

CONVERGENCE OF A FINITE VOLUME SCHEME FOR THE VLASOV–POISSON SYSTEM*

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Abstract. We propose a finite volume scheme to discretize the one-dimensional Vlasov–Poisson system. We prove that, if the initial data are positive, bounded, continuous, and have their first moment bounded, then the numerical approximation converges to the weak solution of the system for the weak topology of L^∞ . Moreover, if the initial data belong to BV , the convergence is strong in $C^0(0, T; L^1_{loc})$. To prove the convergence of the discrete electric field, we obtain an estimation in $W^{1,\infty}(\Omega_T)$. Then we have

$$\begin{aligned} f_h(t, x, v) &\rightharpoonup f(t, x, v) \text{ in } L^\infty(Q_T) \text{ weak-}\star \text{ as } h \rightarrow 0, \\ E_h(t, x) &\rightarrow E(t, x) \text{ in } C^0(\overline{\Omega}_T) \text{ as } h \rightarrow 0, \end{aligned}$$

where (E, f) is the unique weak solution of the Vlasov–Poisson system.

Key words. finite volume schemes, Vlasov–Poisson system, convergence

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1. Introduction. The Vlasov–Poisson system is a model for the motion of a collisionless plasma of electrons in a uniform background of ions and describes the evolution of the distribution function of electrons (solution of the Vlasov equation) under the effects of the transport and self-consistent electric field (solution of the Poisson equation). The coupling between both equations gives a nonlinear problem.

The numerical resolution of the Vlasov equation is most often performed using particle methods (PIC), which consist of approximating the plasma by a finite number of particles. The trajectories of these particles are computed from the characteristic curves given by the Vlasov equation. The interactions with self-consistent and external fields are computed by a numerical method using a mesh of the physical space (see, e.g., Birdsall and Langdon [2] or Cottet and Raviart [5]). This method enables us to get satisfying results with few particles.

Methods relying on a discretization of the phase space have also been proposed (see, e.g., Shoucri and Knorr [15] and Klimas and Farrell [11]) and seem to be more efficient in some cases, for example, when particles in the tail of the distribution play an important physical role, or when the numerical noise due to the finite number of particles becomes too important. Among them, the semi-Lagrangian method [16] consists of directly computing the distribution function on a grid of the phase space. This computation is done by following the characteristic curves at each time step and interpolating the value at the origin of the characteristics by a cubic spline method.

This interpolation method works well for simple geometries of the physical space but does not seem to be well suited to more complex geometries.

To remedy this problem a possible approach is to use the finite volume method which is known to be a robust and computationally cheap method for the discretization of conservation laws (see, e.g., Eymard, Gallouet, and Herbin [9] and the references

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therein, and see Vignal [17]). Finite volume schemes have already been implemented to approximate the solution of the Vlasov equation coupled with the Poisson equation (we refer to Boris and Book [3], Cheng and Knorr [4] and, more recently, Mineau [13]) and with the Maxwell system (see the paper of Fijalkow [8]). The purpose of this work is to prove the convergence of a finite volume scheme for the simplest model problem in plasma physics, namely, the one-dimensional Vlasov–Poisson system with periodic boundary conditions (with respect to the space variable).

Before precisely describing the problem considered here, let us mention related papers where the convergence of a numerical scheme for the Vlasov–Poisson system is investigated. Cottet and Raviart [5] present a precise mathematical analysis of the particle method for solving the one-dimensional Vlasov–Poisson system. We also mention the papers of Wollman and Ozizmir [19] and Wollman [20] on the topic. Ganguly and Victory give a convergence result for the Vlasov–Maxwell system [10]. Schaeffer [14] proves the convergence of a finite difference scheme for the Vlasov–Poisson–Fokker–Planck system, but he discretizes the transport part by a characteristic method and assumes the initial data are three times continuously differentiable. In fact, to the best of our knowledge, no convergence results seem to be available in the literature concerning the numerical resolution of the Vlasov equation by an Eulerian method.

We now recall the Vlasov–Poisson system. Setting $\Omega = (0, L)$ and $\Omega_T = (0, T) \times (0, L)$, denoting by $f(t, x, v)$ the distribution function of electrons in the phase space (with mass normalized to 1 and the charge to +1), and denoting by $E(t, x)$ the self-consistent electric field, the Vlasov–Poisson system reads

$$(1) \quad \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + E(t, x) \frac{\partial f}{\partial v} = 0, \quad (t, x, v) \in (0, T) \times (0, L) \times \mathbb{R},$$

$$(2) \quad \frac{\partial E}{\partial x}(t, x) = \int_{\mathbb{R}} f(t, x, v) dv - 1, \quad (t, x) \in (0, T) \times (0, L),$$

with positive initial data

$$(3) \quad f(0, x, v) = f_0(x, v), \quad (x, v) \in (0, L) \times \mathbb{R}.$$

We impose periodic boundary conditions in x ,

$$(4) \quad f(t, 0, v) = f(t, L, v), \quad (t, v) \in [0, T] \times \mathbb{R},$$

together with the global neutrality of the plasma,

$$(5) \quad \frac{1}{L} \int_0^L \int_{\mathbb{R}} f(t, x, v) dv dx = 1, \quad t \in [0, T].$$

In order to completely determine the electric field $E(t, x)$, we add a zero-average electrostatic condition

$$(6) \quad \int_0^L E(t, x) dx = 0, \quad t \in [0, T],$$

which amounts to assuming that the electric potential is L -periodic.

We first present an upwind finite volume scheme computing the fluxes on the boundary of each cell of the mesh. We obtain the scheme (15) and approximate the electric field using the Green kernel.

From an L^∞ estimate on the first moment of f_h , we obtain a bound on the discrete electric field in $W^{1,\infty}$. We next give a weak BV inequality which will be useful for the convergence of the approximation to the weak solution of the Vlasov–Poisson system. We also prove that if the initial data belong to BV , the approximation remains bounded in BV . From these estimates, we prove

$$\begin{aligned} f_h(t, x, v) &\rightharpoonup f(t, x, v) \text{ in } L^\infty(Q_T) \text{ weak-}\star \text{ as } h \rightarrow 0, \\ E_h(t, x) &\rightarrow E(t, x) \text{ in } C^0(\overline{\Omega}_T) \text{ as } h \rightarrow 0, \end{aligned}$$

where $Q = (0, L) \times \mathbb{R}$, $Q_T = (0, T) \times (0, L) \times \mathbb{R}$, and (f, E) is the unique solution to the Vlasov–Poisson system.

Moreover, if f_0 belongs to $BV(Q)$, then we prove that the convergence of f_h is strong in $C^0([0, T]; L^1_{loc}(Q))$.

2. Regularity and discretization of the Vlasov–Poisson equation. There is quite a number of articles addressing the existence problem in high dimension; see the survey papers of Batt [1] up to 1984 and of DiPerna and Lions [7] for more recent results. Zheng and Majda [21] prove the existence of a solution with a measure as initial data for the one-dimensional case with periodic boundary conditions in x . We mention in particular that Cooper and Klimas [6] proved the global existence and uniqueness of a continuous solution $f(t, x, v)$, with $E(t, x)$ having a bounded derivative $\frac{\partial E}{\partial x}$, if the initial data $f_0(x, v)$ are continuous and their first moment is finite; in other words, there exists a positive function $R(v)$ which is decreasing in $|v|$ such that

$$\exists C > 0, \quad f_0(x, v) \leq C R(v), \text{ and } \int_0^L \int_{\mathbb{R}} |v| R(v) dv dx < +\infty.$$

On the other hand, the result of DiPerna and Lions [7] utilizing “velocity averaging” implies the existence of a renormalized solution if $f_0(x, v)$ is assumed to satisfy the weak condition $f_0 \log^+ f_0 \in L^1((0, L) \times \mathbb{R})$.

In this paper, we assume the initial condition is continuous and belongs to $L^\infty(Q) \cap L^1(Q)$. For simplicity we will consider

$$R(v) = \frac{1}{(1 + |v|)^\lambda} \quad \text{with } \lambda > 2.$$

Then, applying the result of Klimas and Cooper, the system (1)–(6) has a unique weak solution: the coupling of functions (f, E) satisfies $f(t, x, v) \in C^0(Q_T)$, $E(t, x) \in W^{1,\infty}(\Omega_T)$ and, for all $\varphi \in C_c^\infty(Q_T)$,

$$\int_{Q_T} f \left(\frac{\partial \varphi}{\partial t} + v \frac{\partial \varphi}{\partial x} + E(t, x) \frac{\partial \varphi}{\partial v} \right) dx dv dt + \int_Q f_0 \varphi(0, x, v) dx dv = 0,$$

and the electric field $E(t, x)$ is given by the Poisson equation

$$\frac{\partial E}{\partial x}(t, x) = \int_{\mathbb{R}} f(t, x, v) dv - 1.$$

In order to compute a numerical approximation of the solution of the Vlasov–Poisson system, let us define a Cartesian mesh of the phase space \mathcal{M}_h consisting of cells, denoted by $C_{i,j}$, $i \in I = \{0, \dots, n_x - 1\}$, where n_x is the number of subcells of $(0, L)$ and $j \in \mathbb{Z}$.

\mathcal{M}_h is given by an increasing sequence $(x_{i-1/2})_{i \in \{0, \dots, n_x\}}$ of the interval $(0, L)$ and by a second increasing sequence $(v_{j-1/2})_{j \in \mathbb{Z}}$ of \mathbb{R} .

Let $\Delta x_i = x_{i+1/2} - x_{i-1/2}$ be the physical space step and $\Delta v_j = v_{j+1/2} - v_{j-1/2}$ be the velocity space step. The parameter h indicates

$$h = \max_{i,j} \{\Delta x_i, \Delta v_j\}.$$

We assume the mesh is admissible. There exists $\alpha \in (0, 1)$ such that

$$(7) \quad \forall h > 0 \quad \forall (i, j) \in I \times \mathbb{Z}, \quad \alpha h \leq \Delta x_i \leq h, \quad \text{and} \quad \alpha h \leq \Delta v_j \leq h.$$

Finally, we obtain a Cartesian mesh of the phase space constituted of control volumes

$$C_{i,j} = (x_{i-1/2}, x_{i+1/2}) \times (v_{j-1/2}, v_{j+1/2}) \quad \text{for } i \in I \text{ and } j \in \mathbb{Z}.$$

Let Δt be the time step and $t^n = n \Delta t$. We set the discrete initial data $f_{i,j}^0 = \frac{1}{|C_{i,j}|} \int_{C_{i,j}} f_0(x, v) dx dv$ or $f_{i,j}^0 = f_0(x_i, v_j)$, where x_i (resp., v_j) represents the middle of $[x_{i-1/2}, x_{i+1/2}]$ (resp., $[v_{j-1/2}, v_{j+1/2}]$).

The finite volume method consists of integrating the Vlasov equation on each control volume of the mesh, approximating fluxes on the boundary,

$$(8) \quad \frac{1}{|C_{i,j}|} \int_{C_{i,j}} f(t^{n+1}, x, v) dx dv = \frac{1}{|C_{i,j}|} \int_{C_{i,j}} f(t^n, x, v) dx dv - \frac{1}{|C_{i,j}|} (\phi_{i+1/2,j}^n - \phi_{i-1/2,j}^n + \psi_{i,j+1/2}^n - \psi_{i,j-1/2}^n),$$

where $\phi_{i+1/2,j}^n$ and $\psi_{i,j+1/2}^n$ denote the fluxes on the boundary of the cell $C_{i,j}$,

$$\begin{aligned} \phi_{i+1/2,j}^n &= \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} v f(t, x_{i+1/2}, v) dv dt, \\ \psi_{i,j+1/2}^n &= \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} E(t, x) f(t, x, v_{j+1/2}) dx dt. \end{aligned}$$

We approximate the fluxes $\phi_{i+1/2,j}^n$ and $\psi_{i,j+1/2}^n$ by the discrete fluxes $\bar{\phi}_{i+1/2,j}^n$ and $\bar{\psi}_{i,j+1/2}^n$ with a simple upwind scheme

$$\bar{\phi}_{i+1/2,j}^n = \begin{cases} \Delta t \Delta v_j v_j f_{i,j}^n & \text{if } v_j \geq 0, \\ \Delta t \Delta v_j v_j f_{i+1,j}^n & \text{if } v_j < 0, \end{cases}$$

and

$$\bar{\psi}_{i,j+1/2}^n = \begin{cases} \Delta t \Delta x_i E_i^n f_{i,j}^n & \text{if } E_i^n \geq 0, \\ \Delta t \Delta x_i E_i^n f_{i,j+1}^n & \text{if } E_i^n < 0, \end{cases}$$

where E_i^n is an approximation of the electric field on $[x_{i-1/2}, x_{i+1/2}]$ given below by computing an approximate solution of the Poisson equation. The value $f_{i,j}^n$ is assumed to approximate the average of the Vlasov equation solution on the control volume $C_{i,j}$.

Thus, we obtain the discrete version of (8),

$$(9) \quad f_{i,j}^{n+1} = f_{i,j}^n - \frac{1}{|C_{i,j}|} (\bar{\phi}_{i+1/2,j}^n - \bar{\phi}_{i-1/2,j}^n + \bar{\psi}_{i,j+1/2}^n - \bar{\psi}_{i,j-1/2}^n).$$

To complete the scheme, we impose periodic boundary conditions on x (the values $f_{-1,j}^n$ and $f_{n_x,j}^n$ represent an approximation on a “virtual cell”):

$$\begin{aligned} f_{n_x,j}^n &= f_{0,j}^n & \text{if } v_j \geq 0, \\ f_{-1,j}^n &= f_{n_x-1,j}^n & \text{if } v_j < 0. \end{aligned}$$

In order to work with a bounded domain, we will truncate at $|v| = v_h$ (v_h sufficiently large which will go to $+\infty$ as $h \rightarrow 0$). Then we set $J = \{j \in \mathbb{Z}; |v_{j+1/2}| \leq v_h\}$ and impose

$$\bar{\psi}_{i,j+1/2}^n = 0 \quad \forall (i,j) \in I \times \mathbb{Z} \setminus J.$$

Thus, we are able to define the numerical solution approximating the solution of the Vlasov equation on $Q_T = \Omega_T \times \mathbb{R}$ by

$$f_h(t,x,v) = \begin{cases} f_{i,j}^n & \text{if } (t,x,v) \in [t^n, t^{n+1}) \times C_{i,j} \text{ and } (i,j) \in I \times J, \\ 0 & \text{if } |v| > v_h. \end{cases}$$

Computing zeroth and first order moments in v , we define the discrete charge and current densities for $(t,x) \in [t^n, t^{n+1}) \times [x_{i-1/2}, x_{i+1/2})$:

$$\begin{aligned} \rho_h(t,x) &= \int_{\mathbb{R}} f_h(t,x,v) dv = \sum_{j \in \mathbb{Z}} \Delta v_j f_{i,j}^n = \rho_i^n, \\ j_h(t,x) &= \int_{\mathbb{R}} v f_h(t,x,v) dv = \sum_{j \in \mathbb{Z}} \Delta v_j v_j f_{i,j}^n = j_i^n. \end{aligned}$$

To define a continuous approximation of the electric field, we set

$$\bar{\rho}_h(t,x) = \left(1 - \frac{t-t^n}{\Delta t}\right) \rho_h(t^n,x) + \frac{t-t^n}{\Delta t} \rho_h(t^{n+1},x).$$

Now, we are able to explicitly solve the Poisson equation by the corresponding kernel

$$K(x,y) = \begin{cases} \frac{y}{L} - 1 & \text{if } x \leq y \leq L, \\ \frac{y}{L} & \text{if } 0 \leq y \leq x \end{cases}$$

and give the discrete electric field E_h , which is continuous in (t,x) and piecewise linear,

$$(10) \quad E_h(t,x) = \int_0^L K(x,y) (\bar{\rho}_h(t,y) - 1) dy.$$

For example, we may consider the approximation on $(x_{i-1/2}, x_{i+1/2})$, taking the value of the discrete electric field in the middle of the cell,

$$\begin{aligned} E_i^n &= E_h(t^n, x_i) = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} E_h(t^n, x) dx, \\ (11) \quad &= \int_0^L K(x_i, y) (\rho_h(t^n, y) - 1) dy. \end{aligned}$$

We shall now prove the following theorem of convergence for the numerical approximation.

THEOREM 2.1. *Let $f_0(x, v)$ be positive, continuous, and such that*

$$(12) \quad \exists C > 0; \quad f_0(x, v) \leq C R(v) \quad \text{for } (x, v) \in Q,$$

where

$$R(v) = \frac{1}{(1 + |v|)^\lambda} \quad \text{with } \lambda > 2.$$

Let \mathcal{M}_h be a Cartesian mesh of the phase space and Δt be the time step satisfying the CFL condition. Then there exists $\xi \in (0, 1)$ such that

$$(13) \quad \frac{\Delta t}{\Delta x_i \Delta v_j} (\Delta v_j |v_j| + \Delta x_i |E_i^n|) \leq 1 - \xi \quad \forall (i, j) \in I \times J \quad \forall n.$$

If we consider the numerical solution given by the scheme (9), denoted by $f_h(t, x, v)$, and the discrete self-consistent field $E_h(t, x)$ given by (10), then we have

$$\begin{aligned} f_h(t, x, v) &\rightharpoonup f(t, x, v) \text{ in } L^\infty(Q_T) \text{ weak-}\star \text{ as } h \rightarrow 0, \\ E_h(t, x) &\rightarrow E(t, x) \text{ in } C(\overline{\Omega}_T) \text{ as } h \rightarrow 0, \end{aligned}$$

where (f, E) is the unique solution to the Vlasov–Poisson system (1)–(6).

Moreover, if f_0 belongs to $BV(Q)$ and the time step satisfies the CFL condition, then there exists $\xi \in (0, 1)$ such that

$$(14) \quad \frac{\Delta t}{\alpha h} (|v_j| + |E_i^n|) \leq 1 - \xi \quad \forall (i, j) \in I \times J \quad \forall n \geq 0.$$

Then, the convergence of f_h is strong in $C^0([0, T]; L^1_{loc}(Q))$.

REMARK 2.1. *The main idea of the proof is to estimate zeroth and first order moments to have a bound of the discrete electric field in $W^{1,\infty}$. Then we use a compactness argument to prove the strong convergence. For the discrete distribution function an L^∞ estimate is sufficient to have the weak- \star convergence.*

If the initial data belong to $BV(Q)$, we are able to show that $f_h(t)$ remains bounded in $BV(Q)$ and to prove the strong convergence.

3. A priori estimates. In this section, we will give some properties satisfied by the numerical approximation as well as by the solution of the continuous problem. We can check that if the velocity step $\Delta v_j = \Delta v$ for all $j \in \mathbb{Z}$, the numerical scheme preserves zeroth and first order moments. We will first prove a time decreasing property on $\int_Q \phi(f_h(t, x, v)) dx dv$ for all convex functions ϕ , which allows us to have a maximum principle on $f_h(t)$. We will also give an L^∞ estimate on the electric field $E_h(t)$. Then, in Proposition 3.2, we will obtain a uniform bound in x on f_h in order to obtain an L^∞ estimate on the first moment and a $W^{1,\infty}$ estimate on the discrete electric field.

PROPOSITION 3.1. *Let us assume there exists a convex function ϕ such that*

$$\int_0^L \int_{\mathbb{R}} \phi(f_0(x, v)) dx dv < +\infty.$$

Then, under the stability condition (13), the numerical solution is well defined and satisfies

$$\forall t \in \mathbb{R}^+, \tau > 0, \quad \int_0^L \int_{\mathbb{R}} \phi(f_h(t + \tau, x, v)) dx dv \leq \int_0^L \int_{\mathbb{R}} \phi(f_h(t, x, v)) dx dv.$$

Moreover,

$$|E_h(t, x)| \leq \frac{3}{2} L \quad \forall (t, x) \in \mathbb{R}^+ \times (0, L).$$

Proof. Using the scheme (9), we explicitly write the value of the numerical solution at time t^{n+1} , in terms of the values at time t^n ,

$$\begin{aligned} f_{i,j}^{n+1} &= \left(1 - \Delta t \left(\frac{|v_j|}{\Delta x_i} + \frac{|E_i^n|}{\Delta v_j} \right) \right) f_{i,j}^n + \Delta t \frac{v_j^+}{\Delta x_i} f_{i-1,j}^n + \Delta t \frac{v_j^-}{\Delta x_i} f_{i+1,j}^n \\ (15) \quad &+ \Delta t \frac{E_i^{n+}}{\Delta v_j} f_{i,j-1}^n + \Delta t \frac{E_i^{n-}}{\Delta v_j} f_{i,j+1}^n, \end{aligned}$$

where we use the well-known notation $r^+ = \max(r, 0)$ and $r^- = \max(-r, 0)$. Under the stability condition (13), the discrete distribution function $f_{i,j}^{n+1}$ could be written as a convex combination of $f_{i,j}^n, f_{i-1,j}^n, f_{i+1,j}^n, f_{i,j-1}^n, f_{i,j+1}^n$; then considering an arbitrary convex function ϕ , we have

$$\begin{aligned} (16) \quad \phi(f_{i,j}^{n+1}) &\leq \left(1 - \Delta t \left(\frac{|v_j|}{\Delta x_i} + \frac{|E_i^n|}{\Delta v_j} \right) \right) \phi(f_{i,j}^n) + \Delta t \frac{v_j^+}{\Delta x_i} \phi(f_{i-1,j}^n) \\ &+ \Delta t \frac{v_j^-}{\Delta x_i} \phi(f_{i+1,j}^n) + \Delta t \frac{E_i^{n+}}{\Delta v_j} \phi(f_{i,j-1}^n) + \Delta t \frac{E_i^{n-}}{\Delta v_j} \phi(f_{i,j+1}^n). \end{aligned}$$

Then it follows for all $n \in \mathbb{N}$ that

$$\begin{aligned} \int_0^L \int_{\mathbb{R}} \phi(f_h(t^{n+1}, x, v)) dx dv &= \sum_{i,j} \Delta x_i \Delta v_j \phi(f_{i,j}^{n+1}) \leq \sum_{i,j} \Delta x_i \Delta v_j \phi(f_{i,j}^n) \\ &\leq \int_0^L \int_{\mathbb{R}} \phi(f_h(t^n, x, v)) dx dv. \end{aligned}$$

Now, let us prove that the discrete electric field is bounded. The argument is the same as in the continuous case: For (t, x) belonging to Ω_T ,

$$\begin{aligned} |E_h(t, x)| &= \left| \int_0^L K(x, y) (\bar{\rho}_h(t, y) - 1) dy \right| \\ &\leq \left| \int_0^L K(x, y) \bar{\rho}_h(t, y) dy \right| + \left| \int_0^L K(x, y) dy \right| \\ &\leq \|K\|_{L^\infty} \int_0^L \bar{\rho}_h(t, y) dy + \frac{L}{2} \leq L + \frac{L}{2} = \frac{3}{2} L. \end{aligned}$$

Note that the electric field is uniformly bounded in time. □

REMARK 3.1. As a consequence of Proposition 3.1, we consider $\phi(r) = r^-$ (resp., $\phi(r) = (r - \|f_0\|_{L^\infty})^+$) and the initial data is positive (resp., bounded). Then we obtain

that f_h is also positive (resp., bounded). We know the L^p norm of the Vlasov equation solution is preserved over time, but this property does not seem to be satisfied by the numerical approximation; indeed, if we take $\phi(r) = |r|^p$, we are only able to prove that the L^p norm is decreasing (this simple scheme is dissipative).

Now, let us give a uniform bound in (t, x) on f_h and an estimate on first moments ρ_h and j_h .

PROPOSITION 3.2. Assume that $0 \leq f_0(x, v) \leq C R(v) = \frac{C}{(1+|v|)^\lambda}$ for some $\lambda > 2$. Then, there exists a constant C_T depending only on T, L , and f_0 such that

$$0 \leq f_h(t, x, v) \leq C_T R_h(v), \quad (t, x, v) \in (0, T) \times (0, L) \times \mathbb{R},$$

where $R_h(v) = \frac{1}{(1+|v_j|)^\lambda}$ for $v \in [v_{j-1/2}, v_{j+1/2})$, $j \in \mathbb{Z}$.

For h small enough there exists $C_T > 0$,

$$(17) \quad 0 \leq \rho_h(t, x) \leq C_T, \quad |j_h(t, x)| \leq C_T, \quad (t, x) \in \Omega_T.$$

Proof. Notice that there exists $c_0 = c_0(\alpha, \lambda)$ such that

$$\frac{R_h(v_{j+\beta})}{R_h(v_j)} \leq 1 + c_0 \Delta v_j \quad \text{for } \beta = -1 \text{ or } 1.$$

Then we set $A = (1 + \frac{3}{2}L c_0 \Delta t)$ and can easily check that, taking

$$f_{i,j}^0 = \frac{1}{|C_{i,j}|} \int_{C_{i,j}} f_0(x, v) dx dv \quad \text{or} \quad f_{i,j}^0 = f_0(x_i, v_j),$$

we have $f_h(0, x, v) \leq A^0 C R_h(v)$.

If we assume that $f_h(t^n, x, v) \leq A^n C R_h(v)$, using the numerical scheme (9), we obtain

$$\begin{aligned} \frac{f_{i,j}^{n+1}}{C R_h(v_j)} &= \left(1 - \Delta t \frac{\Delta v_j |v_j| + \Delta x_i |E_i^n|}{\Delta x_i \Delta v_j} \right) \frac{f_{i,j}^n}{C R_h(v_j)} + \Delta t \frac{v_j^+}{\Delta x_i} \frac{f_{i-1,j}^n}{C R_h(v_j)} \\ &+ \Delta t \frac{v_j^-}{\Delta x_i} \frac{f_{i+1,j}^n}{C R_h(v_j)} + \Delta t \frac{E_i^{n+}}{\Delta v_j} \frac{f_{i,j-1}^n}{C R_h(v_{j-1})} \frac{R_h(v_{j-1})}{R_h(v_j)} \\ &+ \Delta t \frac{E_i^{n-}}{\Delta v_j} \frac{f_{i,j+1}^n}{C R_h(v_{j+1})} \frac{R_h(v_{j+1})}{R_h(v_j)}. \end{aligned}$$

Under the CFL condition (13) and using the property of $R_h(v)$, we have

$$\begin{aligned} \frac{f_{i,j}^{n+1}}{C R_h(v_j)} &\leq \left(1 - \Delta t \left(\frac{|v_j|}{\Delta x_i} + \frac{|E_i^n|}{\Delta v_j} \right) \right) A^n + \Delta t \frac{|v_j|}{\Delta x_i} A^n + \Delta t \frac{|E_i^n|}{\Delta v_j} A^n (1 + c_0 \Delta v_j) \\ &\leq A^n \left(1 + \frac{3}{2}L c_0 \Delta t \right) = A^{n+1}. \end{aligned}$$

Finally,

$$\forall (i, j) \in I \times \mathbb{Z}, \quad \frac{f_{i,j}^{n+1}}{C R_h(v_j)} \leq A^{n+1}.$$

For a finite time T and for all $n \in \{0, \dots, T/\Delta t\}$, $A^{n+1} < e^{c_0 T}$. Then, as in the continuous case, there exists a majorizing function of the discrete distribution

$$f_h(t, x, v) \leq C_T R_h(v) \quad \text{for } (t, x, v) \in Q_T.$$

In order to prove the inequality (17), we observe

$$\begin{aligned} \int_{\mathbb{R}} R_h(v)dv &= \sum_{j \in \mathbb{Z}} \frac{\Delta v_j}{(1 + |v_j|)^\lambda} \leq 2 \sum_{j \in \mathbb{N}} \frac{h}{(1 + \alpha [j - 1] h)^\lambda} \\ &\leq h + \frac{1}{\alpha} \int_{\mathbb{R}} \frac{dv}{(1 + |v|)^\lambda} < +\infty. \end{aligned}$$

For h small enough, there exists a constant C_T depending only on f_0, α, T, L such that

$$\rho_h(t, x) = \int_{\mathbb{R}} f_h(t, x, v)dv \leq C_T \left(h + \frac{1}{\alpha} \int_{\mathbb{R}} \frac{dv}{(1 + |v|)^\lambda} \right) < +\infty$$

and

$$|j_h(t, x)| \leq \int_{\mathbb{R}} |v| f_h(t, x, v)dv \leq C_T \left(h + \frac{1}{\alpha} \int_{\mathbb{R}} \frac{dv}{(1 + |v|)^{\lambda-1}} \right) < +\infty. \quad \square$$

3.1. Estimation for the derivatives of E_h . In Proposition 3.1, we have already seen that E_h is bounded in L^∞ . Now we give an estimate on the derivatives.

PROPOSITION 3.3. *Under the same assumptions as in Proposition 3.1, for h sufficiently small, there exists a constant C_T , which depends only on the initial data and on the domain, such that*

$$\left| \frac{\partial E_h}{\partial x}(t, x) \right| + \left| \frac{\partial E_h}{\partial t}(t, x) \right| \leq C_T, \quad (t, x) \in \Omega_T.$$

Proof. We first give an estimate of the derivative in x , which is explicitly given in the distribution sense by the Poisson equation: let $(t, x) \in \Omega_T$; then there exists $n \in \{0, \dots, T/\Delta t\}$ such that $t \in [t^n, t^{n+1})$,

$$\begin{aligned} \left| \frac{\partial E_h}{\partial x}(t, x) \right| &= \left| \bar{\rho}_h(t, x) - 1 \right| \leq \left| \bar{\rho}_h(t, x) \right| + 1 \\ &\leq \left(1 - \frac{t - t^n}{\Delta t} \right) \rho_h(t^n, x) + \frac{t - t^n}{\Delta t} \rho_h(t^{n+1}, x) + 1. \end{aligned}$$

By Proposition 3.2, it follows that

$$\left| \frac{\partial E_h}{\partial x}(t, x) \right| \leq C_T \quad \forall (t, x) \in \Omega_T.$$

To obtain an estimate of $\frac{\partial E_h}{\partial t}$, we define a new approximation of the current density denoted by $\bar{j}_h(t, x)$ for $(t, x) \in [t^n, t^{n+1}) \times [x_{i-1/2}, x_{i+1/2})$,

$$\bar{j}_h(t, x) = \bar{j}_i^n + (x - x_{i-1/2}) \frac{\bar{j}_{i+1}^n - \bar{j}_i^n}{\Delta x_i},$$

with

$$\bar{j}_i^n = \sum_{j \in \mathbb{Z}} \Delta v_j (v_j^+ f_{i,j}^n - v_j^- f_{i+1,j}^n)$$

and $\bar{j}_h \in L^\infty(0, T; W^{1,\infty}(\Omega))$. Recalling that $\bar{\rho}_h$ defined previously belongs to the space $W^{1,\infty}(0, T; L^\infty(\Omega))$, we notice that integrating (9) with respect to v yields, as in the continuous case,

$$\forall (t, x) \in \Omega_T, \quad \frac{\partial \bar{\rho}_h}{\partial t}(t, x) + \frac{\partial \bar{j}_h}{\partial x}(t, x) = 0.$$

Now,

$$\begin{aligned} \frac{\partial E_h}{\partial t}(t, x) &= \int_0^L K(x, y) \frac{\partial \bar{\rho}_h}{\partial t}(t, y) dy = \int_0^L -K(x, y) \frac{\partial \bar{j}_h}{\partial x}(t, y) dy \\ &= -\bar{j}_h(t, x) + \frac{1}{L} \int_0^L \bar{j}_h(t, y) dy \end{aligned}$$

and observing that $|\bar{j}_i^n| \leq |j_i^n| + |j_{i+1}^n|$, Proposition 3.2 allows us to complete the proof. \square

3.2. Weak BV estimate for f_h . The following lemma will be useful to obtain the convergence of (E_h, f_h) to the Vlasov equation solution.

LEMMA 3.1. *Under the stability condition (13) on the time step and the condition on the mesh (7), assume the initial data belong to $L^1(Q) \cap L^\infty(Q)$. Consider $R > 0$ and $T > 0$ with $h < R$ and $\Delta t < T$. Let $j_0, j_1 \in \mathbb{Z}$ and $N_T \in \mathbb{N}$ be such that $-R \in (v_{j_0-1/2}, v_{j_0+1/2})$, $R \in (v_{j_1-1/2}, v_{j_1+1/2})$, and $T \in ((N_T - 1)\Delta t, N_T \Delta t)$. We define for all Lipschitzian functions $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$*

$$\begin{aligned} EF_{1h}(\phi) &= \Delta t \sum_{n=0}^{N_T} \sum_{i \in I} \sum_{j=j_0}^{j_1} \Delta x_i \Delta v_j \left[v_j^+ |\phi(f_{i,j}^n) - \phi(f_{i-1,j}^n)| + v_j^- |\phi(f_{i,j}^n) - \phi(f_{i+1,j}^n)| \right. \\ (18) \quad &\quad \left. + E_i^{n+} |\phi(f_{i,j}^n) - \phi(f_{i,j-1}^n)| + E_i^{n-} |\phi(f_{i,j}^n) - \phi(f_{i,j+1}^n)| \right] \end{aligned}$$

and

$$(19) \quad EF_{2h}(\phi) = \Delta t \sum_{n=0}^{N_T} \sum_{i \in I} \sum_{j=j_0}^{j_1} \Delta x_i \Delta v_j |\phi(f_{i,j}^{n+1}) - \phi(f_{i,j}^n)|.$$

Then, there exists $C > 0$ depending only on T, R, f_0, α, ξ such that

$$(20) \quad EF_{1h}(\phi) \leq C h^{1/2} \quad \text{and} \quad EF_{2h}(\phi) \leq C \Delta t^{1/2}.$$

Proof. Let us begin by first proving the result for $\phi(r) = r$. Multiplying the scheme (15) by $\Delta x_i \Delta v_j f_{i,j}^n$ and summing over $i \in \{0, \dots, n_x - 1\}$, $j \in \{j_0, \dots, j_1\}$, and $n \in \{0, \dots, N_T\}$, it follows that

$$B_1 + B_2 = 0,$$

where

$$\begin{aligned} B_1 &= \sum_{n,i,j} \Delta x_i \Delta v_j [f_{i,j}^{n+1} - f_{i,j}^n] f_{i,j}^n. \\ B_2 &= \Delta t \sum_{n,i,j} \left[\Delta v_j v_j^+ [f_{i,j}^n - f_{i-1,j}^n] f_{i,j}^n + \Delta v_j v_j^- [f_{i,j}^n - f_{i+1,j}^n] f_{i,j}^n \right. \\ &\quad \left. + \Delta x_i E_i^{n+} [f_{i,j}^n - f_{i,j-1}^n] f_{i,j}^n + \Delta x_i E_i^{n-} [f_{i,j}^n - f_{i,j+1}^n] f_{i,j}^n \right]. \end{aligned}$$

Noting that

$$[f_{i,j}^{n+1} - f_{i,j}^n] f_{i,j}^n = -\frac{1}{2} [f_{i,j}^{n+1} - f_{i,j}^n]^2 - \frac{1}{2} (f_{i,j}^n)^2 + \frac{1}{2} (f_{i,j}^{n+1})^2,$$

then

$$B_1 = -\frac{1}{2} \sum_{n,i,j} \Delta x_i \Delta v_j [f_{i,j}^{n+1} - f_{i,j}^n]^2 - \frac{1}{2} \sum_{i,j} \Delta x_i \Delta v_j (f_{i,j}^0)^2 + \frac{1}{2} \sum_{i,j} \Delta x_i \Delta v_j (f_{i,j}^{N_T+1})^2.$$

By scheme (15), we have

$$\begin{aligned} & \sum_{n,i,j} \Delta x_i \Delta v_j [f_{i,j}^{n+1} - f_{i,j}^n]^2 \\ &= \sum_{n,i,j} \frac{\Delta t^2}{\Delta x_i \Delta v_j} \left[\Delta v_j v_j^+ [f_{i,j}^n - f_{i-1,j}^n] + \Delta v_j v_j^- [f_{i,j}^n - f_{i+1,j}^n] \right. \\ & \quad \left. + \Delta x_i E_i^{n+} [f_{i,j}^n - f_{i,j-1}^n] + \Delta x_i E_i^{n-} [f_{i,j}^n - f_{i,j+1}^n] \right]^2. \end{aligned}$$

Using the Cauchy–Schwarz inequality and the stability condition (13),

$$\begin{aligned} B_1 &\geq -\frac{1}{2} \Delta t (1 - \xi) \sum_{n,i,j} \left[\Delta v_j v_j^+ [f_{i,j}^n - f_{i-1,j}^n]^2 + \Delta v_j v_j^- [f_{i,j}^n - f_{i+1,j}^n]^2 \right. \\ & \quad \left. + \Delta x_i E_i^{n+} [f_{i,j}^n - f_{i,j-1}^n]^2 + \Delta x_i E_i^{n-} [f_{i,j}^n - f_{i,j+1}^n]^2 \right] \\ & \quad - \frac{1}{2} \sum_{i,j} \Delta x_i \Delta v_j (f_{i,j}^0)^2. \end{aligned}$$

We now study the term B_2 , which may be rewritten as

$$\begin{aligned} B_2 &= \frac{1}{2} \Delta t \sum_{n,i,j} \left[\Delta v_j v_j^+ [f_{i,j}^n - f_{i-1,j}^n]^2 + \Delta v_j v_j^- [f_{i,j}^n - f_{i+1,j}^n]^2 \right. \\ & \quad \left. + \Delta x_i E_i^{n+} [f_{i,j}^n - f_{i,j-1}^n]^2 + \Delta x_i E_i^{n-} [f_{i,j}^n - f_{i,j+1}^n]^2 \right] \\ & \quad + \frac{1}{2} \Delta t \sum_{n,i} \left[\Delta x_i E_i^{n+} [(f_{i,j_1}^n)^2 - (f_{i,j_0-1}^n)^2] + \Delta x_i E_i^{n-} [(f_{i,j_0}^n)^2 - (f_{i,j_1+1}^n)^2] \right]. \end{aligned}$$

Then, since $B_1 + B_2 = 0$ the following inequality holds:

$$\begin{aligned} & \Delta t \sum_{n,i,j} \left[\Delta v_j v_j^+ [f_{i,j}^n - f_{i-1,j}^n]^2 + \Delta v_j v_j^- [f_{i,j}^n - f_{i+1,j}^n]^2 \right. \\ & \quad \left. + \Delta x_i E_i^{n+} [f_{i,j}^n - f_{i,j-1}^n]^2 + \Delta x_i E_i^{n-} [f_{i,j}^n - f_{i,j+1}^n]^2 \right] \\ & \leq \frac{1}{\xi} \sum_{i,j} \Delta x_i \Delta v_j (f_{i,j}^0)^2 + \frac{\Delta t}{\xi} \sum_{n,i} \Delta x_i |E_i^n| [(f_{i,j_0-1}^n)^2 + (f_{i,j_1+1}^n)^2] \\ & \leq \frac{1}{\xi} \int_{Q_T} |f_h(0)|^2 dx dv + \frac{2}{\xi} \|f_0\|_{L^\infty}^2 \|E_h\|_{L^1(\Omega_T)} = \frac{K}{\xi}. \end{aligned}$$

The value K does not depend on h ; indeed,

$$\|E_h\|_{L^1(\Omega_T)} \leq T \|f_0\|_{L^1(Q)} \text{ and } \left(\int_{Q_T} |f_h(0)|^2 dx dv \right)^{1/2} \leq T \|f_0\|_{L^2(Q)}.$$

Finally, the previous inequality and the Cauchy–Schwarz inequality lead to

$$\begin{aligned} EF_{1h}(Id) &\leq \left[\Delta t \sum_{n,i,j} \Delta v_j v_j^+ [f_{i,j}^n - f_{i-1,j}^n]^2 + \Delta v_j v_j^- [f_{i,j}^n - f_{i+1,j}^n]^2 \right. \\ &\quad \left. + \Delta x_i E_i^{n+} [f_{i,j}^n - f_{i,j-1}^n]^2 + \Delta x_i E_i^{n-} [f_{i,j}^n - f_{i,j+1}^n]^2 \right]^{1/2} \\ &\quad \times \left[\Delta t \sum_{n,i,j} \Delta x_i^2 (\Delta v_j |v_j| + \Delta x_i |E_i^n|) \right]^{1/2} \\ &\leq h^{1/2} \left(\frac{K}{\xi} \right)^{1/2} \left[2 T L R \left(R + \frac{3}{2} L \right) \right]^{1/2}. \end{aligned}$$

Now, we prove the second estimate on $EF_{2h}(Id)$, using the scheme (15):

$$\begin{aligned} EF_{2h}(Id) &= \Delta t \sum_{n,i,j} \Delta x_i \Delta v_j \left| f_{i,j}^{n+1} - f_{i,j}^n \right| \\ &\leq \Delta t^2 \sum_{n,i,j} \left[\Delta v_j v_j^+ |f_{i,j}^n - f_{i-1,j}^n| + \Delta v_j v_j^- |f_{i,j}^n - f_{i+1,j}^n| \right. \\ &\quad \left. + \Delta x_i E_i^{n+} |f_{i,j}^n - f_{i,j-1}^n| + \Delta x_i E_i^{n-} |f_{i,j}^n - f_{i,j+1}^n| \right]. \end{aligned}$$

As in the previous case, we use the Cauchy–Schwarz inequality and the stability condition (13). We also recall that the discrete electric field is uniformly bounded:

$$EF_{2h}(Id) \leq \Delta t^{1/2} K^{1/2} \left[2 T L R \frac{1 - \xi}{\xi} \right]^{1/2}.$$

Finally, for all Lipschitzian functions, we have $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$,

$$EF_{1h}(\phi) \leq Lip(\phi) EF_{1h}(Id), \quad EF_{2h}(\phi) \leq Lip(\phi) EF_{2h}(Id).$$

Then, inequality (20) holds for all Lipschitzian functions $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. \square

3.3. Strong BV estimate. In this section, we will assume that the initial data $f_0(x, v)$ belong to $BV(Q)$. In order to obtain the strong convergence in $L^1_{loc}(Q)$, we will obtain an estimation on the total variation of $f_h(t)$.

Preliminary. Since our numerical approximations are functions of several variables, we generalize the definition of the total variation (see, e.g., LeVêque [12]). To simplify, we give the definition for a function with two variables (x, y) .

DEFINITION 3.1. Let $g(x, y)$ be a function defined on \mathbb{R}^2 . The total variation of g is the number given by the following limit:

$$\begin{aligned} TV_{xy}(g) &= \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int |g(x + \varepsilon, y) - g(x, y)| dx dy \\ &\quad + \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int |g(x, y + \varepsilon) - g(x, y)| dx dy. \end{aligned}$$

We can define the total variation of a piecewise constant function g analogously by

$$TV_{xy}(g) = \sum_{i,j} |y_{j+1} - y_j| |g(x_{i+1}, y_j) - g(x_i, y_j)| + |x_{i+1} - x_i| |g(x_i, y_{j+1}) - g(x_i, y_j)|.$$

PROPOSITION 3.4. *Under the stability condition (14) and, if the initial data f_0 belong to $BV(Q)$, then there exists a constant C_T which depends only on f_0 , L , and T such that*

$$\forall n \in \{0, \dots, T/\Delta t\}, \quad TV_{xv}(f_h(t^n)) \leq C_T TV_{xv}(f_0).$$

Proof. Let us write the scheme (9) on cells i and $i + 1$. Calculating the difference between both terms and using the fact

$$E_{i+1}^{n+} = E_{i+1}^{n+} - E_i^{n+} + E_i^{n+}, \\ E_{i+1}^{n-} = E_{i+1}^{n-} - E_i^{n-} + E_i^{n-},$$

we directly obtain

$$f_{i+1,j}^{n+1} - f_{i,j}^{n+1} = \left(1 - \Delta t \left| \frac{v_j^+}{\Delta x_{i+1}} + \frac{v_j^-}{\Delta x_i} + \frac{|E_i^n|}{\Delta v_j} \right| \right) (f_{i+1,j}^n - f_{i,j}^n) + \Delta t \frac{v_j^+}{\Delta x_i} (f_{i,j}^n - f_{i-1,j}^n) + \Delta t \frac{v_j^-}{\Delta x_{i+1}} (f_{i+2,j}^n - f_{i+1,j}^n) + \Delta t \frac{E_i^{n+}}{\Delta v_j} (f_{i+1,j-1}^n - f_{i,j-1}^n) + \Delta t \frac{E_i^{n-}}{\Delta v_j} (f_{i+1,j+1}^n - f_{i,j+1}^n) + \Delta t \frac{E_{i+1}^{n+} - E_i^{n+}}{\Delta v_j} (f_{i+1,j}^n - f_{i+1,j-1}^n) + \Delta t \frac{E_{i+1}^{n-} - E_i^{n-}}{\Delta v_j} (f_{i+1,j+1}^n - f_{i+1,j}^n).$$

Let us multiply by Δv_j , sum over $i \in \{0, \dots, n_x - 1\}$, and have $j \in \mathbb{Z}$:

$$\sum_{i,j} \Delta v_j |f_{i+1,j}^{n+1} - f_{i,j}^{n+1}| \\ (21) \quad \leq \sum_{i,j} \Delta v_j \left(1 - \Delta t \left| \frac{v_j^+}{\Delta x_{i+1}} + \frac{v_j^-}{\Delta x_i} + \frac{|E_i^n|}{\Delta v_j} \right| \right) |f_{i+1,j}^n - f_{i,j}^n| \\ (22) \quad + \Delta t \sum_{i,j} \Delta v_j \frac{v_j^+}{\Delta x_i} |f_{i,j}^n - f_{i-1,j}^n| + \Delta t \sum_{i,j} \Delta v_j \frac{v_j^-}{\Delta x_{i+1}} |f_{i+2,j}^n - f_{i+1,j}^n| \\ (23) \quad + \Delta t \sum_{i,j} E_i^{n+} |f_{i+1,j-1}^n - f_{i,j-1}^n| + \Delta t \sum_{i,j} E_i^{n-} |f_{i+1,j+1}^n - f_{i,j+1}^n| \\ + \Delta t \sum_{i,j} |E_{i+1}^{n+} - E_i^{n+}| |f_{i+1,j}^n - f_{i+1,j-1}^n| \\ + \Delta t \sum_{i,j} |E_{i+1}^{n-} - E_i^{n-}| |f_{i+1,j+1}^n - f_{i+1,j}^n|.$$

We use the fact that f_h is periodic in x to treat terms (21)–(23) and recall that under the stability condition (14) the sum of coefficients in front of the term $|f_{i+1,j}^n - f_{i,j}^n|$ is equal to 1. Finally, we obtain the inequality

$$(24) \quad \sum_{i,j} \Delta v_j |f_{i+1,j}^{n+1} - f_{i,j}^{n+1}| \leq \sum_{i,j} \Delta v_j |f_{i+1,j}^n - f_{i,j}^n|$$

$$(25) \quad + \Delta t \sum_{i,j} |E_{i+1}^{n+} - E_i^{n+}| |f_{i+1,j}^n - f_{i+1,j-1}^n|$$

$$(26) \quad + \Delta t \sum_{i,j} |E_{i+1}^{n-} - E_i^{n-}| |f_{i+1,j+1}^n - f_{i+1,j}^n|.$$

Now, we have to study the terms (25)–(26), which represent a total variation of f_h at time t^n in the function of the velocity variable v . We recall that the discrete electric field is Lipschitz continuous in x ; then

$$\exists c_{1,T} > 0, \quad |E_{i+1}^n - E_i^n| \leq c_{1,T} \Delta x_i,$$

where $c_{1,T}$ is a constant which depends only on the domain, on the initial data, and on the final time T .

We also use the fact that the function $x \mapsto \max(x, 0)$ is Lipschitz continuous with a constant equal to 1.

$$\begin{aligned} \sum_{i,j} \Delta v_j |f_{i+1,j}^{n+1} - f_{i,j}^{n+1}| &\leq \sum_{i,j} \Delta v_j |f_{i+1,j}^n - f_{i,j}^n| \\ &\quad + \Delta t \sum_{i,j} |E_{i+1}^n - E_i^n| (|f_{i+1,j}^n - f_{i+1,j-1}^n|) \\ &\leq \sum_{i,j} \Delta v_j |f_{i+1,j}^n - f_{i,j}^n| + \Delta t \sum_{i,j} c_{1,T} \Delta x_i |f_{i,j+1}^n - f_{i,j}^n|. \end{aligned}$$

We finally obtain an estimate of the total variation in x of the discrete distribution function at time t^{n+1} in the function of the total variation of the discrete distribution at time t^n .

TV_x denotes the total variation in x ; in fact,

$$(27) \quad \begin{aligned} TV_x(f_h(t^{n+1})) &= \sum_{i,j} \Delta v_j |f_{i+1,j}^{n+1} - f_{i,j}^{n+1}| \\ &\leq TV_x(f_h(t^n)) + c_{1,T} \Delta t TV_v(f_h(t^n)). \end{aligned}$$

By a similar argument, using the fact that the mesh is admissible (7),

$$v_{j+1} - v_j \leq \left(\frac{1}{2} + \frac{1}{2\alpha}\right) \Delta v_j,$$

we obtain an estimate of the total variation at time t^{n+1} in v :

$$(28) \quad \exists c_{2,T} > 0, \quad TV_v(f_h(t^{n+1})) \leq TV_v(f_h(t^n)) + c_{2,T} \Delta t TV_x(f_h(t^n)).$$

Thus, with inequalities (27) and (28), we express the total variation estimate at time t^{n+1} in the function of the total variation at time t^n . We set $c_{3,T} = c_{1,T} + c_{2,T}$,

$$(29) \quad TV_{xv}(f_h(t^{n+1})) \leq TV_{xv}(f_h(t^n)) + c_{3,T} \Delta t TV_{xv}(f_h(t^n)).$$

Then,

$$TV_{xv}(f_h(t^n)) \leq \exp(c_{3,T}T) TV_{xv}(f_h(0)) \leq \exp(c_{3,T}T) TV_{xv}(f_0). \quad \square$$

REMARK 3.2. *To achieve the proof of Proposition 3.3, we use the fact that if f_0 belongs to $BV(Q)$, then it satisfies the following inequality:*

$$\sum_{i,j} \Delta v_j |f_{i+1,j}^0 - f_{i,j}^0| + \Delta x_i |f_{i,j+1}^0 - f_{i,j}^0| \leq TV_{xv}(f_0).$$

4. Proof of Theorem 2.1. We consider a sequence of a mesh of the phase space defined as in the beginning of the paper satisfying the condition (7), and we define a time step Δt such that the stability condition (13) is true. This sequence is denoted by $(\mathcal{M}_h)_{h>0}$.

For a given mesh, we are able to construct, by the finite volume scheme (9)–(11), a unique pair (f_h, E_h) . Thus, we set

$$A = \left\{ E_h \in W^{1,\infty}(\Omega_T); \quad E_h \text{ given by (11) for a mesh } \mathcal{M}_h \right\}.$$

On the one hand, in Proposition 3.3 we have proved there exists a constant independent on the mesh \mathcal{M}_h such that

$$\forall E_h \in A, \quad \|E_h\|_{L^\infty} + \left\| \frac{\partial E_h}{\partial t} \right\|_{L^\infty} + \left\| \frac{\partial E_h}{\partial x} \right\|_{L^\infty} \leq C_T.$$

On the other hand, using the fact that the injection from $W^{1,\infty}(\Omega_T)$ to $C^0(\overline{\Omega}_T)$ is compact, there exists a subsequence of $(E_h)_{h>0}$ and a function E belonging to $C^0(\overline{\Omega}_T)$ such that

$$E_h(t, x) \rightharpoonup E(t, x) \quad \text{in } L^\infty(\Omega_T) \text{ weak-}\star \quad \text{as } h \rightarrow 0,$$

$$E_h(t, x) \rightarrow E(t, x) \quad \text{in } C^0(\overline{\Omega}_T) \text{ strong} \quad \text{as } h \rightarrow 0.$$

Moreover, we also know by Proposition 3.1 that the discrete distribution function f_h is bounded in $L^\infty(Q_T)$. Therefore, there exists a subsequence and a function $f \in L^\infty(Q_T)$ such that

$$f_h(t, x, v) \rightharpoonup f(t, x, v) \quad \text{in } L^\infty(Q_T) \text{ weak-}\star \quad \text{as } h \rightarrow 0.$$

The discrete charge ρ_h is bounded in $L^\infty(\Omega_T)$; then up to the extraction of a subsequence, we also have

$$\rho_h(t, x) \rightharpoonup \rho(t, x) \quad \text{in } L^\infty(\Omega_T) \text{ weak-}\star \quad \text{as } h \rightarrow 0.$$

Let us prove that the limit $\rho(t, x)$ is equal to $\int_{\mathbb{R}} f(t, x, v) dv$. Consider $\psi(t, x) \in L^1(\Omega_T)$; then we have

$$\begin{aligned} \int_0^T \int_0^L \left(\rho_h - \int_{\mathbb{R}} f dv \right) \psi(t, x) dx dt &= \int_0^T \int_0^L \int_{|v| \leq r} (f_h - f) \psi(t, x) dv dx dt \\ &\quad + \int_0^T \int_0^L \int_{|v| > r} (f_h - f) \psi(t, x) dv dx dt. \end{aligned}$$

Since $f_h \rightharpoonup f$ in L^∞ weak-*, the first term of the right-hand side goes to zero for every fixed r . Moreover, from the second estimate of Proposition 3.1, we have

$$\int_{|v|>r} |f_h - f|dv \leq \int_{|v|>r} (|f_h| + |f|)dv \leq 2C_T \left(h + \int_{|v|>r} \frac{dv}{(1 + |v|)^\lambda} \right).$$

Then, the second term can be small by choosing r large enough uniformly with respect to h , and thus ρ_h converges to $\int_{\mathbb{R}} f dv$ in $L^\infty(\Omega_T)$ weak-*

Moreover, if we assume the initial data belong to $BV(Q)$, then we construct a new approximation of the distribution function, continuous in time, denoted by \bar{f}_h (it is easy to prove that f_h and \bar{f}_h converge to the same limit), and we set

$$B = \left\{ \bar{f}_h \in C(0, T; L^1_{loc}(Q)); \quad \bar{f}_h \text{ given by (9) for a mesh } \mathcal{M}_h \right\}$$

and

$$B(t) = \left\{ \bar{f}_h(t) \in L^1_{loc}(Q); \quad \bar{f}_h \in B \right\}.$$

A consequence of the Helly compactness theorem and the total variation estimate of the discrete distribution function infers that $B(t) \subset BV(Q)$; then $B(t)$ is relatively compact in $L^1_{loc}(Q)$. Furthermore, using the continuity of \bar{f}_h we can prove that B is uniformly equicontinuous:

$$\forall \varepsilon > 0, \quad \exists \eta > 0, \quad \|f_h(t_1) - f_h(t_2)\|_{L^1_{loc}} \leq \varepsilon, \quad f_h \in B, \quad 0 \leq t_1 \leq t_2 \leq T, |t_1 - t_2| \leq \eta.$$

Then, applying the Ascoli theorem we prove that \bar{f}_h strongly converges to f in $C^0(0, T; L^1_{loc})$.

4.1. Convergence to the weak solution of the Vlasov equation. Let $\varphi \in C^\infty_c(Q_T)$, $R > 0$, and $j_0, j_1 \in \mathbb{Z}$ be such that

$$\text{supp} \left(\varphi(t, x, \cdot) \right) \subset [-R, R]$$

and

$$-R \in (v_{j_0-1/2}, v_{j_0+1/2}) \quad \text{and} \quad R \in (v_{j_1-1/2}, v_{j_1+1/2}).$$

We multiply the finite volume scheme (9) by $\frac{1}{\Delta t \Delta x_i \Delta v_j} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t, x, v) dx dv dt$, sum over $i \in \{0, \dots, nx - 1\}$, $j \in \{j_0, \dots, j_1\}$, and $n \in \{0, \dots, N_T = \frac{T}{\Delta t}\}$,

$$E_1 + E_2 = 0$$

with

$$E_1 = \sum_{n,i,j} (f_{i,j}^{n+1} - f_{i,j}^n) \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t, x, v) dx dv dt,$$

and

$$E_2 = \sum_{n,i,j} \left[\Delta v_j v_j^+ (f_{i,j}^n - f_{i-1,j}^n) + \Delta v_j v_j^- (f_{i,j}^n - f_{i+1,j}^n) + \Delta x_i E_i^{n+} (f_{i,j}^n - f_{i,j-1}^n) + \Delta x_i E_i^{n-} (f_{i,j}^n - f_{i,j+1}^n) \right] \frac{1}{\Delta x_i \Delta v_j} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t, x, v) dx dv dt.$$

Moreover, we denote $E_{1,0}$ and $E_{2,0}$ by

$$E_{1,0} = \int_{Q_T} f_h(t, x, v) \frac{\partial \varphi}{\partial t}(t, x, v) dt dx dv + \int_Q f_0(x, v) \varphi(0, x, v) dx dv$$

and

$$E_{2,0} = \int_{Q_T} f_h(t, x, v) \left(v \frac{\partial \varphi}{\partial x}(t, x, v) + E_h(t, x) \frac{\partial \varphi}{\partial v}(t, x, v) \right) dx dv dt.$$

We will compare E_1 with $E_{1,0}$ and E_2 with $E_{2,0}$ to establish that $E_{1,0} + E_{2,0}$ goes to zero as $h \rightarrow 0$.

Comparison between E_1 and $E_{1,0}$. Let us remark that $E_{1,0}$ can be rewritten as

$$E_{1,0} = \sum_{n,i,j} f_{i,j}^n \int_{C_{i,j}} \left(\varphi(t^{n+1}, x, v) - \varphi(t^n, x, v) \right) dx dv + \int_Q f_0(x, v) \varphi(0, x, v) dx dv.$$

By a discrete integration by parts, it follows that

$$\begin{aligned} E_{1,0} &= - \sum_{n,i,j} \left(f_{i,j}^{n+1} - f_{i,j}^n \right) \int_{C_{i,j}} \varphi(t^{n+1}, x, v) dx dv \\ &\quad - \int_Q \left(f_h(0, x, v) - f_0(x, v) \right) \varphi(0, x, v) dx dv. \end{aligned}$$

Thus,

$$\begin{aligned} |E_1 + E_{1,0}| &\leq \sum_{n,i,j} |f_{i,j}^{n+1} - f_{i,j}^n| \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \left| \frac{\partial \varphi}{\partial t}(t, x, v) \right| dt dx dv \\ &\quad + \int_Q |f_h(0, x, v) - f_0(x, v)| |\varphi(0, x, v)| dx dv \end{aligned}$$

with the discrete initial data defined, for example, by

$$f_h(0, x, v) = \frac{1}{|C_{i,j}|} \int_{C_{i,j}} f_0(x, v) dx dv \quad \forall (x, v) \in C_{i,j}.$$

Using the assumption on the initial data $f_0 \in L^1(Q) \cap L^\infty(Q)$, we then have

$$\lim_{h \rightarrow 0} \int_Q |f_h(0, x, v) - f_0(x, v)| |\varphi(0, x, v)| dx dv = 0.$$

Moreover, from the inequality on the term EF_{2h} given by (20) in Lemma 3.1 (taking $\phi(r) = r$ for $r \in \mathbb{R}^+$), we have

$$\sum_{n,i,j} |f_{i,j}^{n+1} - f_{i,j}^n| \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \left| \frac{\partial \varphi}{\partial t}(t, x, v) \right| dt dx dv \leq C \|\varphi_t\|_{L^\infty} \Delta t^{1/2}.$$

Then,

$$|E_1 + E_{1,0}| \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Comparison between E_2 and $E_{2,0}$. We first introduce the notation

$$E_{2,1} = \sum_{n,i,j} \left[v_j^+ (f_{i,j}^n - f_{i-1,j}^n) \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t, x_{i-1/2}, v) dv dt \right. \\ + v_j^- (f_{i,j}^n - f_{i+1,j}^n) \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t, x_{i+1/2}, v) dv dt \\ + E_i^{n+} (f_{i,j}^n - f_{i,j-1}^n) \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t, x, v_{j-1/2}) dx dt \\ \left. + E_i^{n-} (f_{i,j}^n - f_{i,j+1}^n) \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t, x, v_{j+1/2}) dx dt \right].$$

On the one hand, we compare E_2 and $E_{2,1}$:

$$|E_2 - E_{2,1}| = \left| \sum_{n,i,j} \left[v_j^+ (f_{i,j}^n - f_{i-1,j}^n) \left[\frac{1}{\Delta x_i} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t, x, v) - \varphi(t, x_{i-1/2}, v) dv dt \right] \right. \right. \\ + v_j^- (f_{i,j}^n - f_{i+1,j}^n) \left[\frac{1}{\Delta x_i} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t, x, v) - \varphi(t, x_{i+1/2}, v) dv dt \right] \\ + E_i^{n+} (f_{i,j}^n - f_{i,j-1}^n) \left[\frac{1}{\Delta v_j} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t, x, v) - \varphi(t, x, v_{j-1/2}) dx dt \right] \\ \left. \left. + E_i^{n-} (f_{i,j}^n - f_{i,j+1}^n) \left[\frac{1}{\Delta v_j} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t, x, v) - \varphi(t, x, v_{j+1/2}) dx dt \right] \right] \right|.$$

Using the inequality on EF_{1h} given by (20) in Lemma 3.1 with $\phi(r) = r$, there exists $c > 0$ depending only on $T, R, L, f_0, \alpha, \xi$ such that the following inequality holds:

$$|E_2 - E_{2,1}| \leq c \|\nabla \varphi\|_{L^\infty} h^{1/2}.$$

On the other hand, we estimate $|E_{2,0} + E_{2,1}|$, rewriting the term $E_{2,1}$ as follows (we recall that φ has a compact support):

$$E_{2,1} = - \sum_{n,i,j} f_{i,j}^n \left[v_j \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t, x_{i+1/2}, v) - \varphi(t, x_{i-1/2}, v) dv dt \right. \\ \left. + E_i^n \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t, x, v_{j+1/2}) - \varphi(t, x, v_{j-1/2}) dx dt \right].$$

In the same way,

$$E_{2,0} = \sum_{n,i,j} f_{i,j}^n \left[\int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} v (\varphi(t, x_{i+1/2}, v) - \varphi(t, x_{i-1/2}, v)) dv dt \right. \\ \left. + \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} E_h(t, x) (\varphi(t, x, v_{j+1/2}) - \varphi(t, x, v_{j-1/2})) dx dt \right].$$

Therefore, there exists $c > 0$ depending only on $T, R, L, f_0, \alpha, \xi$:

$$\begin{aligned} |E_{2,0} + E_{2,1}| &\leq \sum_{n,i,j} f_{i,j}^n \left[\int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} |v - v_j| |\varphi(t, x_{i+1/2}, v) - \varphi(t, x_{i-1/2}, v)| dv dt \right. \\ &\quad \left. + \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} |E_h(t, x) - E_i^n| |\varphi(t, x, v_{j+1/2}) - \varphi(t, x, v_{j-1/2})| dx dt \right] \\ &\leq c \|\nabla \varphi\|_{L^\infty} \sum_{n,i,j} \Delta t \Delta x_i \Delta v_j f_{i,j}^n \left[\Delta v_j + \sup |E_h(t, x) - E_i^n| \right] \\ &\leq cT \|\nabla \varphi\|_{L^\infty} \|f_0\|_{L^1} h. \end{aligned}$$

Finally, recalling that $E_1 + E_2 = 0$, we obtain

$$\begin{aligned} \epsilon(\Delta t, h) &= \int_{Q_T} f_h \left(\frac{\partial \varphi}{\partial t} + v \frac{\partial \varphi}{\partial x} + E_h(t, x) \frac{\partial \varphi}{\partial v} \right) dt dx dv + \int_Q f_0(x, v) \varphi(0, x, v) dx dv \\ &= E_{1,0} + E_{2,0} \\ &= E_{1,0} + E_1 + E_{2,0} + E_{2,1} - E_{2,1} + E_2, \end{aligned}$$

and from the previous estimates, we proved there exists a constant C depending only on $\varphi, f_0, L, T, \alpha, \xi$ such that

$$\begin{aligned} |E_{1,0} + E_1| &\leq C (\|f_0 - f_h(0)\|_{L^1} + \Delta t^{1/2}), \\ |E_{2,0} - E_2| &\leq C h^{1/2}, \\ |E_{2,0} + E_{2,1}| &\leq C h. \end{aligned}$$

Then, $\epsilon(\Delta t, h) \rightarrow 0$ as $h \rightarrow 0$.

As we know

$$f_h(t, x, v) \rightharpoonup f(t, x, v) \text{ in } L^\infty(Q_T) \text{ weak-}\star$$

and

$$E_h(t, x) \rightarrow E(t, x) \text{ in } C^0(\overline{\Omega}_T),$$

we have shown that the limit pair (f, E) of a subsequence $(f_h, E_h)_{h>0}$ is a solution of the Vlasov equation (1). To conclude, we have to prove that this couple is also a solution of the Poisson equation (2).

REMARK 4.1. *In practical calculation, we use a large but finite bound M for the velocity space. In this paper, we assume that as $h \rightarrow 0$, the support of the velocity space goes to infinity, and the stability condition (13) imposes on us that*

$$\exists \varepsilon \in (0, 1), \quad v_h \simeq \frac{1}{h^\varepsilon}, \quad \text{and} \quad \Delta t \simeq \frac{h^2}{h^{1-\varepsilon} + h} \simeq h^{1+\varepsilon}.$$

4.2. Convergence to the solution of the Poisson equation. We recall that the discrete electric field defined before is continuous in time, but for a simpler analysis let us consider a new approximation piecewise constant in time:

$$\tilde{E}_h(t, x) = \int_0^L K(x, y) (\rho_h(t, y) - 1) dy.$$

Recalling that $\frac{\partial E_h}{\partial t}$ is uniformly bounded, it is easy to prove that \tilde{E}_h and E_h have the same behavior as h goes to zero. Then \tilde{E}_h converges almost everywhere to E .

Let us prove that $E(t, x)$ is a solution of the Poisson equation. Let $\psi(t, x)$ belong to $L^1(\Omega_T)$,

$$\int_{\Omega_T} \tilde{E}_h(t, x) \psi(t, x) dt dx = \int_{\Omega_T} \left[\int_0^L K(x, y) (\rho_h(t, y) - 1) dy \right] \psi(t, x) dt dx.$$

The discrete charge ρ_h converges to $\rho(t, x) = \int_{\mathbb{R}} f(t, x, v) dv$ in $L^\infty(\Omega_T)$ weak-*, where f is a solution of the Vlasov equation. Thus, using the Fubini theorem we can set $g(t, y) = \int_0^L K(x, y) \psi(t, x) dx$ which belongs to $L^1(\Omega_T)$ and satisfies

$$\int_{\Omega_T} \rho_h(t, y) g(t, y) dt dy \rightarrow \int_{\Omega_T} \rho(t, y) g(t, y) dt dy \quad \text{as } h \rightarrow 0.$$

Thus, we have

$$E(t, x) = \int_0^L K(x, y) (\rho(t, y) - 1) dy \quad \text{and} \quad \rho(t, y) = \int_{\mathbb{R}} f(t, y, v) dv.$$

Then, (f, E) is a solution of the Vlasov–Poisson system.

The weak formulation infers that the solution of the Vlasov–Poisson system belongs to $C^0([0, T]; \mathcal{D}')$, but observing the electric field E is bounded in $W^{1,\infty}(\Omega_T)$ and the initial data are continuous, we see that the distribution function f is also continuous in (x, v) . Let us recall that under our hypothesis, the solution of the Vlasov–Poisson system (1)–(2) is unique; then any subsequence that we considered converges to the same limit and the sequence $(f_h, E_h)_{h>0}$ converges to the unique solution.

5. Error estimates. In this section, we give error estimates on the approximation (f_h, E_h) , assuming the initial data are smooth and have a compact support. We follow the proof of Vila and Villedieu [18]. Let us introduce $\mathcal{M}(Q_T)$, the space of positive measure on Q_T and $W_c^{1,\infty}(Q_T)$, the set of functions in $W^{1,\infty}(Q_T)$, periodic in x and compactly supported in (t, v) .

PROPOSITION 5.1. *Under the stability condition (13) on the time step and the condition on the mesh (7), assume the initial data belong to $L^1(Q) \cap L^\infty(Q)$ and are bounded by the function $R(v)$ defined previously. Then, there exist $\nu_{h,\Delta t}^1$ and $\nu_{h,\Delta t}^2 \in \mathcal{M}(Q_T)$ such that for all $\varphi \in W_c^{1,\infty}(Q_T)$, $\varphi \geq 0$,*

$$\begin{aligned} \int_{Q_T} f_h \left(\frac{\partial \varphi}{\partial t} + v \frac{\partial \varphi}{\partial x} + E_h(t, x) \frac{\partial \varphi}{\partial v} \right) dt dx dv + \int_Q f_0(x, v) \varphi(0, x, v) dx dv \\ \leq \int_Q \varphi(0) d\nu_{h,\Delta t}^1 + \int_{Q_T} (|\varphi_t| + |\nabla_{x,v} \varphi|) d\nu_{h,\Delta t}^1 \end{aligned}$$

and

$$\begin{aligned} - \int_{Q_T} f_h^2 \left(\frac{\partial \varphi}{\partial t} + v \frac{\partial \varphi}{\partial x} + E_h(t, x) \frac{\partial \varphi}{\partial v} \right) dt dx dv - \int_Q f_0^2(x, v) \varphi(0, x, v) dx dv \\ \leq \int_Q \varphi(0) d\nu_{h,\Delta t}^2 + \int_{Q_T} (|\varphi_t| + |\nabla_{x,v} \varphi|) d\nu_{h,\Delta t}^2. \end{aligned}$$

The measures satisfy the properties for all $T > 0$, $R > 0$, and there exists a constant C depending only on $T, R, L, f_0, \alpha, \xi$ such that

$$\nu_{h,\Delta t}^1\left((0, T) \times (0, L) \times B(0, R)\right) \leq C\left(\Delta t^{1/2} + h^{1/2} + \|f_0 - f_h(0)\|_{L^1}\right),$$

$$\nu_{h,\Delta t}^2\left((0, T) \times (0, L) \times B(0, R)\right) \leq C\left(\Delta t^{1/2} + h^{1/2} + \|f_0 - f_h(0)\|_{L^2}\right).$$

Proof. The idea of the proof is to follow the same argument as for the proof of the convergence of the finite volume scheme to the weak solution of the Vlasov equation given above. We use Lemma 3.1 with the convex function $\phi(r) = r$ (resp., $\phi(r) = r^2$) to obtain the bound on the measure $\nu_{h,\Delta t}^1$ (resp., $\nu_{h,\Delta t}^2$). \square

From this proposition we obtain the following theorem which gives us an error estimate on the approximation by the finite volume scheme. Now, we will assume the initial data have a compact support.

THEOREM 5.1. *Let $f_0(x, v)$ belong to $W_c^{1,\infty}(Q)$, let \mathcal{M}_h be a Cartesian mesh of the phase space, and let Δt be the time step satisfying the CFL condition. There exists $0 < \xi < 1$ such that*

$$\frac{\Delta t}{\Delta x_i \Delta v_j} (\Delta v_j |v_j| + \Delta x_i |E_i^n|) \leq 1 - \xi \quad \forall (i, j) \in I \times J \quad \forall n.$$

If we consider the numerical solution given by the scheme (9) denoted by $f_h(t, x, v)$, and the discrete self-consistent field $E_h(t, x)$ given by (10),

$$\int_{Q_T} e^{-\alpha t} |f(t, x, v) - f_h(t, x, v)|^2 dt dx dv \leq C_{1,T} (h^{1/2} + \Delta t^{1/2}) + C_1 \|f_0 - f_h(0)\|_{L^2}.$$

Proof. As we assume that the initial data have a compact support, for a finite time T there exist $\bar{R} > 0$ such that

$$\forall (t, x) \in (0, T) \times (0, L), \quad \text{supp}\left(f(t, x, \cdot)\right) \subset B(0, \bar{R}).$$

Moreover, using the regularity of the initial data, the solution of the Vlasov–Poisson system (E, f) is unique and f belongs to $W^{1,\infty}(Q_T)$. Now, let $\varphi \in W_c^{1,\infty}(Q_T)$. We have

$$\begin{aligned} \int_{Q_T} f^2 \left(\frac{\partial \varphi}{\partial t} + v \frac{\partial \varphi}{\partial x} + E_h(t, x) \frac{\partial \varphi}{\partial v} \right) dt dx dv + \int_Q f_0^2(x, v) \varphi(0, x, v) dx dv \\ = -2 \int_{Q_T} f(E_h - E) \varphi(t, x, v) \frac{\partial f}{\partial v} dt dx dv. \end{aligned}$$

From the first inequality of Proposition 5.1, for all $\varphi \in W_c^{1,\infty}(Q_T)$, $\varphi \geq 0$, we observe that $f \varphi \in W_c^{1,\infty}(Q_T)$ and $f \varphi \geq 0$; then using the regularity of the solution,

$$\begin{aligned} -2 \int_{Q_T} f_h f \left(\frac{\partial \varphi}{\partial t} + v \frac{\partial \varphi}{\partial x} + E_h(t, x) \frac{\partial \varphi}{\partial v} \right) dt dx dv - 2 \int_Q f_0^2(x, v) \varphi(0, x, v) dx dv \\ \geq -2 \langle \nu_{h,\Delta t}^1, \nabla_{t,x,v}(f \varphi) \rangle - 2 \int_{Q_T} f_h (E - E_h) \varphi(t, x, v) \frac{\partial f}{\partial v} dt dx dv. \end{aligned}$$

Moreover, from the second inequality of Proposition 5.1, for all $\varphi \in W_c^{1,\infty}(Q_T)$, $\varphi \geq 0$,

$$\int_{Q_T} f_h^2 \left(\frac{\partial \varphi}{\partial t} + v \frac{\partial \varphi}{\partial x} + E_h(t, x) \frac{\partial \varphi}{\partial v} \right) dt dx dv + \int_Q f_0^2(x, v) \varphi(0, x, v) dx dv \geq -\langle \nu_{h,\Delta t}^2, \nabla_{t,x,v} \varphi \rangle.$$

We finally obtain

$$\begin{aligned} & \int_{Q_T} |f - f_h|^2 \left(\frac{\partial \varphi}{\partial t} + v \frac{\partial \varphi}{\partial x} + E_h(t, x) \frac{\partial \varphi}{\partial v} \right) dt dx dv \\ & \geq -2 \int_{Q_T} (f_h - f) (E_h - E) \varphi \frac{\partial f}{\partial v} dt dx dv \\ (30) \quad & - 2 \|\nabla_{t,x,v} f\|_{L^\infty} \langle \nu_{h,\Delta t}^1, \nabla_{t,x,v} \varphi \rangle - \langle \nu_{h,\Delta t}^2, \nabla_{t,x,v} \varphi \rangle. \end{aligned}$$

We now construct the function φ giving the following result: Let us set $\alpha = 5 \|\frac{\partial f}{\partial v}\|_{L^\infty} L \bar{R}$, $R_1 = L + \bar{R}$, and $\omega = \max(2\bar{R}; \frac{3}{2}L)$ and consider $k \in C^1(\mathbb{R}^+; [0, 1])$ such that

$$k(r) = \begin{cases} 1 & \text{if } r \in [0, R_1 + \omega T), \\ 0 & \text{if } r \in [R_1 + \omega T + 1, +\infty), \end{cases}$$

and $k'(r) \leq 0 \forall r \in \mathbb{R}^+$. Then, we consider

$$\varphi(t, x, v) = \begin{cases} k(|(x, v)| + \omega t) e^{-\alpha t} & \text{if } (t, x, v) \in Q_T, \\ 0 & \text{if } t \geq T. \end{cases}$$

The function φ is not in $W_c^{1,\infty}(Q_T)$, but using a usual regularization technique in time, it may be proved that such a function can be considered [18]. Let us compute each term of the inequality (30). The discrete electric field is computed from the Green kernel, and the following inequality holds:

$$\begin{aligned} |E(t, x) - E_h(t, x)| &= \left| \int_Q K(x, y) [f(t, y, v) - f_h(t, y, v)] dy dv \right|, \\ &\leq \int_Q |f(t, y, v) - f_h(t, y, v)| dy dv. \end{aligned}$$

Thus, using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} & \left| 2 \int_{Q_T} (f_h - f) (E_h - E) \varphi \frac{\partial f}{\partial v} dt dx dv \right| \\ & \leq 2 \left\| \frac{\partial f}{\partial v} \right\|_{L^\infty} \left(\int_{Q_T} |f_h - f|^2 \varphi dt dx dv \right)^{1/2} \left(\int_{Q_T} (E_h - E)^2 \varphi dt dx dv \right)^{1/2} \\ & \leq 2 (2L\bar{R})^{1/2} \left\| \frac{\partial f}{\partial v} \right\|_{L^\infty} \left(\int_{Q_T} |f_h - f|^2 \varphi dt dx dv \right)^{1/2} \\ & \quad \left(\int_{Q_T \times Q} (f_h(t, y, w) - f(t, y, w))^2 \varphi(t, x, v) dt dx dv dy dw \right)^{1/2}. \end{aligned}$$

In the domain of computation $(t, x, v) \in [0, T] \times (0, L) \times B(0, \bar{R})$, the function $k(\cdot)$ is equal to 1; then $\varphi = e^{-\alpha t}$ and the previous inequality can be rewritten as

$$\begin{aligned} & \left| 2 \int_{Q_T} (f_h - f) (E_h - E) \varphi \frac{\partial f}{\partial v} dt dx dv \right| \\ & \leq 4 L \bar{R} \left\| \frac{\partial f}{\partial v} \right\|_{L^\infty} \left(\int_{Q_T} |f_h - f|^2 \varphi dt dx dv \right)^{1/2} \\ & \quad \left(\int_{Q_T} e^{-\alpha t} |f_h(t, y, w) - f(t, y, w)|^2 dt dy dw \right)^{1/2}, \\ & \leq 4 L \bar{R} \left\| \frac{\partial f}{\partial v} \right\|_{L^\infty} \int_{Q_T} |f - f_h|^2 \varphi dt dx dv. \end{aligned}$$

Next, a direct computation gives

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(t, x, v) &= \omega k'(|(x, v)| + \omega t) e^{-\alpha t} - \alpha k(|(x, v)| + \omega t) e^{-\alpha t}, \\ \frac{\partial \varphi}{\partial x}(t, x, v) &= \frac{x}{|(x, v)|} k'(|(x, v)| + \omega t) e^{-\alpha t}, \\ \frac{\partial \varphi}{\partial v}(t, x, v) &= \frac{v}{|(x, v)|} k'(|(x, v)| + \omega t) e^{-\alpha t}. \end{aligned}$$

Replacing the derivatives by their expression, we finally obtain

$$\begin{aligned} & \int_{Q_T} |f - f_h|^2 k'(|(x, v)| + \omega t) e^{-\alpha t} \left(w + \frac{v x + E_h(t, x) v}{|(x, v)|} \right) dt dx dv \\ & \quad - \alpha \int_{Q_T} |f - f_h|^2 k(|(x, v)| + \omega t) e^{-\alpha t} dt dx dv \\ & \geq -4 \left\| \frac{\partial f}{\partial v} \right\|_{L^\infty} L \bar{R} \int_{Q_T} |f - f_h|^2 k(|(x, v)| + \omega t) e^{-\alpha t} dt dx dv \\ & \quad - 2 \left\| \frac{\partial f}{\partial v} \right\|_{L^\infty} \nu_{h, \Delta t}^1 \left((0, T) \times (0, L) \times B(0, \bar{R}) \right) \\ & \quad - \nu_{h, \Delta t}^2 \left((0, T) \times (0, L) \times B(0, \bar{R}) \right). \end{aligned}$$

Since $k' \leq 0$ and $\omega = \max(2 \bar{R}; \frac{3}{2} L)$, we have

$$w + \frac{v x + E_h(t, x) v}{|(x, v)|} \geq 0$$

and therefore, since $k(|(x, v)| + \omega t) = 1$, if $(t, x, v) \in (0, T) \times (0, L) \times B(0, \bar{R})$,

$$\begin{aligned} \int_{Q_T} e^{-\alpha t} |f - f_h|^2 dt dx dv & \leq C_{1, T} \left[\nu_{h, \Delta t}^1 \left((0, T) \times (0, L) \times B(0, \bar{R}) \right) \right. \\ & \quad \left. + \nu_{h, \Delta t}^2 \left((0, T) \times (0, L) \times B(0, \bar{R}) \right) \right]. \end{aligned}$$

From Proposition 5.1, the measures $\nu_{h, \Delta t}^2$ et $\nu_{h, \Delta t}^1$ are bounded:

$$\begin{aligned} \nu_{h, \Delta t}^1 \left((0, T) \times (0, L) \times B(0, R) \right) & \leq C (\Delta t^{1/2} + h^{1/2} + \|f_0 - f_h(0)\|_{L^1}), \\ \nu_{h, \Delta t}^2 \left((0, T) \times (0, L) \times B(0, R) \right) & \leq C (\Delta t^{1/2} + h^{1/2} + \|f_0 - f_h(0)\|_{L^2}). \end{aligned}$$

The proof is complete. \square

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