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Les réseaux de spin et leur géométrie

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LES RÉSEAUX DE SPIN ET LEUR GÉOMÉTRIE

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CHAPTER 1

Introduction

My recent research activity has been concentrated on the study of the relations between geometry and quantum invariants of topological objects as knots, links, trivalent graphs and manifolds. Spin networks may be seen as a central object in these researches and will constitute the unifying "fil rouge" of the present work. In Appendix A I provide a list of all my works including those related to the previous research activity not cited here.

1. A short history of spin networks

The theory of classical spin networks originates from the techniques developed by Giulio Racah ([**51**]) and Eugene Wigner ([**65**]) in the 1950's to perform computations in atomic spectrography, and later used by Roger Penrose ([**49**]) who first gave a general definition of these objects.

A spin network is a number $\langle \Gamma, \mathrm{col}\rangle \in \mathbb{Q}$ associated to a couple (Γ, col) where Γ is a finite trivalent graph equipped with a cyclic ordering of the edges around each vertex and col is an *admissible coloring*, i.e. a map $\mathrm{col} : \{\text{edges of } \Gamma\} \to \mathbb{N}$ such that for each three-uple of edges e_1, e_2, e_3 around a vertex $\mathrm{col}(e_1), \mathrm{col}(e_2), \mathrm{col}(e_3)$ have even sum and satisfy triangular inequalities. This number is defined as a contraction of a tensor with values in finite dimensional representations of $SU(2)$ and using Wigner's $3j$-symbols (we will provide combinatorial definitions in Chapter 2). Besides the initial application to atomic spectrography spin networks have a central role in loop quantum gravity and in the study of quantum field theory.

The interest of spin networks in topology is due to the more recent ideas of quantum field theory and quantum invariants. At the beginning of 1980's, V.G. Drinfeld ([**18**]) and M. Jimbo ([**30**]) introduced the notion of *quantum group* and used the theory of representations of these objects to produce solutions to Yang-Baxter equations and thus representations of braid groups. In the meantime, V. Jones ([**31**]) defined his invariant of knots in S^3 through the theory of representations of Von Neumann algebras and L. Kauffman ([**34**]) found a purely combinatorial definition for this object. Shortly later, A. Kirillov and N. Reshetikhin ([**39**]) generalised the Jones polynomial using the theory of representations of the quantum group $U_q(sl_2)$ and defined the quantum analogues of classical spin networks; these objects are rational functions associated to trivalent graphs embedded in S^3 and equipped with a framing and a coloring (we will recall all the definitions in Chapter 2). Later V. Turaev and N. Reshetikhin ([**53**]) generalized these constructions to produce invariants of graphs associated to the representations of any quantum group and used these invariants ([**54**]) to build invariants of 3-manifolds which are universally accepted to be the rigourous realization of the invariants whose existence was predicted by E. Witten ([**66**]) through Feynmann integrals on the space of $SU(n)$-connections of the Chern-Simons action. Moreover using the rigourous construction of Reshetikhin-Turaev invariants, C. Blanchet, N. Habegger, G. Masbaum and P. Vogel ([**6**]) gave a first rigourous construction of the TQFT underlying these invariants.

One of the main interesting points in the study of these objects is that they allow a direct computation of expectation values of topological quantum field theories which, according to the physical interpretation, are based on infinite dimensional Feynmann integrals. Hence they provide a tool to test conjectures issued from the physical interpretation; instances of these conjectures are those studying the asymptotical behavior of quantum invariants when the *level* goes to infinity, like Witten's asymptotic expansion conjecture and the Volume Conjecture. In the present text we will only deal with the second conjecture which we will recall later on.

2. Physical motivation for spin networks

Spin networks (both in their classical and quantum versions) provide a bridge for mathematicians towards different theories and models in modern physics. We briefly recall three of these connections here below.

2.1. Classical spin networks and Penrose's toy-model for a universe. In [**49**], Roger Penrose proposed a toy model for a universe based on the combinatorial definition of classical spin networks. Roughly speaking according to the model (and to the best of our understanding of the physical statements in the paper), the universe is a spin network with boundary whose topology is unknown and performing an experiment on it corresponds to gluing small graphs along the boundary and observing the probability distributions of the colors on the so-obtained new boundary components of the universe. When the spins (colors) of the new graphs glued are small with respect to those of the universe, and the experiment is repeated enough times one may *define* angles formed by pairs of particles by inverting the solution to the question "If two particles of spin n and m moving in directions forming an angle α interact, what will be the probability distribution of the resulting spins after the interaction?". According to Penrose, no matter how big the universe is, the set of angles one can find through this procedure will always be compatible with a 3-dimensional euclidean geometry, i.e. the matrix whose entries are the cosines of the angles so found (between the directions of the particles represented by the boundary of the universe-graph) will always have rank at most 3. Unfortunately, as Penrose himself points out in the same paper, this toy-model for the universe does not allow a definition of distances between particle thus it provides only "half" of the geometry one would desire; thus the model was immediately abandoned for the more promising theory of twistors.

We cite this model here because we find it quite anticipatory of the ideas which later on showed up in the study of asymptotical behavior of quantum invariants in relation with geometry. It is indeed our opinion that the lessons learned in the last twenty years, namely the use of quantum-groups to enhance classical spin networks and the relations of the resulting objects with geometries other than the Euclidean one, should be used to modify and possibly improve Penrose's model.

2.2. Loop Quantum Gravity and spin networks. Classical spin networks have been extensively used in their gauge enhanced version to provide basis for the Hilbert space of observables in Loop Quantum Gravity (LQG). Very roughly speaking, the starting idea underlying LQG is that even if the position of a point is not well defined up to the action of the group of self-diffeomorphisms of the ambient (three-dimensional) manifold, the topology of a knot is; thus any kind of observable associated to knots (or graphs) could be viewed as an observable for a potential theory of quantum gravity. So a preliminary Hilbert space $\tilde{\mathcal{H}}$ underlying LQG is defined as the span of the functions on the space of $SU(2)$-connections of a 3-dimensional universe M provided by pairs (Γ, col) embedded in M (these pairs, enhanced with the connections, become the "spin networks with connections" defined in Chapter 2, Section 5) which are declared to be a orthonormal basis; the final space of the theory is the quotient $\mathcal{H}$ of $\tilde{\mathcal{H}}$ by the unitary action of the group of self-diffeomorphisms of M. It is claimed ([**55**]) that $\mathcal{H}$ is separable and that isotopy classes of spin networks with holonomies provide bases for it. Even though theoretically speaking some predictions made by LQG could be tested there is today no evidence for or against the theory, which remains thus one of the proposals for a quantum version of gravity. One of the main difficulties in the theory is to show that a suitable semi-classical limit of it recovers general relativity: such a problem may be seen as a generalization of the problem of studying asymptotical behavior of classical spin networks because the limit to be considered is also a limit on the topology of the network (as opposed to a limit only on the colors of a fixed Γ treated later on in this work).

2.3. Chern-Simons quantum field theory. The last (and probably most important) use of spin network in physics we will cite is due to E. Witten ([**66**]) and appears while studying the topological quantum field theory associated to Chern-Simons theory with group $SU(2)$. Given a 3-manifold M and a $SU(2)$-principal bundle P on it, let $\mathcal{A}$ be the space of connections on P. Witten considered the following partition function:

$$Z_r(M) = \int_{\mathcal{A}} \exp(2\pi i r \mathrm{CS}(\omega)) \mathrm{d}\mathcal{A}$$

where $r \geq 3$ is the *level*, $\mathrm{CS}(\omega) \in \mathbb{R}/\mathbb{Z}$ is the Chern-Simons functional of a connection ω and $d\mathcal{A}$ is the (yet undefined at a mathematical level of rigour) Feynmann measure on connections. Introducing a "Wilson line" i.e. a link $L = \sqcup_i L_i$ embedded in M whose components are colored by simple representations of $SU(2)$ (which may be indexed by their dimensions n_i) one may also extend the above partition function to:

$$Z_r(M, L) = \frac{\int_{\mathcal{A}} \exp(2\pi i r \mathrm{CS}(\omega)) \prod_i tr_{n_i}(\mathrm{hol}_{\mathrm{L_i}}(\omega)) \mathrm{d}\mathcal{A}}{Z_r(M)}$$

where tr_{n_i} is the trace in the n_i-dimensional representation and $\mathrm{hol}_{\mathrm{L_i}}(\omega)$ is the (conjugacy class of) the parallel transport of ω along the component L_i of L. As Witten pointed out, for physical renormalization matters the link should be framed. Moreover, one may extend the above "definition" of the partition function to general framed, colored trivalent graphs in M. The resulting invariants where rigourously defined by N. Reshetikin and V. Turaev ([**53**]) using the theory of quantum groups and can also defined in a combinatorial way using the evaluations at suitable roots of unity of the quantum spin-networks (defined in Chapter 2). Even if the physical definitions are still not rigourous the existence of a mathematical definition of the invariants and the combinatorial formulas allowing one to compute their values explicitly make their study very intriguing. In a sense we are able to compute certain Feynmann integrals even without knowing their direct definitions! The quantum field theory associated to the Chern-Simons functional is one of the most important examples of "toy models" for quantum gravity (and this was the initial motivation of Witten) as the action is purely topological and hence invariant under the diffeomorphism group of the manifold.

3. Mathematical problems on spin networks

Spin networks are rational functions $\langle \Gamma, \mathrm{col}\rangle_A$ of a parameter A called the *quantum parameter* and indexed by a framed trivalent graph embedded $\Gamma \subset S^3$ equipped with an integral coloring col of its edges; they are defined as contraction of tensors with values in representations of $U_q(sl_2)$ or via a combinatorial description (we will adopt here the combinatorial point of view, recalled in Chapter 2). In particular, when Γ has no vertices it is a link or a knot. Setting $A = -1$ one gets a number called *classical spin network*: we shall denote it $\langle \Gamma, \mathrm{col}\rangle$. Despite their rigourous definitions, the topological meaning of these objects is still unclear:

PROBLEM 1.1. *Understand the topological meaning of $\langle \Gamma, \mathrm{col}\rangle_A$, at least for planar Γ's.*

Let us try to make more precise the vague statement of the above problem and give an example of how one could attempt to treat it. In the classical case (i.e. $A = -1$) Bruce Westbury ([**64**]) showed that instead of focusing on the meaning of a single $\langle \Gamma, \mathrm{col}\rangle$ one should rather consider the "generating series": $Z(\Gamma) = \sum_{\mathrm{col}} \langle \Gamma, \mathrm{col}\rangle Y^{\mathrm{col}}$ where Y is a multivariable indexed by the edges of Γ. Indeed he was able to compute $Z(\Gamma)$ (up to a suitable renormalization of $\langle \Gamma, \mathrm{col}\rangle$) for planar Γ and showed hence that each $\langle \Gamma, \mathrm{col}\rangle$ is a suitable derivative of a single function associated to Γ whose topological meaning is clear (see Chapter 3 for details). His result was later extended to general Γ (but still in the classical case $A = -1$) by Garoufalidis and van der Veen ([**28**]). We consider this as an example of explanation of the topological meaning of classical spin networks. Recently, jointly with Julien Marché, we provided an extension of these results to the case of classical spin networks equipped with $SL_2(\mathbb{C})$-holonomies (Theorem 3.4). In Chapter 3, we detail our result (which also provides a new and independent proof of Westbury's results) and relate it to the Ising model on one side and to the geometry of $SL_2(\mathbb{C})$-character varieties on the other side. Motivated by Westbury's approach one can then formulate the problem:

PROBLEM 1.2 (Quantum Generating Series). *Compute $S_A(\Gamma) = \sum_{\mathrm{col}} \langle \Gamma, \mathrm{col}\rangle_A Y^{\mathrm{col}}$ where Γ is a graph and Y is a multi variable indexed by the edges of Γ.*

This problem is very difficult if Γ is a knot but some first evidences and results in this direction are presented in the last section of Chapter 3 when Γ is planar. Indeed in Examples 3.24 and 3.25 we show that for $\Gamma =$  and $\Gamma =$  one can still get a result "à la Westbury" computing a suitable non commutative generating series thus interpreting the values $\langle \Gamma, \mathrm{col}\rangle_A$ as "derivatives" of a single element of a non-commutative ring. It is our expectation that such an interpretation does still exist for all planar Γ: this could help clarifying the meaning of quantum spin networks at least for this class of graphs. The partial results in this direction presented in Chapter 3, Section 4 are based on an integrality result for quantum spin networks we proved in [**15**] (Theorem 3.15). In the same section we briefly discuss some of the other problems one might be able to treat by using the theorem and the techniques used to prove it: the problem of *categorification* of spin networks and that of finding an *integral orthogonal base* of the skein algebra of a punctured surface.

Besides the topological meaning, probably the most important open problems and questions in the study of TQFT's and in particular in the study of quantum spin networks are those concerning their "asymptotical behavior":

Problem 1.3. *Understand the behavior for large* $n \in \mathbb{N}$ *of the sequence of numbers obtained by evaluating* $\langle \Gamma, n \cdot \mathrm{col} \rangle_A$ *at* $A = A_n$*, where* $\{A_n\}_{n \in \mathbb{N}}$ *is a given sequence of complex numbers.*

From a physical point of view this study includes the "semi-classical" limit of the theory and should recover classical properties of systems modelled by the networks. From a mathematical point of view the problem of asymptotics has been given different names according to the setting in which the problem was formulated:

- Semi-classical limit of classical spin networks (Wigner's conjecture).
- Melvin-Morton-Rozansky conjecture (which is nowadays a theorem).
- The Volume Conjecture (and its generalizations).

(Another example of such conjectures is Witten's asymptotic expansion conjecture but we will not deal with it in the present work.)

In the study of semi-classical limit of classical spin networks $\{A_n\}_{n \in \mathbb{N}}$ is constantly -1, so one deals only with classical spin networks and tries to understand the asymptotical behavior of the sequence $\langle \Gamma, n \cdot \mathrm{col} \rangle$ (suitably normalized). A particular case of this problem was formulated by E. Wigner who conjectured that if $\Gamma =$  and if the coloring col provides the edge-lengths of a euclidean tetrahedron then the norm of the sequence should go like $(\frac{3}{2}\pi n^3 \mathrm{Vol})^{-\frac{1}{2}}$ times an oscillating correcting factor, where Vol is the volume of the euclidean tetrahedron whose edge lengths are col (a more precise statement of the conjecture will be provided in Chapter 4). There are nowadays many different independent proofs of this conjecture: by J. Roberts [**56**], by J.W. Barrett and C.M. Steele ([**4**]) and a more recent one, based on different techniques by S. Garoufalidis and R. van der Veen ([**28**]). For general Γ the problem is much harder as it is related to the problem of rigidity of euclidean polyhedra: in Chapter 4 we discuss a result (Theorem 4.4) obtained in collaboration with Julien Marché which reduces the study to some algebraic "non-degeneracy" conditions on col. Our result applies to any spin network Γ and in particular can be shown to recover the proof of Wigner's conjecture. In the last section of the chapter we relate the result to the study of classical problems of rigidity of euclidean polyhedra.

Both the Melvin-Morton-Rozansky and the Volume Conjecture (respectively MMR and VC from now on) deal with the case when Γ is a framed knot K. It is customary to state them in terms of the n^{th}*-colored Jones polynomial* of K defined as $J_n(K) = \langle K, n \rangle_A$ and its *normalized* version $J'_n(K) = \frac{J_n(K)}{(-1)^{n+1}[n]}$ (it turns out that $J'_n(K) \in \mathbb{Z}[A^{\pm 1}]$).

The MMR conjecture deals with the evaluations of $J'_n(K)$ at $A_n = exp(\frac{i\pi\alpha}{2n})$ for some fixed $\alpha \in \mathbb{C}$; it was proved by D. Bar Natan and S. Garoufalidis ([**7**]). We state it in a stronger form proved by S. Garoufalidis and T. Q. Le ([**23**]), keeping the name "conjecture" just for the sake of continuity with the nomenclature used in the litterature:

Conjecture 1.4 (Melvin-Morton-Rozansky). *There exists a neighborhood $\mathcal{U}$ of $0 \in \mathbb{C}$ such that $\forall \alpha \in \mathcal{U}$ it holds:*

$$\lim_{n\to\infty} J'_n(K)|_{A=\exp(\frac{i\pi\alpha}{2n})} = \Delta(e^{i\pi\alpha})^{-1} \tag{1}$$

where $\Delta(x)$ is the Alexander polynomial of K normalized so that $\Delta(x^{-1}) = \Delta(x)$ and $\Delta(1) = 1$.

The Volume Conjecture is due to R. Kashaev who defined ([**33**]) an invariant $\mathcal{K}_n(K) \in \mathbb{C}$ which was later shown by H. Murakami and J. Murakami to be equal to the value at $A_n = \exp(\frac{i\pi}{2n})$ of $J'_n(K)$. It states :

Conjecture 1.5 (Volume Conjecture).

$$\lim_{n\to\infty} \frac{2\pi}{n} \log(||J'_n(K)|_{A=\exp(\frac{i\pi}{2n})}||) = \mathrm{Vol}(S^3 \setminus K) \tag{2}$$

where $\mathrm{Vol}(S^3 \setminus K)$ is the sum of the hyperbolic volumes of the hyperbolic pieces in the JSJ decomposition of $S^3 \setminus K$.

The VC in the above form has been formally checked for the Figure Eight knot ([**43**]), the Borromean link ([**24**]), for torus knots ([**40**], [**45**]), their Whitehead doubles ([**68**]) and for the family of "Whitehead chains" ([**62**]). Moreover, there is experimental evidence of its validity for the knots 6_3, 8_9 and 8_{20} ([**46**]). In [**27**], S. Gukov proposed the following extension of the conjecture (and proved it for the figure-eight knot):

Conjecture 1.6 (Generalized Volume Conjecture). *Let $\mathcal{I} = (\mathbb{R} \setminus Q) \cup \{0\}$ and K be a hyperbolic knot; then there exists a neighborhood $\mathcal{U}_K$ of 0 in $\mathcal{I}$ such that $\forall \alpha \in \mathcal{U}_K$ it holds:*

$$\lim_{n\to\infty} \frac{2\pi(1+\alpha)}{n} \log(||J'_n(K)|_{A=\exp(\frac{i\pi(1+\alpha)}{2n})}||) = \mathrm{Vol}_\alpha(S^3 \setminus K) \tag{3}$$

where $\mathrm{Vol}_\alpha(S^3 \setminus K)$ is the volume of the (non-complete) hyperbolic structure on $S^3 \setminus K$ such that the holonomy of the structure around the meridian of K is conjugated to the diagonal matrix with eigenvalues $e^{\pm i\pi\alpha}$.

A different and more general extension has also been proposed by H. Murakami ([**47**]) which agrees with above one but also allows complex values of α.

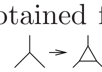
In [**16**], we proposed an extension of the VC to include the case of links in S^3 or in connected sums of copies of $S^2 \times S^1$ and proved it for the infinite family of "fundamental shadow links" (see Theorem 5.11). This is the first proof of the conjecture for an infinite family of hyperbolic links (and knots); moreover these links were already studied in [**17**] because of their remarkable geometric properties; in particular they are *universal* in the sense that every pair (M^3, L) where M^3 is a three-manifold and $L \subset M^3$ is a link can be obtained by an integral Dehn-surgery on some components of a fundamental shadow link (see Theorem 5.10). This feature may allow one to attack the volume conjecture (for

$M^3 = S^3$ and L a knot K) by computing $J_n(K)$ as a weighted sum of the invariants of the fundamental links and then using our asymptotical analysis of the invariants of the links: the difficulties of this approach will be discussed in Chapter 5.

The proof of the Volume Conjecture for fundamental shadow links was based on our computation of the asymptotical behavior of $6j$-symbols of $U_q(sl_2)$. To state it, let us extend the definition of the initial problem of asymptotics to:

Problem 1.7. *Given a sequence $\{\mathrm{col}_n\}_{n\in\mathbb{N}}$ of colorings of Γ with values in $\mathbb{N}$ such that the limit $\lim_{n\to\infty} \frac{\mathrm{col}_n}{n} = \mathrm{col}_\infty$ exists (col_∞ is then a coloring with positive real values) and a sequence $\{A_n\}_{n\in\mathbb{N}}$ of complex numbers, understand the behavior for large $n \in \mathbb{N}$ of the sequence of numbers $\langle \Gamma, \mathrm{col}_n\rangle_{A_n}$ obtained by evaluating $\langle \Gamma, \mathrm{col}_n\rangle_A$ at A_n.*

(We stress here that since $\langle \Gamma, \mathrm{col}\rangle_A$ is a rational function it may have poles at A_n: the notion of "evaluation" is then to be understood as in Definition 5.1.)

In our result we computed the sequence $\langle$$, \mathrm{col}_n\rangle_{A_n}$ at $A_n = \exp(\frac{i\pi}{2n})$ with col_n being n on all the edges (so that $\mathrm{col}_\infty = 1$ on all the edges) and showed that it grows exponentially fast with a growth rate proportional to the volume of a hyperbolic ideal regular octahedron. Later (see Theorem 5.3), we computed the asymptotical behavior of $\langle$$, \mathrm{col}_n\rangle_{A_n}$ at $A_n = \exp(\frac{i\pi}{2n})$ for a full open range of values for the "limit coloring" col_∞ which we called "hyperbolic regime" ([**11**]): such an analysis shows that when the colors satisfy certain inequalities corresponding directly to the inequalities satisfied by the angles of a hyperbolic tetrahedron, the growth of $6j$-symbols is exponential with growth rate given by the volume of the tetrahedron (the regular ideal octahedron being a special case of this). This result may be considered as a "proof of the Volume Conjecture for the tetrahedron"; indeed we formulate a volume conjecture for planar trivalent graphs (Conjecture 5.5) and using the above analysis of the asymptotical behavior of $6j$-symbols we prove it (Theorem 5.6) for all the graphs obtained from the tetrahedron by iterating the operation replacing a vertex by a triangle: . All these results will be stated formally and detailed in Chapter 5.

A promising technique to attack the Volume Conjecture is the use of the so-called *nilpotent spin networks* $\langle \Gamma, \mathrm{col}\rangle_r^{Nil}$: the quantum invariants of framed graphs $\Gamma \subset S^3$ whose edges are colored by a special kind of simple representations of $U_q(sl_2)$ when $q = \exp(i\pi/r)$ is a root of unity (those where $E^r = F^r = 0$ and called *nilpotent* here below), which are parametrized by non integer complex numbers. In [**14**], jointly with Jun Murakami we defined these objects (Theorem 6.1) and provided a "shadow-state sum" model for it (Theorem 6.2). (These invariants were also defined independently by N. Geer and B. Patureau-Mirand in [**25**].) In our work we also showed that if Γ is a knot K then $\mathcal{K}_r(\Gamma) = \lim_{\alpha\to 0}\langle K, \alpha\rangle_r^{Nil}$ where $\mathcal{K}_r(K)$ is the Kashaev invariant of the knot ([**14**], Remark 3.7). Moreover we studied a version of Problem 1.2 for the nilpotent case by proving:

$$\lim_{r\to\infty} \frac{\pi}{2r} \log\left(\langle\right.$$$$, \mathrm{col}\rangle_r^{Nil}\langle$$$$, -\mathrm{col}\rangle_r^{Nil}\left.\right) = \mathrm{Vol}(Tet)$$

where Tet is a hyperbolic truncated tetrahedron whose internal dihedral angles are given by $\pi\mathrm{col}_\infty$ (see Theorem 6.3 for a precise statement). These evidences, exposed in Chapter 6, seem to support the idea that nilpotent spin networks may be a useful tool to tackle the

Volume Conjecture. In the same chapter we also discuss the construction of the analogue of Reshetikhin-Turaev invariants based on nilpotent spin networks: in collaboration with Nathan Geer and Bertrand Patureau-Mirand we obtained invariants $\mathsf{N}(M, T, \omega)$ of three-manifolds M with some graphs $T \subset M$ and a cohomology class $\omega \in H^1(M \setminus T; \mathbb{C}/2\mathbb{Z})$ (see Theorem 6.5). It is our hope that these invariants will allow to extend the Volume Conjecture to knots in general manifolds and maybe to general, empty, manifolds.

4. Structure of the memoire

All the preliminary definitions were concentrated in Chapter 2: experts may skip it or use it as a reference for notations.

Chapter 3 is devoted to the study of generating series of classical and quantum spin networks. There we first discuss the classical case and Theorem 3.4 we proved jointly with J. Marché. In Chapter 3, Section 4 we discuss our integrality result of quantum spin networks and its consequences and potential applications. Then we pass to the study of asymptotics of spin networks and discuss in three separate chapters the classical, the quantum and the nilpotent cases. Using nilpotent networks we also define invariants "à la Reshetikhin-Turaev" of manifolds possibly containing links (or graphs) and equipped with a suitable cohomology class. We summarize the chapters of the memoire by a table:

	Our main results	Bibliographical references
Chapter 3: generating series	Theorems 3.4, 3.5, and 3.15, Examples 3.24 and 3.25	[**13**], [**15**]
Chapter 4: asymptotics of classical spin networks	Theorems 4.3 and 4.4	[**13**]
Chapter 5: asymptotics of quantum spin networks	Theorems 5.3, 5.6, 5.11 and 5.10, Proposition 5.4	[**11**],[**16**],[**17**]
Chapter 6: nilpotent spin networks	Theorems 6.1, 6.2, 6.3, 6.5 and 6.7	[**14**], [**12**]

The first section of each chapter recalls the context of the research on the chapter's topic. The last section (and in Chapter 3 also Section 3) discusses open problems and future directions of research by asking a list of questions and stating open problems. Moreover, we structured the sections so that all the essential informations are contained in the beginning of the section and all the subsections are accessory and contain examples, sketches of proofs, remarks or connections with other problems: the reader may decide to skip all the subsections.

The only exception to the above comments is Chapter 6 whose three sections are meant to detail the three aspects of our research on nilpotent spin networks: the definition and computation of $\langle \Gamma, \mathrm{col} \rangle_r^{Nil}$, their asymptotical behavior and their use to build 3 manifold invariants.

CHAPTER 2

Classical and quantum spin networks, basic definitions and facts

For the sake of self-containedness we recall in this chapter the basic definitions of classical and quantum spin network and some of the key facts we will use later on.

1. Basic definitions and notation

Definition 2.1 (KTG). An *Abstract Trivalent Graph* (ATG) is a finite trivalent graph Γ equipped with a "framing", i.e. the germ of an orientable smooth surface Σ such that Σ retracts on Γ. A *Knotted Trivalent Graph* (KTG) is an ATG embedded in S^3, i.e. a finite trivalent graph $\Gamma \subset S^3$ equipped with a "framing", i.e. the germ of an orientable smooth surface $\Sigma \subset S^3$ such that Σ retracts on Γ. In all the work we will denote by:

- V the set of *vertices* of Γ; $\#V = 2n$ for some $n \in \mathbb{N}$. To specify the edges (or half-edges) touching a vertex v we will write: $v : (g, h, k)$.
- E the set of *edges* of Γ; $\#E = 3n$. To specify the ends of an unoriented edge we will write $e : i \leftrightarrow j$; if e is oriented we will write it as $e : i \to j$.
- H the set of *half-edges* i.e. $H = \{(e, v) | e \in E,\ v \in V \text{ such that } v \in e\}$; $\#H = 6n$. To specify the half-edges g, h contained in an edge e we will write $e : g \leftrightarrow h$ and if e is oriented: $e : g \to h$.
- A the set of *angles* i.e. $A = \{(e, f) | e, f \in E \text{ such that } e \cap f \neq \emptyset, e \neq f\}$; $\#A = 6n$. We will denote angles by greek letters and to specify the two half edges g, h forming an angle α we will write $\alpha : g \leftrightarrow h$; if α is oriented we will use: $\alpha : g \to h$.

Remark 2.2. Note that a ATG is equivalent to a "fat graph" but a KTG is not as we require Γ to be embedded in S^3.

Remark 2.3. We allow Γ to contain some "circle component" which we will alternatively consider as formed either by a single edge of Euler characteristic 0 or by a vertex and an edge (and hence one angle and two half-edges). The cardinalities indicated above are correct only for the union of the components of Γ which contain trivalent vertices.

In order to specify a framing Σ on a KTG or ATG Γ we will only specify (via thin lines as in Figure 1) the edges around which it twists with respect to the blackboard framing in a diagram of Γ, implicitly assuming that Σ will be lying horizontally (i.e. parallel to the blackboard) around Γ out of these twists (in the case of an ATG only the topology of the surface Σ is relevant and not the embedding chosen to draw it). Let us also remark that if D is a diagram of Γ there is a framing Σ_D (called the *blackboard framing*) induced on Γ

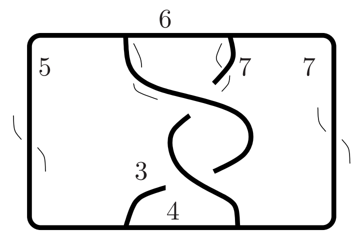


FIGURE 1. An admissibly colored KTG containing 4 vertices and 6 edges.

simply by considering a surface containing Γ and lying almost parallel to the projection plane. Pulling back the orientation of $\mathbb{R}^2$ shows that Σ_D is orientable. The following converse is easily proved:

LEMMA 2.4. *If Γ is a KTG and Σ is the framing on Γ then there exists a diagram D of Γ such that $\Sigma_D = \Sigma$.*

DEFINITION 2.5 (Coloring). A *coloring* of Γ (see Figure 1 for an example) is a map $\mathrm{col} : E \to \mathbb{N}$ (whose values are called *colors*) such that $\forall v \in V$ the following conditions are satisfied:

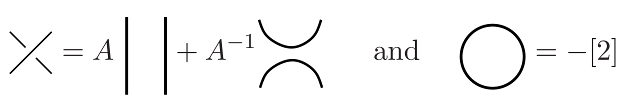
(1) $a_v + b_v + c_v \in 2\mathbb{N}$
(2) $a_v + b_v \geq c_v,\ b_v + c_v \geq a_v,\ c_v + a_v \geq b_v$

where a_v, b_v, c_v are the colors of the edges touching v. We extend an admissible to a map $\mathrm{col} : E \cup A \to \mathbb{N}$ by defining $\mathrm{col}(\alpha) = b_v + c_v - a_v$ if α is the angle formed in v by the edges colored by b_v and c_v.

2. Skein algebras of surfaces

Let us fix a complex number $A \neq 0$, let $A^2 = q$ and let $\{n\} = -\sqrt{-1}(A^{2n} - A^{-2n})$. The *quantum integer* $[n]$ is defined as $[n] \equiv \frac{\{n\}}{\{1\}}$ and the *quantum factorial* as $[n]! \equiv \prod_{j=1}^{n}[j],\ [0]! = 1$. Let also $\begin{bmatrix} n \\ k \end{bmatrix} \equiv \frac{\{n\}!}{\{k\}!\{n-k\}!} \in \mathbb{Z}[A^{\pm 1}]$.

DEFINITION 2.6 (Skein module of an oriented 3-manifold). Let N be an oriented compact three-manifold (possibly with boundary) and let $\mathcal{S}(N)$ be the quotient of the free $\mathbb{Z}[A^{\pm 1}]$-module generated by framed (possibly empty) links in N modulo the ideal generated by the relations induced by isotopy and the following well-known two "Kauffman rules":



where the drawings are meant to represent strands of the links in an embedded ball in N and the framing are horizontal in the drawings. Let $\mathbb{Q}\mathbb{Z}[A^{\pm 1}]$ be the localization of $\mathbb{Z}[A^{\pm 1}]$ by the subset S formed by finite products of quantum integers. Let us also define $\mathcal{S}_{\mathbb{Q}}(N)$ as $\mathcal{S}_{\mathbb{Q}}(N) = \mathbb{Q}\mathbb{Z}[A^{\pm 1}] \otimes_{\mathbb{Z}[A^{\pm 1}]} \mathcal{S}(N)$. The elements of $\mathcal{S}(N)$ are referred to as *skeins*.

REMARK 2.7. If $i : N \hookrightarrow M$ is an embedding then there is an induced map of $\mathbb{Z}[A^{\pm 1}]$-modules $i_* : \mathcal{S}(N) \to \mathcal{S}(M)$.

If $N = S^3$ then $\mathcal{S}(N) = \mathbb{Z}[A^{\pm 1}]$ and the value of a framed link is what is usually called its *Kauffann brackets*, which is essentially identical to its *Jones polynomial*, up to a change of variable and a renormalization. Using Lemma 2.17 it is not difficult to check that for each $k \geq 0$ it holds $\mathcal{S}_{\mathbb{Q}}(\#_k S^2 \times S^1) = \mathbb{QZ}[A, A^{-1}]$. If $N = \Sigma \times [-1, 1]$ then $\mathcal{S}(N)$ can be equipped with the structure of an associative (non commutative) ring (the multiplication being vertical stacking induced by the embedding of two copies of $\Sigma \times [-1, 1]$ one in $\Sigma \times [-1, 0]$ and the other in $\Sigma \times [0, 1]$). We will denote from now on this ring by $\mathcal{S}(\Sigma)$ (or $\mathcal{S}_{\mathbb{Q}}(\Sigma)$ for the ring build analogously from $\mathcal{S}_{\mathbb{Q}}(\Sigma)$). It was proven by Turaev ([**60**]) that $\mathcal{S}(\Sigma)$ is a quantization of Goldman's Poisson algebra.

Given an ATG Γ let $D(\Gamma)$ be the closed orientable three manifold obtained by gluing a solid torus to each boundary component of $\Sigma \times S^1$ so that the meridian is $\{pt\} \times S^1$; it is easy to check that topologically it is $\#_k S^2 \times S^1$ where $k = 1 - \chi(\Gamma)$ ($\chi(\Gamma)$ being the Euler characteristic of Γ). There is a natural embedding $i : \Sigma \hookrightarrow D(\Gamma)$ and using the fact that $\mathcal{S}_{\mathbb{Q}}(\#_k S^2 \times S^1) = \mathbb{QZ}[A, A^{-1}]$ C. Frohman and J. Kania-Bartonszynska ([**20**]) defined by this embedding:

DEFINITION 2.8 (Yang-Mills trace). The Yang-Mills trace is $YM : \mathcal{S}_{\mathbb{Q}}(\Sigma) \to \mathbb{QZ}[A^{\pm 1}]$. It is a trace on the non commutative algebra $\mathcal{S}_{\mathbb{Q}}(\Sigma)$, i.e. $YM(xy) = YM(yx)$, $\forall x, y \in \mathcal{S}_{\mathbb{Q}}(\Sigma)$.

The proof of the statement in the definition is straightforward: the vertical stacking of two skeins in $\Sigma \times [-1, 1]$ can be permuted in $D(\Gamma)$ by isotoping one of the two skeins through the S^1 part. The orientation reversing self-homeomorhism of $\Sigma \times [-1, 1]$ sending $(p, t) \to (p, -t)$ induces an involution on $\bar{\cdot} : \mathcal{S}_{\mathbb{Q}}(\Sigma) \to \mathcal{S}_{\mathbb{Q}}(\Sigma)$. This together with the above trace allows to build a "A-hermitian" product with values in $\mathbb{QZ}[A^{\pm 1}]$ on $\mathcal{S}_{\mathbb{Q}}(\Sigma)$ by $(x, y) \doteq YM(x\overline{y})$ such that $(f(A)x, y) = f(A)(x, y)$ and $(x, f(A)y) = f(A^{-1})(x, y)$, $\forall x, y \in \mathcal{S}_{\mathbb{Q}}(\Sigma)$, $\forall f \in \mathbb{QZ}[A^{\pm 1}]$. In particular if one evaluates in $A \in \mathbb{C}$ with $||A|| = 1$ then the above becomes a standard hermitian product.

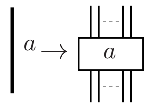
3. Quantum spin networks

Given an ATG Γ and a coloring col on it, we will now recall a standard procedure to associate to (Γ, col) an element $S(\Gamma, \mathrm{col}) \in \mathcal{S}_{\mathbb{Q}}(\Sigma)$ which we will call the *abstract spin network* associated to (Γ, col). Let us first define the Jones-Wenzl projectors $JW_a \in \mathbb{QZ}[A^{\pm 1}][B(a)]$ (the group ring of the braid group on a strands tensored with $\mathbb{QZ}[A^{\pm 1}]$):

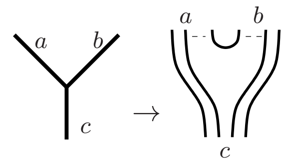
$$JW_a = \boxed{a} \doteq \sum_{\sigma \in \mathfrak{S}_a} \frac{A^{-a(a-1)+3T(\sigma)}}{[a]!} \boxed{\hat{\sigma}} \tag{4}$$

where $\hat{\sigma}$ is the positive braid containing the minimal number ($T(\sigma)$) of crossings and inducing the permutation σ on its endpoints (it is a standard fact that such braid is well defined). Actually JW_a is defined as an element of the Temperley-Lieb algebra, but for the purpose of this presentation we will just consider it as a formal sum of braids. One defines $S(\Gamma, \mathrm{col})$ by the following algorithm:

(1) Cable each edge e of Γ by $JW_{\mathrm{col}(e)}$, i.e., in $\Sigma \times [-1,1]$ replace an edge e colored by a by a formal sum of braids in $B(a)$ according to the above definition of JW_a:



(2) Around each vertex of Γ, connect the (yet free) endpoints of the so-obtained strands in the unique planar way (i.e. contained in Σ) without self returns:



(3) This way one associates to (Γ, col) a formal sum with coefficients $c_i \in \mathbb{Q}(A)$ of links L_i contained in a small neighborhood of Σ in $\Sigma\times[-1,1]$ and therefore framed by annuli running parallel to it. Define $S(\Gamma,\mathrm{col}) = \sum_i c_i L_i \in \mathcal{S}_{\mathbb{Q}}(\Sigma \times [-1,1])$.

Definition 2.9. We will call the skein $S(\Gamma,\mathrm{col}) \in \mathcal{S}_{\mathbb{Q}}(\Sigma)$ defined above the *abstract spin network* associated to (Γ,col).

Definition 2.10 (Quantum spin networks). If Γ is a KTG then the inclusion of Σ in S^3 induces a $\mathbb{QZ}[A^{\pm 1}]$-module map $i_* : \mathcal{S}_{\mathbb{Q}}(\Sigma) \to \mathcal{S}_{\mathbb{Q}}(S^3) = \mathbb{QZ}[A^{\pm1}] \subset \mathbb{Q}(A)$: the *quantum spin network* associated to (Γ,col) is the rational function $\langle \Gamma,\mathrm{col}\rangle_A = i_*(S(\Gamma,\mathrm{col}))$.

Theorem 2.11 (Kauffman, [**36**]). *$\langle\Gamma,\mathrm{col}\rangle_A$ is an invariant up to isotopy of (Γ,col).*

In particular if Γ is a framed link then $\langle\Gamma,\mathrm{col}\rangle_A$ is also known as the *colored Jones polynomial* of Γ (up to a suitable change of variable and renormalization).

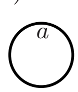
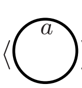
Definition 2.12 (Classical spin networks). If Γ is an ATG then the *classical spin network* $\langle\Gamma,\mathrm{col}\rangle \in \mathbb{Q}$ associated to (Γ,col) is the evaluation at $A=-1$ of the rational function $\langle\Gamma,\mathrm{col}\rangle_A$ where $\langle\Gamma,\mathrm{col}\rangle_A$ is obtained by using any arbitrary embedding of Γ in S^3.

Remark 2.13. Definition 2.12 makes sense because of the following two reasons:

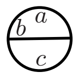
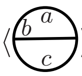
(1) all the denominators appearing in the definition of $L(\Gamma,\mathrm{col})$ are of the form $[a]!$ for some $a\in\mathbb{N}$ and thus are not zero at $A=-1$;
(2) any two embeddings of Γ in S^3 as a KTG are related by isotopies and a finite number of crossing switches. The former transformations do not change $\langle\Gamma,\mathrm{col}\rangle_A$ by Kauffman's theorem and the latter don't change it by Kauffman's rules at $A=-1$.

4. Examples and properties

Example 2.14 (Unknot).

(5)  $\equiv \langle$  $\rangle_A = (-1)^{2a}[2a+1]$

Example 2.15 (Theta graph).

(6)  $\equiv \langle$  $\rangle_A = (-1)^{\frac{a+b+c}{2}}[\frac{a+b+c+2}{2}]!\frac{[\frac{a+b-c}{2}]![\frac{b+c-a}{2}]![\frac{c+a-b}{2}]!}{[a]![b]![c]!}$

Example 2.16 (The tetrahedron or symmetric $6j$-symbol).

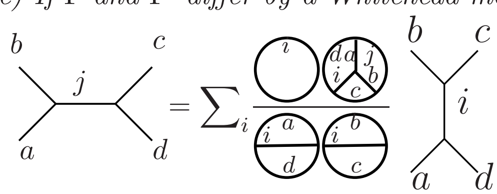
(7)

 $= \frac{\prod_{i=1}^{3}\prod_{j=1}^{4}[Q_i - T_j]!}{[a]![b]![c]![d]![e]![f]!}\sum_{z=MaxT_i}^{z=MinQ_j}(-1)^z \left[\begin{array}{c} z+1 \\ z-T_1, z-T_2, z-T_3, z-T_4, Q_1-z, Q_2-z, Q_3-z, 1 \end{array} \right]$

where $T_1 = \frac{a+b+c}{2}$, $T_2 = \frac{a+e+f}{2}$, $T_3 = \frac{d+b+f}{2}$, $T_4 = \frac{d+e+c}{2}$, $Q_1 = \frac{a+b+d+e}{2}$, $Q_2 = \frac{a+c+d+f}{2}$, $Q_3 = \frac{b+c+e+f}{2}$.

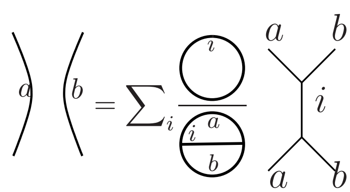
Lemma 2.17 (Properties of quantum spin networks, [**36**],[**48**]). *The following are some of the properties of* $\langle \Gamma, \mathrm{col}\rangle_A$ *:*

1. *(Erasing 0-colored strand) If Γ' is obtained from (Γ, col) by deleting a 0-colored edge, then $\langle \Gamma', \mathrm{col}'\rangle_A = \langle \Gamma, \mathrm{col}\rangle_A$.*
2. *(Half-Twisting) Adding k half-twists to the framing of Γ around an edge colored by a multiplies $\langle \Gamma, \mathrm{col}\rangle_A$ by $\sqrt{-1}^{ak}A^{\frac{k}{2}(a^2+2a)}$.*
3. *(Whitehead move) If Γ and Γ' differ by a Whitehead move then:*

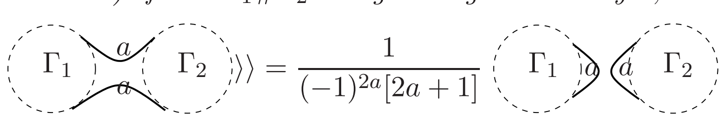


   where i ranges over the all the admissible values.

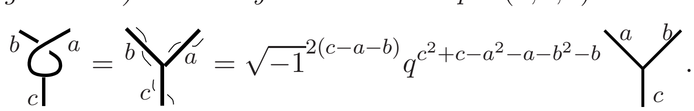
4. *(Fusion rule) In particular, applying the preceding formula to the case $j = 0$ one has:*



5. *(Connected sum) If $\Gamma = \Gamma_1 \# \Gamma_2$ along an edge colored by a, then*



6. *(Twisting vertices) For every admissible 3-uple (a, b, c) it holds:*



Actually all the above properties hold in $\mathcal{S}_{\mathbb{Q}}(N^3)$ for any compact, oriented 3-manifold N^3, where the skein represented by a colored graph $i : (\Gamma, \mathrm{col}) \hookrightarrow N^3$ is $i_*(S(\Gamma, \mathrm{col}))$.

5. Classical spin networks with holonomies

The notion of classical spin network has a natural generalization in the gauge theoretical setting based on the use of connections:

Definition 2.18 (Connections). A *connection* on a graph Γ is a map $\psi : \overrightarrow{E} \to SL_2(\mathbb{C})$ (where $\overrightarrow{E} = \{\text{oriented edges of } \Gamma\}$) such that $\psi(\overrightarrow{e}) = \psi(\overleftarrow{e})^{-1}, \ \forall \overrightarrow{e} \in \overrightarrow{E}$.

Let $\mathcal{S}_{-1}(\Sigma)$ be the evaluation at $A = -1$ of $\mathcal{S}_{\mathbb{Q}}(\Sigma)$ i.e. $\mathcal{S}_{-1}(\Sigma) = \mathbb{Q} \otimes_{\mathbb{Q}\mathbb{Z}[A^{\pm 1}]} \mathcal{S}_{\mathbb{Q}}(\Sigma)$ where $\mathbb{Q}$ is a $\mathbb{Q}\mathbb{Z}[A^{\pm 1}]$-module via the evaluation map at $A = -1$. It was proved by Przyticki that $\mathcal{S}_{\mathbb{Q}}(\Sigma)$ is a free $\mathbb{Q}\mathbb{Z}[A^{\pm}]$-module with a basis formed by the (possibly empty) multicurves in Σ whose components are all essential; hence $\mathcal{S}_{-1}(\Sigma)$ is a free $\mathbb{Q}$-vector space generated by the same set.

A connection ψ on Γ provides a function $\mathrm{hol}_\psi : \mathcal{S}_{-1}(\Sigma) \to \mathbb{C}$ by the following construction. Given a connected essential curve $c \subset \Sigma$, fix an orientation of c and an initial point $p_0 \in c$ and let $\mathrm{hol}_\psi(\mathrm{c}) = -\mathrm{tr}\big(\prod_{\overrightarrow{e} \subset \mathrm{c}} \psi(\overrightarrow{e})\big)$ where the product is ordered according to the edges met while following the orientation of c and starting from p_0; if $c = \cup c_i$ then let $\mathrm{hol}_\psi(\mathrm{c}) = \prod \mathrm{hol}_\psi(\mathrm{c_i})$. Since $tr(A) = tr(A^{-1}), \ \forall A \in SL_2(\mathbb{C})$ the so-obtained value is well defined; extend this definition of $\mathrm{hol}_\psi(\cdot)$ by linearity to the whole $\mathcal{S}_{-1}(\Sigma)$.

Definition 2.19 (Equivalence of connections). Two connections ψ_1, ψ_2 are *equivalent* if $\mathrm{hol}_{\psi_1} = \mathrm{hol}_{\psi_2}$. If they take values in $SU(2)$ this is equivalent to the existence of $g : V \to SU(2)$ such that

$$\psi_1(\overrightarrow{e}) = g(f(\overrightarrow{e}))\psi_2(\overrightarrow{e})g(i(\overrightarrow{e}))^{-1}, \ \forall \overrightarrow{e} \in \overrightarrow{E}$$

where $f(\overrightarrow{e})$ and $i(\overrightarrow{e})$ are respectively the final and initial point of $\overrightarrow{e}$. A *holonomy* on Γ is an equivalence class of connections; the *trivial holonomy* is the class of $\psi(\overrightarrow{e}) = \mathrm{Id}, \ \forall \overrightarrow{e}$.

Definition 2.20 (Classical spin networks with holonomies). Let Γ be equipped with a connection ψ and col be an admissible coloring on Γ. The *classical spin network with holonomy* $\langle \Gamma, \psi, \mathrm{col}\rangle \in \mathbb{C}$ is $\langle \Gamma, \psi, \mathrm{col}\rangle = \mathrm{hol}_{[\psi]}(\mathrm{S}(\Gamma, \mathrm{col})|_{\mathrm{A}=-1})$ where $S(\Gamma, \mathrm{col}) \in \mathcal{S}_{\mathbb{Q}}(\Sigma)$ was defined in 2.9 and $S(\Gamma, \mathrm{col})|_{A=-1}$ is its image in $\mathcal{S}_{-1}(\Sigma)$. In particular $\langle \Gamma, [\mathrm{Id}], \mathrm{col}\rangle = \langle \Gamma, \mathrm{col}\rangle \in \mathbb{Q}$.

5.1. Classical spin networks as contractions of tensors. In this subsection we sketch how to compute $\langle \Gamma, \psi, \mathrm{col}\rangle$ as a contraction of tensors valued in representations of $SL_2(\mathbb{C})$. Pick an embedding of Γ in $\mathbb{R}^3$ such that all the vertices look like $\bigcup\!\!\!\!\mid$ and are placed at different horizontal coordinates. We compute $\langle \Gamma, \psi, \mathrm{col}\rangle$ by assigning morphisms of representations of $SL_2(\mathbb{C})$ at each elementary tangle: we first sketch how to do this when Γ is a link and $\mathrm{col} = 1$ on all its components. Let $V^a = \{P(z, w) | deg(P) = a\}$ be the $a+1$-dimensional irreducible representation of $SL_2(\mathbb{C})$ (the action of a matrix being $A \cdot P(z, w) \doteqdot P(A^{-1}(z, w)))$ and in particular let $V = V^1$. The vector $\cup = \sqrt{-1}(z \otimes w - w \otimes z) \in V \otimes V$ is the only (up to scalar) invariant vector. Similarly, the operator $\cap : V \otimes V \to V^0 = \mathbb{C}$ defined by $\cap(z \otimes w) = \sqrt{-1}$, $\cap(w \otimes z) = -\sqrt{-1}$, $\cap(z \otimes z) = \cap(w \otimes w) = 0$ is the unique (up to scalar) projector from $V \otimes V$ to V^0. One checks that $\bigcirc : V^0 \to V^0$ is the multiplication by -2 and that $(\mathrm{Id} \otimes \cap) \circ (\cup \otimes \mathrm{Id}) = \mathrm{Id} = (\cap \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \cup)$ (planar isotopy invariance). Moreover in order for the Kauffman relation to be satisfied with $A = -1$ (classical case) one needs: $\diagup\!\!\!\!\diagdown(x \otimes y) = -y \otimes x,\ \forall x, y \in V$. This suggests to consider V as an odd vector space. Using the above operators and the invariance of $\cup$ and $\cap$ under the action of $SL_2(\mathbb{C})$ one can move any coupon containing $SL_2(\mathbb{C})$ matrices to a single vertical strand and take their product, thus reducing the coupons to just one per component of Γ. Using the Kauffman relation it is then easy to check that the tensor contraction build by the above tensors is $\langle \Gamma, \psi, \mathrm{col} \rangle$.

When Γ is a graph, using the Jones-Wenzl projectors one can reinterpret the above tensor contraction as one valued in polynomials (thus simplifying the computations); let us sketch how. Having remarked that V can be considered as an odd vector space of polynomials of degree 1 in z, w, then the Jones-Wenzl projectors JW_a (see Equation (4) at $A = -1$) become standard antisimmetrizers of $V^{\otimes a}$ in the super sense, i.e. projectors on the symmetric polynomials $V^a = \{P(z, w) | deg(P) = a\}$ (in the non-super sense). Thus up to replacing each JW_a by $(JW_a)^2$ (they are projectors) and distributing these projectors on the strands near the maxima, the minima and the vertices of the fixed diagram of Γ, one can compute $\langle \Gamma, \mathrm{col} \rangle$ as a contraction of tensors with values in tensor products of V^a's. The tensor product $V^{a_1} \otimes \cdots \otimes V^{a_k}$ can be seen as the space of polynomials in variables $z_1, w_1, z_2, w_2, \ldots, z_k, w_k$ with degree a_j with respect to z_j, w_j. With this convention in mind, it is not difficult to check that if a, b, c are an admissible three-uple then the invariant vector in $V^a \otimes V^b \otimes V^c$ associated by the above construction to the diagram $\cup\!\!\!|$ whose strands are colored by a, b, c and whose half-edges are $(g, h, k) \in H^3$ respectively, is the polynomial:

$$\omega_{a,b,c} = \sqrt{-1}^{a+b+c}(z_g w_h - z_h w_g)^{\frac{a+b-c}{2}}(z_g w_k - z_k w_g)^{\frac{a+c-b}{2}}(z_h w_k - z_k w_h)^{\frac{b+c-a}{2}}.$$

Similarly the operator associated to $\cap_a : V^a \otimes V^a \to V^0 = \mathbb{C}$ is $\cap_a = \frac{\sqrt{-1}^a}{a!^2}(\frac{\partial}{\partial z_1}\frac{\partial}{\partial w_2} - \frac{\partial}{\partial z_2}\frac{\partial}{\partial w_1})^a$ (where z_1, w_2 are the variables representing the polynomials on the left strand and z_2, w_2 those on the right strand). The crossings operate as $\diagup\!\!\!\!\diagdown_{a\,b}(x \otimes y) = (-1)^{ab} y \otimes x,\ \forall x \in V^a,\ y \in V^b$ Finally the role of the holonomy on an edge $e \in E$ is just to act on the polynomials $V^{\mathrm{col}(e)}$ via $V^{\mathrm{col}(e)} \ni P(z, w) \to P(\psi_e^{-1}(z, w))$.

CHAPTER 3

Generating series of spin networks

1. Generating series of classical spin networks: context

Let Γ be an ATG and let:

$$Z(\Gamma) = \sum_{\mathrm{col}} \langle \Gamma, \mathrm{col} \rangle \frac{\prod_{e \in E} \mathrm{col}(e)!}{\prod_{\alpha \in A} \mathrm{col}(\alpha)!} \prod_{e \in E} Y_e^{\mathrm{col}(e)} \in \mathbb{C}[[\{Y_e\}_{e \in E}]] \tag{8}$$

where the sum is taken over all the colorings of Γ. One says that Γ is *planar* if it can be embedded in $\mathbb{R}^2$ so that the cyclic ordering of the edges around its vertices coincides with that induced by the standard orientation of $\mathbb{R}^2$. In 1996, Bruce Westbury proved the following:

Theorem 3.1 (Westbury, [**64**]). *Let $P_\Gamma = \sum_{c \in \Gamma} \prod_{e \in c} Y_e$ where c runs over all the curves (possibly empty or disconnected) embedded in Γ. If Γ is planar then $Z(\Gamma) = P_\Gamma^{-2}$.*

Example 3.2.

$$Z(\bigcirc) = (1+Y)^{-2}$$

$$Z(\ominus) = (1 + Y_1Y_2 + Y_1Y_3 + Y_2Y_3)^{-2}$$

The above result was then extended by S. Garoufalidis and R. van der Veen ([**28**]) to the case of non-planar Γ.

Surprisingly enough, it was never acknowledged in the literature that Westbury's result provides a bridge between the theory of classical spin networks and the study of the Ising model (probably the most studied statistical model): let us do this here below. Let us first rapidly recall what the Ising model is. Let G be a finite graph with vertex set $V(G)$ and edge set $E(G)$. A *spin configuration* on G is a map $\sigma : V(G) \to \{-1, +1\}$. Any positive edge weight system $\{J_e\}_{e \in E(G)}$ determines a probablity measure on the set $\Omega(G)$ of spin configurations: by

$$P(\sigma) = \frac{1}{Z^J(G)} \exp\Big(\sum_{e: u \leftrightarrow v} J_e \sigma_u \sigma_v\Big)$$

where

$$Z^J(G) = \sum_{\sigma \in \Omega(G)} \exp\Big(\sum_{e: u \leftrightarrow v \in E(G)} J_e \sigma_u \sigma_v\Big)$$

is the partition function of the *Ising model* on G with coupling constants $\{J_e\}$. As observed by van der Waerden, the identity $\exp(J_e\sigma_u\sigma_v) = \cosh(J_e)(1+\tanh(J_e)\sigma_u\sigma_v)$ allows to re-express the partition function as:

$$(9)\quad Z^J(G) = \Big(\prod_{e\in E(G)}\cosh(J_e)\Big)\sum_{\sigma\in\Omega(G)}\prod_{e:u\leftrightarrow v}(1+\tanh(J_e)\sigma_u\sigma_v) = \Big(\prod_{e\in E(G)}\cosh(J_e)\Big)2^{\#V(G)}\sum_{\gamma\in\mathcal{E}(G)}\prod_{e\in\gamma}\tanh(J_e)$$

where $\mathcal{E}(G)$ is the set of *even subgraphs* de G i.e. subgraphs γ such that every vertex of G is adjacent to an even number of edges of γ. To solve the Ising model is to compute explicitly the function $Z^J(G)$ for some G.

To go back to the connection with spin networks, let us put $G = \Gamma$. Then $\mathcal{E}(G)$ is the set of curves contained in Γ and, setting $\tanh(J_e) = Y_e$ one recovers:

$$\frac{Z^J(\Gamma)}{2^{\#2n}\prod_{e\in E}\cosh(J_e)} = P_\Gamma$$

so up to a renormalization factor $(2^{\#2n}\prod_{e\in E}\cosh(J_e))$ the partition function of the Ising model on Γ coincides with Westbury's polynomial and thus is the inverse of the square-root of the generating series $Z(\Gamma,\mathrm{Id})$.

This relation opens up many natural and deep questions which we will comment in Chapter 3 Section 3, after exposing our main contributions on the study of generating series of spin networks.

2. A generalization of Westbury's theorem to spin networks with holonomies

In this section we present a recent result obtained jointly with Julien Marché which extends Westbury's theorem to the case of general Γ possibly equipped with non-trivial holonomy ψ. Let:

$$(10)\qquad Z(\Gamma,\psi) = \sum_{\mathrm{col}}\langle\Gamma,[\psi],\mathrm{col}\rangle\frac{\prod_{e\in E}\mathrm{col}(e)!}{\prod_{\alpha\in A}\mathrm{col}(\alpha)!}\prod_{e\in E}Y_e^{\mathrm{col}(e)} \in \mathbb{C}[[\{Y_e\}_{e\in E}]]$$

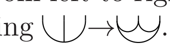
Our result computes explicitly $Z(\Gamma,\psi)$ and, when $\psi = \mathrm{Id}$, recovers Westbury's theorem through a new approach based also on the theory of dimer configurations.

Let us first fix the notation to state our result: fix an auxiliary embedding of Γ in S^3 and a diagram of it in $\mathbb{R}^2$ such that all the vertices look like and are placed at different horizontal coordinates; moreover for each crossing x in the diagram of Γ we fix a rectangular box containing x whose bases are parallel to the coordinate axes and such that the strands forming x are the diagonals of the box. Orient all the edges from left to right in the drawing; this allows to define a *winding number* $w(e) \in \frac{\mathbb{Z}}{2}$ for each edge; for instance $w(\cap) = -\frac{1}{2}$. Let also:

$$(11)\qquad W(\Gamma,\psi) = \sum_{\mathrm{col}}\langle\Gamma,[\psi],\mathrm{col}\rangle\frac{\prod_{e\in E}\mathrm{col}(e)!}{\prod_{\alpha\in A}\mathrm{col}(\alpha)!}\prod_{e\in E}Y_e^{\mathrm{col}(e)}\prod_{x\in\text{crossings}}(-1)^{\mathrm{col}(l(x))\mathrm{col}(r(x))} \in \mathbb{C}[[\{Y_e\}_{e\in E}]]$$

where for each crossing x we let $l(x)$ be the strand running from the lower-left corner of the box containing x to the upper right corner and $r(x)$ be the other strand forming x.

REMARK 3.3. The value of $W(\Gamma,\psi)$ depends on the chosen embedding because of the presence of the factor $\prod_x (-1)^{l(x)r(x)}$. Clearly $Z(\Gamma,\psi)$ does not depend on any choice.

Enumerate the vertices of Γ from left to right and let Γ' be the graph obtained from Γ by blowing up vertices, i.e. replacing . Vertices of Γ' are in 1-to-1 correspondence with the set H of half-edges of Γ and the holonomy may be seen as a map $\psi : H \to \mathrm{SL}_2(\mathbb{C})$ by stipulating that, if h is a half-edge contained in an edge e then $\psi(h) = \psi(e)$ if h contains the initial point of e and $\psi(h) = \mathrm{Id}$ otherwise. Then, we set $F_h = \mathbb{R}^2$ for every half-edge and for any pair of half-edges g, h, we define $b_{g,h} : F_g \times F_h \to \mathbb{C}$ by the formula $b_{g,h}(z_g, w_g, z_h, w_h) = z_g w_h - z_h w_g$.

We define the following $\mathbb{C}[[\{Y_e^{\pm\frac{1}{2}}\}_{e\in E}]]$-valued quadratic form on $\bigoplus_{h\in H} F_h$:

$$\mathcal{Q}(x) = 2\sum_{\alpha:g\to h} i b_{g,h}(\psi_g^{-1}x_g, \psi_h^{-1}x_h)\sqrt{Y_{e(g)}Y_{e(h)}} + 2\sum_{e:g\to h} i^{-2w(e)} b_{g,h}(x_g, x_h).$$

In the above expression $\alpha : g \to h$ means that α is an angle between the half-edges g and h such that vertex of Γ' corresponding to g is at the left of that corresponding to h, and $e(\cdot) : H \to E$ is the map assigning to each half-edge the edge it is contained in. In the same way, $e : g \to h$ means that the edge e contains the half-edges g and h and g is the "left half-edge" (i.e. g contains the initial point of e) while h is the "right half edge" of e. Using Gaussian integrals, we prove the following result:

THEOREM 3.4 (Generating series of spin networks with holonomy, Costantino-Marché [**13**]). *Let Γ be a graph with holonomy $\psi : H \to \mathrm{SL}_2(\mathbb{C})$, and let $\mathcal{Q}$ be constructed as above. Then $\det(\mathcal{Q})$ is a polynomial in $\{Y_e\}_{e\in E}$, its evalution at $\{Y_e = 0\ \forall e\}$ is 1 and the following formula holds: $W(\Gamma,\psi) = \det(\mathcal{Q})^{-1/2}$. In particular if Γ is planar then $Z(\Gamma,\psi) = \det(\mathcal{Q})^{-1/2}$.*

In the above result, the determinant is computed in the canonical basis of $\oplus_{h\in H} F_h$. We remark that strictly speaking, the formula makes sense only for holonomies in $\mathrm{SL}_2(\mathbb{R})$ because of the indeterminacy in the square root. By analytic continuation, the formula holds in general.

In the case when ψ takes values in the group of diagonal matrices $\mathrm{D}\subset\mathrm{SL}_2(\mathbb{R})$, we provided a combinatorial interpretation of our formula which extends Westbury's formula: let t be a map $t : H \to \mathbb{R}^*$ such that for all h, one has $\psi_h = \begin{pmatrix} t_h & 0 \\ 0 & t_h^{-1}\end{pmatrix}$ in the basis (z_h, w_h) and let $C(\Gamma)$ be the set of all oriented curves immersed in Γ which pass over an edge of Γ either $0, 1$ or 2 times, in the latter case with opposite orientations (and without "self returns"). Given $\gamma \in C(\Gamma)$ we denote by $\mathrm{cr}(\gamma)$ the number of crossings modulo 2 of the corresponding immersion.

Let $Y^\gamma = \prod_{e\subset\gamma} Y_e$ (hence each edge may appear 0, 1 or 2 times) and $tr(\gamma) = \prod_{h\subset\gamma} t_h^{\epsilon(\gamma,h)}$ ($\epsilon(\gamma,h)$ being 1 if γ crosses h in the positive direction, -1 otherwise). In the following result we generalized Westbury's result to the case of holonomies with values in diagonal matrices:

THEOREM 3.5 ([**13**], Theorem 3.1). *Let* Γ *be a planar graph equipped with an abelian holonomy* ψ*. Then:*

$$Z(\Gamma,\psi) = \Big(\sum_{\gamma\in C(\Gamma)} (-1)^{\mathrm{cr}(\gamma)} tr(\gamma) Y^{\gamma}\Big)^{-1}$$

The non-planar case of Theorem 3.4 (and Theorem 3.5), can be easily treated as follows. One needs to fix the sign $\prod_x (-1)^{l(x)r(x)}$ in $W(\Gamma,\psi)$; to do this use the identity

$$(-1)^{ab} = \frac{1}{2}(1+(-1)^a+(-1)^b-(-1)^{a+b}),\ \forall a,b\in\mathbb{Z}.$$

More explicitly, for each edge e let $\mathrm{Op}_e : \mathbb{C}[[\{Y_e\}_{e\in E}]] \to \mathbb{C}[[\{Y_e\}_{e\in E}]]$ be the automorphism that switches the sign of Y_e. Then, for a crossing x between edges e_1 and e_2 define $S_x : \mathbb{C}[[\{Y_e\}_{e\in E}]] \to \mathbb{C}[[\{Y_e\}_{e\in E}]]$ as $S_x = \frac{1}{2}(id + \mathrm{Op}_{e_1} + \mathrm{Op}_{e_2} - \mathrm{Op}_{e_1}\circ \mathrm{Op}_{e_2})$. Then we recover $Z(\Gamma,\psi)$ as

$$Z(\Gamma,\psi) = S_{x_1}\circ\cdots\circ S_{x_k}(W(\Gamma,\psi)).$$

In particular, when ψ is trivial this recovers Garoufalidis and Van der Veen's extension to non-planar graphs of Westbury's theorem [**28**].

2.1. Recovering Westbury's theorem: dimers and the Ising model. Comparing the statement of Theorem 3.4 and that of Westbury's result (3.1) one sees that in the former, $Z(\Gamma,\mathrm{Id})$ is the inverse of a square-root as opposed to the inverse of a square. To match the two, one can observe that writing the bilinear form $\mathcal{Q}$ of Theorem 3.4 as a matrix in the basis $z_1,\dots,z_{6n},w_1,\dots w_{6n}$ (where the indices correspond to the half-edges of Γ ordered from left to right in the fixed diagram of Γ), when $\psi=\mathrm{Id}$ is planar, one gets:

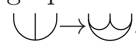
$$\mathcal{Q} = \begin{pmatrix} 0 & M^t \\ M & 0 \end{pmatrix}$$

where M is antisimmetric, thus $\det(\mathcal{Q}) = \det(M)^2$ (because the number of columns of M is $6n$, where $n=\frac{\#V}{2}\in\mathbb{N}$). Moreover since $M=-M^t$ then $\det(M)=\mathrm{Pf(M)}^2$, thus one is left to prove that $\mathrm{Pf(M)} = \mathrm{P}_\Gamma$ where P_Γ is provided in Theorem 3.1.

Thanks to Kasteleyn's works, we know that the Pf(M) counts *dimer configurations* on a graph whose vertices are in $1-1$ correspondence with the columns of M and whose oriented edges have weights given by $w(e:i\to j)=M_{i,j}$:

DEFINITION 3.6 (Dimer configurations). Given a finite graph G a *dimer configuration* on G is a subset $\mathcal{C}$ of {edges of G} such that every vertex of G is contained in exactly one edge in $\mathcal{C}$. If the edges are equipped with weights $w(e)\in R$ in an abelian ring R, the weight $w(\mathcal{C})$ of $\mathcal{C}$ is $\prod_{e\in\mathcal{C}} w(e)$. The generating series of dimer configurations is $\sum_{\mathcal{C}} w(\mathcal{C})$.

THEOREM 3.7 (Kasteleyn, [**35**]). *Given a graph* G *and weights* $w(e:i\to j)=M_{i,j}$ *such that* $M_{i,j}=-M_{j,i}$ *it holds:* $\mathrm{Pf(M)} = \sum_{\mathcal{C}} \pm \mathrm{w}(\mathcal{C})$*. The signs can be fixed if* G *is equipped with a so-called* Kasteleyn orientation *of the edges.*

We apply the above theorem to the case when G is the graph Γ' obtained from Γ by applying the following local transformation to each vertex: . The vertices of Γ' are indexed by the half-edges of Γ and the oriented edges are equipped with weights: $w(g \to h) = M_{g,h}$ where M was given above. The idea to recover Westbury's theorem is to set $\psi = \mathrm{Id}$ and apply Kasteleyn's theorem to Γ' (whose edges are equipped by the entries $M_{i,j}$), and to observe that there is a bijection $\mathrm{bij} : \{\text{dimer configurations on } \Gamma'\} \to \{\text{curves in } \Gamma\}$ such that the weight $w(d)$ of a dimer configuration d on Γ' is equal to the $\prod_{e\in \mathrm{bij(d)}} Y_e$. We refer to [**13**] for further details.

2.2. Examples.

2.2.1. *The unknot.* Let $\psi \in SL_2(\mathbb{C})$ be a fixed matrix and let λ, λ^{-1} be its eigenvalues. Then $Z(\bigcirc, \psi) = (1+\lambda Y)^{-1}(1+\lambda^{-1}Y)^{-1}$. This can be checked directly by using the standard diagram of the unknot (here we consider $\Gamma = \bigcirc$ as a graph with one 2-valent vertex and one edge composed of two half-edges), and writing: $\mathcal{Q} = 2(\sqrt{-1}(\lambda^{-1}z_1w_2 - \lambda z_2 w_1)Y + \sqrt{-1}(z_1w_2 - z_2w_1))$ which, in the basis z_1, z_2, w_1, w_2 is represented by the matrix:

$$Q = \sqrt{-1}\begin{pmatrix} 0 & 0 & 0 & 1+\lambda^{-1}Y \\ 0 & 0 & -(1+\lambda Y) & 0 \\ 0 & -(1+\lambda Y) & 0 & 0 \\ 1+\lambda^{-1}Y & 0 & 0 & 0 \end{pmatrix}$$

2.2.2. *The abelian theta-graph.* Let $\Gamma = \ominus$ and let $\psi_i = \begin{pmatrix} t_i & 0 \\ 0 & t_i^{-1}\end{pmatrix}$, $i = 1,2,3$ be the values of the holonomies of the edges of Γ. Using Theorem 3.5 one can check that, letting $T_i = \frac{t_j}{t_k} + \frac{t_k}{t_j}$ (with $i \neq j \neq k$) then $Z(\Gamma, \psi)^{-1} = 1 + T_1Y_2Y_3 + T_2Y_1Y_3 + T_3Y_1Y_2 + T_1Y_1^2Y_2Y_3 + T_2Y_2^2Y_3Y_1 + T_3Y_3^2Y_1Y_2 + Y_1^2Y_2^2 + Y_2^2Y_3^2 + Y_1^2Y_3^2$, which for $t_i = 1$ (and hence $T_i = 2$) equals Westbury's formula: $(1 + Y_1Y_2 + Y_1Y_2 + Y_3Y_2)^2$. Of course one could also use Theorem 3.4, in which case one would compute the determinant of a matrix $\mathcal{Q} = \begin{pmatrix} 0 & M^t \\ M & 0 \end{pmatrix}$ where M is:

$$M = \sqrt{-1}\begin{pmatrix} 0 & -\sqrt{Y_1Y_2} & -\sqrt{Y_1Y_3} & 0 & 0 & -1 \\ \sqrt{Y_1Y_2} & 0 & -\sqrt{Y_2Y_3} & 0 & -1 & 0 \\ \sqrt{Y_1Y_3} & \sqrt{Y_2Y_3} & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\sqrt{Y_3Y_2} & -\sqrt{Y_3Y_1} \\ 0 & 1 & 0 & \sqrt{Y_3Y_2} & 0 & -\sqrt{Y_2Y_1} \\ 1 & 0 & 0 & \sqrt{Y_3Y_1} & \sqrt{Y_2Y_1} & 0 \end{pmatrix}$$

2.3. Sketch of the proof of Theorem 3.4. Using the ideas and the notation sketched in Subsection 5.1 we can compute $\langle \Gamma, \psi, \mathrm{col}\rangle$ as a differential operator applied to a polynomial in variables $z_1, \dots z_{6n}, w_1, \dots w_{6n}$, where z_i, w_i are associated to the i^{th} half-edge of Γ (the numbering being induced by the embedding chosen at the beginning). Suppose for the moment that ψ is trivial. Using the machinery introduced in Subsection 5.1, one can remark that:

$$W(\Gamma,\mathrm{col}) = \sum_{\mathrm{col}}\Big(\prod_{e:g\to h}\mathrm{col}(e)!\cap_{\mathrm{col}(e)}\Big)\circ\Big(\prod_{v\in V}\frac{\omega_{a_v,b_v,c_v}}{(a_v+b_v-c_v)!(b_v+c_v-a_v)!(c_v+b_v-a_v)!}\Big)\prod_{e\in E}Y_e^{\mathrm{col}(e)}$$

where for every vertex we let a_v, b_v, c_v be the colors of the three strands surrounding it (from left to right). But, after a little manipulation, the latter can be also re-expressed as the evaluation in $\{z_h = w_h = 0,\ \forall h\}$ of the function obtained by applying the differential operator

$$\exp(\frac{P^{op}}{2}) = \exp(\sum_{e:g\to h}\sqrt{-1}^{wind(e)}(\frac{\partial}{\partial z_g}\frac{\partial}{\partial w_h} - \frac{\partial}{\partial z_h}\frac{\partial}{\partial w_g}))$$

to the function

$$\exp(\frac{Q}{2}) = \exp\Big(\sum_{v\in V,\ v:(h,g,k)}\sqrt{-1}((z_g w_h - z_h w_g)\sqrt{Y_gY_h} + (z_g w_k - z_k w_g)\sqrt{Y_gY_k} + (z_h w_k - z_k w_h)\sqrt{Y_kY_h})\Big). \tag{12}$$

Thus to evaluate $Z(\Gamma,\mathrm{Id}) = \big(\exp(\frac{P^{op}}{2})\circ\exp(\frac{Q}{2})\big)|_0$ one remarks that both P^{op} and Q have degree 2 (they are elements respectively of $S^2(\oplus F_h)$ and of $S^2(\oplus F_h^*)$) and uses the following is an ubiquitous generalization of the Gaussian integration formula:

Proposition 3.8 (Fourier transforms of Gaussian functions). *Let V be a real vector space equipped with a basis, $P\in S^2(V^*)$ and let $Q\in S^2(V^*)$ be a non-degenerate $\mathbb{R}$-bilinear form; let $Q^{-1}\in S^2(V)$ be its inverse (i.e. represented in the dual basis by the inverse of the matrix representing Q). Then using Lebesgue integration in the given basis:*

$$\int_V P(x)\exp(-\frac{1}{2}Q(x))\mathrm{d}\mu = \frac{(2\pi)^{n/2}}{\sqrt{\det(Q,\mu)}}\big(P^{op}\exp(\frac{1}{2}Q^{-1})\big)|_0 \tag{13}$$

where P^{op} is P seen as a differential operator on $C^\infty(V^*,\mathbb{C})$.

Consider a deformation Q_ϵ of Q which is non-degenerate and has positive real part. For concreteness, we can pick $Q_0 = \sum_{h\in H}(z_h^2 + w_h^2)$ and set $Q_\epsilon = Q + \epsilon Q_0$. We define $Z_\epsilon = \big(\exp(\frac{1}{2}P)^{op}\exp(\frac{1}{2}Q_\epsilon)\big)|_0$ so that $Z(\Gamma,1) = \lim_{\epsilon\to 0} Z_\epsilon$. Then replacing Q by Q_ϵ^{-1} and $\frac{1}{2}P$ by $\exp(\frac{1}{2}P)$, Formula (13) gives

$$Z_\epsilon = (2\pi)^{-n/2}\det(Q_\epsilon^{-1})^{1/2}\int_{F^*}\exp(\frac{1}{2}P(x) - \frac{1}{2}Q_\epsilon^{-1}(x))\mathrm{d}x.$$

We apply now gaussian integration, noting that the quadratic form $Q_\epsilon^{-1} - P$ is still non-degenerate and has positive real part. Hence, we have

$$Z_\epsilon^2 = \frac{\det(Q_\epsilon^{-1})}{\det(Q_\epsilon^{-1} - P)} = \det(Q_0 - Q_\epsilon P)^{-1}$$

Letting ϵ go to 0, we find that $Z(\Gamma,1) = \det(Q_0 - QP)^{-1/2}$.

Suppose now that ψ is represented by a discrete connection on Γ with values in $SL_2(\mathbb{R})$. To adapt the construction: for each angle α connecting two half-edges g and h, we need to replace $b_{g,h}: F_g\times F_h\to\mathbb{C}$ by $b_{g,h}(\psi_g^{-1}x_g, \psi_h^{-1}x_h)$. We denote by Q_ψ the resulting quadratic form. By the assumption that ψ lives in $\mathrm{SL}_2(\mathbb{R})$, P takes again only imaginary values. Hence, the argument above repeats exactly and we obtain $Z(\Gamma,\psi)=\det(Q_0-Q_\psi P)^{-1/2}$. The general case, that is, for ψ taking values in $\mathrm{SL}_2(\mathbb{C})$ follows by analytic continuation.

One can simplify this formula by remarking that the matrix of P in the canonical basis satisfy $P^{-1}=-P$ and moreover, we have $\det(P)=1$. We obtain the formula of Theorem 3.4: $Z(\Gamma,\psi)=\det(\mathcal{Q})^{-1/2}$.

3. Perspectives on generating series of classical spin networks

In this section we discuss some of the many questions left open in the study of generating series of classical spin networks: as it will be clear most of them tend to relate spin-networks to geometry and to statistical models.

Let us first remark that given Γ and a coloring col on it, the values $\langle\Gamma,\cdot,\mathrm{col}\rangle$ form a function from the set of equivalence classes of connections to $\mathbb{C}$. By the very definition of the notion of holonomy (Definition 2.19) such a class is a point in the $SL_2(\mathbb{C})$-character variety of Σ:

$$\chi_\Sigma^{SL_2(\mathbb{C})}=\mathrm{Hom}(\pi_1(\Sigma),SL_2(\mathbb{C}))//SL_2(\mathbb{C}).$$

Hence it is very natural to see spin-networks as maps $\langle\Gamma,\cdot,\mathrm{col}\rangle:\chi_\Sigma^{SL_2(\mathbb{C})}\to\mathbb{C}$. Moreover, let us define $\chi_\Sigma^{SU(2)}=\mathrm{Hom}(\pi_1(\Sigma),SU(2))/SU(2)$; it is not difficult to check that actually $\chi_\Sigma^{SU(2)}\subset\chi_\Sigma^{SL_2(\mathbb{C})}$ (two $SU(2)$ representations which are conjugated in $SL_2(\mathbb{C})$ are also conjugated in $SU(2)$), that $\chi_\Sigma^{SU(2)}$ is compact (it is homeomorphic to the quotient of $SU(2)^{\#E}$ by the action of $SU(2)^{\#V}$ acting as explained in Definition 2.19), and that $\chi_\Sigma^{SU(2)}$ is the real part of the involution on $\chi_\Sigma^{SL_2(\mathbb{C})}$ which sends a morphism ρ to $(\rho^*)^{-1}$. Equipping $\chi_\Sigma^{SU(2)}$ with the measure induced by the Haar measure on $SU(2)^{\#E}$, one can prove, using Peter-Weyl's theorem, the following:

Theorem 3.9 ([**13**], Section 4.2). *The set of functions $\{\langle\Gamma,\cdot,\mathrm{col}\rangle\}$ where* col *ranges over all the admissible colorings of* Γ*, forms an orthogonal Hilbert basis of $L^2(\chi_\Sigma^{SU(2)},\mathbb{C})$. Moreover*

$$||\langle\Gamma,\cdot,\mathrm{col}\rangle||^2=|\prod_{v\in V}\underset{c_v}{\overset{a_v}{\ominus_{b_v}}}\prod_{e\in E}\left(\bigcirc^{\mathrm{col}(e)}\right)^{-1}|$$

Thus, reinterpreting Theorem 3.4 in view of the above theorem one sees that for character varieties of orientable surfaces with boundary one can pick Hilbert bases of $\chi_\Sigma^{SU(2)}$ naturally associated to spines of Σ (our initial Γ) *sum them up and get a nice function of additional formal variables indexed by the edges of the spine.* This phenomenon is quite surprising as in general, given a Hilbert basis of $L^2(X,\mathbb{C})$ for some space X one is not able to sum explicitly the generating series of the basis.

Example 3.10. To better seize the above peculiar fact let us apply Theorem 3.4 to the case of $\Gamma=\bigcirc$, when $Z(\Gamma,\psi)=(1+\lambda Y)^{-1}(1+\lambda^{-1}Y)^{-1}$ where λ,λ^{-1} are the eigenvalues of ψ (see Subsection 2.2). Letting $x=\lambda+\lambda^{-1}$ then one has $Z(\Gamma,\psi)=(1+xY+Y^2)^{-1}$: in view of the above theorem we are computing the generating series of the trace functions on the conjugacy classes of $SL_2(\mathbb{C})$ i.e. the generating series of Tchebischev polynomials.

So a natural question (but vague) to ask is:

QUESTION 3.11. *What is the geometric meaning (if any) of the formal variables Y_e used to build the generating series? And what is the meaning of $Z(\Gamma, \cdot)$ itself?*

In all the above discussion Γ was kept fixed, but it is well known that any two spines Γ and Γ' of a surface Σ can be connected by a finite sequence of Whitehead moves. Indeed any spine Γ of Σ is dual to an ideal triangulation $T(\Gamma)$ and any two such triangulations, thanks to a theorem independently proved by Penner and Harer, are connected by flips (which are dual to Whitehead moves). So another natural question to ask is:

QUESTION 3.12. *How does $Z(\Gamma, \cdot)$ change after a flip?*

Moreover there exist other, more geometrically meaningful, bases of functions of $L^2(\chi_\Sigma^{SL_2(\mathbb{C})}, \mathbb{C})$: namely the Chekhov-Fock coordinates (and the monomials in these coordinates) based on the triangulation $T(\Gamma)$ (extended holomorphically from the Teichmuller space to the whole $\chi_\Sigma^{SL_2(\mathbb{C})}$). So it is natural to ask:

QUESTION 3.13. *What is the change of base from the spin networks to the set of monomials in the Chekhov-Fock coordinates?*

All these questions are susceptible of relating the theory of spin networks to the study of geometrical properties of Teichmuller spaces.

Moreover, the relation we pointed out between the Ising model and the generating series of spin networks are proven only when ψ is the trivial connection:

QUESTION 3.14. *What is the statistical model generalizing the Ising model whose partition function is the inverse of the square root of $Z(\Gamma, \psi)$ for general ψ? Is it phisically meaningful? What is the physical quantity it is modeling?*

4. Quantum spin networks: integrality, categorification and generating series

Until now, in this chapter we have focused on classical spin networks. In this last section we discuss the more mysterious quantum case by examining three problems related to an integrality result we obtained in [**15**].

In the litterature different kind of renormalizations of the invariant $\langle \Gamma, \mathrm{col} \rangle \in \mathbb{Q}(A)$ have been used; of these let us mention:

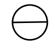
- The *skein* (or "Kauffman bracket") normalization: $\langle \Gamma, \mathrm{col} \rangle_A$, of Definition 2.10.
- The *integral* renormalization:

$$\langle \Gamma, \mathrm{col} \rangle_A^{\mathrm{Int}} = \langle \Gamma, \mathrm{col} \rangle_A \frac{\prod_{e \in E}([2\mathrm{col}(e)]!)^{\chi(e)}}{\prod_{\alpha \in A}[\mathrm{col}(\alpha)]!} \tag{14}$$

  where $\chi(e)$ is 0 if e is a closed (circle) component and 1 otherwise.
- The *unitary* renormalization:

$$\langle \Gamma, \mathrm{col} \rangle_A^U = \langle \Gamma, \mathrm{col} \rangle_A \prod_{v \in V} \sqrt{-1}^{a_v+b_v+c_v} \sqrt{\frac{[a_v+b_v-c_v]![b_v+c_v-a_v]![c_v+a_v-b_v]!}{[a_v+b_v+c_v+1]!}}$$

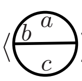
  where we use the notation of Definition 2.5.

In general, when Γ is not a link, neither $\langle \Gamma, \mathrm{col}\rangle$ nor $\langle \Gamma, \mathrm{col}, \rangle_A^U$ is a Laurent polynomial: this can easily be checked already for the case of $\Gamma =$  (the so-called *θ-graph*). On contrast, the following was proved in [**15**] for the case of the integral normalization:

THEOREM 3.15 (Integrality of the renormalized Kauffman brackets). *There exist $m, n \in \mathbb{Z}$ such that $\langle \Gamma, \mathrm{col}\rangle_A^{\mathrm{Int}} \in (\sqrt{-1})^m A^n \mathbb{Z}[A^2, A^{-2}]$.*

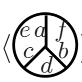
It is worth remarking that the normalization $\langle \cdot, \rangle_A^{\mathrm{Int}}$ depends only on the abstract colored graph (Γ, col) and not on its embedding and that the rational function used to renormalize is a q-hypergeometric balanced function: it is a ratio of quantum factorials such that the sums of the arguments of the factorials in the numerator and in the denominator are equal. The integrality of the following examples is manifest from the integrality of quantum binomials:

EXAMPLE 3.16 (Theta graph).

(15) $\langle$  $\rangle_A^{\mathrm{Int}} = (-1)^{a+b+c}[a+b+c+1]\begin{bmatrix} a+b+c \\ a+b-c, b+c-a, c+a-b \end{bmatrix}$

EXAMPLE 3.17 (The tetrahedron or symmetric $6j$-symbol).

(16)

$\langle$  $\rangle_A^{\mathrm{Int}} = \sum_{z=MaxT_i}^{z=MinQ_j} (-1)^z \begin{bmatrix} z+1 \\ z-T_1, z-T_2, z-T_3, z-T_4, Q_1-z, Q_2-z, Q_3-z, 1 \end{bmatrix}$

where $T_1 = a+b+c,\ T_2 = a+e+f,\ T_3 = d+b+f,\ T_4 = d+e+c,\ Q_1 = a+b+d+e,\ Q_2 = a+c+d+f,\ Q_3 = b+c+e+f$.

In our view, Theorem 3.15 may be relevant to three main problems detailed below:

(1) Search for integral bases for skein algebras;
(2) Categorification of quantum spin networks;
(3) Study of generating series of quantum spin networks.

Integral bases for skein algebras. The study of the skein algebra of a surface and the action of the mapping class group on it is an important, still developing subject. It is well known that for a fixed Γ, letting Σ be the orientable surface formed by its framing, the skeins $S(\Gamma, \mathrm{col}) \in \mathcal{S}_{\mathbb{Q}}(\Sigma)$ (see Definition 2.9) form a basis of $\mathcal{S}_{\mathbb{Q}}(\Sigma)$ when col ranges over all the colorings on Γ. This basis has good properties with respect to the other natural one formed by multicurves: all its elements are mutually orthogonal with respect to the bilinear product induced by the Yang-Mills measure on Σ and the structure constants of the algebra are easy to compute in this basis; on contrast both statements are false for the base of multicurves. But this is known at present only in $\mathcal{S}_{\mathbb{Q}}(\Sigma)$ and not in $\mathcal{S}(\Sigma)$. Multicurves clearly form a basis also of the latter algebra but in general $S(\Gamma, \mathrm{col}) \notin \mathcal{S}(\Sigma)$. So defining $S^{\mathrm{Int}}(\Gamma, \mathrm{col}) \equiv S(\Gamma, \mathrm{col}) \frac{\prod_{e\in E}([2\mathrm{col}(e)]!)^{\chi(e)}}{\prod_{\alpha\in A}[\mathrm{col}(\alpha)]!}$, in view of Theorem 3.15 it is natural to ask:

QUESTION 3.18. *Is it true that $S^{\mathrm{Int}}(\Gamma, \mathrm{col}) \in \mathcal{S}(\Sigma)$? Do they form a basis of $\mathcal{S}(\Sigma)$ as $\mathbb{Z}[A^{\pm 1}]$-module?*

Theorem 3.15 gives a strong evidence for a positive answer to the first question: indeed it implies that given an embedding $i : \Gamma \to S^3$, the image $i_*(S^{\mathrm{Int}}(\Gamma, \mathrm{col}))$ is integral (i.e. it is in $\mathbb{Z}[A^{\pm 1}]$), *and this holds for any embedding i.*

Categorification of quantum spin networks. Roughly speaking to categorify a polynomial invariant of knotted objects in S^3 is to find an algebraic complex whose Poincaré polynomial provides the invariant and whose homology is an invariant of the object. Such a complex was build first by M. Khovanov for the Jones polynomials of knots. The construction of the categorification of quantum spin networks strictly includes that case: it was recently solved by I. Frenkel, C. Stroppel and J. Sussan ([**22**]) who build an infinite complex whose Poincaré polynomial is $\langle \Gamma, \mathrm{col}\rangle_A$. Unfortunately, since in general $\langle \Gamma, \mathrm{col}\rangle_A$ is not a Laurent polynomial, the complex is necessarily going to be infinite dimensional and the computation of its homology is hard. Theorem 3.15 may provide a path to a finite dimensional categorification of quantum spin networks: since $\langle \Gamma, \mathrm{col}\rangle_A^{\mathrm{Int}}$ is a Laurent polynomial one may hope to find a finite dimensional complex categorifying it.

QUESTION 3.19. *Can one categorify $\langle \Gamma, \mathrm{col}\rangle_A^{\mathrm{Int}}$ by a finite dimensional complex?*

In the proof of Theorem 3.15 one shows that the morphisms associated to a decomposition of Γ into "elementary pieces" (maxima, minima, crossings and vertices) are integral: it is our hope that this may be used to answer the above question by categorifying the elementary morphisms one at a time. This would mean to find suitable categories associated to colored points (most probably, according to the ideas of [**21**], categories of modules over cohomology algebras of suitable flag manifolds) and bimodules associated to each elementary piece such that tensoring with the bimodules gives "elementary functors" whose induced maps at the level of the Grothendieck groups of the categories is the map associated to the elementary piece. This program is most probably very ambitious in the presence of crossings, but may turn out to be manageable in the case of planar graphs:

PROBLEM 3.20. *Find a categorification of the planar elementary pieces such that the functors obtained by composing the elementary functors are invariant under planar isotopies.*

Study of generating series of quantum spin networks. In terms of generating series of spin networks, observe that, already in the classical case (when $A = -1$), one has:

$$Z(\Gamma, \mathrm{Id}) = \sum_{\mathrm{col}} \langle \Gamma, \mathrm{col}\rangle_{A=-1}^{\mathrm{Int}} \prod_{e \in E} Y_e^{\mathrm{col}(e)}$$

that is: the integral normalization is exactly that used in Westbury's result. Letting $R = \mathbb{C}[A^{\pm 1}]$, it is therefore natural to ask:

QUESTION 3.21. *Working in the completion of the commutative ring $R[\{Y_e\}_{e\in E}]$ with respect to the degree filtration, what is the value of $Z_A(\Gamma) = \sum_{\mathrm{col}} \langle \Gamma, \mathrm{col}\rangle_A^{\mathrm{Int}} \prod_{e\in E} Y_e^{\mathrm{col}(e)}$?*

EXAMPLE 3.22. $Z_A(\bigcirc) = (1 + A^2 Y)^{-1}(1 + A^{-2}Y)^{-1}$.

REMARK 3.23 (Integrality is a necessary condition to a positive answer). If one expects an answer "à la Westbury" i.e. of the form $(1+P_A(Y))^{-1}$ for some "quantum polynomial"

$P_A(Y)$ then a necessary condition for this to be possible is the integrality of $\langle\Gamma,\mathrm{col}\rangle_A^{\mathrm{Int}}$: from this point of view Theorem 3.15 may be considered as a positive evidence for such an expectation.

Actually it is our opinion that Question 3.21 is not the "correct one" and that rather one should work in a suitable non-commutative ring. We motivate our opinion here below.

Given a planar graph Γ let $c_1,\dots c_k$ be the (finitely many) curves embedded in it and let $R[c_1,\dots c_k]$ be *as a R-module* the algebra of polynomials in $c_1,\dots c_k$ with coefficients in R; let also $\phi: R[c_1,\dots c_k]\to R[\{Y_e\}_{e\in E}]$ be the morphism of R-modules defined on monomials by:

$$\phi_w(c_1^{\alpha_1}\cdots c_k^{\alpha_k}) = A^{w(\alpha_1,\dots\alpha_k)}\prod_{e\in E} Y_e^{\mathrm{col}(e)}$$

where col is the coloring which on an edge $e\in E$ takes the value $\mathrm{col}(e)=\sum_{i|e\subset c_i}\alpha_i$ and $w(\alpha_1,\dots,\alpha_k)$ is a polynomial function of the α_i's depending on Γ. In the following two examples we show that, if one equips with a suitable algebra structure the R-module $R[c_1,\dots c_k]$ and finds a suitable $w(\alpha_1,\cdots\alpha_k)$ one can recover $Z_A(\Gamma)$ by first performing computations in $R[c_1,\dots,c_k]$ (more precisely in its completion with respect to the degree filtration) and then applying ϕ_w.

EXAMPLE 3.24. Let c_1,c_2,c_3 be such that $c_ic_j=A^4c_jc_i,\forall j>i$. It holds:

$$X_a(\ominus)\equiv\sum_{\alpha,\beta,\gamma}\langle \ominus_{a,b,c}\rangle_A^{\mathrm{Int}}A^{2(\alpha\beta+\alpha\gamma+\beta\gamma)}c_1^{\alpha}c_2^{\beta}c_3^{\gamma}=\frac{1}{(1+A^2(c_1+c_2+c_3))}\frac{1}{(1+A^{-2}(c_1+c_2+c_3))}$$

where $a=\beta+\gamma, b=\alpha+\gamma, c=\beta+\alpha$. Moreover $\phi_w(X_A(\ominus))=Z(\ominus)$ if one sets $w(\alpha_1,\alpha_2,\alpha_3)=-2\sum_{i<j}\alpha_i\alpha_j$.

EXAMPLE 3.25. Let $c_1,\dots,c_7$ be such that $c_ic_j=A^4c_jc_i,\ \forall j>i$. It holds:

$$X_a(\ )\equiv\sum_{\mathbb{N}^7}\langle\ ,\mathrm{col}\rangle_A^{\mathrm{Int}}A^{2\sum_{i<j}\alpha_i\alpha_j}c_1^{\alpha_1}\cdots c_7^{\alpha_7}=\frac{1}{(1+A^2(c_1+\cdots+c_7))}\frac{1}{(1+A^{-2}(c_1+\cdots+c_7))}$$

where the summation is on α_i's and for each 7-uple of values for α_i one defines a coloring col as $\mathrm{col}(e)=\sum_{i|e\subset c_i}\alpha_i$. Moreover $\phi_w(X_A(\ ))=Z(\ )$ if one sets $w(\alpha_1,\cdots,\alpha_7)=-2\sum_{i<j}\alpha_i\alpha_j$.

The above examples (which can be proved using the A^4-multinomial theorem and the formulas in Example 3.16 and 3.17) suggest that for planar Γ the definition of $Z_A(\Gamma)$ is not as natural as that of $X_A(\Gamma)$ and that the "correct" set of variables to be used to index the colorings may be related to the set of simple curves embedded in Γ: this is clearly in line with Westbury's theorem. We formulate hence the following:

QUESTION 3.26. *Given a planar* Γ*, using the notation introduced above, can one find an* R*-algebra structure on* $R[c_1,\dots c_k]$ *and a polynomial function* $w(c_1,\dots c_k)$ *such that, letting* $X_A(\Gamma)\equiv\sum_{\mathbb{N}^k}\langle\Gamma,\mathrm{col}(\alpha)\rangle A^{-w(\alpha_1,\cdots,\alpha_k)}\prod_i c_i^{\alpha_i}$ *it holds* $X_A(\Gamma)=\frac{1}{1+A^2(c_1+\cdots+c_k)}\frac{1}{1+A^{-2}(c_1+\cdots+c_k)}$*?*

A positive answer would provide clear "meaning" to planar quantum spin networks (and would imply that one could compute $Z_A(\Gamma)$ by applying ϕ_w to $X_A(\Gamma)$). Even with this formulation, Theorem 3.15 is in line with a positive answer in general: Remark 3.23 still holds. We are currently working on Question 3.26 by trying to generalize our proof of Westbury's theorem in non-commutative algebras: indeed one can use a notion of quantum calculus to interpret the variables $z_h, w_h,\ h \in H$ as elements of a module-algebra in the braided category of $U_q(sl_2)$-modules.

CHAPTER 4

Asymptotics of classical spin networks

In this chapter we discuss the problem of computing the asymptotical behavior of classical spin networks: first we recall the problem, Wigner's conjecture and the known results. Then we detail a result obtained jointly with Julien Marché ([**13**]) which computes the asymptotical behaviour for a general spin network provided some non-degeneracy conditions are satisfied. We then conclude with a list of open problems and questions which relate the semi-classical analysis of spin-networks to the surprisingly hard problem of rigidity of euclidean polyhedra.

1. Semi-classical analysis of classical spin network

Given a spin network (Γ, col) define

$$[\Gamma, \mathrm{col}] = \frac{\langle \Gamma, \mathrm{col}\rangle^2}{\prod_{v:(e_1,e_2,e_3)} \langle \Theta, \mathrm{col}(e_1), \mathrm{col}(e_2), \mathrm{col}(e_3)\rangle}$$

where by $v : (e_1, e_2, e_3)$ we indicate the edges touching the vertex v and

$$\langle \Theta, \mathrm{col}(e_1), \mathrm{col}(e_2), \mathrm{col}(e_3)\rangle$$

is the value of the Θ-graph colored by $\mathrm{col}(e_1), \mathrm{col}(e_2), \mathrm{col}(e_3)$. The problem of computing the asymptotical behavior of spin networks is the following:

Problem 4.1. *Given a pair* (Γ, col)*, compute the asymptotical behavior of the sequence* $s_n = [\Gamma, n\mathrm{col}]$ *as* n *goes to* ∞*.*

This behaviour corresponds to a classical limit of quantum mechanics and is expected to be related to euclidean geometric quantities. Wigner's conjecture (now proved) dealt with the case $\Gamma =$ ☮:

Conjecture 4.2 (Wigner's conjecture). *Let* Tet *be the tetrahedron whose edge lengths are* $\ell(e) = \mathrm{col}(e)$, $\forall e \in E$*. If* Tet *is euclidean (i.e. the Cayley-Menger determinant is positive), then it holds:*

$$[☮, n\mathrm{col}] \sim_{n\to\infty} \frac{2}{3\pi n^3 \mathrm{Vol}} \Big(cos\big(\frac{\pi}{4} + \sum_{e\in E} (n\mathrm{col}(e) + 1)\frac{\theta_e}{2}\big)\Big)^2 \tag{17}$$

where θ_e *is the exterior dihedral angle at* e *(note that* Tet *is unique up to isometry) and* Vol *is the euclidean volume of* Tet*. If* Tet *is minkowskian (i.e. the Cayley-Menger determinant is negative) then* $[☮, n\mathrm{col}]$ *decays exponentially.*

The first rigorous proof of the above conjecture was obtained (in the euclidean case) by Roberts [**56**] and then re-obtained and extended to the minkowskian and flat case using different techniques [**9, 28**]. (For the sake of expliciteness: the value of $[\Gamma, n\mathrm{col}]$ can be easily computed using Formulas 6 and 7, after setting $A = -1$, i.e. replacing "quantum numbers" by standard integers and "quantum multinomials" by standard multinomials.)

In the 1990's physicists used spin networks in the spin foam models for quantum gravity ([**3**]) and the study of the asymptotical behavior was extended from the 3j () and 6j () to more complicated networks as the 9j (complete bipartite 3,3 graph) 10j, 15j (skeleta of the 4 and 5 simplices), see [**4, 19**]. For general graphs, Garoufalidis and Van der Veen proved that the generating series of the sequence $k \mapsto \langle \Gamma, kc\rangle$ is a G-function, implying that the sequence $\langle \Gamma, kc\rangle$ is of Nilsson type and thus that the asymptotic behavior does exist [**28**]. Abdesselam obtained estimates on the growth of spin-network evaluations, specially for generalized drum graphs, see [**1**].

2. Our results on general networks

In his book [**65**], Wigner showed that the square of a $6j$-symbol may be computed by a simple integral formula over 4 copies of $G = SU(2)$. Barrett observed that this formula may be generalized to any graph [**4**]. In [**13**] Section 4, in collaboration with Julien Marché, we extended it to the case of spin networks equipped with holonomies and computed the generating series of squares of spin networks:

Theorem 4.3. *Let $G = SU(2)$ equipped with its Haar measure and $(\Gamma, \mathrm{col}, \psi)$ be a spin network equipped with a holonomy with values in $SL_2(\mathbb{C})$. Define*

$$[\Gamma, \mathrm{col}, \psi] = \frac{\langle \Gamma, \mathrm{col}, \psi\rangle^2}{\prod_{v:(e_1,e_2,e_3)} \langle \Theta, \mathrm{col}(e_1), \mathrm{col}(e_2), \mathrm{col}(e_3)\rangle}$$

where by $v : (e_1, e_2, e_3)$ we indicate the edges touching v and $\langle \Theta, \mathrm{col}(e_1), \mathrm{col}(e_2), \mathrm{col}(e_3)\rangle$ is the value of the Θ-graph colored by $\mathrm{col}(e_1), \mathrm{col}(e_2), \mathrm{col}(e_3)$. It holds:

$$[\Gamma, \mathrm{col}, \psi] = \int_{G^V} \prod_{e:v\leftrightarrow w} tr_{\mathrm{col}(e)}(\psi_{e,v} g_v \psi_{e,v}^{-1} \psi_{e,w} g_w^{-1} \psi_{e,w}^{-1}) \mathrm{d}g \tag{18}$$

Considering the generating series $W(\Gamma, \psi) = \sum_{\mathrm{col}} [\Gamma, \mathrm{col}, \psi]\, Y^{\mathrm{col}(e)}$, the following holds:

$$W(\Gamma, \psi) \;=\; \int_{G^V} \frac{\mathrm{d}g}{\prod_{e=h_1\leftrightarrow h_2=v\leftrightarrow w} \det(\psi_{h_2} g_w \psi_{h_2}^{-1} - \psi_{h_1} g_v \psi_{h_1}^{-1} Y_e)}$$

Using Kirillov's trace formula, the integral formula (18) may be transformed in order to apply the stationary phase approximation and try to compute the asymptotical behavior of $[\Gamma, n\mathrm{col}, \mathrm{Id}]$ when $n \to \infty$. This method was applied in [**3, 19**] for some specific spin networks but it faces some technical difficulties because of the existence of so-called "degenerate configurations". In [**13**] Section 5, we use a different transformation which allows us to treat uniformly all configurations corresponding to critical points: in Subsection 2.1 we briefly recall the transformation we use.

Using this transformation, in [**13**] we describe precisely the critical points of the integrand of (18) and compute the associated Hessian. Under suitable genericity hypotheses on Γ and col described here below, we compute the dominating terms in the asymptotical development of $[\Gamma, n\mathrm{col}, \mathrm{Id}]$ for general Γ.

We now describe our result leaving further details for Subsection 2.1. Let I be the set of maps P from oriented edges of Γ to S^2 which satisfy the following relations:

- Denoting by $-e$ the edge e with opposite orientation, we have $P_{-e} = -P_e$
- For all vertex v with outgoing edges e_1, e_2, e_3 we have $\sum_i c_{e_i} P_{e_i} = 0$

and seen up to the diagonal action of $SO(3)$. Given $P \in I$ we define $r_P(\xi) = \sum_e c_e ||P_e \times \xi||^2$ for $\xi \in \mathbb{R}^3$ and $q_P(\xi_v) = \sum_{e:(v,w)} c_e ||P_e \times (\xi_v - \xi_w)||^2$ for $(\xi_v) \in \bigoplus_{v\in V} \mathbb{R}^3$. In these formulas, we denoted by $\times$ the bracket in the cross-product in $\mathbb{R}^3$.

Given a pair (P, Q) of non-isometric elements of I, in [**13**] Subsection 5.2 we show how to associate to each edge a value $\theta_e \in [0, \pi[$ and the associated *phase function* $\tau_e = \exp(i\theta_e)$ (since the definition of θ_e is a little technical we will skip it here). Then we set for $\kappa \in \mathbb{R}_+$ and $(\xi_v) \in \bigoplus_{v\in V} \mathbb{R}^3$:

$$q^{\kappa}_{P,Q}(\xi_v) = \sum_e c_e\Big(\frac{\kappa^2\tau_e^2 + 1}{\kappa^2\tau_e^2 - 1}||Q_e \times (\xi_v - \xi_w)||^2 + 2i\langle Q_e, \xi_v \times \xi_w\rangle\Big).$$

Then, given a quadratic form q on $\mathbb{R}^n$, we denote by $\det'(q)$ the determinant of the restriction of the quadratic form to the orthogonal of the kernel of q, that is the product of all non-zero eigenvalues of the matrix of q. We also set $\det'(q_{P,Q}) = \lim_{\kappa\to 1}(\kappa - 1)^{-3}\det'(q^{\kappa}_{P,Q})$.

The following are the non-degeneracy conditions we shall need to impose to state our theorem:

(1) For all $P \in I$ it holds $\det(r_P) \neq 0$.
(2) For all $P \in I$ it holds $\det'(q_P) \neq 0$.
(3) For each pair $(P, Q) \in I^2, P \neq Q$ it holds $\det'(q_{P,Q}) \neq 0$.

THEOREM 4.4. *Let* (Γ, col) *be a colored graph satisfying the above non-degeneracy conditions. Then denoting by* χ *the Euler characteristic of* Γ*, one has*

$$\begin{aligned}[\Gamma, n\mathrm{col}(c)] \quad &= \quad \frac{(2\chi)^{3/2}}{(\pi n^3)^{-\chi-1}}\Big(\sum_{P\in I}\frac{\det(r_P)^{1/2}}{\det'(q_P)^{1/2}} \\ &\quad + \sum_{(P,Q)\in I^2/\pm 1, P\neq Q} 2Re\Big(\frac{i^{-\chi}\det(r_P)^{1/2}e^{i\sum_e(n\mathrm{col}(e)+1)\theta_e}}{\det'(q_{P,Q})^{1/2}\prod_e \sin(\theta_e)}\Big) + O(n^{-1})\Big)\end{aligned}$$

The above formula is completely general and can be tested on a computer; for instance we tested it in the case of the tetrahedron and of the prism with triangular base and of course its results are in line with the known asymptotics ([**56**], [**28**], [**4**], [**9**]), thus it provides yet another proof of Wigner's conjecture in the euclidean case. Surprisingly enough, in that case the ratio $\frac{\det(r_P)^{1/2}}{\det'(q_P)^{1/2}}$ provides the volume term showing up in Formula 17.

2.1. Some ideas of the proof of Theorem 4.4. Let V be the 2-dimensional representation of G and for any $v, w \in V$ let $h(v,w) = \langle v, w\rangle$ be the hermitian product on V. Denote by S^3 the unit sphere of V and $\mathrm{d}v$ the Haar measure on it (i.e. such that $\int_{S^3} 1 dv = 1$). One can prove that for any $g \in G$, it holds $tr_n(g) = (n+1)\int_{S^3}\langle v, \rho(g)v\rangle^n \mathrm{d}v$ where tr_n is the trace in the $n+1$-dimensional representation of G. Then, combining this with Formula 18 we deduce the following formula:

$$[\Gamma, \mathrm{col}] = [\Gamma, \mathrm{col}, \mathrm{Id}] = \prod_e (\mathrm{col}(e)+1)\int_{G^V}\int_{(S^3)^E}\prod_{e:v\to w}\langle g_v u_e, g_w u_e\rangle^{c_e}\mathrm{d}g\mathrm{d}u \tag{19}$$

where the notation $e : v \to w$ means that e is an oriented edge joining v to w, u_e is an element of S^3 corresponding to the edge e.

Let X be the following subset of $G^V \times (S^3)^E$:

$$\begin{aligned} X \quad = \quad & \{(g_v, u_e) \in G^V \times (S^3)^E \text{ such that } \forall e : v \to w, \langle g_v u_e, g_w u_e\rangle \neq 0 \\ & \text{and the family } (\pi(u_e))_{e\in E} \text{ has rank at least 2 in } \mathfrak{g}\} \end{aligned}$$

Let $F : X \to \mathbb{C}/2i\pi\mathbb{Z}$ be the map $F(g,u) = \sum\limits_{e:v\to w} c_e \ln\langle g_v u_e, g_w u_e\rangle$; clearly by (19) it holds:

$$[\Gamma, \mathrm{col}] = \prod_e(\mathrm{col}(e)+1)\int_X \exp(F(g,u)). \tag{20}$$

The integral (20) presents some symmetries; the group $(S^1)^E \times G \times G$ acts on the integrand as follows: $(\alpha_e, g, h)\cdot(g_v, u_e) = (g g_v h^{-1}, h(\alpha_e u_e))$ and F is constant on the orbits. Notice that the stabilizer of the action of $(S^1)^E \times G \times G$ on X is $\{\pm 1\}$, hence the quotient Y is a smooth manifold of dimension $12\chi - 6$. We denote by $\tilde{F} : Y \to \mathbb{C}/2i\pi\mathbb{Z}$ the induced map.

The main tool of the proof of Theorem 4.4 is stationary phase approximation when replacing the coloring col by $n$col and letting n go to infinity in $\int_Y \exp(n\tilde{F})\mathrm{d}\mu$. In this formula, the measure μ is obtained from the Haar measure on $(S^3)^{2\chi}\times(S^3)^{3\chi}$ by integration over the action of $(S^1)^E \times G \times G/\{\pm 1\}$, equipped with its Haar density.

The non-degeneracy hypotheses imply that any critical point x is isolated in Y. Provided that $\mathrm{H}ess(\tilde{F})_x$ is non-degenerate, we can apply the stationary phase expansion theorem to $\tilde{F}$ which is a smooth function with non-positive real part, and obtain that the local contribution of x to $[\Gamma, n\mathrm{col}]$ when n goes to infinity is:

$$I(x) = \prod_e \langle e\rangle\ e^{n\tilde{F}(x)}\left(\frac{(2\pi)^{12\chi-6}}{\det(-n\mathrm{H}ess(\tilde{F})_x, \mu)}\right)^{1/2} = \prod_e\langle e\rangle\ e^{n\tilde{F}(x)} I(nq(x)). \tag{21}$$

The proof of Theorem 4.4 consists then in:

(1) finding all the critical points of $\tilde{F}$ in Y;
(2) computing the Hessian of F at a critical point y;
(3) computing the gaussian integral $I(\tilde{q}(y))$ where $q(y)$ is the opposite of the Hessian at y;
(4) summing up all the contributions.

The most delicate point in the above list is the third. Indeed instead of working on Y one works in X (and thus with F) and computes $I(-\mathrm{H}ess(\tilde{F})_y)$ as a quotient of $I(-\mathrm{H}ess(F)_x)$ at a preimage x of y deformed by a quadratic form which is non-degenerate on the kernel of $\mathrm{H}ess(F)_x$, and takes the limit when the deformation is small. To build this form one has to deal with the Haar density on the kernel induced by the identification of the kernel with the tangent space to $(S^1)^E \times G \times G$. All the details are treated in [**13**], Section 5.

3. Open questions and perspectives

The geometric meaning of the non-degeneracy conditions of Theorem 4.4 is still not completely clarified. First of all let us note that a $P \in I$ provides a *geometric realization* in $\mathbb{R}^3$ of the polyhedron whose edges are in correspondence with those of Γ (and have lengths given by col) and whose faces are triangles (corresponding to vertices of Γ); in particular if Γ is planar, it is a (possibly non-convex) geometric realization of the dual of Γ. Trying to interpret the non-degeneracy conditions of Theorem 4.4 in these terms one sees that:

(1) The quantity $\det(r_P)$ is zero only if the configuration P is planar which can occur only for very special values of c.
(2) The non-vanishing of $\det'(q_P)$ is equivalent to the infinitesimal rigidity of the configuration P. In particular, the hypothesis does not hold if the set I is not discrete: this happens for instance for the regular cube or more generally, the cube whose edges are colored by the lengths of Bricard's flexible octahedron.
(3) We do not have a geometric interpretation of the determinant $\det'(q_{P,Q})$ but in our numerical experiments on the spin-networks formed by the 1-skeleton of a tetrahedron and of a triangular prism this determinant was non-zero. In any case, the conditions define a Zariski open set of configurations.

We believe that describing when the non-degeneracy conditions hold is a very difficult task as it contains the problem of the flexibility of polyhedra, a notoriously hard problem. We recall here that:

- by Cauchy's theorem a convex euclidean polyhedron with triangular faces, fixed combinatorial structure and fixed edge lengths is rigid;
- dropping the convexity hypothesis one may get flexible polyhedra (the easiest example being Bricard's octahedron);
- in the latter case the volume of all the deformed polyhedra is constant (it is a solution to an algebraic equation in the edge lengths as proved in [**57**]).

We expect that for planar graphs whose colors correspond to the lengths of a *generic convex* configuration of the dual graph, there is a simple geometric condition ensuring that the non-degeneracy conditions hold, so a natural (and classical!) question derived by the above analysis is:

QUESTION 4.5. *Suppose Γ is planar, is it true that for a* generic *(in a suitable sense) coloring $\mathbb{R}_+$-valued coloring of the edges of Γ the set of non-isometric geometric realizations I is finite? What if the coloring takes values in $\mathbb{N}$?*

The matrices $r_P, q_P, q_{P,Q}$ showing up in Theorem 4.4 are defined geometrically (and combinatorially). Moreover the fact that in the case of the tetrahedron the ratio of determinants $\sqrt{\frac{\det(r_P)}{\det'(q_P)}}$ is related to the volume is an intriguing fact for which we have no conceptual explanation:

Question 4.6. *What is the geometric meaning of the ratios* $\sqrt{\frac{\det(r_P)}{\det'(q_P)}}$ *and* $\sqrt{\frac{\det(r_P)}{\det'(q_{P,Q})}}$*? Are they related to the volume of the configuration P and/or Q?*

Consider the case when (Γ, col) is the cube with all colors equal to 2: it is not difficult to check that this case is not covered by our theorem as the space of geometric configurations contains flexible realizations. Moreover in [**28**] it is shown that in this case the highest order term goes like n^{-4} (instead of a n^{-9} term predicted by our result when the degeneracy hypotheses hold). So a hard general problem is:

Problem 4.7. *Compute the highest order terms of the expansion of* $[\Gamma, n\mathrm{col}]$ *when the non-degeneracy hypotheses do not hold: what is the rate of decay (as a power of n)? What is the coefficient of the highest order term?*

CHAPTER 5

Asymptotical behavior of quantum spin networks

In this chapter we first comment on the volume conjecture on knotted objects different from knots. Then we discuss our result ([**11**]) computing the asymptotical behavior of $6j$-symbols of $U_q(sl_2)$ in what we called the "hyperbolic regime" and compare it with a previoulsy known result by Woodward and Taylor. Based on our result, we formulate a volume conjecture for planar trivalent graphs which we prove for an infinite family of such graphs (and currently being studied in collaboration with François Guéritaud and Roland van der Veen). We then pass to knots and links and state a result proving the conjecture for an infinite family of hyperbolic links ([**16**]). We conclude by discussing the perspectives open by these results and the many open problems.

1. Volume Conjecture for links or graphs

The analysis of the Volume Conjecture for quantum spin networks other than knots is more delicate. One of reason is that the VC, as stated in the Introduction, is known to be false for links: indeed $J_n(K_1 \sqcup K_2) = J_n(K_1) \cdot J_n(K_2)$ and so $J'_n(K_1 \sqcup K_2) = [n]J'_n(K_1)J'_n(K_2)$ so that $\mathcal{K}_n(K_1 \sqcup K_2) = J'_n(K_1 \sqcup K_2)|_{A=\exp(\frac{i\pi}{2n})} = 0$. Moreover T.Q. Le and A.T. Tran ([**42**]) showed that the Kashaev invariant of some non trivial cables of a knot are 0 for odd n (in our notation). So to correct the statement of the conjecture and make it plausible in the case of links one may use the following notion of "evaluation" as follows:

Definition 5.1. Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be a meromorphic function. We denote $ev_n(f)$ the first non-zero coefficient of the Laurent series expansion of f around $A = \exp(\frac{i\pi}{2n})$.

Then in Conjecture 1.5 replace $J'_n(K)|_{A=\exp(\frac{i\pi}{2n})}$ by $ev_n(J_n(K))$ and restrict to even n: at the best of our knowledge there are no known counterexamples to this version of the conjecture, even considering links. This conjecture was proved for Whitehead doubles of torus knots ([**68**]), Whitehead chains ([**62**]) and for certain 2-cables of the figure eight knot ([**58**]).

Another reason why the VC is more delicate for general spin networks is that as soon as there are more than one edge in the object considered, there are infinitely many ways of rescaling a coloring to ∞. In the above statement we were coloring all the components of a link by n but a geometric meaning could be assigned also to different rescaling of the colors. This is evident in the the most relevant result concerning graphs we are aware of which is to be considered as an answer to the problem of asymptotical behavior of graphs in the case of $\Gamma =$ due to Y. Taylor and C. Woodward ([**61**]) for a certain range of colors:

THEOREM 5.2. *Let* col *be an admissible coloring of* $\Gamma =$ *and let* $r \in \mathbb{N}$ *be greater than* $\frac{1}{2}\max(T_1, T_2, T_3, T_4)$ *(using the notation of Example 2.16). Let* τ^n *be the spherical tetrahedron whose edge-lenghts are* $\ell_e^n = \frac{\pi(n\mathrm{col}(e)+1)}{n(r-2)+2}$ *and* θ_e^n *be the internal dihedral angles of* τ^n *and let* $G(n)$ *be the Gram matrix of* τ^n *(i.e.* 4×4 *matrix whose columns and rows correspond to vertices of* Γ *and whose* $(i,j)^{th}$ *entry is* $\cos(\ell_e^n)$ *with* $e : i \leftrightarrow j$*). Then it holds:*

$$\langle \ , n \cdot \mathrm{col}\rangle^U|_{A=\exp(\frac{i\pi}{2nr})} \underset{n\to\infty}{\sim} \frac{2\pi \cos\left(\frac{\pi}{4} + \frac{n(r-2)+2}{2\pi}\left(\sum_e \ell_e^n \theta_e^n - 2\mathrm{Vol}(\tau^n)\right)\right)}{(n(r-2)+1)^{\frac{3}{2}} \det(G(n))^{\frac{1}{4}}}$$

Where $\langle \Gamma, \mathrm{col}\rangle_A^U$ is the unitary renormalization introduced in Chapter 2 Section 4.

2. A Volume Conjecture for planar graphs and for links in $\#_k S^2 \times S^1$

Let us compare the Melvin-Morton-Rozansky and the Volume Conjecture:

	MMR-Conjecture	Generalized VC
Values of α:	Near to 0	Near to 1
Asymptotical behavior:	Polynomial	Exponential
Geometry appearing:	$SU(2)$	$PSL(2,\mathbb{C})$

(The last row of the table refers to the fact that the Alexander polynomial of is related to representations of $\pi_1(S^3 \setminus K)$ into $SU(2)$, and even $U(1)$, while hyperbolic structures provide morphisms into $PSL(2,\mathbb{C})$.) In Theorem 5.2 Γ is colored by an initial coloring col and one evaluates at the $(r \cdot n)^{th}$-root fo unity the colors $\mathrm{col}_n = n \cdot \mathrm{col}$ so that the limit $\lim_{n\to\infty} \frac{\mathrm{col}_n}{rn} = \frac{\mathrm{col}}{r}$ takes values in $[0,1]$. So according to the philosophy underlying the above table, Theorem 5.2 should rather be viewed as an analogue of the Melvin-Morton-Rozansky conjecture than of the Volume Conjecture as the ratio between the colors of Γ and the order of the root of unity used in the evaluation is "small" (i.e. less than 1). Our analysis of the asymptotical behavior of $6j$-symbols of $U_q(sl_2)$ deals with a completely disjoint set of cases where the behavior is in line with the right column of the table. Our main result (proved with a different notation and wording in [**11**]) states :

THEOREM 5.3. *Let* τ *be the hyperbolic hyperideal tetrahedron whose exterior dihedral angles are* θ_e *and let* $\mathrm{col}^\infty(e) = 2 - \frac{\theta_e}{\pi}$*. Then for any sequence* $\{\mathrm{col}_n\}_{n\in\mathbb{N}}$ *of colorings on the* 1*-skeleton of* τ *such that* $\lim_{n\to\infty} \frac{\mathrm{col}_n}{n} = \mathrm{col}_\infty$ *it holds:*

$$\lim_{n\to\infty} \frac{2\pi}{n} \log\left(|ev_n(\langle \ , \mathrm{col}_n\rangle_{A=\exp(\frac{i\pi}{2n})})|\right) = 2\mathrm{Vol}(\tau)$$

In the above result by "hyperbolic hyperideal polyhedron" we mean the (non-compact) polyhedron obtained in the Klein model by considering the convex-hull of the intersection of $\overline{\mathbb{H}}^3$ with a convex polyhedron whose vertices lie outside $\mathbb{H}^3$ and such that all its edges touch $\overline{\mathbb{H}}^3$ (we allow tangent edges). Comparing Theorems 5.3 and 5.2 one sees various differences:

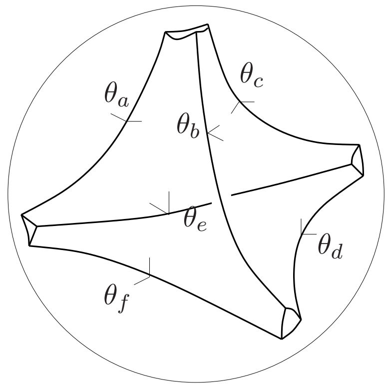


FIGURE 1. A hyperideal tetrahedron and its exterior dihedral angles in the ball model.

	Theorem 5.2	Theorem 5.3
Range of col^{∞} to which it applies	$\mathrm{col}_{\infty}(e) < 1$	$\mathrm{col}_{\infty}(e) > 1$
Asymptotical behavior:	Polynomial	Exponential
Geometry appearing:	Spherical	Hyperbolic
Order of evaluation:	0	-5
Meaning of colors:	Edge lengths	Dihedral angles

In the table by "order of evaluation" we mean the order of the zero (or pole) of the function $\langle$☮$, \mathrm{col}_n\rangle_A$ at $A_n = \exp(\frac{i\pi}{2n})$: one can show that in the hyperbolic case there is a pole of order 5. Another striking difference between the two results is that the limit considered in Theorem 5.2 is much more delicate: the statement applies to a precise sequence col_n such that $\lim_{n\to\infty} \frac{\mathrm{col}_n}{n} = \mathrm{col}_{\infty}$ not to any such sequence and moreover one has to evaluate only at roots of unity which are multiples of the initial r.

We will not provide the proof of the Theorem 5.3 but we will bound ourselves to remark that the proof exploits a peculiar behavior of $6j$-symbols when col_{∞} is in the range given by the inequalities satisfied by the angles of a hyperbolic tetrahedron: they are finite sums positive real numbers each of which is growing exponentially fast. To compute the overall exponential growth of the sum it is therefore sufficient to identify the summands with maximal growth rate. These numbers are obtained by applying Lemma 5.7 to the binomials of Formula (7). The rest of the proof shows that the maximal growth rate coincides with the Murakami-Yano formula for the volume of a truncated hyperbolic hyperideal tetrahedron plus a term which corresponds to the volume of the truncation half-prisms.

Another (easier) instance of the above dichotomy in the asymptotic behavior and of the appearence of geometry is the following (unpublished):

PROPOSITION 5.4. *Let* $\mathrm{col}_{\infty} = (a, b, c) \in \mathbb{R}^3_+$ *be a real-valued coloring on* ⊖. *For every sequence* col_n *such that* $\lim_{n\to\infty} \frac{col_n}{n} = \mathrm{col}_{\infty}$ *it holds:*

$$|ev_n(\langle \ominus, \mathrm{col}_n\rangle_A)| \underset{n\to\infty}{\sim} \exp\left(\frac{n}{\pi} V(a,b,c)\right)$$

where $V(a,b,c) = \Lambda(\pi a)+\Lambda(\pi b)+\Lambda(\pi c)-\Lambda(\pi\frac{a+b+c}{2})-\Lambda(\pi\frac{a+b-c}{2})-\Lambda(\pi\frac{a-b+c}{2})-\Lambda(\pi\frac{-a+b+c}{2})$ *and* $\Lambda(x) = -\int_0^x \log(2|sin(t)|)dt$ *is the Lobatchevski function. Moreover, if* $a = 2-\frac{\theta_a}{\pi}$, $b = 2-\frac{\theta_b}{\pi}$, $c = 2-\frac{\theta_c}{\pi}$ *where* $\theta_a, \theta_b, \theta_c$ *are the exterior dihedral angles of an ideal hyperbolic triangular prism* P *(see Figure 2) then* $V(a,b,c) = \mathrm{Vol}(P)$*. On contrast if* $a, b, c < 1$ *and* $a+b+c > 2$ *then* $V(a,b,c) < 0$ *and the behavior is exponentially decaying.*

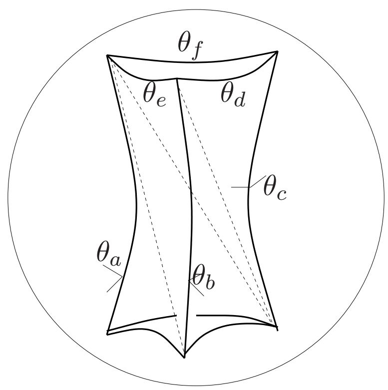


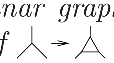
FIGURE 2. A hyperideal prism, its exterior dihedral angles and its decomposition into three ideal tetrahedra in the ball model. It holds: $\theta_e = \pi - \frac{\theta_a+\theta_b-\theta_c}{2}, \theta_f = \pi - \frac{\theta_a+\theta_c-\theta_b}{2}, \theta_d = \pi - \frac{\theta_c+\theta_b-\theta_a}{2}$.

Comparing the statements of Proposition 5.4 and of Theorem 5.3 we formulated the following version of the Volume Conjecture for planar trivalent graphs:

CONJECTURE 5.5 (VC for planar graphs). *Let* Γ *be the* 1*-skeleton of an hyperideal hyperbolic polyhedron* $P(\Gamma)$ *and let* $\theta = E \to]0,\pi]$ *be the map associating to each edge the exterior dihedral angle of* $P(\Gamma)$ *at the edge. Then for any sequence* $\{\mathrm{col}_n\}_{n\in\mathbb{N}}$ *of colorings on* Γ *such that* $\mathrm{col}_\infty = \lim_{n\to\infty}\frac{\mathrm{col}_n}{n} = 2-\frac{\theta}{\pi}$ *(as maps* $E \to [1,2[$*) it holds:*

$$ev_n\big(|\langle \Gamma, \mathrm{col}_n\rangle_A|\big) \underset{n\to\infty}{\sim} \exp\big(\frac{2n}{\pi}\mathrm{Vol}(P)\big) \tag{22}$$

The combination of Proposition 5.4 and of Theorem 5.3 gives as a direct consequence (unpublished):

THEOREM 5.6. *The Volume Conjecture for planar graphs is true for all the graphs obtained from the tetrahedron by a finite sequence of*  .

Indeed it is sufficient to use the equality:

$$\langle \boxed{\Gamma} \equiv \boxed{\Gamma'} \rangle_A = \langle \boxed{\Gamma} \supset\!\in \boxed{\Gamma'} \rangle_A \, (\ominus)^{-1} \tag{23}$$

for the vertex connected sum of two spin networks and to remark that the polyhedron obtained by gluing two hyperideal ones along a prism is obtained by cutting out of the two polyhedra two half-prisms and then gluing: by Proposition 5.4 the volume of the two half-prisms cut out is exactly taken into account by the $\ominus^{-1}$.

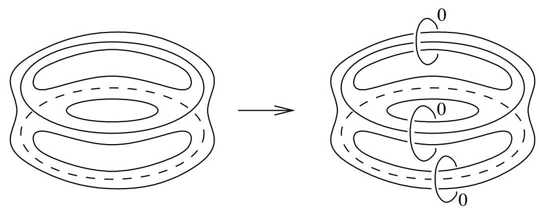


FIGURE 3. An instance of construction of fundamental hyperbolic link.

2.1. Sketch of proof of Proposition 5.4. The proof of the first statement of the proposition is a direct application to the formula in Example 2.15 of the following fundamental lemma (whose proof is a straightforward analysis) which may be seen as on of the key reasons why hyperbolic geometry shows up when considering certain asymptotic limits of quantum factorials:

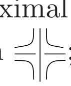
LEMMA 5.7. *Let* $\alpha \in \mathbb{R}_+$, $\{b^n\}_{n\in\mathbb{N}}$ *be a sequence such that* $\lim_{n\to\infty} \frac{b^n}{n} = \alpha$. *Let also* $\{k\} = -i(A^{2k} - A^{-2k})$ *(so that* $[k] = \frac{\{k\}}{\{1\}}$*) and* $\{k\}! = \prod_{j=1}^{k}\{j\}$. *Then for* n *big enough* $\{b_n\}!$ *has a zero of order* $\lfloor\alpha\rfloor$ *at* $A = \exp(\frac{i\pi}{2n})$*; moreover* $ev_n\{b_n\}! \in \mathbb{R}$ *and* $|ev_n\{b_n\}!| \underset{n\to\infty}{\sim} \exp\big(-\frac{n}{\pi}\Lambda(\pi\alpha)\big)$.

The second statement of Proposition 5.4 is a consequence of Milnor's formula for the volume of ideal tetrahedra, applied to three tetrahedra decomposing the prism (the formula provides a sum of 9 evaluations of the Λ function of which two are opposite). The last statement is obtained by re-expressing $V(a,b,c)$ in terms of internal dihedral angles $\alpha_a, \alpha_b, \alpha_c$ of P, observing that the conditions $a, b, c, < 1, a+b+c > 2$ are equivalent to $a = 1 - \frac{\alpha_a}{\pi}, b = 1 - \frac{\alpha_b}{\pi}, c = 1 - \frac{\alpha_c}{\pi}$ (and so on) and finally noting that $V(-\alpha_a, -\alpha_b, -\alpha_c) = -V(\alpha_a, \alpha_b, \alpha_c)$ (because each summand changes sign).

3. The volume conjecture for links in $\#_k S^2 \times S^1$

In [**16**] we extended the VC to links in $\#_k S^2 \times S^1$ by remarking that $\mathbb{Q}(A) \otimes_{Z[A^{\pm 1}]} \mathcal{S}(\#_k S^2 \times S^1) = \mathbb{Q}(A)$ and thus one may define $\langle \Gamma, \mathrm{col}\rangle_A \in \mathbb{Q}(A)$ for any colored $\Gamma \subset \#_k S^2 \times S^1$. We used this for a family of links we studied with D. Thurston in [**17**]:

DEFINITION 5.8 (Fundamental hyperbolic links). A fundamental hyperbolic link is one obtained by the following procedure:

(1) pick a 4-valent graph G in S^3 and a maximal tree $T \subset G$;
(2) replace each vertex of G by the diagram ;
(3) connect the so-obtained three-uples of boundary points (with any permutation) following the edges of G;
(4) put 0-framed meridians around the three-uple of strands passing along the edges of $G \setminus T$.

EXAMPLE 5.9. Using as G the graph with 2 vertices and 4 edges connecting them, for one of the possible choices of gluing the strands along the edges of G one gets the link in Figure 3. The link is contained in the connected sum of 3 copies of $S^2 \times S^1$.

THEOREM 5.10. *[[**17**]] A fundamental hyperbolic link obtained from a connected graph with c vertices is contained in $\#_{c+1}S^2 \times S^1$ and its complement admits a hyperbolic complete structure with volume $c\mathrm{Vol}(Oct)$ where Oct is the regular ideal octahedron. Moreover the family of fundamental hyperbolic links is* universal *in the sense that for every pair (M^3, L) where M^3 is a compact orientable 3 manifold and $L \subset M$ is a link, there exists a fundamental hyperbolic link F such that (M, L) is the result of an integral Dehn-filling on some components of F (the non-filled ones forming L); moreover F can be chosen to be obtained by a graph with c vertices where $h||M|| < c < k||M||^2$ where h and k are two universal constants (not depending on (M, L)) and $||\cdot||$ is the Gromov norm.*

(Through this theorem we also proved that "3-manifolds efficiently bound 4-manifolds" i.e. for any M triangulated with t tetrahedra there exist a 4-manifold W bounded by M admitting a triangulation with less than kt^2 simplexes.)

Using Theorem 5.10 we proved (in [**16**] for the complete case and in [**11**] for the non complete one) the following result which is the first proof of the VC for an infinite family of hyperbolic links and knots:

THEOREM 5.11. *The VC is true for each fundamental hyperbolic link F, namely: there exists a neighborhood $\mathcal{U}$ of $(0, \ldots, 0) \in \mathbb{R}^{\#F}$ such that for any sequence of even colorings $\{\mathrm{col}_n\}$ on F with $\lim_{n\to\infty} \frac{\mathrm{col}_n(e)}{n} = \mathrm{col}_\infty(e) \in 1 + \mathcal{U},\ \forall e \in E$ it holds:*

$$\lim_{n\to\infty} \frac{2\pi}{n} ev_n(\langle F, \mathrm{col}_n\rangle_n) = \mathrm{Vol}_{\mathrm{col}_\infty - 1}(F)$$

where by $\mathrm{Vol}_{\mathrm{col}_\infty-1}(F)$ we denote the hyperbolic volume of the complement of F in the $\#_k S^2 \times S^1$ equipped with the (not necessarily complete) structure whose holonomy around the meridian of a component e of F has eigenvalues $\exp\big(\pm i\pi(\mathrm{col}_\infty(e) - 1)\big)$.

4. Open questions and perspectives

The volume conjecture for planar graphs opens a possibly easier problem which directly relates to hyperbolic geometry of polyhedra. For fixed Γ the range of col_∞ to which the conjecture should apply is well understood: thanks to a result of I. Rivin ([**52**]) we know completely which are the possible sets of dihedral angles on a polyhedron $P(\Gamma)$. Even if the evidence given by Theorem 5.6 is encouraging, the case where Γ is a cube (we state it here in the symmetric completely ideal case) is very hard:

QUESTION 5.12. *Let Γ be the 1 skeleton of a cube and let col_n be the uniform color n on all the edges. Is it true that $\lim_{n\to\infty} ev_n\big(\langle\Gamma, \mathrm{col}_n\rangle_A\big) \underset{n\to\infty}{\sim} \exp\big(\frac{2n}{\pi}\mathrm{Vol}(D)\big)$ where D is the regular ideal dodecahedron?*

Numerical experiments seem to support a positive answer but the complexity of calculations grows too fast to allow any confidence on these evidences. In work in progress with François Guéritaud and Roland van der Veen we are proving that a set of recurrence relations on the different colored invariants of a fixed planar graph may be "taken to the limit" and produce a set of differential equations "à la Schäfli" on the set of functions of the angles of $P(\Gamma)$ which would be satisfied by the exponential growth function (if it existed!). We are also able to show that $\mathrm{Vol}(P)$ viewed as a function of the dihedral angles is a solution to all these differential equations: the uniqueness of the solution, though is still unclear.

On the side of links, the universal property of fundamental hyperbolic links may be used to attack the VC for a general knot K in S^3: indeed it can be shown that K can be obtained from a fundamental shadow link F by successively Dehn-filling all the components of F but one so that at every step the resulting link is still contained in a connected sum of copies of $S^2 \times S^1$. Then at every step the "colored Jones polynomial" is defined and one may study the VC for the link. So one is reduced to attacking the VC by increasing complexity: the initial step (that of fundamental shadow links) being already completely treated (Theorem 5.11). The second step would be to perform a single hyperbolic surgery on a component of F and to prove the conjecture for this case. The invariant of the resulting link is a linear combination of the invariants of F; since we know that the invariants of F have an exponential behavior given by the volume of a deformed hyperbolic structure, the linear combination way be seen as an average or integration over a path of hyperbolic structures on F of an integrand which is proportional to $\exp(n\mathrm{Vol}_\alpha(F))$ (where α parametrizes some non-complete structure on the complement of F). This kind of idea is similar in spirit to that proposed by Yokota ([**67**]) but may be easier to study as the geometry of fundamental shadow links is quite clear. Of course one gets more and more complicated formulas when dealing with the subsequent surgeries but all the difficulties of the Volume Conjecture are already present at this first surgery level. Moreover one can show that from this point of view the knots in S^3 are not the simplest examples: they cannot be obtained with less than 2 surgeries on a fundamental shadow link, while it is not difficult to provide examples of hyperbolic knots in $S^2 \times S^1$ obtained by a single surgery. So the next step to attack the VC according to this programme would be:

PROBLEM 5.13. *Let F be a fundamental shadow link and F' be obtained by a single integer Dehn-surgery on a component of F. Suppose that F' is contained in a connected sum of copies of $S^2 \times S^1$ and that its complement is hyperbolic. Prove the VC for F'.*

CHAPTER 6

Nilpotent spin networks and Reshetikhin-Turaev invariants

In this chapter we discuss the theory of spin-networks colored by irreducible representations of $U_q(sl_2)$ at $q = \exp(\frac{i\pi}{r})$ ($r \in \mathbb{N}$) of a special type called *nilpotent*, i.e. such that $E^r = F^r = 0$. In order to avoid recalling the details of the representation theory of $U_q(sl_2)$ and to keep the constructions as combinatorial as possible, we define these objects by providing a set of characterizing properties they satisfy and cite a result (joint with Jun Murakami, proved also independently by Nathan Geer and Bertrand Patureau-Mirand) ensuring the actual existence of these objects. Then we cite a result (joint with Jun Murakami) analyzing the asymptotical behavior (when $r \to \infty$) of these networks for a certain family of tetrahedral networks and relate it to the geometry of hyperbolic tetrahedra. We conclude by discussing a construction of invariants of triples (M^3, Γ, ω) where M^3 is a closed three-manifold, $\Gamma \subset M$ is a (possibly empty) KTG and $\omega \in H^1(M \setminus \Gamma; \mathbb{C}/2\mathbb{Z})$: these invariants (constructed jointly with Bertrand Patureau-Mirand and Nathan Geer) are the analogue of Reshetikhin-Turaev invariants of 3-manifolds but constructed by means of the theory of the nilpotent representations of $U_q(sl_2)$.

1. Nilpotent spin-networks

From now on Γ will be a KTG, the edges of Γ will be oriented and $q = \exp(\frac{i\pi}{r})$ ($r \geq 3 \in \mathbb{N}$). For $x \in \mathbb{C}$ we extend the notation q^x by setting $q^x = \mathrm{e}^{i\pi x/r}$ and we keep the notation $\{x\} = q^x - q^{-x}$. For $x \in \mathbb{N}$ we let $\{x\}! = \prod_{j=1}^{x}\{j\}$ and for $a, b \in \mathbb{C}$ such that $a - b \in \{0, 1, \dots, r-1\}$ we also let $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\prod_{c=b+1}^{c=a}\{c\}}{\{a-b\}!}$.

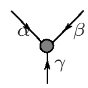
Let $H_r = \{1-r, 3-r, \dots, r-3, r-1\}$, $X_r = \mathbb{Z} \cup \setminus r\mathbb{Z}$ and let the "modified dimension" $\mathsf{d} : \mathbb{C} \setminus X_r \to \mathbb{C}$ be given by $\mathsf{d}(a) = (-1)^{r-1} \begin{bmatrix} a+r-1 \\ a \end{bmatrix}^{-1}$. A (nilpotent) *coloring* of Γ from now on will be a map $\mathrm{col} : E \to \mathbb{C} \setminus X_r$.

The $2r^{\text{th}}$ nilpotent $\mathfrak{sl}(2)$ invariant $\langle \cdot \rangle_r$ of isotopy classes of colored framed oriented graphs has been first considered for the class of colored links by Y. Akutsu, T. Deguchi and T. Ohtsuki in [**2**]. Jointly with Jun Murakami ([**14**], Theorem 3.4) we extended the construction to all framed colored trivalent graphs; this was also done independently by Nathan Geer and Bertrand Patureau-Mirand ([**25**]). Here below we summarize these results (together with all the properties of the invariant) using the middle weight notation for colors as opposed to the highest weight notation used in [**14**] (i.e. a color a in [**14**] corresponds to a color $\alpha = 2a + 1 - r$ here):

THEOREM 6.1 (Nilpotent spin networks, [**14**], [**25**]). *There is a unique invariant* $\langle \Gamma, \mathrm{col}\rangle_r^{Nil} \in \mathbb{C}$ *of framed, colored, oriented graphs in* S^3 *which enjoys the properties* $1) \cdots 9)$ *here below.*

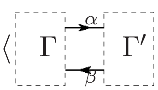
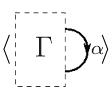
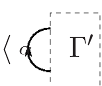
(1) If (Γ', col') is obtained from (Γ, col) by simultaneously changing the orientation of an edge and its color α to $-\alpha$ then $\langle \Gamma', \mathrm{col}'\rangle_r^{Nil} = \langle \Gamma, \mathrm{col}\rangle_r^{Nil}$. This property allows us to extend the coloring to all orientations of a fixed graph.

(2) $\langle \Gamma, \mathrm{col}\rangle_r^{Nil} = 0$ if for one of the vertex of Γ, the sum of the colors of the incoming edges is not in H_r. We then say that the coloring of Γ is trivially null:

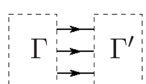
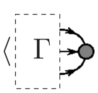
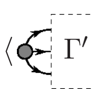
(24a) $\langle$  $\rangle_r^{Nil} = 0$ if $\alpha + \beta + \gamma \notin H_r$

(3) Let $C(\Gamma)$ be the set of non trivially null colorings of a fixed oriented trivalent framed graphs Γ. $C(\Gamma)$ is a Zariski open subset of a disjoint union of affine subspaces of $\mathbb{C}^E$. Then as a function on $C(\Gamma)$, $\langle \Gamma, \cdot\rangle_r^{Nil}$ is holomorphic on each connected component of $C(\Gamma)$.

(4) If $\Gamma\sharp\Gamma'$ denotes the connected sum of two colored graphs along an edge colored by α and $\mathrm{col}\sharp\mathrm{col}'$ the induced coloring on $\Gamma\sharp\Gamma'$ then $\langle \Gamma\sharp\Gamma', \mathrm{col}\sharp\mathrm{col}'\rangle_r^{Nil} = \mathsf{d}(\alpha)^{-1}\langle \Gamma, \mathrm{col}\rangle_r^{Nil}\langle \Gamma', \mathrm{col}'\rangle_r^{Nil}$:

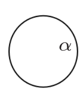
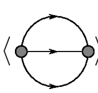
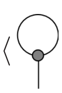
(24b) $\langle$  $\rangle_r^{Nil} = \delta_\alpha^\beta\, \mathsf{d}(\alpha)^{-1}\, \langle$  $\rangle_r^{Nil} \langle$  $\rangle_r^{Nil}$

(5) If Γ'' denotes the connected sum of two colored graphs Γ, Γ' along a vertex with compatible incident colored edges then $\langle \Gamma'', \mathrm{col}''\rangle_r^{Nil} = \langle \Gamma, \mathrm{col}\rangle_r^{Nil}\langle \Gamma', \mathrm{col}'\rangle_r^{Nil}$:

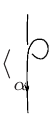
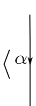
(24c) $\langle$  $\rangle_r^{Nil} = \langle$  $\rangle_r^{Nil}$ $\langle$  $\rangle_r^{Nil}$

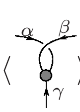
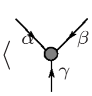
(6) $\langle \cdot \rangle_r^{Nil}$ is zero on split graphs: $\langle \Gamma \sqcup \Gamma', \mathrm{col} \sqcup \mathrm{col}'\rangle_r^{Nil} = 0$.

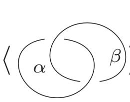
(7) Normalisation (the "Θ" graph is assumed not to have trivially null coloring):

(24d) $\langle$  $\rangle_r^{Nil} = \mathsf{d}(\alpha)$, $\quad\langle$  $\rangle_r^{Nil} = 1$, $\quad\langle$  $\rangle_r^{Nil} = 0$.

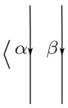
(8) The framing twist. Letting $t_\alpha = \frac{\alpha^2 - (r-1)^2}{2}$ it holds:

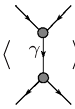
(24e) $\langle$  $\rangle_r^{Nil} = q^{t_\alpha} \langle$  $\rangle_r^{Nil}$.

(24f) $\langle$  $\rangle_r^{Nil} = q^{\frac{t_\gamma - t_\alpha - t_\beta}{2}} \langle$  $\rangle_r^{Nil}$.

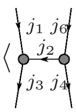
(24g) $\langle$  $\rangle_r^{Nil} = (-1)^{r-1} r q^{\alpha\beta}$.

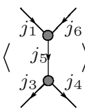
(9) The following two relations hold whenever all appearing colors are in $\mathbb{C} \setminus X_r$:

(24h) $\langle$ 

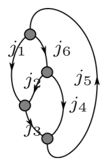
$\rangle_r^{Nil} = \sum_{\gamma \in \alpha+\beta+H_r} \mathsf{d}(\gamma) \langle$ 

$\rangle_r^{Nil}$

(24i) $\langle$ 

$\rangle_r^{Nil} = \sum_{j_5 \in j_1+j_6+H_r} \mathsf{d}(j_5)^{-1} \begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix} \langle$ 

$\rangle_r^{Nil}$

where the 6j-symbols $\begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix} = \langle$ 

$\rangle_r^{Nil}$ are given by:

$$\begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix} = (-1)^{n-1+B_{165}} \frac{\{B_{345}\}!\,\{B_{123}\}!}{\{B_{246}\}!\,\{B_{165}\}!} \begin{bmatrix} j_3+n-1 \\ A_{123}+1-n \end{bmatrix} \begin{bmatrix} j_3+n-1 \\ B_{354} \end{bmatrix}^{-1} \times$$
$$\times \sum_{z=m}^{M} (-1)^z \begin{bmatrix} A_{165}+1 \\ j_5+z+n \end{bmatrix} \begin{bmatrix} B_{156}+z \\ B_{156} \end{bmatrix} \begin{bmatrix} B_{264}+B_{345}-z \\ B_{264} \end{bmatrix} \begin{bmatrix} B_{453}+z \\ B_{462} \end{bmatrix}$$

where $A_{xyz} = \frac{j_x+j_y+j_z+3(r-1)}{2}$, $B_{xyz} = \frac{j_x+j_y-j_z+r-1}{2}$, $m = \max(0, \frac{\mathrm{j}_3+\mathrm{j}_6-\mathrm{j}_2-\mathrm{j}_5}{2})$ and $M = \min(\mathrm{B}_{435}, \mathrm{B}_{165})$. (Here are using the formula provided in Theorem 1.12 of [**14**] taking into account that a color $a \in \mathbb{C}$ in [**14**] corresponds to the color $2a+1-r$ here.)

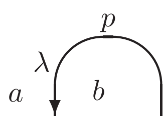
1.1. IRF (or shadow-state) model for $\langle \Gamma, \mathrm{col} \rangle_r^{Nil}$. Let (Γ, col) be a colored, oriented, framed trivalent graph. In [**14**] we provided a shadow-state sum model for computing $\langle \Gamma, \mathrm{col} \rangle_r^{Nil}$. We recall it in this subsection. Let us cut open a diagram of Γ along an edge $e_1 \in E$ and put it in a $(1,1)$-tangle like position so that the two strands which were contained in e_1 are directed towards the bottom. The diagram T_Γ just constructed splits the plane into *regions* $R_0, \ldots, R_k$, where we let R_0 and R_1 be respectively the leftmost and the rightmost regions. Let $a_0, a_1 \in \mathbb{C} \setminus X_r$ be complex numbers satisfying $a_0 + \mathrm{col}(e_1) - a_1 \in H_r$. We define a *state* of T_Γ as a mapping $\varphi : \{R_0, R_1, R_2, \cdots, R_d\} \longrightarrow \mathbb{C}$ which satisfies the following conditions.

(1) $\varphi(R_0) = a_0$, $\varphi(R_1) = a_1$,
(2) If R_i and R_j are adjacent along e_k, R_i is on the left of e_k (with respect to the reader) and R_j is on the right of e_k, then $\varphi(R_i) + \mathrm{col}(e_k)$ (or $\varphi(R_i) - \mathrm{col}(e_k)$ if e_k is oriented upwards) $= \varphi(R_j) + l$, for some $l \in H_r$.
(3) $\forall i\ \varphi(R_i) \notin X_r$.

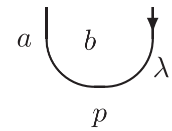
Let $Z_{a_0,a_1}(T_\Gamma)$ be the following state sum.

$$Z_{a_0,a_1}(T_\Gamma) = \sum_{\substack{\varphi: \\ \text{states}}} \prod_{p\,:\,\text{maximum}} W_{\max}(p) \prod_{p\,:\,\text{minimum}} W_{\min}(p) \prod_{p\,:\,\text{crossing}} W_{\mathrm{c}}(p) \prod_{p\,:\,\text{vertex}} W_{\mathrm{v}}(p) \tag{25}$$

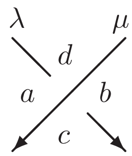
where $W_{\max}(p)$, $W_{\min}(p)$, $W_{\text{crossing}}(p)$, $W_{\text{vertex}}(p)$ are given as follows (in the pictures we denote by λ, μ, η the colors of the edges of Γ and by a, b, c the states of the regions):



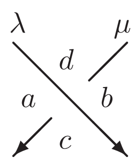
$\xrightarrow[W_{\max}(p)]{} \mathsf{d}(a)^{-1}$,



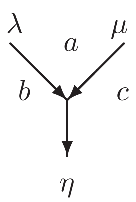
$\xrightarrow[W_{\min}(p)]{} \mathsf{d}(b)$.



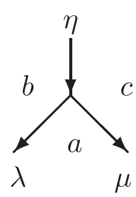
$$\xrightarrow[W_{\text{crossing}}(p)]{} q^{t_a+t_b-t_c-t_d}\ \mathsf{d}(d) \begin{vmatrix} \mu & a & c \\ \lambda & b & d \end{vmatrix}$$



$$\xrightarrow[W_{\text{crossing}}(p)]{} q^{-t_a-t_b+t_c+t_d}\ \mathsf{d}(d) \begin{vmatrix} \mu & a & c \\ \lambda & b & d \end{vmatrix}$$



$\xrightarrow[W_{\text{vertex}}(p)]{} \mathsf{d}(a) \begin{vmatrix} \mu & \lambda & \eta \\ b & c & a \end{vmatrix}$,



$\xrightarrow[W_{\text{vertex}}(p)]{} \begin{vmatrix} b & \lambda & a \\ \mu & c & \eta \end{vmatrix}$

Theorem 6.2 (Face model for $\langle \Gamma, \text{col} \rangle_r^{Nil}$, [**14**]). *It holds:* $\langle \Gamma, \text{col} \rangle_r^{Nil} = \mathsf{d}(\text{col}(e_1))\ Z_{a_0,a_1}(T_\Gamma)$.

2. Asymptotical behavior of nilpotent spin-networks

The invariant provided in Theorem 6.1 depends continuously on the colors of the graph; moreover in [**14**] (Remark 3.7) we proved that the invariant is still defined for a link L whose colors are 0 and in that case it coincides with the Kashaev invariant. Thus one can hope to attack the volume conjecture for a hyperbolic knot K by studying $\langle K \rangle_r^{Nil}$.

Indeed the computation of the Kashaev invariant is in general not easy: the colored Jones polynomials may be computed either through the R-matrix approach or through the so-called *shadow-state* sums but only in their *unnormalized* form (i.e. $J_r(\bigcirc) = (-1)^r[r]$) and the Kashaev invariant is the evaluation at $q = \exp(\frac{i\pi}{r})$ of $\frac{J_r(K)}{J_r(\bigcirc)}$. Typically shadow-state sums provide much simpler models, but standard quantum $6j$-symbols involved in these sums may have poles at the roots of unity at which the volume conjecture is stated; clearly these poles cancel out at the level of the whole state-sum (because the colored Jones polynomials are polynomials!) but not at a term-by-term level in the sum. Thus it may turn out to be useful to have shadow state-sums for the Kashaev invariants whose summands are all regular at the root of unity.

It seems to be the case if one uses the model provided in Subsection 1.1. Indeed it was proved in [**14**] that the nilpotent $6j$-symbols: $\begin{vmatrix} a & b & e \\ d & c & f \end{vmatrix}$, which a-priori are defined only for colors in $\mathbb{C} \setminus X_r$, are actually Laurent polynomials in $q^a, \ldots, q^f$ and thus can be continuously extended on all $\mathbb{C}$. Moreover the study of the asymptotical behavior (when $r \to \infty$) of these symbols was studied and proved to be related to hyperbolic geometry, consistently with the expectations of the Volume Conjecture:

THEOREM 6.3 (Asymptotics of nilpotent $6j$-symbols, [**14**], Theorem 2.2). *Let T be the hyperbolic truncated tetrahedron with oriented labeled edges as in Figure 1, and let $0 < \theta_a$, θ_b, θ_c, θ_d, θ_e, $\theta_f < \pi$ be the internal dihedral angles at the edges. Let $a_r, b_r, c_r, d_r, e_r, f_r$ be sequences of integers such that $\lim_{r\to\infty} \frac{\pi a_r}{r} = \theta_a, \ldots, \lim_{r\to\infty} \frac{\pi f_r}{r} = \theta_f$. Put $\overline{a_r} = -a_r$, $\cdots$, $\overline{f_r} = -f_r$. Using these parameters, the volume of T is given as follows:*

$$\mathrm{Vol}(T) = \lim_{r\to\infty} \frac{\pi}{2\,r} \log \left(\left| \begin{matrix} a_r & b_r & e_r \\ d_r & c_r & f_r \end{matrix} \right| \left| \begin{matrix} \overline{a_r} & \overline{b_r} & \overline{e_r} \\ \overline{d_r} & \overline{c_r} & \overline{f_r} \end{matrix} \right| \right).$$

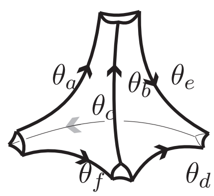


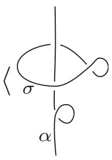
FIGURE 1. A truncated hyperbolic tetrahedron T.

The above result is a first evidence showing that the volume conjecture may be attacked through the study of nilpotent spin networks, and that hyperbolic geometry actually shows up when considering asymptotical limits of quantum invariants at roots of unity. We will not provide the proof of the theorem but we will bound ourselves to remark that the proof exploits the peculiar behavior of $6j$-symbols outlined in Lemma 1.15 of [**14**]: they are finite sums positive real numbers each of which is growing exponentially fast. To compute the overall exponential growth of the sum is therefore sufficient to identify the summands with maximal growth rate. This key point is what makes relatively easy to compute the asymptotical behavior in our case: in general one has to deal with oscillating complex valued sums whose behavior is quite complicated. This property is similar to that we used in [**11**] for $6j$-symbols associated to the representation theory of $U_q(sl_2)$ for generic q to prove an analogue of the generalized volume conjecture for tetrahedra. The rest of the proof shows that the maximal growth rate, expressed in terms of the angles of the tetrahedron, satisfies the same Schläfli differential equation as the volume function, thus they differ by a constant and finally that in a special case (the case of an ideal tetrahedron) the two functions are equal.

3. Nilpotent Reshetikhin-Turaev invariants

In a joint work with Nathan Geer and Bertrand Patureau-Mirand we used nilpotent invariants of links to define invariants of 3-manifolds obtained by surgery on framed links in S^3, and possibly containing an additional framed oriented graph Γ. It turns out that the colorings on the surgery links induce a cohomology class on the surgered manifold $\omega \in H^1(M \setminus \Gamma, \mathbb{C}/2\mathbb{Z})$ and that the invariants we define are actually invariants of such three-uples (M^3, Γ, ω). Our invariant are defined "à la Reshetikhin-Turaev" and are the surgery counterpart of the Turaev-Viro like invariants defined by N. Geer, B. Patureau-Mirand and V. Turaev in [**26**].

In the rest of this section we let $G = \mathbb{C}/2\mathbb{Z}$ (as an additive group), $\overline{X} = \mathbb{Z}/2\mathbb{Z} \subset G$, and Δ_+ be the scalar defined by the following equality:

$\langle$  $\rangle_r^{Nil} = \Delta_+ \langle$  $\rangle_r^{Nil}$

let also $\Delta_- = \overline{\Delta_+}$. It turns out that if $r \notin 4\mathbb{Z}$ then $\Delta_\pm \neq 0$, thus we suppose $r \notin 4\mathbb{Z}$ from now on.

Let M be a compact oriented 3-manifold M and $\Gamma \subset M$ be a framed, oriented, trivalent graph whose edges are colored by elements of $\mathbb{C} \setminus X_r$. Let also $\omega \in H^1(M \setminus \Gamma; G)$.

If M is presented as integral surgery over a link L in S^3, then we can see Γ as a subset of $S^3 \setminus L$ and ω induces a G-coloring g_ω on $L \cup \Gamma$ defined as $g_\omega(e_i) = \omega(m_i)$, for each edge e_i of $L \cup \Gamma$ where m_i is the meridian of e_i oriented to have positive linking with e_i (here we are implicitly using the fact that m_i induces a well-defined class in $H_1(M; \mathbb{Z})$ because it is contained in $S^3 \setminus L$).

Definition 6.4. We say that (M, Γ, ω) is *a compatible triple* if for each edge e of Γ the image in G of the $\mathbb{C}$-color of Γ associated to e is $g_\omega(e)$. We say that a surgery presentation via $L \subset S^3$ for a compatible 3-uple (M, Γ, ω) is *computable* if $g_\omega(e) \in \mathbb{C}/2\mathbb{Z} \setminus \overline{X}$ for all edges e of L.

Let V_α be the nilpotent r-dimensional simple representation of $U_q(sl_2)$ whose highest weight is $\alpha + r - 1$ and on which the twist acts as $q^{\frac{\alpha^2 - (r-1)^2}{2}}$ Id. If $\bar{\alpha} \in G$ then we say that $\Omega_{\bar{\alpha}} = \sum_{k \in H_r} \mathsf{d}(\alpha + k) V_{\alpha+k}$ is a Kirby color of degree $\overline{\alpha}$ if $\alpha \in \mathbb{C}$ is such that the image of α in $\mathbb{C}/2\mathbb{Z}$ is $\overline{\alpha}$. We can "color" a knot K with a Kirby color $\Omega_{\bar{\alpha}}$: let $K(\Omega_{\bar{\alpha}})$ be the formal linear combination $\sum_{k \in H_r} \mathsf{d}(\alpha + k) K(\alpha + k)$ where $K(\alpha + k)$ is the knot K colored by $\alpha + k$. One acts similarly on a link L. We extend by linearity the definition of $\langle \rangle_r^{Nil}$ to links whose components are colored with some Kirby color $\Omega_{\bar{\alpha}}$. The following is proved in [**12**]:

Theorem 6.5. *If L is a colored link which gives rise to a computable surgery presentation of a compatible triple (M, Γ, ω), then*

$$\mathsf{N}(M, \Gamma, \omega) = \frac{\langle L \cup \Gamma \rangle_r^{Nil}}{\Delta_+^p \ \Delta_-^s}$$

is a well defined topological invariant (i.e. depends only of the homeomorphism class of the triple (M, Γ, ω)), where (p, s) is the signature of the linking matrix of the surgery link L and each component L_i is colored by a Kirby color $\Omega_{g_\omega(L_i)}$.

The computability hypothesis is just a technical one: every compatible triple either has a computable surgery presentation or is closely related to a triple which has a computable surgery presentation. The following result takes care of this technical point allowing to extend the definition of the invariant to almost any three-uple (M, Γ, ω).

PROPOSITION 6.6. *Let (M, Γ, ω) be a compatible triple and L be a link which gives rise to a surgery presentation of M. If $L \cup \Gamma$ has an edge e such that $g_\omega(e) \in (\mathbb{C} \setminus \mathbb{Z})/2\mathbb{Z}$ then there exists a surgery presentation of (M, Γ, ω) which is computable. In particular, the triple (M, Γ, ω) has computable surgery presentation if $\Gamma \neq \emptyset$ and Γ has an edge whose color is in $\mathbb{C} \setminus \mathbb{Z}$.*

In [**12**] we also showed how to extend the definition of the invariant when $\Gamma \neq \emptyset$ but all the edges of Γ are colored by integers: we do not detail this extension here because its importance is mainly technical. On contrast, it is worth mentioning what happens in the case when $\Gamma = \emptyset$ and all the edges of L are colored by an integer:

THEOREM 6.7. *Let (M, Γ, ω) be compatible triple. For $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ let $\Gamma \sqcup u_\alpha$ be the disjoint union of Γ with the unknot u_α colored with α. Let $\omega \sqcup \omega_\alpha$ be the unique element of $H^1(M \setminus (\Gamma \sqcup u_\alpha), \mathbb{C}/2\mathbb{Z})$ such that $\omega \sqcup \omega_\alpha$ restricts to ω and $(M, \omega \sqcup \omega_\alpha, \Gamma \sqcup u_\alpha)$ is a compatible triple. Define*

$$\mathsf{N}^0(M, \omega, \Gamma) = \frac{\mathsf{N}(M, \omega \sqcup \omega_\alpha, \Gamma \sqcup u_\alpha)}{\mathsf{d}(\alpha)}.$$

Then $\mathsf{N}^0(M, \omega, \Gamma)$ is a well defined topological invariant (i.e. depends only of the homeomorphism class of the compatible triple (M, Γ, ω)). Moreover, if (M, Γ, ω) has a computable surgery presentation (i.e. if N is defined) then $\mathsf{N}^0(M, \Gamma, \omega) = 0$.

Notice that the above theorems can be used to define an invariant for every compatible triple (M, Γ, ω). In particular, if the invariant N is well defined then also N^0 is but it is zero, otherwise (for instance when $\Gamma = \emptyset$ and ω is the reduction of an integer class) N is not defined but N^0 is (and it is not necessarily 0: it is non-trivial for the Poincaré sphere). In Subsection 3.2 we give examples to show that N and N_0 are computable and non-trivial.

3.1. Key points about the above invariant. The construction of $\mathsf{N}(M, \Gamma, \omega)$ mimicks that of Reshetikhin-Turaev invariants, so let us spell which are the main differences and difficulties. First of all the underlying theory of representations has an infinite number of simple modules V_α, all of dimension r and indexed by $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, where $\alpha + r - 1$ is the highest weight of V_α; in the case of standard Reshetikhin-Turaev invariant the simple modules are only $V_1, \ldots, V_r$ with $dim(V_i) = i$. The modules V_α and $V_{\alpha+2r}$ are isomorphic as $U_q(sl_2)$-modules but distinguished as objects in the braided category of such modules: indeed the twist acts by $\exp(\frac{2\pi i(\alpha^2-(r-1)^2)}{2r})\,\mathrm{Id}$ on each of them and these scalars are different. The fact that the number of simple modules is not finite, makes the definition of the "Kirby color" (used to perform surgery) more delicate: for each $\overline{\alpha} \in G$ one choses $\alpha \in \mathbb{C}$ representing $\overline{\alpha}$ and defines $\omega_{\overline{\alpha}} = \sum_{k \in H_r} \mathsf{d}(\alpha + k) V_{\alpha+k}$. It turns out that if a coloring of the union of a link $L \subset S^3$ and a KTG $\Gamma \subset S^3 \setminus L$ is compatible (and this condition is crucial) then the choices of the $\alpha's$ to represent $\overline{\alpha}'s$ on each component of L is irrelevant.

The role of the coloring is clarified once one introduces a cohomology class on the manifold obtained by surgery on L: the compatibility condition is equivalent to stating that there exists a cohomology class $\omega \in H^1(M \setminus \Gamma; G)$ such that for each edge e of $\Gamma \cup L$ it holds $\omega(m) = \mathrm{col}(e)$ where m is the positive meridian of the edge.

A second delicate point is the computability of the colors: if some color is in $\mathbb{Z}$ then the invariant $\langle L \cup \Gamma, \mathrm{col}\rangle_r^{Nil}$ is not defined; thus, making sure that one can actually find a presentation of (M, Γ, ω) whose coloring is computable and showing that two such presentations are related by Kirby moves through computable presentations is technically more complicated than in the standard case.

3.2. Examples and applications.

EXAMPLE 6.8 (Distinguishing lens spaces). In [**29**], Remark 3.9, it was observed that the standard $U_q(sl_2)$ Reshetikhin-Turaev invariants cannot distinguish the lens spaces $L(65, 8)$ and $L(65, 18)$: it turns out that the invariants $N(M, \Gamma, \omega)$ can and, as a consequence, are independent on the standard Reshetikhin-Turaev invariants. Indeed one can present $L(65, 8)$ and $L(65, 18)$ as surgeries over chain links and, using computer algebra software, compute numerically $S_r(L(p, q)) = \sum_{\omega \in H^1(L(p,q);G), \omega \neq 0} \mathsf{N}(L(p, q), \emptyset, \omega)$. It turns out that $S_3(L(65, 8)) \neq S_3(L(65, 18))$.

EXAMPLE 6.9 (Studying self-diffeomorphisms of a rational homology sphere). Let M be an oriented rational homology sphere, $c_1, \ldots c_n$ and $d_1, \ldots d_n$ be two distinct minimal sets of generators of $H_1(M; \mathbb{Z})$ such that the order of c_i and of d_i are the same for each i. Let $\omega \in H^1(M; G)$ and let $\omega' \in H^1(M; G)$ be defined by $\omega'(c_i) = \omega(d_i)$. If $N(M, \emptyset, \omega') \neq N(M, \emptyset, \omega)$ then there exists no positive self-diffeomorphism $\phi : M \to M$ such that $\phi_*(c_i) = d_i$. In particular one can apply this argument to distinguish, for instance, the following two generators of $H_1(L(5, 1); \mathbb{Z})$: present $L(5, 1)$ as the surgery over a 5-framed unknot in S^3; one generator c_1 is represented by the meridian of the unknot, and another, d_1, by its double.

4. Open questions and perspectives.

In this section we discuss the many questions which are still open about the nilpotent Reshetikhin-Turaev invariant defined above. As stated above, $\mathsf{N}(M, \Gamma, \omega)$ is independent on the standard Reshetikhin-Turaev invariants so a natural general question is:

QUESTION 6.10. *What are the relations between $\mathsf{N}(M, \Gamma, \omega)$ and the other known quantum and classical invariants of the three-uple? And in particular with the Turaev-Viro type invariants introduced in* [**26**]*?*

Motivated by Example 6.8 for $L(65, 8)$ and $L(65, 18)$ one may ask:

QUESTION 6.11. *Does N distinguish all the lens spaces for r big enough? Does the whole set of N (for all r)?*

A particular case which can always be studied is when $\Gamma = \emptyset$ and $\omega = 0$:

QUESTION 6.12. *What are the relations between $\mathsf{N}^0(M, \emptyset, 0)$ and the other known quantum and classical invariants of the three-uple?*

Although at the moment there are no experimental evidences, motivated by the relations of $\langle L, 0\rangle_r^{Nil}$ with the r^{th} Kashaev invariant of L, one may try and ask whether a general version of the Volume Conjecture may be formulated for three manifolds (instead than for knots in S^3):

QUESTION 6.13. *What is the asymptotical behavior (if any) of* $\mathsf{N}^0(M, \emptyset, 0)$ *as* $r \to \infty$*? More in general what is (if any) the asymptotical of* $\mathsf{N}(M, \Gamma, r\omega)$ *as* $r \to \infty$*?*

Theorem 6.3 may be considered as a first positive evidence for the second question in the case when $M = S^3$, Γ is the tetrahedral graph and ω is the unique element of $H^1(S^3 \setminus \Gamma; G)$ such that for each edge e of Γ, the value of ω on the meridian of the edge is $\omega(m) = \mathrm{col}(e)$.

The main interest of the above question is that the only extension of the Volume Conjecture for manifolds is due to Benedetti and Baseilhac and it involves the Quantum Hyperbolic Invariants which often may be complicated to compute. On contrast the invariants N and N^0 are defined through surgery presentations and thus effectively computable.

Even without studying the asymptotical behavior of nilpotent invariants, one may ask what is the functor (if any) underlying these invariants:

QUESTION 6.14. *What is the TQFT (or its generalization) underlying the construction of* $\mathsf{N}(M, \Gamma, \omega)$*?*

Preliminary observations seem to show that a HQFT structure underlies the invariants: we are planning to work on the preceding questions in future work joint with Nathan Geer and Bertrand Patureau-Mirand.

Nilpotent representations are not the most general ones when q is a root of unity: there are cyclic ones (on which E^r and F^r do not act as 0).

QUESTION 6.15. *Can one build link invariants and* 3*-manifold invariants using general (cyclic) modules of* $U_q(sl_2)$ *when* q *is a root of unity?*

There are first positive evidences to the above question due to Kashaev and Reshetikhin, but at present no 3-manifold invariant is known based on surgery presentations and cyclic modules.

APPENDIX A

Complete list of publications

The complete list of our works including those not discussed in the present memoire:

- “Nilpotent Reshetikhin-Turaev invariants of 3-manifolds”, with N. Geer et B. Patureau-Mirand, to be made available in Summer 2011.
- “Generating series of classical spin networks‘”, with J. Marché, arXiv:1103:5644, 44 pages (2011).
- “On $SL(2,\mathbb{C})$ quantum $6j$-symbol and its relation to the hyperbolic volume”, with Jun Murakami, to appear in Quantum Topology (2011).
- “Integrality of Kauffman brackets of trivalent graphs”, arxiv:0908.0542, 25 pages (submitted).
- “$6j$-symbols, hyperbolic structures and the Volume Conjecture”, Geometry & Topology 11 (2007), 1831–1853.
- “Colored Jones invariants of links in $\#S^2 \times S^1$ and the Volume Conjecture”, J. London Math. Soc. (2) 76 (2007), no. 1, 1–15.
- “3-manifolds efficiently bound 4-manifolds”, with D. Thurston, J. Topol, 1 (2008), no. 3, 703–745.
- “Triangulations of 3-manifolds, hyperbolic relative handlebodies, and Dehn filling”, with R. Frigerio, B. Martelli et C. Petronio, Commentarii Mathematici Helvetici 87 (2007), no. 4, 903–933.
- “Stein domains and branched shadows of 4-manifolds”, Geometriae Dedicata, 121, no. 1 (2006), 89–111.
- “Branched shadows and complex structures of 4-manifolds”, Journal of Knot Theory and its Ramifications 17 (2008), no. 11, 1429–1454.
- “Complexity of 4-manifolds”, Experimental Mathematics 15, no. 2 (2006), 237–249.
- “A calculus for branched spines of 3-manifolds”, Mathematische Zeitschrift 251 (2005), 2, 427–442.
- “A short introduction to shadows of 4-manifolds”, Fundamenta Mathematicae 188 (2005), 271–291.
- “Shadows and branched shadows of 3 and 4-manifolds”, Ph.D. Thesis, published in“Tesi di Perfezionamento della Scuola Normale Superiore di Pisa” (2005).
- “On a new proof of the JSJ Theorem”, Rendiconti del Seminario Matematico dell’Università e del Politecnico di Torino, vol. 60 n. 3 (2002).
- “Euler structures and Reidemeister torsion of three manifolds”, with R. Benedetti, C. Petronio, and G. Amendola, Rendiconti dell’Istituto Matematico dell’Università di Trieste, Supp. Vol. XXXII pp. 1–23 (2001).

Bibliography

1. A. ABDESSELAM, "On the volume conjecture for classical spin networks", arXiv:0904.1734 (2009).
2. Y. AKUTSU, T. DEGUCHI AND T. OHTSUKI, "Invariants of colored links", J. Knot Theory Ramifications (1992), 1, 161–184.
3. J. BAEZ, "Spin foam models" Class. Quantum Grav. 15 (1998) 1827–1858.
4. J.W. BARRETT, C. M. STEELE, "Asymptotics of relativistic spin networks", Class. Quantum Grav. 20 (2003) 1341–1361.
5. J. BERNSTEIN, I. FRENKEL, M. KHOVANOV, " A categorification of the Temperley-Lieb algebra and Schur quotients of $U(sl_2)$ via projective and Zuckerman functors", Selecta Math. (N.S.) 5 (1999), 2, 199–241.
6. C. BLANCHET, N. HABEGGER,G. MASBAUM, P. VOGEL, "Topological quantum field theories derived from the Kauffman bracket" Topology 34 (1995), no. 4, 883–927.
7. D. BAR-NATAN,S. GAROUFALIDIS,"On the Melvin-Morton-Rozansky conjecture", Invent. Math. 125 (1996), no. 1, 103–133.
8. L. CRANE, I. FRENKEL, " Four-dimensional topological quantum field theory, Hopf categories and the canonical bases", J. Math. Phys. 35 (1994), 10, 5136–5154.
9. L. CHARLES, "On the quantization of polygon spaces", Asian J. Math. 14 (2010), no. 1, 109–152.
10. J. S. CARTER, D. E. FLATH, M. SAITO, "The classical and quantum $6j$-symbols", Math. Notes, Princeton University Press (1995).
11. F. COSTANTINO, "$6j$-symbols hyperbolic structures and the Volume Conjecture", Geom. Topol. 11, (2007), 1831–1853.
12. F. COSTANTINO, N. GEER, B. PATUREAU-MIRAND, "Nilpotent Reshetikhin-Turaev invariants of 3-manifolds", to be made available during Summer (2011).
13. F. COSTANTINO,J. MARCHÉ, "Generating series and asymptotics of classical spin networks", arXiv:1103:5644, 44 pages (2011).
14. F. COSTANTINO,J. MURAKAMI, "On $SL_2(\mathbb{C})$-quantum $6j$-symbol and its relation to the hyperbolic volume", to appear in Quantum Topology (2011), arXiv:1005.4277, 30 pages.
15. F. COSTANTINO, "Integrality of Kauffman brackets of trivalent graphs", arxiv:0908.0542 (2009), 25 pages.
16. F. COSTANTINO, "Colored Jones invariants of links in $\#S^2 \times S^1$ and the Volume Conjecture", J. London Math. Soc. (2) 76, (2007), no 1, 1–15.
17. F. COSTANTINO, D. P. THURSTON, "3-manifolds efficiently bound 4-manifolds", J. Topology (1), 3, (2008).
18. V. G. DRINFELD, "Hopf algebras and the quantum Yang-Baxter equation", Dokl. Akad. Nauk SSSR 283 (1985), no. 5, 1060–1064.
19. L. Freidel, D. Louapre, "Asymptotics of 6j and 10j symbols", Class.Quant.Grav. 20 (2003) 1267–1294.
20. C. FROHMAN, J. KANIA-BARTONSZYNSKA, "Shadow-world evaluation of the Yang-Mills measure", Alg. Geom & Topol. 4, (2004), 311–332.
21. I. FRENKEL, M. KHOVANOV, C. STROPPEL, "A categorification of finite dimensional representations of quantum $\mathfrak{sl}_2$ and their tensor products", Sel. Math. New ser. 12 (2006), Birkhäuser Verlag, Basel.

22. I. Frenkel, C. Stroppel, J Sussan, "Categorifying fractional Euler characteristics, Jones-Wenzl projector and $3j$-symbols with applications to exts of Harish-Chandra bimodules", preprint: arXiv:1007.4680, 64 pages (2010).
23. S. Garoufalidis, T. Le, "Asymptotics of the colored Jones function of a knot", xxx.arXiv.org/math.GT/0508100 (2007).
24. S. Garoufalidis, T. Q. Le, "The colored Jones function is q-holonomic", Geom. Topol. 9 (2005), 1253–1293 (electronic).
25. N. Geer, B. Patureau-Mirand, "Polynomial 6-symbols and states sums", preprint: arxiv 09111353, 33 pages (2009).
26. N. Geer, B. Patureau-Mirand and V. Turaev, "Modified $6j$-symbols and 3-manifold invariants", preprint: arxiv 0910.1624, 37 pages (2009).
27. S. Gukov, "Three dimensional quantum gravity, Chern-Simons theory and the A-polynomial", Commun. Math. Phys. 255 (2005), 577–627.
28. S. Garoufalidis, R. Van der Veen, "Asymptotics of classical spin networks", www.arXiv.org/0902.3113 (2009).
29. L. Jeffrey, "Chern-Simons-Witten invariants of lens spaces and torus bundles, and the semiclassical approximation", Comm. Math. Phys. 147 (1992), no. 3, 563–604.
30. M. Jimbo, "A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation", Letter. Math. Phys. 10 (1985), 63–69.
31. V. Jones, "A polynomial invariant for knots via von Neumann algebras", Bull. Amer. Math. Soc. (N.S.) 12 (1985), no. 1, 103–111.
32. R. Kashaev, "The hyperbolic volume of knots from quantum dilogarithm", Modern Phys. Lett. A, 39 (1997), 269–275.
33. R. Kashaev, "A link invariant from quantum dilogarithm", Mod. Phys. Lett. A 10 (1995), 1409–1418.
34. L. H. Kauffman, "State models and the Jones polynomial", Topology 26 (1987), no. 3, 395–407.
35. P.W. Kasteleyn, "The statistics of dimers on a lattice", Physica 27 (1961), 1209–1225.
36. L. Kauffman, S. Lins, " Temperley-Lieb recoupling theory and invariants of 3-manifolds", Annals of Mathematics Studies, 134. Princeton University Press, Princeton, NJ, 1994.
37. R. Kirby, P. Melvin, "The 3-manifold invariants of Witten and Reshetikhin-Turaev for $sl(2,\mathbb{C})$", Invent. Math. 105, 473–545 (1991).
38. M. Khovanov, "A categorification of the Jones polynomial", Duke Math. J. 101 (2000), no. 3, 359–426.
39. A. N. Kirillov, N. Yu. Reshetikhin, "Representations of the algebra $U_q(\mathrm{sl}(2))$, q-orthogonal polynomials and invariants of links", Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), 285–339, Adv. Ser. Math. Phys., 7, World Sci. Publ., Teaneck, NJ, 1989.
40. R. Kashaev, O. Tirkkonnen, "A proof of the volume conjecture on the torus knots", J. Math. Sci. 115 (2003), 2033–2036.
41. T.Q. Le, "Integrality and symmetry of quantum link invariants", Duke Math. J. 102 (2000), no. 2, 273–306.
42. T.Q. Le and A.T. Tran, "On the Volume Conjecture for cables of knots", Journal of Knot Theory and its Ramifications 19 (2010), 1673–1791.
43. H. Murakami, J. Murakami, "The colored Jones polynomial and the simplicial volume of a knot", Acta Math. 186 (2001), 85–104.
44. H. Murakami, J. Murakami, "Some limits of the colored Jones polynomial of the Figure eight knot", Kyungpook Math. J. 44 (2004) 369–383.
45. H. Murakami, J. Murakami, "Asymptotic behaviors of the colored Jones polynomial of a torus knot" xxx.arXiv.org/math.GT/0405126 (2004).
46. H. Murakami, J. Murakami, M. Okamoto, T. Takata, Y. Yokota, "Kashaev's conjecture and the Chern-Simons invariants of knots and links", Exper. Math. 11 (2002), 427–435.
47. H. Murakami, "A version of the Volume Conjecture", Adv. Math. 211, N. 2 (2007), 678–683.
48. G. Masbaum , P. Vogel, "Three-valent graphs and the Kauffman bracket", Pac. J. Math. Math. 164, N. 2 (1994).

49. R. Penrose, “Applications of negative dimensional tensors”, in Combinatorial Mathematics and its Applications, (1971) 221–244.
50. G. Ponzano, T. Regge, “ Semi-classical limit of Racah coefficients”, Spectroscopic and group theoretical methods in physics, North Holland (1968) 1–58.
51. G. Racah, “Theory of complex spectra I, II, III,IV”, Phys. Rev. 61 (1942) 186–197, ibid 62 (1942) 438–462, ibid 63 (1943) 367–382, ibid 76 (1949) 1352-1365.
52. I. Rivin, “ A characterization of ideal polyhedra in hyperbolic 3-space”, Ann. of Math. 143 (1996), 51–70.
53. N.Yu. Reshetikhin, V. G. Turaev, “Ribbon graphs and their invariants derived from quantum groups”, Comm. Math. Phys. 127 (1990), no. 1, 1–26.
54. N. Reshetikhin, V. Turaev, “Invariants of 3-manifolds via link polynomials and quantum groups”, Invent. Math. 103, (1991), 547–597.
55. C. Rovelli, “ Loop quantum gravity”, Living Rev. Relativ. 1 (1998), 1998-1, 68 pp. (electronic).
56. J. Roberts, “Classical 6j-symbols and the tetrahedron”, Geom. Topol. 3 (1999), 21–66.
57. I.Kh. Sabitov, “The volume of a polyhedron as a function of its metric”, Fundam. Prikl. Mat. 2(4), (1996), 1235–1246.
58. T. Q. Le,A. T. Tran, “On the Volume Conjecture for cables of knots”, arXiv:0907.0122 (2009).
59. V. G. Turaev, “Quantum invariants of knots and 3-manifolds”, de Gruyter Studies in Mathematics, Vol. 18, Walter de Gruyter & Co., Berlin, 1994.
60. V. G. Turaev, “Skein quantization of Poisson algebras”, Ann. Scient. Éc. Norm. Sup. 4^{eme} série, 24, 1991, 535–704.
61. Y.U. Taylor,C.T. Woodward, “$6j$ symbols for $U_q(sl_2)$ and non-euclidean tetrahedra”, Selecta Math. (N.S.) 11 (2005), no.3-4, 539–571.
62. R. Van der Veen, “The volume conjecture for Whitehead chains”, xxx.arXiv.org/math.GT/0611181 (2006).
63. O. Viro, “Quantum relatives of the Alexander polynomial”, St. Petersburg Math. J. 18 (2007), 391–457.
64. B. Westbury, “A generating function for spin network evaluations”, Knot theory (Warsaw, 1995), 447–456, Banach Center Publ., 42, Polish Acad. Sci., Warsaw, 1998.
65. E. P. Wigner, “Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren”, J. W. Edwards, Ann Arbor, Michigan, 1944. viii+332 pp.
66. E. Witten, “Quantum field theory and the Jones polynomial”, Comm. Math. Phys. 121 (1989), no. 3, 351–399.
67. Y. Yokota, “On the potential function functions for the hyperbolic structures of a knot complement”, Geom. & Topol. Monographs 4 (2002), 303–311.
68. H. Zheng, “Proof of the volume conjecture for Whitehead doubles of a family of torus knots”, xxx.arXiv.org/math.GT/0508138 (2005).

Nous passons en revue plusieurs aspects et résultats connus sur les réseaux de spin classiques et quantiques. Nous commençons par l'étude des séries génératrices de ces objets dans le cas classique, et puis nous discutons les problèmes d'intégralité dans le cas quantique. Puis nous passons à l'étude du comportement asymptotique des réseaux de spin, d'abord dans le cas classique et puis dans le cas quantique, en reliant ce comportement à la géométrie des polyèdres. Nous terminons par une exposition des résultats connus à présent sur les réseaux de spin nilpotents et sur leur utilisation pour la définition d'invariants de variétés de dimension 3 "à la Reshetikhin-Turaev".

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