Introduction to Brownian motion

François Chapon

Contents

1. Introduction and definition 1
2. Properties of the Brownian motion 3
2.1. Markov property 3
2.2. Path properties 5
2.3. Reflection principle 8
2.4. Zero-one laws 11
2.5. Zeroes of Brownian motion 12
3. Law of iterated logarithm 13
Appendix A. Classical 0–1 laws 15

1. Introduction and definition

Brownian motion is a fundamental object in Probability theory. It is named after the botanist Robert Brown, who described the following phenomenon in 1827: looking through a microscope of pollen immersed in water, he observed that particles moved erratically and completely at random. The stochastic process now known as Brownian motion was formally modeled by Louis Bachelier in 1900 and by Albert Einstein in 1905. One way to introduce it, is to consider the diffusive limit of a random walk.

So consider a random walk on $\mathbb{Z}$: let $(\varepsilon_k)_{k\geq 1}$ be a sequence of i.i.d. random variables distributed according to the Bernoulli distribution $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$, and define $S_0 = 0$ a.s. and for all $n \geq 1$,

$$S_n = \varepsilon_1 + \cdots + \varepsilon_n.$$

The central limit theorem (CLT) gives us the asymptotic position of the random walk at time n: roughly speaking, one has in distribution for large n,

$$S_n \approx \sqrt{n}N,$$

where N is a $\mathcal{N}(0,1)$ random variable. We want a more precise information on the *trajectory* of the walk, i.e. on the whole process $(S_n\,;\, n \geq 0)$. It is thus natural to consider different times $0 = t_0 < t_1 < \cdots < t_k$ and to consider the walk at these different times. Define

$$n_i = \lfloor nt_i \rfloor,$$

so that $\frac{n_i}{n} \underset{n\to\infty}{\longrightarrow} t_i$. The increments

$$S_{n_i} - S_{n_{i-1}} = \varepsilon_{n_{i-1}+1} + \cdots + \varepsilon_{n_i}$$

for $1 \leq i \leq k$, are thus independent. Using now the multidimensional central limit theorem, one has

$$\left(\frac{S_{n_i} - S_{n_{i-1}}}{\sqrt{n}}\,;\, 1 \leq i \leq k\right) \xrightarrow[n\to\infty]{\text{(d)}} (N_1, \ldots, N_k),$$

where $N_1, \ldots, N_k$ are independent random variables such that

$$N_i \sim \mathcal{N}(0, t_i - t_{i-1}).$$

Equivalently, this can be stated as

$$\left(\frac{S_{n_i}}{\sqrt{n}}\,;\, 1 \leq i \leq k\right) \xrightarrow[n\to\infty]{\text{(d)}} (N_1, N_1 + N_2, \ldots, N_1 + \cdots + N_k).$$

This gives the asymptotic position of the walk for a finite number of times. It is thus natural to consider the walk as a function of time: define

$$S_t^{(n)} := \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}, \quad t \geq 0$$

(or better: a linear interpolation of the above). The previous convergence is thus the convergence of the "finite dimensional" distribution of the (continuous time) stochastic process $(S_t^{(n)}\,;\, t \geq 0)$: for all k, and all $t_1 < \cdots < t_k$,

$$(S_{t_1}^{(n)}, \ldots, S_{t_k}) \xrightarrow[n\to\infty]{\text{(d)}} (B_{t_1}, \ldots B_{t_k}),$$

where the stochastic process $(B_t\,;\, t \geq 0)$ satisfies:

- $B_0 = 0$ a.s. ;
- for all k, for all $0 = t_0 < t_1 < \cdots < t_k$, the random variables
$$\left(B_{t_i} - B_{t_{i-1}}\,;\, 1 \leq i \leq k\right)$$
are independent and such that
$$B_{t_i} - B_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1});$$
- $t \mapsto B_t$ is almost surely continuous (from $\mathbb{R}_+$ to $\mathbb{R}$).

We now put the following definition:

Definition 1.1. *The above process $B = (B_t\,;\, t \geq 0)$ is called a standard Brownian motion (BM for short).*

Remark 1.1.
- The Brownian motion is thus a stochastic process with independent and stationary increments (like the Poisson process) and with a.s. continuous paths (unlike the Poisson process!). It can be understood as a random walk in continuous time and is nothing more than a very long random walk.
- The a.s. continuity of paths is not obvious, and it is a non trivial fact that this process exists! We will admit its existence in this course, a proof will be seen next year in the stochastic program of the M2RI.
- The term standard refers to the fact that the Brownian motion starts from 0. We can start the Brownian motion from any $x \in \mathbb{R}$ by putting $B_t^x = B_t + x$.

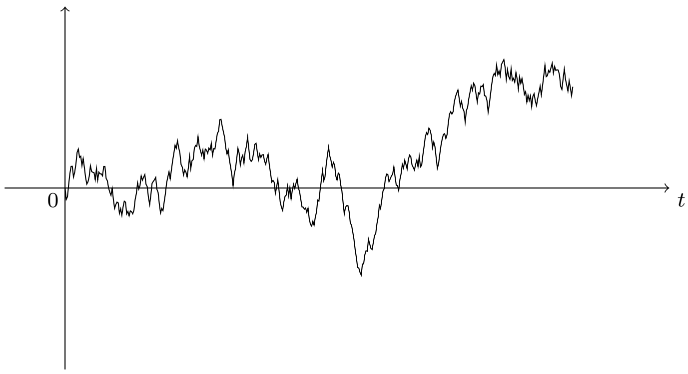

Figure 1. A Brownian trajectory.

- The above convergence of "finite dimensional" distributions is not sufficient in general. One wants to consider function of the whole trajectory $(S_t^{(n)}\,;\,t \geq 0)$, so that for instance $\sup_{t\in[0,1]} S_t^{(n)}$ should converge towards $\sup_{t\in[0,1]} B_t$. One thus needs a "good notion" of convergence in distribution for the the whole process $(S_t^{(n)}\,;\,t \geq 0)$, which again will be seen in the M2RI: this is the content of Donsker's invariance principle which states that
$$(S_t^{(n)}\,;\,t \geq 0) \underset{n\to\infty}{\longrightarrow} (B_t\,;\,t \geq 0)$$
in distribution (to be made precise!). Note that as in the CLT, the limit object is universal and does not depend on the law of the random variables ε_k.
- Thus in this course, we assume that BM exists and that no problem of measurability occurs, and we are going to play a bit with this process.

Remark 1.2. Another characterization of the finite dimensional distributions of the Brownian motion is the following. For any k, and any $t_1, \ldots, t_k$, the vector $(B_{t_1}, \ldots, B_{t_k})$ is a centered Gaussian vector with covariance given by
$$\mathbb{E}(B_t B_s) = s \wedge t.$$
Indeed, assume $t > s$. Then
$$\begin{aligned}\mathbb{E}(B_t B_s) &= \mathbb{E}\left[(B_t - B_s + B_s)B_s\right] = \mathbb{E}\left[(B_t - B_s)B_s\right] + \mathbb{E}\left[B_s^2\right]\\ &= \mathbb{E}\left[(B_t - B_s)\right]\mathbb{E}\left[B_s\right] + \mathbb{E}\left[B_s^2\right],\end{aligned}$$
since the increments $B_t - B_s$ and B_s are independent. Since $\mathbb{E}\left[(B_t - B_s)\right] = 0$ and $\mathbb{E}\left[B_s^2\right] = s$, the claim follows.

2. Properties of the Brownian motion

In the following, $B = (B_t\,;\,t \geq 0)$ denotes a standard Brownian motion. We also use the notation $(B(t)\,;\,t \geq 0)$ or $(B_t)_{t\geq 0}$.

2.1. Markov property. First, it is easy to see that BM enjoys the Markov property:

Theorem 2.1 (Time homogeneity). *For all $s > 0$, the process $(B_{t+s} - B_s\,;\,t \geq 0)$ is a Brownian motion independent of $\sigma(B_u\,;\,u \leq s)$.*

In fact, we also have the strong Markov property:

Theorem 2.2 (Strong Markov property). *For any a.s. finite stopping time T, the process $(B_{t+T} - B_T\,;\,t \geq 0)$ is a Brownian motion independent of the σ-algebra $\mathcal{F}_T$ of T-past events.*

PROOF. In both cases, this is just a reformulation of the independence and the stationarity of the increments and for the strong Markov property that BM is a.s. continuous (right continuity suffices as we have seen for the Poisson process). $\square$

PROPOSITION 2.1 (Symmetry). *The process $(-B_t\,;\,t \geq 0)$ is a BM.*

PROOF. Trivial. $\square$

PROPOSITION 2.2 (Scaling invariance). *Let $a > 0$. Then the process X defined by $X_t = \frac{1}{a}B_{a^2t}$ is a BM.*

PROOF. It is clear that $X_0 = 0$, that the increments of X are independent and that $X_{t+s} - X_s = \frac{1}{a}(B_{a^2(t+s)} - B_{a^2s}) \sim \mathcal{N}(0, t-s)$. Moreover, the continuity of X follows from that of B. $\square$

PROPOSITION 2.3 (Time inversion). *Let X defined by*

$$X_t = \begin{cases} 0, & \text{if } t = 0, \\ tB_{\frac{1}{t}}, & \text{if } t > 0. \end{cases}$$

Then $X = (X_t\,;\,t \geq 0)$ is a Brownian motion.

PROOF. Clearly, for any $t_1, \ldots, t_k$, the vector $(X_{t_1}, \ldots, X_{t_k})$ is a centered Gaussian vector. Let us compute its covariance: assume $t > s$, then

$$\mathbb{E}(X_t X_s) = ts\,\mathbb{E}(B_{1/t}B_{1/s}) = ts\frac{1}{t} = s.$$

It remains to prove the continuity of the paths. For $t > 0$, X is clearly continuous at t by the continuity of B, so we only have to prove the continuity at $t = 0$. We have that

$$\left\{\lim_{t\to 0} X_t = 0\right\} = \bigcap_{n\geq 1}\bigcup_{m\geq 1}\bigcap_{q\in(0,\frac{1}{m})\cap\mathbb{Q}}\left\{|X_q| \leq \frac{1}{n}\right\}$$

where we use continuity of X at positive times. Since the distributions of X and B agree at positive times (so in particular at positive rational times), one has (why?)

$$\mathbb{P}\left(\bigcap_{q\in(0,\frac{1}{m})\cap\mathbb{Q}}\left\{|X_q| \leq \frac{1}{n}\right\}\right) = \mathbb{P}\left(\bigcap_{q\in(0,\frac{1}{m})\cap\mathbb{Q}}\left\{|B_q| \leq \frac{1}{n}\right\}\right),$$

hence

$$\mathbb{P}\left(\lim_{t\to 0} X_t = 0\right) = \mathbb{P}\left(\lim_{t\to 0} B_t = 0\right) = 1,$$

by the a.s. continuity of B.

$\square$

Time inversion is a useful property to relate properties at infinity to properties in a neigborhood of the origin. For instance, one has:

PROPOSITION 2.4 (Law of large numbers).

$$\lim_{t\to\infty}\frac{B_t}{t} = 0 \quad a.s.$$

PROOF. Consider the BM X defined by time inversion: $X_t = \begin{cases} 0, & \text{if } t = 0, \\ tB_{\frac{1}{t}}, & \text{if } t > 0 \end{cases}$. Then, a.s.

$$\lim_{t\to\infty}\frac{B_t}{t} = \lim_{t\to\infty}\frac{X_t}{t} = \lim_{t\to\infty} B_{1/t} = 0,$$

by continuity. $\square$

2.2. Path properties. The paths of Brownian motion are very erratic. First, we have:

THEOREM 2.3. *Almost surely, Brownian motion is not monotone on any interval.*

PROOF. Let $[a, b]$ be an interval, with $a < b$. Suppose that a.s., B is monotone on $[a, b]$. Let $a = a_0 < a_1 < \cdots < a_n < a_{n+1} = b$ be a subdivision of $[a, b]$. The increments $B_{a_{i+1}} - B_{a_i}$, $i = 0, \ldots, n$ have thus the same sign. By independence of the increments, and the fact that they are normally distributed, one has

$$\begin{aligned}
\mathbb{P}(B_{a_1} - B_{a_0}, \ldots, B_{a_{n+1}} - B_{a_0} &\text{ have the same sign}) \\
&= 2\,\mathbb{P}(B_{a_1} - B_{a_0} \geq 0, \ldots, B_{a_{n+1}} - B_{a_0} \geq 0) \\
&= 2\prod_{i=0}^{n} \mathbb{P}(B_{a_{i+1}} - B_{a_i} \geq 0) \\
&= 2\frac{1}{2^{n+1}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{P}(B \text{ is monotone on } [a, b]) \\
&\quad = \mathbb{P}(\text{for all subdivision } a = a_0 < a_1 < \cdots < a_n < a_{n+1} = b, \\
&\qquad\qquad\qquad B_{a_{i+1}} - B_{a_i},\ i = 0, \ldots, n \text{ have the same sign}) \\
&\quad = \lim_{n\to\infty} \mathbb{P}(B_{a_1} - B_{a_0}, \ldots, B_{a_{n+1}} - B_{a_0} \text{ have the same sign}) \\
&\quad = 0
\end{aligned}$$

Hence, taking a countable union over intervals with rational endpoints, we get that

$$\mathbb{P}\left(\exists\, [a, b], \text{with } a, b \in \mathbb{Q} \text{ such that } B \text{ is monotone on } [a, b]\right) = 0.$$

Now if there were a real interval $[a, b]$ such that B is monotone on $[a, b]$, it will contain an interval with rational endpoints such that B is monotone on it, hence

$$\mathbb{P}\left(\exists\, [a, b], \text{with } a, b \in \mathbb{R} \text{ such that } B \text{ is monotone on } [a, b]\right) = 0,$$

that is, a.s., Brownian motion is not monotone on any interval. ☐

Hence, Brownian motion is an example of a (random) function that is continuous but nowhere monotone. We will see that it is also nowhere differentiable, but first:

PROPOSITION 2.5. *Almost surely,*

$$\limsup_n \frac{B_n}{\sqrt{n}} = +\infty \quad \textit{and} \quad \liminf_n \frac{B_n}{\sqrt{n}} = -\infty.$$

PROOF. Let $c > 0$. By (reverse) Fatou's lemma,

$$\begin{aligned}
\mathbb{P}(B_n > c\sqrt{n} \text{ infinitely often}) &= \mathbb{P}(\limsup_n \{B_n > c\sqrt{n}\}) \\
&\geq \limsup_n \mathbb{P}(B_n > c\sqrt{n}).
\end{aligned}$$

By the scaling property, $\mathbb{P}(B_n > c\sqrt{n}) = \mathbb{P}(B_1 > c) > 0$. Now, let $X_n = B_n - B_{n-1}$. Then the X_n's are independent and $B_n = \sum_{k=1}^{n} X_k$. Thus,

$$\limsup_n \{B_n > c\sqrt{n}\} = \limsup_n \left\{ \sum_{k=1}^{n} X_k > c\sqrt{n} \right\}$$

belongs to the exchangeable σ-algebra generated by $(X_k)_k$ which is trivial by Hewitt-Savage 0–1 law (see the appendix). Thus, since

$$\mathbb{P}(\limsup_n \{B_n > c\sqrt{n}\}) > 0,$$

it must be equal to 1. Since,

$$\left\{ \limsup_n \frac{B_n}{\sqrt{n}} = +\infty \right\} = \bigcap_{c>0} \left\{ \limsup_n \frac{B_n}{\sqrt{n}} > c \right\},$$

we get that $\limsup_n \frac{B_n}{\sqrt{n}} = +\infty$ a.s. The analogous statement for the lim inf follows using the fact that $(-B_s)_s$ is a BM. □

Proposition 2.6. *For all $t \geq 0$, almost surely, BM is not differentiable at t.*

Proof. Let X be the BM defined by time inversion. Then,

$$\limsup_{n\to\infty} \frac{X_{1/n} - X_0}{1/n} = \limsup_{n\to\infty} nX_{1/n} \geq \limsup_{n\to\infty} \sqrt{n}X_{1/n} = \limsup_{n\to\infty} \frac{B_n}{\sqrt{n}} = +\infty.$$

Hence, BM is not differentiable at $t = 0$. Now, for $t > 0$, consider the Brownian motion W defined by $W_s = B_{s+t} - B_t$. Then, W being differentiable at $s = 0$ is the same that B being differentiable at t!. Hence, for all $t \geq 0$, a.s. BM is not differentiable at t. □

With more work, we can in fact exchange the $\forall\, t$ and the a.s. in the previous proposition:

Theorem 2.4 (Paley, Wiener, Zygmud (1933)). *Almost surely, BM is nowhere differentiable.*

Proof. Using time homogeneity, it suffices to prove that BM is not differentiable at any $t \in [0, 1]$. Suppose that it is not the case. Hence, there exists $t_0 \in [0, 1]$ such that

$$\limsup_{h\to 0} \left| \frac{B(t_0 + h) - B(t_0)}{h} \right| < \infty,$$

that is, there exists $M > 0$, there exists $\delta \in (0, 1)$ such that for all $h \in (0, \delta)$,

$$|B(t_0 + h) - B(t_0)| \leq Mh$$

Moreover, since BM is bounded on $[0, 2]$ by continuity, this implies that there exists $M > 0$ such that for all $h \in [0, 1]$

$$|B(t_0 + h) - B(t_0)| \leq Mh.$$

Indeed, say that for all $t \in [0, 2]$, $|B(t)| \leq M'$. Then, $|B(t_0 + h) - B(t_0)| \leq 2M'$. Hence, let $h \in [0, 1]$. If $h \in (0, \delta)$,

$$|B(t_0 + h) - B(t_0)| \leq Mh$$

and if $h \in [\delta, 1]$,

$$|B(t_0 + h) - B(t_0)| \leq \frac{2M'}{\delta}\delta \leq \frac{2M'}{\delta}h$$

since $\delta \leq h$. Letting $M'' = \max(M, \frac{2M'}{\delta})$, one gets that for all $h \in [0, 1]$,

$$|B(t_0 + h) - B(t_0)| \leq M''h.$$

We want to show that this event has probability 0. To that end, we are going to show that there are no three consecutive increments that belong to that event.

If $t_0 \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$, then

$$\begin{aligned}
\left|B\left(\frac{k+1}{2^n}\right) - B\left(\frac{k}{2^n}\right)\right| &\leq \left|B\left(\frac{k+1}{2^n}\right) - B(t_0)\right| + \left|B(t_0) - B\left(\frac{k}{2^n}\right)\right| \\
&= \left|B\left(t_0 + \frac{k+1}{2^n} - t_0\right) - B(t_0)\right| + \left|B(t_0) - B\left(t_0 + \frac{k}{2^n} - t_0\right)\right| \\
&\leq M\left(\frac{k+1}{2^n} - t_0\right) + M\left(\frac{k}{2^n} - t_0\right) \\
&\leq \frac{3M}{2^n},
\end{aligned}$$

using that $\frac{k}{2^n} - t_0 \leq \frac{1}{2^n}$. Likewise, one has

$$\left|B\left(\frac{k+2}{2^n}\right) - B\left(\frac{k+1}{2^n}\right)\right| \leq \frac{5M}{2^n} \quad \text{and} \quad \left|B\left(\frac{k+3}{2^n}\right) - B\left(\frac{k+2}{2^n}\right)\right| \leq \frac{7M}{2^n}.$$

Let, for all n and all k,

$$A_k = \bigcap_{j=1,2,3} \left\{\left|B\left(\frac{k+j}{2^n}\right) - B\left(\frac{k+j-1}{2^n}\right)\right| \leq \frac{(2j+1)M}{2^n}\right\}.$$

By independence of the increments, we have

$$\begin{aligned}
\mathbb{P}(A_k) &= \prod_{j=1,2,3} \mathbb{P}\left(\left|B\left(\frac{k+j}{2^n}\right) - B\left(\frac{k+j-1}{2^n}\right)\right| \leq \frac{(2j+1)M}{2^n}\right) \\
&\leq \prod_{j=1,2,3} \mathbb{P}\left(\left|B\left(\frac{k+j}{2^n}\right) - B\left(\frac{k+j-1}{2^n}\right)\right| \leq \frac{7M}{2^n}\right).
\end{aligned}$$

The trick now is to used Brownian scaling: for all k and all j, $B\left(\frac{k+j}{2^n}\right) - B\left(\frac{k+j-1}{2^n}\right) \sim \mathcal{N}(0, \frac{1}{2^n})$. Hence, we get that

$$\begin{aligned}
\mathbb{P}(A_k) &\leq \left(\int_{-7M/2^n}^{7M/2^n} \frac{1}{\sqrt{2\pi}} 2^{n/2} e^{-2^n x^2/2} dx\right)^3 \\
&\leq \left(\frac{14M}{\sqrt{2\pi}}\right)^3 \left(\frac{1}{2^{n/2}}\right)^3.
\end{aligned}$$

This holds for any $k = 1, \ldots, 2^n - 3$. Define

$$A^{(n)} = \bigcup_{k=1}^{2^n-3} A_k.$$

Hence,

$$\begin{aligned}
\mathbb{P}\left(A^{(n)}\right) &\leq \sum_{k=1}^{2^n-3} \mathbb{P}(A_k) \\
&\leq \left(\frac{14M}{\sqrt{2\pi}}\right)^3 2^n \left(\frac{1}{2^{n/2}}\right)^3 \\
&= \left(\frac{14M}{\sqrt{2\pi}}\right)^3 \frac{1}{2^{n/2}},
\end{aligned}$$

which is sommable over n. (One can remark that it would not have worked if we had considered only two increments!). Hence, by Borel-Cantelli lemma, we get that

$$\mathbb{P}\left(\limsup_n A^{(n)}\right) = 0.$$

But since

$$\left\{\exists\, t_0 \in [0,1] \text{ s.t. } \sup_{h\in[0,1]} \left|\frac{B(t_0+h)-B(t_0)}{h}\right| \leq M\right\} \subset \limsup_n A^{(n)},$$

it concludes the proof. □

2.3. Reflection principle. First recall the reflection principle for the simple random walk on $\mathbb{Z}$, i.e. $S_0 = 0$, and for $n \geq 1$, $S_n = \varepsilon_1 + \cdots + \varepsilon_n$, where $(\varepsilon_k)_{k\geq 1}$ are i.i.d. with distribution $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$.

PROPOSITION 2.7. *Define, for all $n \geq 1$, $M_n = \sup_{0\leq k\leq n} S_k$. Let $a \in \mathbb{N}^*$. Then, we have*

$$\mathbb{P}(M_n \geq a) = 2\,\mathbb{P}(S_n \geq a+1) + \mathbb{P}(S_n = a).$$

PROOF. We claim that for all $v \in \mathbb{Z}$, we have

$$\mathbb{P}(M_n \geq a, S_n = v) = \begin{cases} \mathbb{P}(S_n = v) & \text{if } v \geq a, \\ \mathbb{P}(S_n = 2a - v) & \text{if } v < a. \end{cases}$$

Indeed, if $v \geq a$, then if the end point of the walk is v, it must have reach level a, hence $\{S_n = v\} \subset \{M_n \geq a\}$. The $v < a$ case is where the reflection principle enters. Reflecting

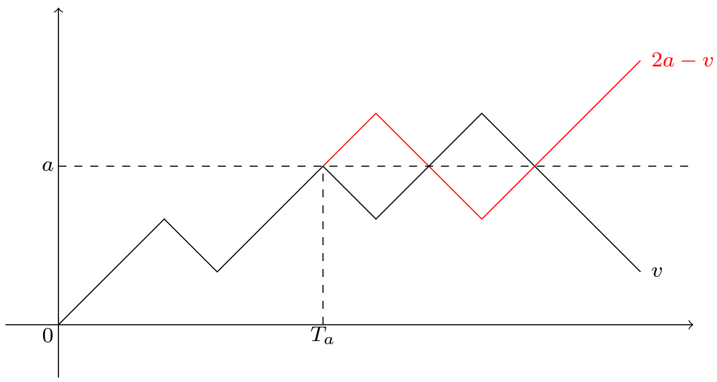

FIGURE 2. The reflection principle for the random walk.

the walk with respect to the line $y = a$ at the first hitting time of a gives a one-to-one correspondance between paths from $(0,0)$ to (n,v) reaching a and paths from $(0,0)$ to $(n, 2a-v)$ (since $v < a$, one has $2a - v > a$), see Fig. 2. Hence, we have, by independence of the increments and the fact that they are uniformly distributed on $\{-1,1\}$,

$$\begin{aligned}\mathbb{P}(M_n \geq a, S_n = v) &= \left(\frac{1}{2}\right)^n \times \#\{\text{paths } (0,0) \to (n,v) \text{ reaching } a\} \\ &= \left(\frac{1}{2}\right)^n \times \#\{\text{paths } (0,0) \to (n, 2a-v)\} \\ &= \mathbb{P}(S_n = 2a - v).\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mathbb{P}(M_n \geq a) &= \sum_{v\in\mathbb{Z}} \mathbb{P}(M_n \geq a, S_n = v) \\
&= \sum_{v\geq a} \mathbb{P}(S_n = v) + \sum_{v<a} \mathbb{P}(S_n = 2a - v) \\
&= \mathbb{P}(S_n \geq a) + \sum_{u>a} \mathbb{P}(S_n = u) \\
&= \mathbb{P}(S_n \geq a) + \mathbb{P}(S_n > a) \\
&= 2\,\mathbb{P}(S_n \geq a+1) + \mathbb{P}(S_n = a),
\end{aligned}$$

which proves the proposition. □

A similar result holds for the Brownian motion. Define the running maximum process M by

$$M_t = \sup_{s\in[0,t]} B_s, \quad \text{for all } t \geq 0.$$

Denote by T_a the first hitting time of a by the Brownian motion, that is $T_a = \inf\{t \geq 0 \,|\, B_t = a\}$. This is a stopping time relative to the natural filtration of the Brownian motion.

Theorem 2.5 (Reflection principle). *Let $a > 0$. Then,*

$$\mathbb{P}(T_a \leq t) = \mathbb{P}(M_t \geq a) = 2\,\mathbb{P}(B_t \geq a) = \mathbb{P}(|B_t| \geq a).$$

Remark 2.1. Hence, for all $t \geq 0$, we have that $M_t \overset{(d)}{=} |B_t|$. This is not true for the whole process as M is non-decreasing while $|B|$ is not.

Proof. It is clear that

$$\{T_a \leq t\} = \{M_t \geq a\}$$

by continuity. The last equality is also clear by symmetry. Now, for the middle equality, write

$$\begin{aligned}
\mathbb{P}(M_t \geq a) &= \mathbb{P}(M_t \geq a, B_t < a) + \mathbb{P}(M_t \geq a, B_t \geq a) \\
&= \mathbb{P}(M_t \geq a, B_t < a) + \mathbb{P}(B_t \geq a),
\end{aligned}$$

since $\{B_t \geq a\} \subset \{M_t \geq a\}$. Now, conditioning by $\mathcal{F}_{T_a}$,

$$\mathbb{P}(M_t \geq a, B_t < a) = \mathbb{P}(T_a \leq t, B_t < a) = \mathbb{E}\left[\mathbb{1}_{\{T_a\leq t\}}\,\mathbb{P}(B_t < a \,|\, \mathcal{F}_{T_a})\right]$$

we get, since $B_{T_a} = a$,

$$\begin{aligned}
\mathbb{P}(M_t \geq a, B_t < a) &= \mathbb{E}\left[\mathbb{1}_{\{T_a\leq t\}}\,\mathbb{P}(B_t - B_{T_a} < 0 \,|\, \mathcal{F}_{T_a})\right] \\
&= \frac{1}{2}\,\mathbb{P}(T_a \leq t)
\end{aligned}$$

since by the strong Markov inequality, $(B_{T_a+s} - B_{T_a}\,;\, s \geq 0)$ is a Brownian motion independent of $\mathcal{F}_{T_a}$ and thus, on $\{T_a \leq t\}$, one has

$$\mathbb{P}(B_t - B_{T_a} < 0 \,|\, \mathcal{F}_{T_a}) = \frac{1}{2}.$$

Eventually, we get

$$\mathbb{P}(M_t \geq a) = \frac{1}{2}\,\mathbb{P}(M_t \geq a) + \mathbb{P}(B_t \geq a),$$

giving the result. □

A more general version of the reflection principle can be stated as follows:

Theorem 2.6 (Reflection principle). *Let T be a finite stopping time (with respect to the natural filtration of the Brownian motion). Then, the process $(B_t^* \,;\, t \geq 0)$ defined by*

$$B_t^* = \begin{cases} B_t & \text{if } 0 \leq t \leq T, \\ 2B_T - B_t & \text{if } t > T, \end{cases}$$

is a Brownian motion, called the reflected Brownian motion at T.

Proof. By the strong Markov property and symmetry, the process $Y = (-(B_{t+T} - B_T)\,;\, t \geq 0)$ is a Brownian motion, independent of $\mathcal{F}_T$, and in particular of $(B_t\,;\, t \in [0, T])$. By concatenating the two, that is by considering the process

$$\begin{cases} B_t & t \in [0, T] \\ Y_{t-T} + B_T & t > T \end{cases}$$

we get exactly the process B^* which is thus a Brownian motion, see Fig. 3. $\square$

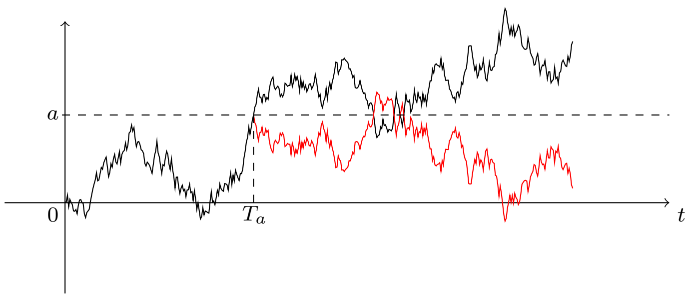

Figure 3. The reflected Brownian motion.

As a corollary of the reflection principle, we can compute the distribution of T_a.

Corollary 2.1. *The random variable T_a has density with respect to Lebesgue measure given by*

$$f_a(t) = a \frac{e^{-\frac{a^2}{2t}}}{\sqrt{2\pi t^3}} \mathbb{1}_{]0,+\infty[}(t).$$

Proof. By the previous theorem, we have

$$\mathbb{P}(T_a \leq t) = 2\,\mathbb{P}(B_t \geq a) = 2\int_a^\infty \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx.$$

By differentiating with respect to t (and using Lebesgue theorem), the density of T_a is on $]0, +\infty[$ equal to

$$\begin{aligned} f_a(t) &= \frac{1}{\sqrt{2\pi}} 2 \int_a^\infty \left(-\frac{1}{2}\right) \frac{1}{t^{3/2}} e^{-x^2/2t} dx + \frac{1}{\sqrt{2\pi}} 2 \int_a^\infty \frac{1}{\sqrt{t}} \frac{x^2}{2t^2} e^{-x^2/2t} dx \\ &= \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{t^{3/2}} \int_a^\infty e^{-x^2/2t} dx + \frac{1}{t^{5/2}} \int_a^\infty x^2 e^{-x^2/2t} dx \right). \end{aligned}$$

Using integration by parts to compute the second integral, one gets

$$\int_a^\infty x^2 e^{-x^2/2t} dx = ate^{-a^2/2t} + t \int_a^\infty e^{-x^2/2t} dx.$$

Hence, the two remaining integrals cancel out, and we get

$$f_a(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-a^2/2t}.$$

$\square$

2.4. Zero-one laws. The natural filtration of BM is defined by $\mathcal{F}_t = \sigma(B_u\,;\, 0 \le u \le t)$. Define now

$$\mathcal{F}_t^+ = \bigcap_{s>t} \mathcal{F}_s.$$

Then, $(\mathcal{F}_t^+)_{t\ge 0}$ is again a filtration and $\mathcal{F}_t \subset \mathcal{F}_t^+$, but $\mathcal{F}_t^+$ is a bit larger and "allows an additional infinitesimal glance into the future". Note that

$$\mathcal{F}_0^+ = \bigcap_{s>0} \mathcal{F}_s$$

is the σ-algebra of all events on an infinitesimal small interval to the right of the origin.

We have the following slightly improved Markov property:

THEOREM 2.7. *For all $s \ge 0$, the process $(B_{t+s} - B_s\,;\, t \ge 0)$ is a Brownian motion independent of $\mathcal{F}_s^+$.*

PROOF. By continuity, one has

$$B(t+s) - B(s) = \lim_{n\to\infty} B(t+s_n) - B(s_n),$$

where $(s_n)_n \searrow s$. By the Markov property (i.e. time homogeneity), we have that

$$B(t+s_n) - B(s_n)$$

is independent of $\mathcal{F}_{s_n}$, hence its limit is independent of $\mathcal{F}_s^+ = \bigcap_{u>s} \mathcal{F}_u$. □

THEOREM 2.8 (Blumenthal's 0–1 law). *For all $A \in \mathcal{F}_0^+$, one has $\mathbb{P}(A) = 0$ or 1.*

PROOF. By the above Markov property, $(B_{t+s} - B_s\,;\, t \ge 0)$ is independent of $\mathcal{F}_s^+$. In particular, for $s = 0$, one has $(B_t\,;\, t \ge 0)$ is independent of $\mathcal{F}_0^+$. But since $\mathcal{F}_0^+ \subset \sigma(B_t\,;\, t \ge 0)$, we have that $\mathcal{F}_0^+$ is independent of itself. □

REMARK 2.2. The following strong Markov property also holds. Let T be a stopping time with respect to the filtration $(\mathcal{F}_t^+)_{t\ge 0}$. Define

$$\mathcal{F}_T^+ = \{A \in \mathcal{F} \mid A \cap \{T \le t\} \in \mathcal{F}_t^+, \text{ for all } t \ge 0\}.$$

the σ-algebra of T past events. Then, if $T < \infty$ a.s., the process $(B_{T+t} - B_T\,;\, t \ge 0)$ is a Brownian motion independent of $\mathcal{F}_T^+$ (exercise).

Now consider the tail σ-algebra of Brownian motion: let

$$\mathcal{G}_t = \sigma(B_u\,;\, u \ge t),$$

and let

$$\mathcal{T} = \bigcap_{t\ge 0} \mathcal{G}_t$$

be the σ-algebra of tail events. Then:

THEOREM 2.9 (0–1 law for tail events). *For all $A \in \mathcal{T}$, one has $\mathbb{P}(A) = 0$ or 1.*

PROOF. Using time inversion of Brownian motion, $\mathcal{T}$ is mapped onto $\mathcal{F}_0^+$, hence the result follows from Blumenthal's 0–1 law. □

2.5. Zeroes of Brownian motion. Consider the random set of the zeroes of Brownian motion:

$$\mathcal{Z} = \{t \geq 0 \,|\, B_t = 0\}.$$

Note that it is the section of the set

$$\{(t,\omega) \in \mathbb{R}_+ \times \Omega \,|\, B_t(\omega) = 0\},$$

and thus is a measurable set.

PROPOSITION 2.8. *We have that $\lambda(\mathcal{Z}) = 0$ a.s. (where λ denotes Lebesgue measure).*

PROOF. Indeed,

$$\begin{aligned}\mathbb{E}\left[\lambda(\mathcal{Z})\right] &= \mathbb{E}\left[\int_0^\infty \mathbb{1}_{B_t=0}dt\right] \\ &= \int_0^\infty \mathbb{P}(B_t = 0)dt,\end{aligned}$$

by Fubini's theorem. Since $\mathbb{P}(B_t = 0) = 0$, one gets $\lambda(\mathcal{Z}) = 0$ a.s. □

PROPOSITION 2.9. *We have that $\mathcal{Z}$ is closed a.s.*

PROOF. This is clear since $t \mapsto B_t$ is continuous a.s. □

LEMMA 2.1. *Let $\tau = \inf\{t > 0 \,|\, B_t > 0\}$ and $\sigma = \inf\{t > 0 \,|\, B_t = 0\}$. Then,*

$$\mathbb{P}(\tau = 0) = \mathbb{P}(\sigma = 0) = 1.$$

PROOF. We have

$$\begin{aligned}\{\tau = 0\} &= \{\forall \eta > 0, \exists\, 0 < \varepsilon < \eta, B_\varepsilon > 0\} \\ &= \bigcap_n \bigcup_{\varepsilon \in (0,\frac{1}{n})\cap\mathbb{Q}} \{B_\varepsilon > 0\} \\ &\in \mathcal{F}_0^+.\end{aligned}$$

Hence, by Blumenthal's 0-1 law, $\mathbb{P}(\tau = 0) \in \{0,1\}$, so we have to prove that it has positive probability. Since $\{B_t > 0\} \subset \{\tau \leq t\}$, one has

$$\mathbb{P}(\tau \leq t) \geq \mathbb{P}(B_t > 0) = \frac{1}{2}.$$

Hence,

$$\mathbb{P}(\tau = 0) = \lim_{t\to 0} \mathbb{P}(\tau \leq t) \geq \frac{1}{2} > 0,$$

so $\mathbb{P}(\tau = 0) = 1$. The same holds replacing B by $-B$. Hence, by continuity and the intermediate value theorem, we have

$$\mathbb{P}(\sigma = 0) = 1.$$

□

PROPOSITION 2.10. *Almost surely, for all $\varepsilon > 0$, BM has infinitely many zeroes in $(0,\varepsilon)$. In particular, $\mathcal{Z}$ is infinite a.s.*

PROOF. By the previous lemma, a.s. for all $\varepsilon > 0$, there exists $t \in (0,\varepsilon)$ such that $B_t = 0$. Suppose that the set of zeroes in $(0,\varepsilon)$ is finite and denote

$$t_0 = \min\{t \in (0,\varepsilon) \,|\, B_t = 0\}.$$

But B must have a zero on $(0,t_0)$ which contradicts the minimality of t_0. Hence, B has infinitely many zeroes in $(0,\varepsilon)$. □

PROPOSITION 2.11. *Almost surely, $\mathcal{Z}$ has no isolated point.*

Proof. Recall that a point x is isolated if the singleton $\{x\}$ is open. Equivalently, there exists $\varepsilon > 0$ such that $]x-\varepsilon, x+\varepsilon[\setminus\{x\} = \varnothing$.

Let $q \in \mathbb{Q}$ and consider

$$\tau_q = \inf\{t \geq q \mid B_t = 0\}.$$

Then τ_q is a stopping time, and is a.s. finite (since BM crosses 0 for arbitrarily large time t). Moreover the infimum is a minimum, since $\mathcal{Z}$ is a closed set. Hence, $\tau_q \in \mathcal{Z}$. Now, by the strong Markov property, $(B_{\tau_q+t}\,;\, t \geq 0)$ is a Brownian motion. Since BM crosses 0 for any small interval to the right of the origin, τ_q is not isolated (from the right). Hence, a.s., for all $q \in \mathbb{Q}$, τ_q is not an isolated point.

It remains to prove that any other point is not isolated either. Let $z \in \mathcal{Z} \setminus \{\tau_q \mid q \in \mathbb{Q}\}$. Consider an increasing sequence of rationals q_n such that $q_n \nearrow z$. Then, $q_n \leq \tau_{q_n} < z$ (since $q_n < z$ and $z \neq \tau_{q_n}$). Letting $n \to \infty$, one has that $\tau_{q_n} \to z$, so z is not isolated (from the left). $\square$

Proposition 2.12. *Almost surely, $\mathcal{Z}$ is uncountable.*

Proof. This is a classical consequence of Baire's theorem: in a non-empty (separable) complete metric space X, every countable union of closed sets with empty interior has empty interior. Thus if moreover X has no isolated point, X is uncountable: since it has no isolated point, the singletons have empty interior, hence if it were countable $X = \bigcup_{x\in X}\{x\}$ would have empty interior, which is not possible since it is open and non-empty.

Thus, since $\mathcal{Z}$ is closed in $\mathbb{R}$, it is complete as a metric space and is not empty, so by the above $\mathcal{Z}$ is uncountable. $\square$

3. Law of iterated logarithm

Lemma 3.1. *Let $X \sim \mathcal{N}(0,1)$. For all $x > 0$, one has*

$$\frac{x}{x^2+1}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \leq \mathbb{P}(X > x) \leq \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

Proof. Exercise.

For the right inequality,

$$\mathbb{P}(X > x) = \int_x^\infty e^{-u^2/2}\frac{1}{\sqrt{2\pi}}du \underset{\substack{\uparrow \\ \frac{u}{x}\geq 1}}{\leq} \frac{1}{x}\frac{1}{\sqrt{2\pi}}\int_x^\infty u e^{-u^2/2}du = \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

For the left inequality, put

$$f(x) = xe^{-x^2/2} - (x^2+1)\int_x^\infty e^{-u^2/2}du.$$

Then,

$$f'(x) = 2e^{-x^2/2} - 2x\int_x^\infty e^{-u^2/2}du,$$

which is non-negative by the right inequality. Hence, f is non-decreasing, $f(0) < 0$ and $\lim_{x\to\infty} f(x) = 0$, so $f(x) \leq 0$ for all $x > 0$. $\square$

Theorem 3.1 (Law of iterated logarithm). *Almost surely,*

$$\limsup_{t\to\infty}\frac{B_t}{\sqrt{2t\log\log t}} = 1 \quad \text{and} \quad \liminf_{t\to\infty}\frac{B_t}{\sqrt{2t\log\log t}} = -1.$$

Proof. Put $\Psi(t) = \sqrt{2t\log\log t}$. The statement for the liminf is obtained using the statement for the limsup applied to the Brownian motion $-B$.

We first prove the upper bound: a.s. $\limsup_{t\to\infty}\frac{B_t}{\Psi(t)} \leq 1$.

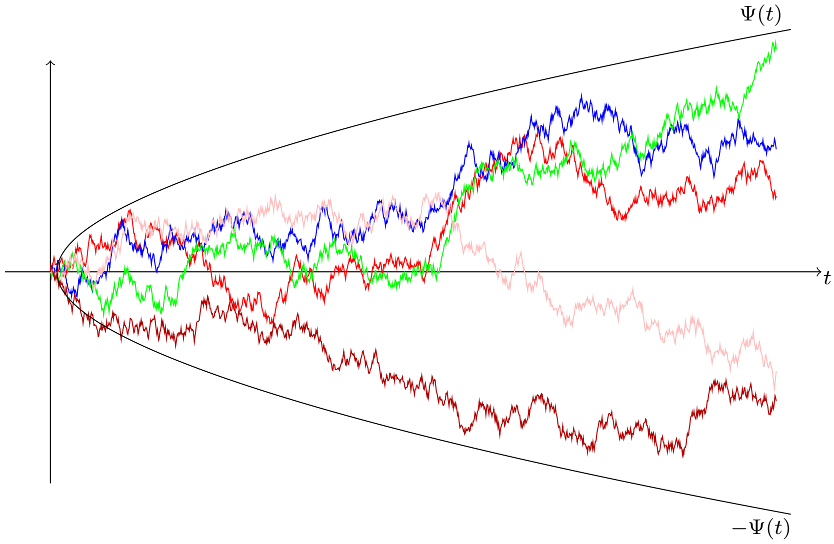

Figure 4. The law of iterated logarithm.

Let $\varepsilon > 0$ and $q > 1$. Define

$$A_n = \left\{ \max_{0 \le t \le q^n} B_t \ge (1+\varepsilon)\Psi(q^n) \right\}.$$

By the reflection principle, $\max_{0\le s\le t} B_s \overset{(d)}{=} |B_t|$. Therefore,

$$\begin{aligned}\mathbb{P}(A_n) &= \mathbb{P}\left(|B_{q^n}| \ge (1+\varepsilon)\Psi(q^n)\right)\\ &\le 2\exp(-(1+\varepsilon)^2 \log\log q^n) = \frac{2}{(n\log q)^{(1+\varepsilon)^2}},\end{aligned}$$

where we used the fact that if $Z \sim \mathcal{N}(0,1)$, $\mathbb{P}(Z > x) \le e^{-x^2/2}$, for all $x > 0$. Hence, we get that

$$\sum_n \mathbb{P}(A_n) < \infty,$$

so by Borel-Cantelli lemma, $\mathbb{P}(\limsup_n A_n) = 0$. Hence, a.s., there exists n_0 such that for all $n \ge n_0$, for all $0 \le t \le q^n$,

$$B_t \le (1+\varepsilon)\Psi(q^n).$$

For large t, choose n large enough such that $q^{n-1} \le t \le q^n$. Then we have,

$$\begin{aligned}\frac{B_t}{\Psi(t)} &= \frac{B_t}{\Psi(q^n)}\frac{\Psi(q^n)}{q^n}\frac{t}{\Psi(t)}\frac{q^n}{t}\\ &\le (1+\varepsilon)q,\end{aligned}$$

since $\frac{q^n}{t} \le q$ and $\frac{\Psi(t)}{t}$ is decreasing (for t large enough). Hence, a.s.

$$\limsup_{t\to\infty} \frac{B_t}{\Psi(t)} \le (1+\varepsilon)q,$$

which holds for all $\varepsilon > 0$, and all $q > 1$, proving the upper bound.

We now pass to the lower bound: a.s. $\limsup_{t\to\infty} \frac{B_t}{\Psi(t)} \le 1$.

We want to use the reverse Borel-Cantelli lemma. Let $q > 1$ and let

$$D_n = \left\{ B(q^n) - B(q^{n-1}) \ge \Psi(q^n - q^{n-1}) \right\}.$$

Hence, using the lower bound estimate of the normal distribution given in the previous lemma, one has, for some constant C,

$$\begin{aligned}\mathbb{P}(D_n) &\geq C\frac{e^{-\log\log(q^n-q^{n-1})}}{\sqrt{2\log\log(q^n-q^{n-1})}}\\ &\leq \frac{C'}{n\log n},\end{aligned}$$

for another constant C', where we use that $\log(q^n - q^{n-1}) < n\log q$. Hence,

$$\sum_n \mathbb{P}(D_n) = \infty$$

and since the D_n's are independent by the independence of the increments of the Brownian motion, one has, by the second Borel-Cantelli lemma, that

$$\mathbb{P}(\limsup_n D_n) = 1.$$

Hence, a.s., infinitely often,

$$\begin{aligned}B(q^n) &\geq B(q^{n-1}) + \Psi(q^n - q^{n-1})\\ &\geq -2\Psi(q^{n-1}) + \Psi(q^n - q^{n-1}),\end{aligned}$$

using the upper bound applied to the Brownian motion $-B$. Hence,

$$\begin{aligned}\frac{B(q^n)}{\Psi(q^n)} &\geq \frac{-2\Psi(q^{n-1}) + \Psi(q^n - q^{n-1})}{\Psi(q^n)}\\ &\geq \frac{-2}{\sqrt{q}} + \frac{q^n - q^{n-1}}{q^n} = 1 - \frac{2}{\sqrt{q}} - \frac{1}{q},\end{aligned}$$

using that

$$\frac{\Psi(q^{n-1})}{\Psi(q^n)} = \frac{\Psi(q^{n-1})}{\sqrt{q^{n-1}}}\frac{\sqrt{q^n}}{\Psi(q^n)}\frac{1}{\sqrt{q}} \leq \frac{1}{\sqrt{q}}$$

since $\frac{\Psi(t)}{\sqrt{t}}$ is increasing, and using that $\frac{\Psi(t)}{t}$ is decreasing for large t. Hence, a.s.,

$$\limsup_{t\to\infty} \frac{B_t}{\Psi(t)} \geq 1 - \frac{2}{\sqrt{q}} - \frac{1}{q},$$

so letting $q \to \infty$ gives the lower bound. □

Appendix A. Classical 0–1 laws

Let $(X_n)_{n\geq 1}$ be a sequence of random variables. Define, for all $n \geq 1$,

$$\mathcal{F}_n = \sigma(X_k, 1 \leq k \leq n)$$

the σ-algebra generated by $X_1, \ldots, X_n$. Define also, for all $n \geq 0$,

$$\mathcal{G}_n = \sigma(X_{n+1}, X_{n+2}, \ldots).$$

The tail σ-algebra $\mathcal{T}$ is defined as

$$\mathcal{T} = \bigcap_{n\geq 0} \mathcal{G}_n.$$

Intuitively, the σ-algebra of tail events consists of events which do not depend on the first finitely many times of the process $(X_n)_{n\geq 1}$. For instance, the random variables $\limsup_n X_n$ and $\liminf_n X_n$ are $\mathcal{T}$-measurable.

Theorem A.1 (Kolomogorov's 0–1 law). *Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables. The tail σ-algebra $\mathcal{T}$ is trivial: for all $A \in \mathcal{T}$, $\mathbb{P}(A) = 0$ or 1.*

PROOF. We have that

$$\bigcup_{n\geq 1} \mathcal{F}_n$$

is a π-system that generates $\sigma(X_1, X_2, \ldots)$ $(= \mathcal{G}_0)$. Since $(X_n)_{n\geq 0}$ are independent random variables, $\mathcal{F}_n$ is independent of $\mathcal{G}_n$. Since $\mathcal{T} \subset \mathcal{G}_n$, for all $n \geq 0$, $\mathcal{T}$ is independent of $\mathcal{F}_n$, for all n, hence $\mathcal{T}$ is independent of $\sigma(X_1, X_2, \ldots)$. But obviously, $\mathcal{T} \subset \sigma(X_1, X_2, \ldots)$, so $\mathcal{T}$ is independent of itself. Hence, for all $A \in \mathcal{T}$,

$$\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A)\,\mathbb{P}(A)$$

that is, $\mathbb{P}(A) \in \{0, 1\}$. $\square$

As a corollary, we get that if $(X_n)_{n\geq 0}$ is a sequence of independent random variables, then $\limsup_n X_n$ and $\liminf_n X_n$ are almost surely constant.

Consider the random walk $S_n = X_1 + \cdots + X_n$. Consider the event

$$\{S_n = 0 \text{ i.o.}\} = \limsup_n \{S_n = 0\} = \bigcap_k \bigcup_{n\geq k} \{S_n = 0\}.$$

This is NOT a tail event in general (note that this different than $\{\limsup_n S_n = 0\}$! Take $S_n > 0$ and $S_n \to 0$ a.s. for example). But it is an exchangeable event:

Loosely speaking, an event $A \subset \mathbb{R}^\infty$ is exchangeable if it is invariant under finite permutations, that is for all n, for all permutation $\sigma \in \mathcal{S}_n$,

$$(\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots) \in A \Leftrightarrow (\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)}, \omega_{n+1}, \ldots) \in A$$

The exchangeable σ-algebra $\mathcal{E}$ is the class of all such events. If $(X_n)_n$ is a sequence of random variables,

$$\mathcal{E} = \bigcap_n \mathcal{E}_n,$$

where $\mathcal{E}_n$ is the σ-algebra generated by $X_{n+1}, X_{n+1}, \ldots$ and symmetric functions of $(X_1, \ldots, X_n)$.

Compared to the tail σ-algebra $\mathcal{T}$ which consists of events that do not depend on the first coordinates, the exchangeable σ-algebra consists of events that are invariant by permutations of the first coordinates. For instance, in the case of the random walk S_n,

$$\limsup_n \{S_n = 0\} \in \mathcal{E},$$

but not in $\mathcal{T}$. Indeed,

$$\omega \in \limsup_n \{S_n = 0\}$$

is equivalent to: for all k, there exists $n \geq k$ such that $\sum_{j=1}^n X_j(\omega) = 0$. Hence, changing the values of the first coordinates $X_1, \ldots, X_n$ will affect the value of the sum. On the other hand, permuting finitely many indices will not affect the value of S_n.

Note that it is clear that

$$\mathcal{T} \subset \mathcal{E}.$$

THEOREM A.2 (Hewitt-Savage's 0–1 law). *Let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables. The exchangeable σ-algebra $\mathcal{E}$ is trivial: for all $A \in \mathcal{E}$, $\mathbb{P}(A) = 0$ or 1.*

PROOF USING BACKWARDS MARTINGALES. Let F be a bounded symmetric function on $\mathbb{R}^{\mathbb{N}}$, that is F is invariant under any permutation with finite support. Let

$$Z = F(X_1, X_2, \ldots),$$

so Z is measurable with respect to the exchangeable σ-algebra $\mathcal{E}$. We are going to prove that Z is a.s. constant.

Define,

$$Z_n = \mathbb{E}(Z \,|\, \mathcal{F}_n) \quad \text{and} \quad Y_n = \mathbb{E}(Z \,|\, \mathcal{G}_n).$$

Hence $(Z_n)_n$ is a closed martingale, thus by the martingale convergence theorem, it converges a.s. and in L^1:

$$Z_n \to \mathbb{E}(Z \mid \mathcal{F}_\infty) = Z$$

since Z is $\mathcal{F}_\infty$-measurable. Indeed,

$$\mathcal{F}_\infty = \sigma(X_1, X_2, \ldots)$$

and obviously $\mathcal{E} \subset \mathcal{F}_\infty$.

On the other hand, $(Y_n)_n$ is a backwards martingale, so by the convergence theorem for backwards martingales, a.s. and in L^1, one has

$$Y_n \to \mathbb{E}(Z \mid \mathcal{T}).$$

But since $\mathcal{T}$ is trivial by Kolmogorov's 0–1 law, $\mathbb{E}(Z \mid \mathcal{T})$ is constant a.s., hence $\mathbb{E}(Z \mid \mathcal{T}) = \mathbb{E}(Z)$. Hence, for all $\varepsilon > 0$, and for n large enough,

$$\mathbb{E}(|Z_n - Z|) < \varepsilon \quad \text{and} \quad \mathbb{E}(|Y_n - \mathbb{E}(Z)|) < \varepsilon,$$

Now, since Z_n is $\mathcal{F}_n$-measurable, one can write

$$Z_n = g(X_1, \ldots, X_n),$$

for some bounded function $g \colon \mathbb{R}^n \to \mathbb{R}$. The above first inequality writes then

$$\mathbb{E}\left[|F(X_1, X_2, \ldots) - g(X_1, \ldots, X_n)| \right] < \varepsilon.$$

Since $(X_n)_{n\geq 1}$ are i.i.d. random variables, $(X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n}, X_{2n+1}, \ldots)$ has the same distribution than $(X_{n+1}, \ldots, X_{2n}, X_1, \ldots, X_n, X_{2n+1}, \ldots)$. Hence, the above bound is also

$$\mathbb{E}\left[|F(X_{n+1}, \ldots, X_{2n}, X_1, \ldots, X_n, X_{2n+1}, \ldots) - g(X_{n+1}, \ldots, X_{2n})| \right] < \varepsilon.$$

But since F is invariant under finite permutation, this gives

$$\mathbb{E}\left[|Z - g(X_{n+1}, \ldots, X_{2n})| \right] < \varepsilon.$$

Hence, we have, since $g(X_{n+1}, \ldots, X_{2n})$ is $\mathcal{G}_n$-measurable,

$$\begin{aligned}
\mathbb{E}\left[|Y_n - g(X_{n+1}, \ldots, X_{2n})|\right] &= \mathbb{E}\left[|\,\mathbb{E}(Z - g(X_{n+1}, \ldots, X_{2n}) \mid \mathcal{G}_n)|\right] \\
&\leq \mathbb{E}\left[|Z - g(X_{n+1}, \ldots, X_{2n})|\right] \\
&< \varepsilon.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
\mathbb{E}\left[|Z - \mathbb{E}(Z)|\right] &\leq \mathbb{E}\left[|Z - g(X_{n+1}, \ldots, X_{2n})|\right] + \mathbb{E}\left[|g(X_{n+1}, \ldots, X_{2n}) - Y_n|\right] + \mathbb{E}\left[|Y_n - \mathbb{E}(Z)|\right] \\
&< 3\varepsilon.
\end{aligned}$$

Letting ε goes to zero, we get that $Z = \mathbb{E}(Z)$ almost surely. $\square$