \in \mathbb{L}^6(\mathbb{R}^3)$ such that $E_T = a + b$. Recall that $\mathcal{A}_0$ is defined by (12). Therefore, we conclude

$$\left| \int_{\mathbb{R}^3} E_T \, g \, dx \right| \leq C(f_0) \, \varepsilon^{1/2} (1 + \varepsilon^{1/6}) \, \|g\|_{\mathbb{L}^{6/5} \cap \mathbb{L}^2} + \varepsilon^{1/2} \, \sigma \, \mathcal{A}_0 \, \|E_T\|_{\mathbb{L}^2 + \mathbb{L}^6} \, \|g\|_{\mathbb{L}^{6/5} \cap \mathbb{L}^2},$$

for any $g \in \mathbb{L}^{6/5} \cap \mathbb{L}^2$. By duality, we have

$$\|E_T\|_{\mathbb{L}^2 + \mathbb{L}^6} \leq C(f_0) \, (1 + \varepsilon^{1/6}) \, \varepsilon^{1/2} \, + \varepsilon^{1/2} \, \sigma \, \mathcal{A}_0 \, \|E_T\|_{\mathbb{L}^2 + \mathbb{L}^6}.$$

Consequently, as soon as the initial datum f_0 fulfils

$$\sigma \, \varepsilon^{1/2} \mathcal{A}_0 < 1,$$

we have

$$\|E_T(t)\|_{\mathbb{L}^2 + \mathbb{L}^6} \leq C(f_0) \; \varepsilon^{1/2}.$$

The proof of Proposition 3 is then complete. □

Remark 1 *Let us stress here that the constant $C(f_0)$ occurring in Proposition 3 only depends on $\|f_0\|_{L^\infty}$, $\|f_0\|_{L^1}$ and $\mathcal{E}(0)$.*

4 Regularization of the Vlasov-Darwin system.

The goal of this section is to construct a sequence of solutions of suitable approximations of the Vlasov-Darwin system such that the *a priori* estimates derived in the previous section hold uniformly.

Consider an initial datum f_0 satisfying (10), (11) and the smallness condition (14). There exist a sequence $f_0^n \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and a positive constant C_0 such that

$$\|f_0^n\|_{L^1} + \|f_0^n\|_{L^\infty} + \mathcal{E}^n(0) \leq C_0, \quad \varepsilon^{1/2} \, \sigma \, \mathcal{A}_0^n < 1, \quad \forall n \geq 1, \tag{56}$$

where $\mathcal{A}_0^n$ and $\mathcal{E}^n(0)$ denote the constant defined by (12) and the initial total energy corresponding to f_0^n respectively. Next, we consider a non negative and radially symmetric function $\theta_n \in C^\infty(\mathbb{R}^3_x)$ such that,

$$supp \, \theta_n \subset B(0, 1/n), \qquad \int_{\mathbb{R}^3} \theta_n dx = 1.$$

As the parameter ε does not matter in this section, we will omit it and assume that $\varepsilon = 1$. As in [6], we consider the following regularized system

$$\frac{\partial f}{\partial t} + v(\xi) \cdot \nabla_x f + (E_L + E_T + v(\xi) \times B) * \theta_n \cdot \nabla_\xi f = 0, \tag{57}$$

with the initial datum

$$f(0, x, \xi) = f_0^n(x, \xi), \quad (x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3,$$

and the electromagnetic fields (E_T, E_L, B) are given by

$$\begin{cases} \dfrac{\partial E_L}{\partial t} - \nabla \times B = -j * \theta_n, \\ \\ \dfrac{\partial B}{\partial t} + \nabla \times E_T = 0, \\ \\ \nabla \cdot E_L = \rho * \theta_n, \quad \nabla \cdot B = 0, \\ \\ E = E_T + E_L \text{ and } \quad \nabla \cdot E_T = 0, \quad \nabla \times E_L = 0, \end{cases} \tag{58}$$

with

$$\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, \xi)\, d\xi, \quad j(t, x) = \int_{\mathbb{R}^3} v(\xi)\, f(t, x, \xi)\, d\xi.$$

Here, the symbol $*$ denotes the convolution with respect to the x variable

Proposition 4 *There exists a smooth solution, denoted by (f^n, E_T^n, E_L^n, B^n), to the system (57)-(58) such that*

$$f^n \in C^1(\mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3),$$
$$E_L^n, B^n \in C^1(\mathbb{R}^+; I\!H^1(\mathbb{R}^3)) \text{ and } E_T^n \in L^\infty(\mathbb{R}^+; Y),$$

where Y is given by (24). In addition, $f^n(t, ., .)$ is compactly supported for each $t \geq 0$.

The proof of this result is performed by classical arguments (see [12] for details and the references in [6]). Of course, the regularity properties enjoyed by f^n heavily depend on n. Nevertheless, since the smallness condition (56) is satisfied uniformly with respect to n, we may argue as in the previous section and using the Young inequality, we easily check that the following estimates hold uniformly.

Lemma 6 *There exists a positive constant C, depending only on f_0 such that, for $t \geq 0$,*

$$\|f^n(t)\|_{L^1} + \|f^n(t)\|_{L^\infty} \leq C,$$
$$\|\rho^n(t)\|_{L^{4/3}} + \|j^n(t)\|_{\mathbb{L}^{4/3}} \leq C,$$
$$\|B^n(t)\|_{\mathbb{L}^2} + \|E_L^n(t)\|_{\mathbb{L}^2} + \|E_T^n(t)\|_{\mathbb{L}^2 + \mathbb{L}^6} \leq C.$$

5 Proof of Theorem 1.

Here, we keep the assumptions and notations introduced in the previous section.

Since $\mathbb{L}^2 + \mathbb{L}^6$ is continuously embedded in $\mathbb{L}^2_{loc}$, it follows from Lemma 6 that

$$\forall\ R\ >\ 0, \quad \exists\ C_R > 0, \quad \int_{B(0,R)} |E_T^n|^2\, dx \leq C_R. \tag{59}$$

The bounds of Lemma 6 and (59) give a good functional framework to analyze the product $(E_L^n + E_T^n + v(\xi) \times B^n)\, f^n$ in L^1_{loc} since every term belongs to L^2_{loc}. Indeed, these estimates allow to extract a subsequence of (f^n, E_T^n, E_L^n, B^n) (not relabelled) which converges weakly in L^2_{loc} towards (f, E_T, E_L, B). In order to pass to the limit in the nonlinear term

$$(E_L^n + E_T^n + v(\xi) \times B^n)\, f^n$$

strong compactness is needed and is provided by the classical velocity averaging lemma, which we recall now [6, 22].

Lemma 7 *Consider* $\Omega = (0,T) \times \mathbb{R}^3$ *and* $a(.) \in L^\infty_{loc}(\mathbb{R}^3;\mathbb{R}^3)$*, satisfying*

$$\forall \omega \in S^2, \quad \forall u \in \mathbb{R}; \quad |\{\xi \in \mathbb{R}^3; \quad a(\xi).\omega = u\}| = 0.$$

Let $(f^n)_{n\in I\!N}$ *be a bounded sequence in* $L^2_{loc}(\Omega \times \mathbb{R}^3_\xi)$ *which satisfies: for any* $\psi \in \mathcal{D}(\mathbb{R}^3_\xi)$*, the sequence*

$$\left\{ \int_{\mathbb{R}^3} \left(\frac{\partial f^n}{\partial t} + \nabla_x \cdot a(\xi)\, f^n \right) \psi(\xi) d\xi \right\} \tag{60}$$

is relatively compact in $H^{-1}_{loc}(\Omega)$*. Then the sequence*

$$\left\{ \int_{\mathbb{R}^3} f^n(t,x,\xi)\, \psi(\xi)\, d\xi \right\}$$

is relatively compact in $L^2_{loc}(\Omega)$*.*

By (57), f^n satisfies

$$\int_{\mathbb{R}^3} \left(\frac{\partial f^n}{\partial t} + v(\xi) \cdot \nabla f^n \right) \psi(\xi)\, d\xi = \int_{\mathbb{R}^3} g^n \cdot \nabla \psi(\xi)\, d\xi,$$

with $g^n = (E_L^n + E_T^n + v(\xi) \times B^n) * \theta_n\, f^n$, for any $\psi \in \mathcal{D}(\mathbb{R}^3_\xi)$. Since E_L^n, E_T^n and B^n are bounded in $\mathbb{L}^2_{loc}((0,T)\times\mathbb{R}^3)$ and f^n is bounded in $L^\infty((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$, $g^n \cdot \nabla\psi(\xi)$ is bounded in $L^2_{loc}((0,T) \times \mathbb{R}^3 \times B(0,R))$, for all T, R. We are then in a position to apply Lemma 7 to deduce that

$$\forall \psi \in \mathcal{D}(\mathbb{R}^3_\xi), \quad \int_{\mathbb{R}^3} f^n(t,x,\xi)\, \psi(\xi) d\xi \to \int_{\mathbb{R}^3} f(t,x,\xi)\, \psi(\xi) d\xi, \quad \text{in} \quad L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^3).$$

It is now straightforward to pass to the limit in (57) as $n \to \infty$: indeed, for $\psi \in \mathcal{D}(\mathbb{R}^3_\xi)$ and $\varphi \in \mathcal{D}((0,T) \times \mathbb{R}^3_x)$, we have

$$\int_0^T \int_{\mathbb{R}^3} (E_L^n + E_T^n)\, \varphi(t,x) \left(\int_{\mathbb{R}^3} f^n(t,x,\xi)\, \psi(\xi) d\xi \right) dx\, dt \longrightarrow \int_0^T \int_{\mathbb{R}^3} (E_L + E_T)\, \varphi(t,x) \left(\int_{\mathbb{R}^3} f(t,x,\xi)\, \psi(\xi) d\xi \right) dx\, dt,$$

and

$$\int_0^T \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} f^n(t,x,\xi)\, \psi(\xi)\, v(\xi) d\xi \right) \times B^n\, \varphi(t,x)\, dx\, dt \longrightarrow \int_0^T \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} f(t,x,\xi)\, \psi(\xi)\, v(\xi)\, d\xi \right) \times B\, \varphi(t,x)\, dx\, dt.$$

The fact that (f, E_L, E_T, B) is a weak solution to the Vlasov-Darwin model then readily follows.

6 Proof of Theorem 2.

In this section, we fix a non negative initial datum f_0 (which does not depend on ε) and satisfies (10), (11) and

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0\, |\xi|^2 d\xi dx < +\infty.$$

Clearly, the smallness condition (14) is fulfilled by f_0 for ε small enough. For such ε, we denote by $(f^\varepsilon, E_L^\varepsilon, E_T^\varepsilon, B^\varepsilon)$ the solution of the Vlasov-Darwin system (4)-(6) constructed in Theorem 1. Observe that

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\gamma(\xi) - 1}{\varepsilon^2}\, f_0\, dx\, d\xi \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\xi|^2\, f_0\, dx\, d\xi$$

and we infer from Proposition 2 and (16) that

$$\|j^\varepsilon(t)\|_{\mathbb{L}^1} + \|\rho^\varepsilon(t)\|_{L^1} + \|\rho^\varepsilon(t)\|_{L^{4/3}} \leq C(f_0), \tag{61}$$

$$\|f^\varepsilon(t)\|_{L^1} + \|f^\varepsilon(t)\|_{L^\infty} \leq C(f_0). \tag{62}$$

We next prove some estimates on the electromagnetic fields.

Proposition 5 *The sequence $(E_L^\varepsilon)_{\varepsilon > 0}$ is relatively compact in $\mathcal{C}([0,T], \mathbb{L}^{5/4}_{loc}(\mathbb{R}^3))$ and*

$$\varepsilon^{1/2}\, \|E_T^\varepsilon\|_{\mathbb{L}^2 + \mathbb{L}^6} + \|B^\varepsilon\|_{\mathbb{L}^2} \leq C\, \varepsilon.$$

Before giving the proof of Proposition 5, we establish a uniform bound on the current density j^ε.

Lemma 8 *There exists a positive constant $C(f_0) > 0$, depending only on the initial data f_0, such that, for $t \geq 0$,*

$$\|j^\varepsilon(t)\|_{\mathbb{L}^{5/4}} \;\leq\; C(f_0).$$

Proof. We define f_1^ε and f_2^ε by

$$f_1^\varepsilon = \begin{cases} f^\varepsilon & \text{if } |\xi| \leq 1/\varepsilon, \\ 0 & \text{otherwise} \end{cases}$$

and $f_2^\varepsilon = f^\varepsilon - f_1^\varepsilon$. We also put

$$j_1^\varepsilon = \int_{\mathbb{R}^3} \frac{\xi}{(1+\varepsilon^2|\xi|^2)^{1/2}} f_1^\varepsilon d\xi \quad \text{and} \quad j_2^\varepsilon = \int_{\mathbb{R}^3} \frac{\xi}{(1+\varepsilon^2|\xi|^2)^{1/2}} f_2^\varepsilon d\xi.$$

On the one hand, Proposition 1 and (16) imply

$$\int_{\mathbb{R}^6} |\xi|^2 \, f_1^\varepsilon(t) d\xi \, dx \leq (1+\sqrt{2}) \; \mathcal{E}(0) \quad \text{and} \quad \|f_1^\varepsilon(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty}.$$

Using an interpolation inequality in the spirit of Lemma 1, we get

$$\|j_1^\varepsilon(t)\|_{\mathbb{L}^{5/4}} \leq C \, \|f_1^\varepsilon(t)\|_{L^\infty}^{1/5} \left(\int_{\mathbb{R}^6} |\xi|^2 \, f_1^\varepsilon(t) d\xi \, dx \right)^{4/5} \leq C(f_0).$$

On the other hand, Proposition 1 and (16) indicate that j_2^ε and f_2^ε are bounded in $\mathbb{L}^1(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^6)$ by

$$\|j_2^\varepsilon(t)\|_{\mathbb{L}^1} \leq \int_{\mathbb{R}^6} f_2^\varepsilon(t)|\xi| \, d\xi \, dx \leq \varepsilon \; (1+\sqrt{2}) \; \mathcal{E}(0) \quad \text{and} \quad \|f_2^\varepsilon(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty}. \tag{63}$$

Then, applying the Hölder inequality, we have

$$\|j_2^\varepsilon(t)\|_{\mathbb{L}^{5/4}} \leq \|j_2^\varepsilon(t)\|_{\mathbb{L}^1}^{1/5} \; \|j_2^\varepsilon(t)\|_{\mathbb{L}^{4/3}}^{4/5}. \tag{64}$$

An estimate of $\|j_2^\varepsilon\|_{\mathbb{L}^{4/3}}$ is now needed. Let us set

$$\rho_2^\varepsilon(t,x) = \int_{\mathbb{R}^3} f_2^\varepsilon(t,x,\xi) \, d\xi.$$

By an interpolation inequality and (63), we obtain

$$\varepsilon \, \|j_2^\varepsilon(t)\|_{\mathbb{L}^{4/3}} \leq \|\rho_2^\varepsilon(t)\|_{L^{4/3}} \leq C \, \|f_2^\varepsilon(t)\|_{L^\infty}^{1/4} \left(\int_{\mathbb{R}^6} f_2^\varepsilon(t)|\xi| d\xi \, dx \right)^{3/4} \leq \; C(f_0) \; \varepsilon^{3/4}.$$

Then, substituting this estimate in (64), we have

$$\begin{aligned} \|j_2^\varepsilon(t)\|_{\mathbb{L}^{5/4}} &\leq C\,\|j_2^\varepsilon(t)\|_{\mathbb{L}^1}^{1/5}\ \|j_2^\varepsilon(t)\|_{\mathbb{L}^{4/3}}^{4/5} \leq (\varepsilon\,\mathcal{E}(0))^{1/5}\ \left(C(f_0)\ \varepsilon^{-1/4}\right)^{4/5} \\ &\leq C(f_0)\ (\mathcal{E}(0))^{1/5}\,. \end{aligned}$$

Gathering the previous estimates on j_1^ε and j_2^ε, we conclude that j^ε is uniformly bounded in $L^\infty(\mathbb{R}^+, \mathbb{L}^{5/4}(\mathbb{R}^3))$. □

Proof of Proposition 5. The longitudinal component of the electric field may be written as

$$E_L^\varepsilon = -\nabla\phi^\varepsilon, \quad \text{where} \quad -\Delta\phi^\varepsilon = \rho^\varepsilon.$$

Consequently from Lemma 2 and (61), there exists a positive constant $C(f_0) > 0$ such that

$$\|E_L^\varepsilon(t)\|_{\mathbb{L}^p} \leq C(f_0), \quad \text{for } 3/2 < p \leq\ 12/5, \tag{65}$$

$$\|\nabla E_L^\varepsilon(t)\|_{\mathbb{L}^q} \leq C(f_0), \quad \text{for } 1 < q \leq 4/3. \tag{66}$$

From (6), E_L^ε also satisfies the following equation

$$\varepsilon\ \frac{\partial E_L^\varepsilon}{\partial t} - \nabla\times B^\varepsilon = -\varepsilon\, j^\varepsilon.$$

Since the magnetic field B^ε belongs to $\mathbb{L}^2(\mathbb{R}^3)$ and satisfies

$$-\Delta B^\varepsilon = \varepsilon\,\nabla\times j^\varepsilon,$$

it follows from Lemma 2 and Lemma 8 that

$$\|\nabla\times B^\varepsilon\|_{\mathbb{L}^{5/4}} \leq C\,\varepsilon\ \|j^\varepsilon(t)\|_{\mathbb{L}^{5/4}} \leq\ C(f_0)\varepsilon.$$

whence,

$$\left\|\frac{\partial E_L^\varepsilon}{\partial t}\right\|_{\mathbb{L}^{5/4}} \leq C\,\|j^\varepsilon(t)\|_{\mathbb{L}^{5/4}} \leq C(f_0). \tag{67}$$

Owing to (65), (66) and (67), the sequence $(E_L^\varepsilon)_{\varepsilon>0}$ is relatively compact in $\mathcal{C}(\mathbb{R}^+, \mathbb{L}^{5/4}_{loc}(\mathbb{R}^3))$.

Finally, Lemma 2 and Proposition 2 yield

$$\varepsilon^{1/2}\,\|E_T^\varepsilon\|_{\mathbb{L}^2+\mathbb{L}^6} + \|B^\varepsilon\|_{\mathbb{L}^2} \leq C\,\varepsilon,$$

which completes the proof of Proposition 5. □

Now it follows from (62) and Proposition 5 that there exist a subsequence of $(f^\varepsilon, E_L^\varepsilon, E_T^\varepsilon, B^\varepsilon)_{\varepsilon>0}$ (not relabelled) and a couple (f, E) such that

$$f^\varepsilon \rightharpoonup f \text{ weakly in } L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3 \times \mathbb{R}^3)), \quad \text{when } \varepsilon \to 0,$$

and

$$\begin{aligned} E_L^\varepsilon &\to E \text{ strongly in } L^\infty(\mathbb{R}^+, \mathbb{L}^{5/4}(B(0,R))), \\ E_T^\varepsilon &\to 0 \text{ strongly in } L^\infty(\mathbb{R}^+, \mathbb{L}^2(B(0,R))), \\ B^\varepsilon &\to 0 \text{ strongly in } L^\infty(\mathbb{R}^+, \mathbb{L}^2(B(0,R))), \end{aligned}$$

for any R. It is then straightforward to pass to the limit as ε goes to zero in the Vlasov-Darwin system and check that (f, E) is a weak solution to the Vlasov-Poisson system.

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