

About the HUM method and its application to the numerical approximation of controls of parabolic PDEs

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- Two Hilbert spaces : the state space $(E, \langle \cdot, \cdot \rangle)$ and the control space $(U, [\cdot, \cdot])$.
- $\mathcal{A} : D(\mathcal{A}) \subset E \mapsto E$ is some *elliptic* operator such that $-\mathcal{A}$ generates an analytic semigroup in E .
- $\mathcal{B} : U \mapsto D(\mathcal{A}^*)'$ the control (bounded) operator, \mathcal{B}^* its adjoint.
- **COMPATIBILITY ASSUMPTION** : we assume that

$$\left(t \mapsto \mathcal{B}^* e^{-t\mathcal{A}^*} \psi \right) \in L^2(0, T; U), \text{ and } \left\| \mathcal{B}^* e^{-\cdot \mathcal{A}^*} \psi \right\|_{L^2(0, T; U)} \leq C \|\psi\|, \quad \forall \psi \in E.$$

Our controlled parabolic problem is

$$(S) \quad \begin{cases} \partial_t y + \mathcal{A}y = \mathcal{B}v & \text{in }]0, T[, \\ y(0) = y_0, \end{cases}$$

Here, $y_0 \in E$ is the initial data, $v \in L^2(]0, T[, U)$ is the control we are looking for.

THEOREM (WELL-POSEDNESS OF (S) IN A DUAL SENSE)

For any $y_0 \in E$ and $v \in L^2(0, T; U)$, there exists a unique $y = y_{v, y_0} \in \mathcal{C}^0([0, T], E)$ such that

$$\langle y(t), \psi \rangle - \langle y_0, e^{-t\mathcal{A}^*} \psi \rangle = \int_0^t \left[v(s), \mathcal{B}^* e^{-(t-s)\mathcal{A}^*} \psi \right] ds, \quad \forall t \in [0, T], \forall \psi \in E.$$

NOTATION : $\mathcal{L}_T(v|y_0) \stackrel{\text{def}}{=} y_{v, y_0}(T).$

$$(S) \begin{cases} \partial_t y + \mathcal{A}y = \mathcal{B}v & \text{in }]0, T[, \\ y(0) = y_0. \end{cases}$$

For a given (fixed) control time $T > 0$ and any $\delta \geq 0$, we set

$$\text{Adm}(y_0, \delta) \stackrel{\text{def}}{=} \left\{ v \in L^2(0, T; U), \text{ s.t. } \|\mathcal{L}_T(v|y_0)\| \leq \delta \right\}.$$

APPROXIMATE CONTROL PROBLEM FROM THE INITIAL DATA y_0

Do we have

$$\text{Adm}(y_0, \delta) \neq \emptyset, \quad \forall \delta > 0 ?$$

NULL-CONTROL PROBLEM FROM THE INITIAL DATA y_0

Do we have

$$\text{Adm}(y_0, 0) \neq \emptyset ?$$

(Fattorini-Russel, '71) (Lebeau-Robbiano, '95)

(Fursikov-Imanuvilov, '96) (Alessandrini-Escauriaza, '08)

(Ammar-Khodja, Benabdallah, González-Burgos, de Teresa, '11)

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(Lions, '88) (Glowinski–Lions, '90s)

IDEAS

- To formulate control problems as constrained optimisation problems.
- To write the associated **unconstrained** dual optimisation problem.
- To find conditions for the solvability of the dual problem and prove that there are satisfied.

COST OF THE CONTROL We set

$$F(v) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T \llbracket v(t) \rrbracket^2 dt, \quad \forall v \in L^2(0, T; U),$$

and for any $\delta \geq 0$, we define (if it exists !), v^δ to be the unique minimiser

$$F(v^\delta) = \inf_{v \in \text{Adm}(y_0, \delta)} F(v). \quad (P^\delta)$$

DUAL PROBLEMS

- The dual pb of (P^0) is not coercive in the natural space E . We need to introduce a **big** abstract space obtained as the completion of E with respect to a suitable norm.
- The dual pb of (P^δ) , $\delta > 0$ is coercive in E but is not smooth.

PRIMAL PROBLEM

$$F_\varepsilon(v) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T \llbracket v(t) \rrbracket^2 dt + \frac{1}{2\varepsilon} \|\mathcal{L}_T(v|y_0)\|^2, \quad \forall v \in L^2(0, T; U),$$

we consider the following problem : to find $v_\varepsilon \in L^2(0, T; U)$ such that

$$F_\varepsilon(v_\varepsilon) = \inf_{v \in L^2(0, T; U)} F_\varepsilon(v). \quad (P_\varepsilon)$$

PROPOSITION

For any $\varepsilon > 0$, the functional F_ε is strictly convex, continuous and coercive. Therefore, it admits a unique minimiser $v_\varepsilon \in L^2(0, T; U)$.

DUAL PROBLEM

(Fenchel-Rockafellar duality theorem)

$$J_\varepsilon(q^F) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T \left\| \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} q^F \right\|^2 dt + \frac{\varepsilon}{2} \|q^F\|^2 + \left\langle y_0, e^{-T\mathcal{A}^*} q^F \right\rangle, \quad \forall q^F \in E.$$

PROPOSITION

For any $\varepsilon > 0$, the functional J_ε is strictly convex, continuous and coercive. Therefore, it admits a unique minimiser $q_\varepsilon^F \in E$.

REMARK

We do not require any particular assumption on the operators \mathcal{A} and \mathcal{B} .
In particular **we do not assume** that the PDE (S) is (or is not) controllable.

PROPOSITION (DUALITY PROPERTIES PRECISED)

For any $\varepsilon > 0$, the minimisers v_ε and q_ε^F of the functionals F_ε and J_ε respectively, are related through the formulas

$$v_\varepsilon(t) = \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} q_\varepsilon^F, \text{ for a.e. } t \in]0, T[,$$

and

$$\mathcal{L}_T(v_\varepsilon | y_0) = y_{v_\varepsilon, y_0}(T) = -\varepsilon q_\varepsilon^F.$$

As a consequence, we have

$$\inf_{L^2(0,T;U)} F_\varepsilon = F_\varepsilon(v_\varepsilon) = -J_\varepsilon(q_\varepsilon^F) = -\inf_E J_\varepsilon.$$

THEOREM

- ① Problem (S) is *approximately controllable from the initial data y_0* if and only if

$$\mathcal{L}_T(v_\varepsilon | y_0) = y_{v_\varepsilon, y_0}(T) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

- ② Problem (S) is *null-controllable from the initial data y_0* if and only if

$$M_{y_0}^2 \stackrel{\text{def}}{=} 2 \sup_{\varepsilon > 0} \left(\inf_{L^2(0, T; U)} F_\varepsilon \right) = 2 \sup_{\varepsilon > 0} F_\varepsilon(v_\varepsilon) < +\infty.$$

IN THE NULL-CONTROLLABLE CASE

$$\|v_\varepsilon\|_{L^2(0, T; U)} \leq M_{y_0}, \quad \text{and} \quad \|\mathcal{L}_T(v_\varepsilon | y_0)\| \leq M_{y_0} \sqrt{\varepsilon}.$$

Moreover we have $\|v^0\|_{L^2(0, T; U)} = M_{y_0}$ and

$$v_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} v^0, \quad \text{strongly in } L^2(0, T; U), \quad \text{and} \quad \frac{\mathcal{L}_T(v_\varepsilon | y_0)}{\sqrt{\varepsilon}} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

where v^0 is the unique HUM null-control (that is the one of minimal L^2 -norm).

Non observable adjoint states : $Q_F \stackrel{\text{def}}{=} \left\{ q^F \in E, \text{ s.t. } \mathcal{B}^* e^{-t\mathcal{A}^*} q^F = 0, \forall t \geq 0 \right\}.$

THEOREM (CONVERGENCE OF THE PENALISED HUM FINAL STATE)

For any $y_0 \in E$, the penalised-HUM sequence of controls $(v_\varepsilon)_\varepsilon$ satisfies

$$\mathcal{L}_T(v_\varepsilon | y_0) \xrightarrow{\varepsilon \rightarrow 0} \mathbb{P}_{Q_F} \left(e^{-T\mathcal{A}} y_0 \right).$$

PROPOSITION (SELFADJOINT CASE)

Assume that \mathcal{A} is selfadjoint, and set $Y_T \stackrel{\text{def}}{=} e^{-T\mathcal{A}^*} Q_F) e^{-T\mathcal{A}} Q_F$ then

$$\mathbb{P}_{Q_F} \left(e^{-T\mathcal{A}} y_0 \right) = e^{-T\mathcal{A}} \left(\mathbb{P}_{\overline{Y_T}} y_0 \right).$$

Therefore, the system is approximately controllable from y_0 if and only if $\mathbb{P}_{\overline{Y_T}} y_0 = 0$.

- The set of (approximately) controllable initial data is Y_T^\perp .
- For any $y_0 \in Y_T$ we have

$$v_\varepsilon = 0, \quad \forall \varepsilon > 0,$$

$$\text{Adm}(y_0, \delta) \neq \emptyset \Leftrightarrow \delta \geq \left\| e^{-T\mathcal{A}} y_0 \right\|.$$

Non observable adjoint states : $Q_F \stackrel{\text{def}}{=} \left\{ q^F \in E, \text{ s.t. } \mathcal{B}^* e^{-t\mathcal{A}^*} q^F = 0, \forall t \geq 0 \right\}.$

THEOREM (CONVERGENCE OF THE PENALISED HUM FINAL STATE)

For any $y_0 \in E$, the penalised-HUM sequence of controls $(v_\varepsilon)_\varepsilon$ satisfies

$$\mathcal{L}_T(v_\varepsilon | y_0) \xrightarrow{\varepsilon \rightarrow 0} \mathbb{P}_{Q_F} \left(e^{-T\mathcal{A}} y_0 \right).$$

COROLLARY (APP. CONTROLLABILITY AND UNIQUE CONTINUATION)

The system (S) is *approximately controllable from the initial data y_0 if and only if*

$$\left[\mathcal{B}^* e^{-(T-t)\mathcal{A}^*} q^F = 0, \quad \forall t \in [0, T] \right] \implies \left\langle y_0, e^{-T\mathcal{A}^*} q^F \right\rangle = 0. \quad (\text{UC})$$

PROPOSITION (APP. CONTROLLABILITY AND WEAK OBSERVABILITY)

The property (UC) is equivalent to the following **weak observability inequality**

$$\left| \left\langle y_0, e^{-T\mathcal{A}^*} q^F \right\rangle \right|^2 \leq \mathbf{C}_{\varepsilon, y_0}^2 \left\| \mathcal{B}^* e^{-(T-\cdot)\mathcal{A}^*} q^F \right\|_{L^2(0, T; U)}^2 + \varepsilon \left\| q^F \right\|^2, \quad \forall q^F \in E, \forall \varepsilon > 0.$$

THEOREM (NULL-CONTROLLABILITY AND OBSERVABILITY)

Problem (S) is *null-controllable from y_0 if and only if*, there exists $\tilde{M}_{y_0} \geq 0$ such that

$$\left| \left\langle y_0, e^{-T\mathcal{A}^*} q^F \right\rangle \right|^2 \leq \tilde{M}_{y_0}^2 \left\| \mathcal{B}^* e^{-(T-\cdot)\mathcal{A}^*} q^F \right\|_{L^2(0,T;U)}^2, \quad \forall q^F \in E.$$

Moreover, the best constant \tilde{M}_{y_0} is equal to the cost of the HUM control $\llbracket v^0 \rrbracket_{L^2(0,T;U)}$.

For each $\varepsilon > 0$, let $y_{0,\varepsilon} \in E$ such that $(y_{0,\varepsilon})_\varepsilon$ is bounded in E and

$$e^{-T\mathcal{A}}y_{0,\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} e^{-T\mathcal{A}}y_0.$$

ASSOCIATED HUM FUNCTIONALS

$$\tilde{F}_\varepsilon(v) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T \|v(t)\|^2 dt + \frac{1}{2\varepsilon} \|\mathcal{L}_T(v|y_{0,\varepsilon})\|^2, \quad \forall v \in L^2(0, T; U),$$

$$\tilde{J}_\varepsilon(q^F) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T \|\mathcal{B}^* e^{-(T-t)\mathcal{A}^*} q^F\|^2 dt + \frac{\varepsilon}{2} \|q^F\|^2 + \langle y_{0,\varepsilon}, e^{-T\mathcal{A}^*} q^F \rangle, \quad \forall q^F \in E.$$

We denote by \tilde{v}_ε the unique minimiser of \tilde{F}_ε .

CONTROLLABILITY CONDITIONS

$$(S) \text{ is app. cont. from } y_0 \iff \mathcal{L}_T(\tilde{v}_\varepsilon|y_{0,\varepsilon}) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

$$\sup_{\varepsilon > 0} \left(\inf_{L^2(0, T; U)} \tilde{F}_\varepsilon \right) < +\infty \implies (S) \text{ is null-controllable from } y_0.$$

$$\left. \begin{array}{l} (S) \text{ is null-controllable from } y_0 \\ \sup_{\varepsilon > 0} \frac{1}{\varepsilon} \|e^{-T\mathcal{A}}(y_0 - y_{0,\varepsilon})\|^2 < +\infty \end{array} \right\} \implies \sup_{\varepsilon > 0} \left(\inf_{L^2(0, T; U)} \tilde{F}_\varepsilon \right) < +\infty.$$

For each $\varepsilon > 0$, let $y_{0,\varepsilon} \in E$ such that $(y_{0,\varepsilon})_\varepsilon$ is bounded in E and

$$e^{-T\mathcal{A}}y_{0,\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} e^{-T\mathcal{A}}y_0.$$

ASSOCIATED HUM FUNCTIONALS

$$\tilde{F}_\varepsilon(v) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T \|v(t)\|^2 dt + \frac{1}{2\varepsilon} \|\mathcal{L}_T(v|y_{0,\varepsilon})\|^2, \quad \forall v \in L^2(0, T; U),$$

$$\tilde{J}_\varepsilon(q^F) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T \|\mathcal{B}^* e^{-(T-t)\mathcal{A}^*} q^F\|^2 dt + \frac{\varepsilon}{2} \|q^F\|^2 + \langle y_{0,\varepsilon}, e^{-T\mathcal{A}^*} q^F \rangle, \quad \forall q^F \in E.$$

We denote by \tilde{v}_ε the unique minimiser of \tilde{F}_ε .

CONTROLLABILITY CONDITIONS

$$\left. \begin{array}{l} \text{(S) is null-controllable from } y_0 \\ \sup_{\varepsilon > 0} \frac{1}{\varepsilon} \|e^{-T\mathcal{A}}(y_0 - y_{0,\varepsilon})\|^2 < +\infty \end{array} \right\} \implies \sup_{\varepsilon > 0} \left(\inf_{L^2(0,T;U)} \tilde{F}_\varepsilon \right) < +\infty.$$

DISCUSSION : Assume $\mathcal{A} = \mathcal{A}^*$ and $Q_F \neq \{0\}$, then take $y_{0,\varepsilon} = \varepsilon^\alpha z$, $z \in e^{-T\mathcal{A}^*} Q_F$

$$\inf_{L^2(0,T;U)} \tilde{F}_\varepsilon = \frac{\varepsilon^{2\alpha-1}}{2} \|e^{-T\mathcal{A}}z\|^2 \xrightarrow{\varepsilon \rightarrow 0} +\infty, \quad \text{as soon as } \alpha < 1/2.$$

$$y_{0,\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \textcircled{0} \iff \text{this initial data is indeed null-controllable !!}$$

For each $\varepsilon > 0$, let $y_{0,\varepsilon} \in E$ such that $(y_{0,\varepsilon})_\varepsilon$ is bounded in E and

$$e^{-T\mathcal{A}}y_{0,\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} e^{-T\mathcal{A}}y_0.$$

ASSOCIATED HUM FUNCTIONALS

$$\tilde{F}_\varepsilon(v) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T \|v(t)\|^2 dt + \frac{1}{2\varepsilon} \|\mathcal{L}_T(v|y_{0,\varepsilon})\|^2, \quad \forall v \in L^2(0, T; U),$$

$$\tilde{J}_\varepsilon(q^F) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T \|\mathcal{B}^* e^{-(T-t)\mathcal{A}^*} q^F\|^2 dt + \frac{\varepsilon}{2} \|q^F\|^2 + \langle y_{0,\varepsilon}, e^{-T\mathcal{A}^*} q^F \rangle, \quad \forall q^F \in E.$$

We denote by \tilde{v}_ε the unique minimiser of \tilde{F}_ε .

PROPOSITION (RELAXED OBSERVABILITY INEQUALITY)

Assume that

$$\sup_{\varepsilon > 0} \frac{1}{\varepsilon} \|e^{-T\mathcal{A}}(y_0 - y_{0,\varepsilon})\|^2 < +\infty.$$

The system (S) is null-controllable from the initial data y_0 if and only if

$$\left| \langle y_{0,\varepsilon}, e^{-T\mathcal{A}^*} q^F \rangle \right|^2 \leq M \left(\left\| \mathcal{B}^* e^{-(T-\cdot)\mathcal{A}^*} q^F \right\|_{L^2(0,T;U)}^2 + \varepsilon \|q^F\|^2 \right), \quad \forall q^F \in E.$$

We do not require the system to be null-controllable from any of the $(y_{0,\varepsilon})_\varepsilon$.

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FRAMEWORK

For any $h > 0$, we are given

- A discrete state space $(E_h, \langle \cdot, \cdot \rangle_h)$.
- An approximate operator \mathcal{A}_h on E_h .
- A discrete control space $(U_h, [\cdot, \cdot]_h)$.
- A linear operator $\mathcal{B}_h : U_h \rightarrow E_h$, \mathcal{B}_h^* being its adjoint $\langle \mathcal{B}_h u, x \rangle_h = [\mathcal{B}_h^* x, u]_h$.

The semi-discrete control problem is (S_h) $\begin{cases} \partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h, \\ y_h(0) = y_{0,h}. \end{cases}$

Its solution is referred to as $t \mapsto y_{v_h, y_{0,h}}(t) \in E_h$ and we set

$$\mathcal{L}_T^h(v_h | y_{0,h}) \stackrel{\text{def}}{=} y_{v_h, y_{0,h}}(T).$$

QUESTIONS

Assume that $(y_{0,h})_h$ are, in some sense, approximations of a $y_0 \in E$.

- 1 Can we relate the controllability properties of (S) starting from y_0 to the ones of (S_h) starting from $y_{0,h}$?
- 2 Can we obtain uniform bounds (w.r.t. h) for the associated controls v_h ?

- ① It may happen that (S_h) is not controllable even if (S) is.

EXAMPLE : the 2D 5-point discrete Laplace operator \mathcal{A}_h .

(Kavian, Zuazua)

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There exists a non trivial $\psi_h \in E_h$ such that

$$\mathcal{A}_h^* \psi_h = \mu_h \psi_h, \text{ and } \mathcal{B}_h^* \psi_h = 0.$$

$$\implies \psi_h \in \mathcal{Q}_{F,h} \stackrel{\text{def}}{=} \{q_h^F \in E_h, \text{ s.t. } \mathcal{B}_h^* e^{-t\mathcal{A}_h^*} q_h^F = 0, \forall t \geq 0\},$$

For any control $v_h \in L^2(0, T; U_h)$, $\frac{d}{dt} \langle y_h(t), \psi_h \rangle_h + \mu_h \langle y_h(t), \psi_h \rangle = 0$,

and thus

$$\left\langle \mathcal{L}_T^h(v_h | y_{0,h}), \psi_h \right\rangle_h = \langle y_h(T), \psi_h \rangle_h = e^{-\mu_h T} \langle y_{0,h}, \psi_h \rangle_h. \quad (1)$$

REMARK : The eigenvalue μ_h is very large $\sim \frac{C}{h^2}$ thus $\langle \mathcal{L}_T^h(v_h | y_{0,h}), \psi_h \rangle_h$ is exponentially small.

- ② Even if (S) and (S_h) are both controllable, it is not necessarily desirable to compute a null-control v_h of (S_h) to obtain a suitable approximation of a null-control of (S) .

$$F_{\varepsilon,h}(v_h) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T \|v_h(t)\|_h^2 dt + \frac{1}{2\varepsilon} \left\| \mathcal{L}_T^h(v_h|y_{0,h}) \right\|_h^2, \quad \forall v_h \in L^2(0,T; U_h),$$

$$J_{\varepsilon,h}(q_h^F) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T \left\| \mathcal{B}_h^* e^{-(T-t)\mathcal{A}_h^*} q_h^F \right\|_h^2 dt + \frac{\varepsilon}{2} \left\| q_h^F \right\|_h^2 + \left\langle y_{0,h}, e^{-T\mathcal{A}_h^*} q_h^F \right\rangle_h, \quad \forall q_h^F \in E_h.$$

- For each value of $h > 0$, all the previous results apply.
- We denote by $v_{\varepsilon,h}$ the unique minimiser of $F_{\varepsilon,h}$.

GOAL

One would like to let $(\varepsilon, h) \rightarrow (0, 0)$ but this should be done with some care.

COMMENTS

- 1 Even if (S) is controllable from y_0 , in the cases where $Q_{F,h} \neq \{0\}$ we may have

$$\lim_{\varepsilon \rightarrow 0} \left\| \mathcal{L}_T^h(v_{\varepsilon,h}|y_{0,h}) \right\|_h \neq 0, \quad \forall h > 0$$

- 2 One can prove that for any $h > 0$

$$\sup_{\varepsilon > 0} \|v_{\varepsilon,h}\|_{L^2(0,T;U_h)} < +\infty.$$

Let $h \in]0, +\infty[\mapsto \phi(h) \in]0, +\infty[$ be given such that $\lim_{h \rightarrow 0} \phi(h) = 0$.

DEFINITION

For a given family of initial data $Y_0 = (y_{0,h})_h \in \prod_{h>0} E_h$, we say that the family of problems (S_h) is $\phi(h)$ -null controllable from Y_0 , if there exists a $h_0 > 0$ such that

$$M_{Y_0}^2 \stackrel{\text{def}}{=} 2 \sup_{0 < h < h_0} \left(\inf_{L^2(0,T;U_h)} F_{\phi(h),h} \right) < +\infty,$$

where $F_{\phi(h),h}$ is built upon $y_{0,h}$.

THEOREM (RELAXED OBSERVABILITY)

For a given $Y_0 \in E_{\text{init}}$, the problems (S_h) are $\phi(h)$ -null-controllable from Y_0 if and only if there exists $h_0 > 0$ and $\tilde{M}_{Y_0} > 0$, such that, for any $0 < h < h_0$

$$\left| \left\langle y_{0,h}, e^{-T\mathcal{A}_h^*} q_h^F \right\rangle_h \right|^2 \leq \tilde{M}_{Y_0}^2 \left(\left\| \mathcal{B}_h^* e^{-(T-\cdot)\mathcal{A}_h^*} q_h^F \right\|_{L^2(0,T;U_h)}^2 + \phi(h) \left\| q_h^F \right\|_h^2 \right), \quad \forall q_h^F \in E_h.$$

In such case, the best constant \tilde{M}_{Y_0} is equal to M_{Y_0} and

$$\left\| v_{\phi(h),h} \right\|_{L^2(0,T;U_h)} \leq M_{Y_0}, \text{ and } \left\| \mathcal{L}_T^h(v_{\phi(h),h} | y_{0,h}) \right\|_h \leq M_{Y_0} \sqrt{\phi(h)}, \quad \forall 0 < h < h_0.$$

Let $h \in]0, +\infty[\mapsto \phi(h) \in]0, +\infty[$ be given such that $\lim_{h \rightarrow 0} \phi(h) = 0$.

DEFINITION

For a given family of initial data $Y_0 = (y_{0,h})_h \in \prod_{h>0} E_h$, we say that the family of problems (S_h) is $\phi(h)$ -null controllable from Y_0 , if there exists a $h_0 > 0$ such that

$$M_{Y_0}^2 \stackrel{\text{def}}{=} 2 \sup_{0 < h < h_0} \left(\inf_{L^2(0,T;U_h)} F_{\phi(h),h} \right) < +\infty,$$

where $F_{\phi(h),h}$ is built upon $y_{0,h}$.

PROPOSITION

Assume that, for some $C_{\text{obs}} > 0$, the following relaxed observability inequality holds

$$\left\| e^{-T\mathcal{A}_h^*} q_h^F \right\|_h^2 \leq C_{\text{obs}}^2 \left(\left\| \mathcal{B}_h^* e^{-(T-\cdot)\mathcal{A}_h^*} q_h^F \right\|_{L^2(0,T;U_h)}^2 + \phi(h) \left\| q_h^F \right\|_h^2 \right), \quad \left| \begin{array}{l} \forall q_h^F \in E_h, \\ \forall 0 < h < h_0 \end{array} \right.$$

then for any **bounded** family Y_0 , the problems (S_h) are $\phi(h)$ -null-controllable from Y_0 and we have

$$M_{Y_0} \leq C_{\text{obs}} \left(\sup_{0 < h < h_0} \|y_{0,h}\|_h \right).$$

(Lasiecka-Triggiani, '00) (Labbé-Trélat, '06)

- We suppose given $\tilde{P}_h : E_h \rightarrow D((\mathcal{A}^*)^{\frac{1}{2}})$ and $\tilde{Q}_h : U_h \rightarrow U$ such that

$$\|y_h\|_h = \left\| \tilde{P}_h y_h \right\|, \forall y_h \in E_h, \text{ and } \llbracket u_h \rrbracket_h = \left\llbracket \tilde{Q}_h u_h \right\rrbracket.$$

- We set $P_h = (\tilde{P}_h)^* : D((\mathcal{A}^*)^{\frac{1}{2}})' \rightarrow E_h$ and $Q_h = (\tilde{Q}_h)^* : U \rightarrow U_h$ and we assume that

$$P_h \tilde{P}_h = \text{Id}_{E_h}, \text{ and } Q_h \tilde{Q}_h = \text{Id}_{U_h}.$$

- We define now \mathcal{A}_h and \mathcal{B}_h through their adjoints by the formulas

$$\mathcal{A}_h^* = P_h \mathcal{A}^* \tilde{P}_h, \quad \mathcal{B}_h^* = Q_h \mathcal{B}^* \tilde{P}_h.$$

- + Standard approximation properties ...

EXAMPLE : Finite element Galerkin approximation.

(Labbé-Trélat, '06)

THEOREM

Assume that (S) is null-controllable at time T .

There exists a $\beta > 0$, **depending on the approximation properties of E_h and U_h** such that the relaxed-observability inequality holds as soon as

$$\liminf_{h \rightarrow 0} \frac{\phi(h)}{h^\beta} > 0.$$

In that case, for any $y_0 \in E$, we can define $y_{0,h} = P_h y_0$ and build the associated penalised HUM discrete controls $v_{\phi(h),h}$.

Then, there is a null-control $v \in \text{Adm}(y_0, 0)$ such that, up to a subsequence, we have

$$\tilde{Q}_h v_{\phi(h),h} \xrightarrow{h \rightarrow 0} v, \quad \text{in } L^2(0, T; U), \quad \text{and} \quad \tilde{P}_h y_h \xrightarrow{h \rightarrow 0} y_{v, y_0}, \quad \text{in } L^2(0, T; E).$$

- The limit control v may not be the HUM control.
- Proving **strong convergence** of the discrete control is very difficult.
- In practice, the power β is low : for the 1D heat equation, Neumann boundary control, \mathbb{P}^1 finite element, we get $\beta = 0.45$. It means that

$$\|y_h(T)\|_h \approx_0 \sqrt{\phi(h)} = h^{0.225} \quad \Leftarrow \text{Very poor convergence.}$$

(B.-Hubert-Le Rousseau, '09-'11)

THEOREM

We assume that \mathcal{A}_h is the usual finite difference approximation of $-\operatorname{div}(\gamma \nabla \cdot)$ for a smooth γ on a regular Cartesian mesh and that $\mathcal{B}_h = 1_\omega$. Then, there exists $h_0 > 0$, $C > 0$ such that, the relaxed observability inequality holds as soon as the function ϕ satisfies

$$\liminf_{h \rightarrow 0} \frac{\phi(h)}{e^{-C/h^2}} > 0.$$

Thus, for any bounded family of initial data $Y_0 \in E_{\text{init}}$, and for any $0 < h < h_0$ we have

$$\|v_{\phi(h),h}\|_{L^2(0,T;U_h)} \leq C_{\text{obs}} \|y_{0,h}\|_h, \text{ and } \left\| \mathcal{L}_T^h(v_{\phi(h),h} | y_{0,h}) \right\|_h \leq C_{\text{obs}} \|y_{0,h}\|_h \sqrt{\phi(h)}.$$

CONSEQUENCE : The $\phi(h)$ -null-controllability holds for any $\phi(h) \geq e^{-C/h^2}$.

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GENERAL PRINCIPLE : Minimise dual functionals $J_{\varepsilon,h}$ or $J_{\varepsilon,h,\delta t}$ (with $\varepsilon = \phi(h)$).

PROPOSITION (GRADIENTS AND GRAMIAN OPERATORS)

For any $h > 0, \delta t > 0, \varepsilon > 0$ and any $q_h^F \in E_h$, we have

$$\begin{aligned}\nabla J_{\varepsilon,h}(q_h^F) &= \underbrace{\mathcal{L}_T^h \left(\mathcal{B}_h^* e^{-(T-\cdot)\mathcal{A}_h^*} q_h^F \middle| 0 \right)}_{\stackrel{\text{def}}{=} \Lambda^h q_h^F} + \varepsilon q_h^F + \mathcal{L}_T^h(0|y_{0,h}), \\ \nabla J_{\varepsilon,h,\delta t}(q_h^F) &= \underbrace{\mathcal{L}_T^{h,\delta t} \left(\mathcal{B}_h^* \mathcal{L}_T^{*,h,\delta t}(q_h^F) \middle| 0 \right)}_{\stackrel{\text{def}}{=} \Lambda^{h,\delta t} q_h^F} + \varepsilon q_h^F + \mathcal{L}_T^{h,\delta t}(0|y_{0,h}),\end{aligned}$$

where $\mathcal{L}_T^{*,h,\delta t}(q_h^F)$ is the solution of the adjoint fully-discrete pb associated with q_h^F .

COMPUTATION OF GRAMIAN OPERATORS

The computation of $\Lambda_{\bullet} q_h^F$ amounts to

- ① Solve a backward parabolic problem.
- ② Apply \mathcal{B}_h^*
- ③ Solve a forward parabolic problem with the control previously computed.

GENERAL PRINCIPLE : Minimise dual functionals $J_{\varepsilon,h}$ or $J_{\varepsilon,h,\delta t}$ (with $\varepsilon = \phi(h)$).

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where $\mathcal{L}_T^{*,h,\delta t}(q_h^F)$ is the solution of the adjoint fully-discrete pb associated with q_h^F .

EQUATIONS TO SOLVE

The semi/fully-discrete controls are computed by solving the equations

$$\begin{aligned}(\Lambda^h + \varepsilon \text{Id})q_h^F &= -\mathcal{L}_T^h(0|y_{0,h}) , \\ (\Lambda^{h,\delta t} + \varepsilon \text{Id})q_h^F &= -\mathcal{L}_T^{h,\delta t}(0|y_{0,h}) .\end{aligned}$$

In practice, we use a conjugate gradient algorithm.

GENERAL PRINCIPLE : Minimise dual functionals $J_{\varepsilon,h}$ or $J_{\varepsilon,h,\delta t}$ (with $\varepsilon = \phi(h)$).

PROPOSITION (GRADIENTS AND GRAMIAN OPERATORS)

For any $h > 0, \delta t > 0, \varepsilon > 0$ and any $q_h^F \in E_h$, we have

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where $\mathcal{L}_T^{*,h,\delta t}(q_h^F)$ is the solution of the adjoint fully-discrete pb associated with q_h^F .

CONDITION NUMBER

$$\text{Basic estimate : } \varepsilon \left\| q_h^F \right\|_h \leq \left\| (\Lambda^\bullet + \varepsilon \text{Id}) q_h^F \right\|_h \leq (C + \varepsilon) \left\| q_h^F \right\|_h.$$

$$\text{Cond}(\Lambda^\bullet + \varepsilon \text{Id}) \sim \frac{1}{\varepsilon}.$$

TWO MAIN PRINCIPLES

- 1 $\varepsilon = \phi(h)$ should not be too small in order to maintain a reasonable condition number (i.e. computational cost)

$$\text{Cond}(\Lambda^\bullet + \phi(h)\text{Id}) \sim \frac{1}{\phi(h)}.$$

- 2 The size of the computed solution at time T is

$$\|y_h(T)\|_h \approx C_{\text{obs}} \sqrt{\phi(h)}.$$

It seems reasonable to choose

$$\phi(h) \sim_{h \rightarrow 0} h^{2p},$$

where p is the order of accuracy of the numerical method under study.

REMARKS

- Computing a null-control for (S_h) , i.e. taking $\varepsilon = \phi(h) = 0$, is not possible in general.
- Choosing $\phi(h)$ much smaller than h^{2p} (like e^{-C/h^2}) is a useless computational effort.

We set $E = E_h = \mathbb{R}$, $\mathcal{A} = \lambda > 0$, $\mathcal{A}_h = (\lambda + \delta_h) \in \mathbb{R}$ with $\delta_h \xrightarrow{h \rightarrow 0} 0$, $\mathcal{B} = \mathcal{B}_h = 1$.

$$(S) \begin{cases} y' + \lambda y = v, \\ y(0) = 1, \end{cases} \quad \text{and} \quad (S_h) \begin{cases} y_h' + (\lambda + \delta_h)y_h = v_h, \\ y_h(0) = 1. \end{cases}$$

Uncontrolled solution $e^{-T\mathcal{A}_h}y_{0,h} = e^{-(\lambda+\delta_h)T}$.

GRAMIAM “OPERATORS”

$$\Lambda_h q^F = \frac{1 - e^{-2(\lambda+\delta_h)T}}{2(\lambda + \delta_h)} q^F, \quad \text{and} \quad \Lambda q^F = \frac{1 - e^{-2\lambda T}}{2\lambda} q^F, \quad \forall q^F \in \mathbb{R},$$

PROPOSITION

The corresponding semi-discrete penalised and exact HUM controls are

$$v_{\varepsilon,h}(t) = -e^{-(T-t)(\lambda+\delta_h)} \frac{2(\lambda + \delta_h)e^{-(\lambda+\delta_h)T}}{1 - e^{-2(\lambda+\delta_h)T} + \boxed{2\varepsilon(\lambda + \delta_h)}},$$

$$v(t) = -e^{-(T-t)\lambda} \frac{2\lambda e^{-\lambda T}}{1 - e^{-2\lambda T}}.$$

$$(S) \begin{cases} y' + \lambda y = v, \\ y(0) = 1, \end{cases} \quad \text{and} \quad (S_h) \begin{cases} y'_h + (\lambda + \delta_h)y_h = v_h, \\ y_h(0) = 1. \end{cases}$$

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$$v(t) = -e^{-(T-t)\lambda} \frac{2\lambda e^{-\lambda T}}{1 - e^{-2\lambda T}}.$$

ERROR ESTIMATES

$$\|v - v_{\varepsilon,h}\|_{L^2(0,T;U)} \leq C(\lambda, T)(|\delta_h| + \varepsilon), \quad \text{for } \delta_h \text{ and } \varepsilon \text{ small,}$$

$$\mathcal{L}_T(v_{\varepsilon,h}|1) = C_1(\lambda, T)\delta_h + C_2(\lambda, T)\varepsilon + O(\varepsilon^2 + \delta_h^2),$$

with $C_i(\lambda, T) > 0$.

CONCLUSION : The *optimal* choice is to take $\varepsilon = \phi(h) \sim \delta_h$.

$$\partial_t y - \partial_x^2 y = \mathbf{1}_\Omega v, \quad \text{in } \Omega =]0, 1[,$$

in the particular case where $\omega = \Omega$.

STANDARD FINITE DIFFERENCE APPROXIMATION ON A UNIFORM GRID

$$\partial_t y_i - \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = v_i, \quad \forall i \in \{1, \dots, N\}.$$

EIGENFUNCTIONS OF \mathcal{A}

$$\phi_k(x) = \sin(k\pi x), \quad \lambda_k = k^2 \pi^2, \quad \forall k \geq 1.$$

EIGENFUNCTIONS OF \mathcal{A}_h

$$\phi_{k,h} = (\sin(k\pi x_i))_i, \quad \lambda_{k,h} = \frac{4 \sin^2\left(\frac{k\pi h}{2}\right)}{h^2}, \quad \forall 1 \leq k \leq 1/h.$$

EQUATIONS FOR THE k -TH EIGENMODE

$$y' + \lambda_k y = v, \quad y_h' + \lambda_{k,h} y_h = v_h.$$

Here

$$\delta_{k,h} = \lambda_{k,h} - \lambda_k \underset{h \rightarrow 0}{\sim} -\frac{k^4 \pi^4}{12} h^2.$$

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We assume that \mathcal{A}_h is SPD and let $(\psi_{j,h}, \mu_{j,h})_j$ its eigenelements.

ASSUMPTION : DISCRETE LEBEAU-ROBBIANO SPECTRAL INEQUALITY

There exists $h_0 > 0$, $\alpha \in [0, 1)$, $\beta > 0$, and $\kappa, \ell > 0$ such that, for any $h < h_0$ and for any $(a_j)_j \in \mathbb{R}^{\mathbb{N}}$, we have

$$\left\| \sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \right\|_h^2 \leq \kappa e^{\kappa \mu^\alpha} \left\| \mathcal{B}_h^* \left(\sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \right) \right\|_h^2, \quad \forall \mu < \frac{\ell}{h^\beta}. \quad (\mathcal{H}_{\alpha,\beta})$$

THEOREM

Assume that assumption $(\mathcal{H}_{\alpha,\beta})$ holds, then there exists $h_0 > 0$, $C > 0$ such that, the relaxed observability inequality holds as soon as the function ϕ satisfies

$$\liminf_{h \rightarrow 0} \frac{\phi(h)}{e^{-C/h^\beta}} > 0.$$

Thus, for any bounded family of initial data $Y_0 \in E_{\text{init}}$, and for any $0 < h < h_0$ we have

$$\left\| v_{\phi(h),h} \right\|_{L^2(0,T;U_h)} \leq C_{\text{obs}} \|y_{0,h}\|_h, \text{ and } \left\| \mathcal{L}_T^h(v_{\phi(h),h} | y_{0,h}) \right\|_h \leq C_{\text{obs}} \|y_{0,h}\|_h \sqrt{\phi(h)}.$$

Let $E_{\mu,h} = \text{Span}\{\psi_{j,h}, \mu_{j,h} \leq \mu\}$. Assumption $(\mathcal{H}_{\alpha,\beta})$ says that

$$\left\| q_h \right\|_h^2 \leq \kappa e^{\kappa \mu^\alpha} \left\| \mathcal{B}_h^* q_h \right\|_h^2, \quad \forall q_h \in E_h, \forall \mu < \frac{\ell}{h^\beta}.$$

④ NON-UNIFORM PARTIAL OBSERVABILITY

$$\left\| e^{-\tau \mathcal{A}_h^*} q_h^F \right\|_h^2 \leq C \frac{C e^{C \mu^\alpha}}{\tau} \int_0^\tau \left\| \mathcal{B}_h^* e^{-s \mathcal{A}_h^*} q_h^F \right\|_h^2 ds, \quad \forall q_h^F \in E_{\mu,h}, \forall \tau > 0.$$

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② PARTIAL CONTROLLABILITY

For any $y_{0,h} \in E_{\mu,h}$, and any $\tau > 0$ there exists a $v_h \in L^2(0, \tau; U_h)$ such that

$$\begin{cases} \|v_h\|_{L^2(0,\tau;U_h)} \leq C \frac{e^{C\mu^\alpha}}{\tau^{1/2}} \|y_{0,h}\|_h, \\ \partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h, \quad y_h(0) = y_{0,h}, \end{cases} \quad \mathbb{P}_{E_{\mu,h}} y_h(\tau) = 0.$$

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2 PARTIAL CONTROLLABILITY

For any $y_{0,h} \in E_{\mu,h}$, and any $\tau > 0$ there exists a $v_h \in L^2(0, \tau; U_h)$ such that

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3 CONSTRUCTION OF THE CONTROL : Time slicing procedure.



Let $E_{\mu,h} = \text{Span}\{\psi_{j,h}, \mu_{j,h} \leq \mu\}$. Assumption $(\mathcal{H}_{\alpha,\beta})$ says that

$$\|q_h\|_h^2 \leq \kappa e^{\kappa\mu^\alpha} \left\| \mathcal{B}_h^* q_h \right\|_h^2, \quad \forall q_h \in E_h, \forall \mu < \frac{\ell}{h^\beta}.$$

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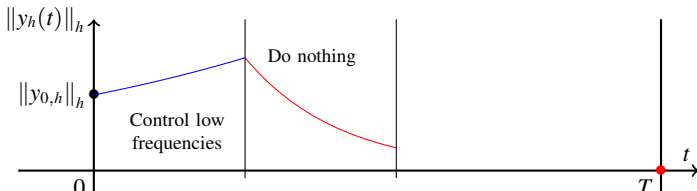
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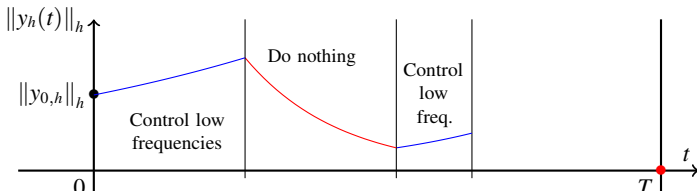
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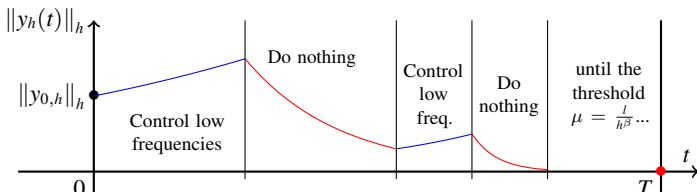
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3 CONSTRUCTION OF THE CONTROL : Time slicing procedure.



$$\left\| \sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \right\|_h^2 \leq \kappa e^{\kappa \mu^\alpha} \left[\mathcal{B}_h^* \left(\sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \right) \right]_h^2, \quad \boxed{\forall \mu < \frac{\ell}{h^\beta}}. \quad (\mathcal{H}_{\alpha,\beta})$$

IMPORTANT OBSERVATION

Excepted in very particular cases, the assumption $(\mathcal{H}_{\alpha,\beta})$ has no chance to hold true without restriction on μ , for dimension reasons.

See also the counter-example of Kavian.

FINITE DIFFERENCE FRAMEWORK

We assume now that \mathcal{A}_h is the finite-difference discretisation of $-\nabla \cdot (\gamma \nabla \cdot)$, γ being a Lipschitz continuous coefficient and $\mathcal{B}_h = 1_\omega$ with $\omega = \subset \Omega$.

$$\left\| \sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \right\|_h^2 \leq \kappa e^{\kappa \mu^\alpha} \left\| \mathcal{B}_h^* \left(\sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \right) \right\|_h^2, \quad \forall \mu < \frac{\ell}{h^\beta}. \quad (\mathcal{H}_{\alpha,\beta})$$

THE CARLEMAN ESTIMATE

For a suitable weight function φ , $\varepsilon_0 > 0$, $s_0 > 0$ and $C > 0$.

For any $s \geq s_0$ such that $sh \leq \varepsilon_0$ and any $u \in \mathcal{C}^2([0, \tau], E_h)$, with $u(0) = 0$, we have

$$\begin{aligned} & s^3 \int_0^\tau \|e^{s\varphi} u\|_h^2 dt + s \left\| e^{s\varphi(0,\cdot)} \partial_t u(0) \right\|_h^2 + s e^{s\varphi(\tau)} \|\partial_t u(\tau)\|_h^2 + s^3 e^{2s\varphi(\tau)} \|u(\tau)\|_h^2 \\ & \leq C \int_0^\tau \left\| e^{s\varphi} (-\partial_t^2 + \mathcal{A}_h) u \right\|_h^2 dt + C s e^{2s\varphi(\tau)} \langle \mathcal{A}_h u(\tau), u(\tau) \rangle_h + C s \left\| \mathcal{B}_h^* e^{s\varphi(0,\cdot)} \partial_t u(0) \right\|_h^2. \end{aligned}$$

$$\left\| \sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \right\|_h^2 \leq \kappa e^{\kappa \mu^\alpha} \left[\mathcal{B}_h^\star \left(\sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \right) \right]_h^2, \quad \forall \mu < \frac{\ell}{h^\beta}. \quad (\mathcal{H}_{\alpha,\beta})$$

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$$s^3 e^{2s\varphi(\tau)} \|u(\tau)\|_h^2 \leq C s e^{2s\varphi(\tau)} \langle \mathcal{A}_h u(\tau), u(\tau) \rangle_h + C s e^{Cs} \|\mathcal{B}_h^\star \partial_t u(0)\|_h^2,$$

as soon as $(-\partial_t^2 + \mathcal{A}_h)u = 0$.

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WE TAKE : $u(t) = \sum_{\mu_{j,h} \leq \mu} a_j \frac{\sinh(\sqrt{\mu_{j,h}} t)}{\sqrt{\mu_{j,h}}} \psi_{j,h}$, sol. of the elliptic Cauchy problem.

$$\|u(\tau)\|_h^2 \geq \frac{1}{\mu} \sum_{\mu_{j,h} \leq \mu} |a_j|^2 \sinh^2(\sqrt{\mu_{j,h}} \tau), \text{ and } \langle \mathcal{A}_h u(\tau), u(\tau) \rangle_h = \sum_{\mu_{j,h} \leq \mu} |a_j|^2 \sinh^2(\sqrt{\mu_{j,h}} \tau)^2,$$

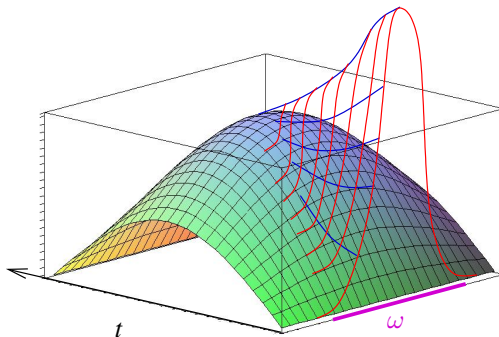
$$\|\partial_t u(0)\|_h^2 = \left[\mathcal{B}_h^\star \left(\sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \right) \right]_h^2.$$

Finally, we need $s \sim \sqrt{\mu}$ which gives the condition $\sqrt{\mu} h \leq \varepsilon_0$.

THE CONTINUOUS CASE

We consider a weight function ψ in $\mathcal{C}^2([0, \tau] \times \overline{\Omega}, \mathbb{R})$, such that

$$\begin{aligned} |\nabla \psi| &\geq c \text{ and } \psi > 0 \text{ in }]0, \tau[\times \Omega, \quad \partial_{n_x} \psi(t, x) < 0 \text{ in } (0, \tau) \times \partial\Omega, \\ \partial_t \psi &\geq c \text{ on } \{0\} \times (\Omega \setminus \omega), \quad \nabla_x \psi(T, \cdot) = 0 \text{ and } \partial_t \psi(T, \cdot) \leq -c \text{ on } \Omega. \end{aligned}$$



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We set $\varphi = e^{\lambda \psi}$ and $\rho = e^{s\varphi}$, and

$$P = -\partial_t^2 - \nabla \cdot (\gamma \nabla \cdot).$$

WE COMPUTE THE COMMUTATOR BETWEEN P AND ρ

ASSUME $d = 1$ AND $\gamma = 1$

$$f = Pu = -\partial_t^2 u - \partial_x^2 u,$$

becomes (with $r = \rho^{-1}$)

$$\underbrace{\partial_t^2 v + \partial_x^2 v + r(\partial_t^2 \rho)v + r(\partial_x^2 \rho)v}_{=Av} + \underbrace{2r(\partial_t \rho)(\partial_t v) + r(\partial_x \rho)(\partial_x v)}_{=Bv} = -rf.$$

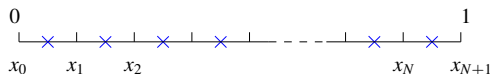
Then

$$2(Av, Bv)_{L_{t,x}^2} \leq \|rf\|_{L_{t,x}^2}.$$

We conclude by integrating by parts in the term (Av, Bv) and using the properties of the weights (and by choosing λ large enough). The inequality then holds for any s large enough.

THE 1D UNIFORM DISCRETISATION CASE

✓ for smooth Cartesian multi-D geometry



$$\mathfrak{M} = (x_i)_{1 \leq i \leq N}$$

$$\overline{\mathfrak{M}} = (x_{i+\frac{1}{2}})_{0 \leq i \leq N}$$

- Primal/Dual discrete functions

$$u = (u_i)_{1 \leq i \leq N} \in \mathbb{R}^{\mathfrak{M}}, \quad v = (v_{i+\frac{1}{2}})_{0 \leq i \leq N} \in \mathbb{R}^{\overline{\mathfrak{M}}}.$$

- Discrete derivatives (Dirichlet BC : $\Leftrightarrow u_0 = u_{N+1} = 0$)

$$\left((Du)_{i+\frac{1}{2}} = \frac{u_{i+1} - u_i}{h} \right)_{0 \leq i \leq N} \in \mathbb{R}^{\overline{\mathfrak{M}}}, \quad \left((\bar{D}v)_i = \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{h} \right)_{1 \leq i \leq N} \in \mathbb{R}^{\mathfrak{M}},$$

- Discrete integrals and norms

$$\|u\|_h^2 = \int_{\Omega} u^2 = \sum_{i=1}^N h u_i^2, \quad \int_{\Omega} v = \sum_{i=0}^N h v_{i+\frac{1}{2}}, \dots$$

The discrete operator reads $\mathcal{A}_h = -\bar{D}(\gamma D \cdot)$

- We consider **the same weight** as in the continuous case which is sampled on the meshes.
- Discrete integration by parts and discrete derivation of products lead to new terms

$$\underbrace{\partial_i^2 v + \textcolor{red}{r}\bar{\rho} \bar{D} D v + r(\partial_i^2 \rho) v + r(\bar{D} D \rho) \bar{v}}_{=A_v} + \underbrace{2r(\partial_i \rho)(\partial_i v) + 2r\bar{D}\rho \bar{D} v}_{=B_v} = -rf + \dots$$

- All the additional terms are **small** with respect to h but contain **high powers of the parameters** s and λ .
- We shall need a condition relating s and h of the following form

$$sh \leq \varepsilon_0,$$

where ε_0 only depends on the data.

- In the proof of the discrete Lebeau-Robbiano, we take $s \sim \sqrt{\mu}$, which gives the threshold $\mu \leq \frac{\varepsilon_1}{h^2}$.

PRODUCTS DERIVATIVES

$$f_i \in \mathbb{R}^{\mathfrak{M}}, g_i \in \mathbb{R}^{\overline{\mathfrak{M}}}$$

$$D(f_1 f_2) = (Df_1) \overline{f_2} + (Df_2) \overline{f_1},$$

$$\bar{D}(g_1 g_2) = (\bar{D}g_1) \overline{g_2} + (\bar{D}g_2) \overline{g_1}.$$

PRODUCTS AVERAGES

$$\overline{f_1 f_2} = \overline{f_1} \overline{f_2} + \frac{h^2}{4} (Df_1)(Df_2).$$

REPEATED AVERAGES

$$\bar{\bar{f}} = f + \frac{h^2}{4} \bar{D} D f,$$

Typical term (remember that $r = \rho^{-1}$)

$$r \bar{\bar{\rho}} = 1 + \frac{h^2}{4} r (\bar{D} D \rho) \approx 1 + \frac{h^2}{4} r \partial_x^2 \rho \approx 1 + \frac{h^2 s^2}{4} (\partial_x \phi)^2 \underbrace{r \rho}_{=1} + \dots$$

DISCRETE INTEGRATION BY PARTS

$$f \in \mathbb{R}^{\overline{\mathfrak{M}}} \text{ and } g \in \mathbb{R}^{\mathfrak{M}}$$

$$\int_{\Omega} f(Dg) = - \int_{\Omega} (\bar{D}f) g + f_{N+1} g_{N+\frac{1}{2}} - f_0 g_{\frac{1}{2}}.$$

GOALS

- To obtain relaxed observability estimates for some time depend coefficients parabolic equations.
- To study some semi-linear case.
- To derive discrete parabolic Carleman estimates (*a la Fursikov-Imanuvilov*).

CARLEMAN WEIGHTS

- Standard weights in space $\varphi = e^{\lambda\psi} - e^{\lambda\|\psi\|_\infty}$.
- Singular weights in time $\theta(t) = (t + \alpha h)^{-1}(T - t + \alpha h)^{-1}$.

THEOREM

For any $\tau > 0$ large enough, there exists $\alpha > 0$ and $h_0 > 0$ such that for any function $q_h \in C^1([0, T], E_h)$ and any $h < h_0$ we have

$$\begin{aligned} & \|(\tau\theta)^{\frac{1}{2}} e^{-\tau\theta\varphi} Dq_h\|_{L^2(0,T;E_h)}^2 + \|(\tau\theta)^{\frac{3}{2}} e^{-\tau\theta\varphi} q_h\|_{L^2(0,T;E_h)}^2 \\ & \leq C \left(\|e^{-\tau\theta\varphi} (-\partial_t + \mathcal{A}_h^*) q_h\|_{L^2(0,T;E_h)}^2 + \|(\tau\theta)^{\frac{3}{2}} e^{-\tau\theta\varphi} \mathcal{B}_h^* q_h\|_{L^2(0,T;E_h)}^2 \right) \\ & \quad + Ch^{-2} \left(\left\| e^{-\tau\theta\varphi} q_h(0) \right\|_h^2 + \left\| e^{-\tau\theta\varphi} q_h(T) \right\|_h^2 \right) \end{aligned}$$

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THEOREM (RELAXED OBSERVABILITY INEQUALITY)

There exists $C > 0$ s.t. for any function $a_h \in L^\infty([0, T[, E_h)$, and any $h \leq \min(h_0, h_1)$ with $h_1 \sim \|a_h\|_\infty^{-\frac{2}{3}}$, any solution of $-\partial_t q_h + \mathcal{A}_h^* q_h + a_h q_h = 0$ satisfies

$$\|q_h(0)\|_h^2 \leq C_{\text{obs}} \|\mathcal{B}_h^* q_h\|_{L^2(0, T; E_h)}^2 + e^{-\frac{C-1}{h} + T\|a_h\|_\infty} \|q_h(T)\|_h^2.$$

with $C_{\text{obs}} = e^{C(1 + \frac{1}{T} + T\|a_h\|_\infty + \|a_h\|_\infty^{\frac{2}{3}})}$.

(Fernández-Cara – Zuazua, '00), (B. – Le Rousseau, '14)

$$\partial_t y_h + \mathcal{A}_h y_h + g(y_h) y_h = \mathbf{1}_\omega v_h, \quad y_h(0) = y_{0,h},$$

SUBLINEAR CASE : $|g(s)| \leq M$

There exists

$$C = e^{C_1(1 + \frac{1}{T} + T\|g\|_\infty + \|g\|_\infty^{\frac{2}{3}})}$$

such that for any initial data $y_{0,h} \in E_h$, and any $h < h_0$, there exists a semi-discrete control v_h such that

$$\|v_h\|_{L^2(0,T;U_h)} \leq C \|y_{0,h}\|_h, \quad \text{and} \quad \|y_h(T)\|_h \leq C e^{-\frac{C-1}{h}} \|y_0\|_h.$$

- Uniform estimates with respect to h :
 - Weak convergence of the contro towards a null-control of the semi-linear problem. At least for a subsequence ...
 - Strong convergence of the controlled solution
- We can replace $e^{-C^{-1}/h}$ by any $\phi(h)$ that goes to 0 not too fast.
- In practice, we use a Picard fixed-point procedure to compute the approximate solution but other non-linear solvers can be useful.

(Fernández-Cara – Zuazua, '00), (B. – Le Rousseau, '14)

$$\partial_t y_h + \mathcal{A}_h y_h + g(y_h) y_h = \mathbf{1}_\omega v_h, \quad y_h(0) = y_{0,h},$$

SUPERLINEAR CASE : $|g(s)| \leq M \ln(1 + |s|)^r$, $r < 3/2$

- In 1D : For any initial data $y_{0,h} \in E_h$ such that $\|Dy_{0,h}\|_h \leq M$ and $h < h_0$ there exists a v_h such that

$$\|v_h\|_{L^2(0,T;U_h)} \leq C_M, \quad \text{and} \quad \|y_h(T)\|_h \leq C_M e^{-\frac{C-1}{h}}.$$

- In multi-D : similar result (no need of a H^1 estimate) but with a non-uniform bound of the control

$$\|v_h\|_{L^2(0,T;U_h)} \leq C \|y_{0,h}\|_h h^{-\alpha}, \quad \text{and} \quad \|y_h(T)\|_h \leq C \|y_{0,h}\|_h e^{-\frac{C-1}{h}}.$$

BONUS : UNIFORM LOCAL CONTROLLABILITY RESULT IN 1D FOR ANY g

LINEARIZATION + TRUNCATURE + SCHAUDER FIXED-POINT PROCEDURE

$$(S_{z_h}) : \partial_t y_h + \mathcal{A}_h y_h + \underbrace{g(T_{R_h}(z_h))}_{=a_h} y_h = \mathbf{1}_\omega v_h,$$

$\Lambda_h : z_h \in \text{Some space } Z_h \longmapsto v_h \text{ the HUM-pen. control for } (S_{z_h})$
 $\longmapsto y_h \in \text{the same space } Z_h.$

MAIN ISSUE : to find a suitable space Z_h

- Sublinear case : one can take for some R_0

$$Z_h \stackrel{\text{def}}{=} \{z_h \in L^2(0, T; U_h), \quad \|z_h\|_{L^2(0, T; U_h)} \leq R_0\}.$$

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MAIN ISSUE : to find a suitable space Z_h

- Sublinear case :
- Superlinear case : We need an L^∞ estimate

$$Z_h \stackrel{\text{def}}{=} \{z_h \in L^\infty(0, T; U_h), \|z_h\|_{L^\infty(0, T; U_h)} \leq R_h\}.$$

Then we take $R_h = Ch^{-\alpha}$, $\alpha > d/2$ in order to satisfy

$$\begin{cases} h \lesssim \|a_h\|_\infty^{-\frac{2}{3}} \sim \ln(1 + R_h)^{-\frac{2r}{3}} \Rightarrow \text{Carleman} \\ h^{-d/2} e^{C_T + \ln(1 + R_h)^{-\frac{2r}{3}}} \|y_{0,h}\|_h \leq R_h \Rightarrow \text{stability estimate} \\ + \text{a good estimate on the control time } T_h \end{cases}$$

since we want that

$$\|y_h\|_{L^\infty} \leq h^{-d/2} \|y_h\|_{L^\infty(0, T; E_h)} \lesssim h^{-d/2} e^{C_T + \ln(1 + R_h)^{-\frac{2r}{3}}} \|y_{0,h}\|_h \leq R_h.$$

LINEARIZATION + TRUNCATURE + SCHAUDER FIXED-POINT PROCEDURE

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MAIN ISSUE : to find a suitable space Z_h

- Sublinear case :
- Superlinear case :
- Improvement in 1D :

$$\text{Discrete Sobolev embedding } \|y_h\|_\infty \leq C \|Dy_h\|_h.$$

LEMMA (UNIFORM REGULARISING EFFECT)

Assume that $y_{0,h} \in E_h$ is such that $\|y_{0,h}\|_\infty \leq M_0$ for some $M_0 > 0$ then there exists $t_1 > 0$ depending only on M_0 , and g , such that the solution y_h of

$$\partial_t y_h + \mathcal{A}_h y_h + g(y_h) y_h = 0, \text{ and } y_h(0) = y_{0,h},$$

exists on $[0, t_1]$ and satisfies $\|Dy_h(t_1)\|_h \leq C(M_0, g)$.

- 1 INTRODUCTION
- 2 SOME FACTS ABOUT THE HILBERT UNIQUENESS METHOD AND ITS PENALIZED VERSION
- 3 **THE HUM APPROACH IN THE DISCRETE FRAMEWORK**
 - The semi-discrete setting
 - Practical considerations
 - Discrete Carleman inequalities and applications
 - **The fully discrete setting**
 - Few words about control to the trajectories
 - Error analysis in time
- 4 NUMERICAL RESULTS
 - 1D Scalar equations
 - 1D Parabolic systems
 - Some 2D results
- 5 CONCLUSIONS / PERSPECTIVES

We have introduced and analyzed the $\phi(h)$ -null-controllability hold for

$$(S_h) \quad \begin{cases} \partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h, \\ y_h(0) = y_{0,h}. \end{cases}$$

WHAT ABOUT TIME DISCRETIZATION OF SUCH A SYSTEM ?

We study **unconditionally stable schemes** : the θ -scheme with $\theta \in [1/2, 1]$

$$(S_{h,\delta t}) \quad \begin{cases} \frac{y_h^{n+1} - y_h^n}{\delta t} + \mathcal{A}_h(\theta y_h^{n+1} + (1-\theta)y_h^n) = \mathcal{B}_h v_h^{n+1}, \quad \forall n \in \llbracket 0, M-1 \rrbracket, \\ y_h^0 = y_{0,h} \in E_h, \end{cases}$$

where, $\delta t = T/M$, $v_{h,\delta t} = (v_h^n)_{1 \leq n \leq M} \in (U_h)^M$ is a fully-discrete control function whose cost is defined by

$$\llbracket v_{h,\delta t} \rrbracket_{L^2_{\delta t}(0,T;U_h)} \stackrel{\text{def}}{=} \left(\sum_{n=1}^M \delta t \llbracket v_h^n \rrbracket_h^2 \right)^{\frac{1}{2}}.$$

The value at the final time iteration of the controlled solution of $(S_{h,\delta t})$ is denoted by

$$\mathcal{L}_T^{h,\delta t}(v_{h,\delta t} | y_{0,h}) \stackrel{\text{def}}{=} y_h^M.$$

THE PENALISED HUM PRIMAL FUNCTIONAL

$$F_{\varepsilon,h,\delta t}(v_h,\delta t) \stackrel{\text{def}}{=} \frac{1}{2} \llbracket v_h,\delta t \rrbracket_{L^2_{\delta t}(0,T;U_h)}^2 + \frac{1}{2\varepsilon} \left\| \mathcal{L}_T^{h,\delta t}(v_h,\delta t|y_{0,h}) \right\|_h^2.$$

DEFINITION (DUAL FUNCTIONAL)

We define the functional

$$J_{\varepsilon,h,\delta t}(q_h^F) \stackrel{\text{def}}{=} \frac{1}{2} \left\| \mathcal{B}_h^* \mathcal{L}_T^{*,h,\delta t}(q_h^F) \right\|_{L^2_{\delta t}(0,T;U_h)}^2 + \frac{\varepsilon}{2} \left\| q_h^F \right\|_h^2 - \left\langle y_{0,h}, q_h^1 - \delta t(1-\theta)\mathcal{A}_h q_h^1 \right\rangle_h, \quad \forall q_h^F \in E_h,$$

where $\mathcal{L}_T^{*,h,\delta t}(q_h^F) = (q_h^n)_{1 \leq n \leq M}$ is the solution of the following adjoint problem

$$\left\{ \begin{array}{l} q_h^{M+1} = q_h^F, \\ \frac{q_h^M - q_h^{M+1}}{\delta t} + \theta \mathcal{A}_h q_h^M = 0, \\ \frac{q_h^n - q_h^{n+1}}{\delta t} + \mathcal{A}_h(\theta q_h^n + (1-\theta)q_h^{n+1}) = 0, \quad \forall n \in \llbracket 1, M-1 \rrbracket. \end{array} \right. \quad (S_{h,\delta t}^*)$$

THE PENALISED HUM PRIMAL FUNCTIONAL

$$F_{\varepsilon,h,\delta t}(v_h,\delta t) \stackrel{\text{def}}{=} \frac{1}{2} \|v_h,\delta t\|_{L^2_{\delta t}(0,T;U_h)}^2 + \frac{1}{2\varepsilon} \left\| \mathcal{L}_T^{h,\delta t}(v_h,\delta t|y_{0,h}) \right\|_h^2.$$

$$J_{\varepsilon,h,\delta t}(q_h^F) \stackrel{\text{def}}{=} \frac{1}{2} \left\| \mathcal{B}_h^* \mathcal{L}_T^{*,h,\delta t}(q_h^F) \right\|_{L^2_{\delta t}(0,T;U_h)}^2 + \frac{\varepsilon}{2} \|q_h^F\|_h^2 \\ - \left\langle y_{0,h}, q_h^1 - \delta t(1-\theta)\mathcal{A}_h q_h^1 \right\rangle_h, \quad \forall q_h^F \in E_h,$$

THEOREM (DUALITY)

The functionals $F_{\varepsilon,h,\delta t}$ and $J_{\varepsilon,h,\delta t}$ are in duality, in the sense that their respective minimisers $v_{\varepsilon,h,\delta t} \in L^2(0,T;U_h)$ and $q_{\varepsilon,h,\delta t}^F \in E_h$ satisfy

$$\inf_{L^2_{\delta t}(0,T;U_h)} F_{\varepsilon,h,\delta t} = F_{\varepsilon,h,\delta t}(v_{\varepsilon,h,\delta t}) = -J_{\varepsilon,h,\delta t}(q_{\varepsilon,h,\delta t}^F) = -\inf_{E_h} J_{\varepsilon,h,\delta t},$$

and moreover

$$v_{\varepsilon,h} = \mathcal{B}_h^* \mathcal{L}_T^{*,h,\delta t}(q_{\varepsilon,h,\delta t}^F).$$

THEOREM (CASE $\theta \in]1/2, 1[$)

Assume that the discrete Lebeau-Robbiano inequality $(\mathcal{H}_{\alpha,\beta})$ holds and let ϕ be such that

$$\liminf_{h \rightarrow 0} \frac{\phi(h)}{e^{-C/h^\beta}} > 0.$$

Then, there exists $h_0 > 0$, $C_T > 0$, $C_{\text{obs}} > 0$ such that for any $0 < h < h_0$ and any $\delta t \leq C_T |\log \phi(h)|^{-1}$, the following relaxed observability inequality holds

$$\left\| q_h^1 - \delta t(1 - \theta) \mathcal{A}_h q_h^1 \right\|_h^2 \leq C_{\text{obs}}^2 \left(\left\| \mathcal{B}_h^* q_h^n \right\|_{L_{\delta t}^2(0,T;U_h)}^2 + \phi(h) \left\| q_h^F \right\|_h^2 \right), \quad \forall q_h^F \in E_h.$$

Thus, for any such δt and h and any initial data $y_{0,h} \in E_h$, the full-discrete control $v_{\phi(h),h,\delta t}$, obtained by minimising $F_{\phi(h),h,\delta t}$ (or equivalently $J_{\phi(h),h,\delta t}$) satisfies

$$\left\| v_{\phi(h),h,\delta t} \right\|_{L_{\delta t}^2(0,T;U_h)} \leq C_{\text{obs}} \|y_{0,h}\|_h,$$

$$\left\| \mathcal{L}_T^{h,\delta t}(v_{\phi(h),h,\delta t} | y_{0,h}) \right\|_h \leq C_{\text{obs}} \sqrt{\phi(h)} \|y_{0,h}\|_h.$$

CASE $\theta = 1/2$: We need the (much stronger) condition $\delta t \rho(\mathcal{A}_h) \leq \delta$ for some δ .

MAIN IDEA : ADAPT THE LEBEAU-ROBBIANO ORIGINAL STRATEGY

STEP 1 : Use the discrete L.R. inequality to prove controllability of frequency modes less than μ with cost $e^{C\mu^\alpha} \|y_0\|_h$.

STEP 2 : Construct a suitable full discrete control by a discrete **finite** time slicing procedure :

$$\{0, \dots, M\} = \bigsqcup_{j=1}^J \{M'_j, \dots, M'_j + 2M_j\}. \quad (\star)$$

- Between discrete times M'_j and $M'_j + M_j$:

Use a control for frequencies less than $2^{j/\alpha}$ (Step 1).

- Between discrete times $M'_j + M_j + 1$ and $M'_j + 2M_j$:

Let the system evolve without control and take advantage of the parabolic dissipation since the solution only contains frequencies greater than $2^{j/\alpha}$.

NEW DIFFICULTIES

- δt has to be small enough (i.e. M large enough) in order to construct a suitable slicing (\star).
- The full-discrete heat semi-group

$$(\text{Id} + \theta \delta t \mathcal{A}_h)^{-1} (\text{Id} + (1 - \theta) \delta t \mathcal{A}_h)$$

do not have the same dissipation properties as the semi-discrete semi-group

$$e^{-\delta t \mathcal{A}_h}.$$

THE θ -SCHEME FOR $\theta > 1/2$

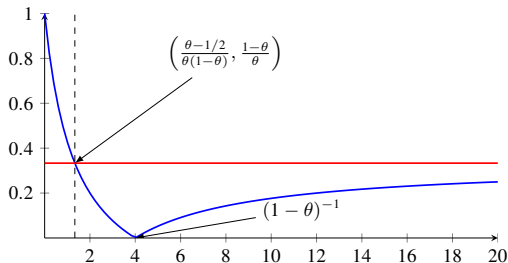
The iteration matrix for the system is

$$\mathcal{C}_{h,\delta t} = (\text{Id} + \theta \delta t \mathcal{A}_h)^{-1} (\text{Id} - (1 - \theta) \delta t \mathcal{A}_h).$$

- Let us analyse $\text{Sp}(\mathcal{C}_{h,\delta t})$:

Image of $\text{Sp}(\delta t \mathcal{A}_h)$ through

$$x \mapsto \frac{1 - (1 - \theta)x}{1 + \theta x}$$



- In practice, $\rho(\delta t \mathcal{A}_h) \sim C \frac{\delta t}{h^p}$, for some p (e.g. $p = 2$ for classical FD)

THE θ -SCHEME FOR $\theta > 1/2$

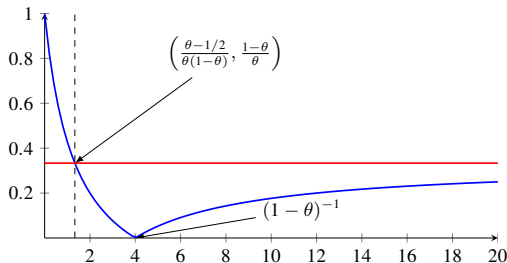
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- In practice, $\rho(\delta t \mathcal{A}_h) \sim C \frac{\delta t}{h^p}$, for some p (e.g. $p = 2$ for classical FD)
 \rightsquigarrow Case (a) : For $\delta t \times \mu_{i,h}$ less than $\frac{\theta - 1/2}{\theta(1 - \theta)}$: we have exponential damping.

THE θ -SCHEME FOR $\theta > 1/2$

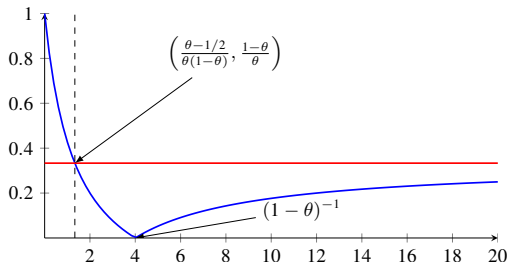
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$$\mathcal{C}_{h,\delta t} = (\text{Id} + \theta \delta t \mathcal{A}_h)^{-1} (\text{Id} - (1 - \theta) \delta t \mathcal{A}_h).$$

- Let us analyse $\text{Sp}(\mathcal{C}_{h,\delta t})$:

Image of $\text{Sp}(\delta t \mathcal{A}_h)$ through

$$x \mapsto \frac{1 - (1 - \theta)x}{1 + \theta x}$$



- In practice, $\rho(\delta t \mathcal{A}_h) \sim C \frac{\delta t}{h^p}$, for some p (e.g. $p = 2$ for classical FD)
 \rightsquigarrow Case (b) : For $\delta t \times \mu_{i,h}$ greater than $\frac{\theta-1/2}{\theta(1-\theta)}$ (possibly $\rightarrow +\infty$) the damping factor can be $\sim (1 - \theta)/\theta < 1$ **but we assumed that $\delta t \leq C_T |\log \phi(h)|$** :

$$\left(\frac{1 - \theta}{\theta} \right)^M \leq \left(\frac{1 - \theta}{\theta} \right)^{\frac{M \delta t}{C_T |\log(\phi(h))|}} = e^{-\xi \frac{M \delta t}{h^\gamma}} \sim e^{-\xi \frac{T}{C_T |\log \phi(h)|}} \sim \phi(h).$$

THE CRANK-NICOLSON SCHEME ($\theta = 1/2$)

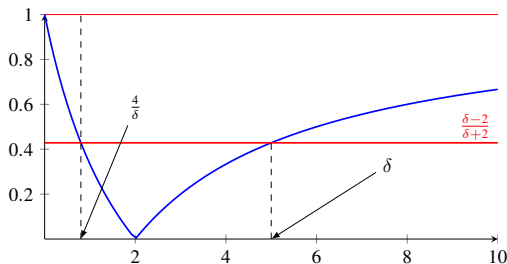
The iteration matrix for the system is

$$\mathcal{C}_{h,\delta t} = \left(\text{Id} - \frac{\delta t}{2} \mathcal{A}_h \right)^{-1} \left(\text{Id} + \frac{\delta t}{2} \mathcal{A}_h \right).$$

- Let us analyse $\text{Sp}(\mathcal{C}_{h,\delta t})$:

Image of $\text{Sp}(\delta t \mathcal{A}_h)$ through

$$x \mapsto \frac{1 - x/2}{1 + x/2}$$



- For large $\delta t \times \mu_{i,h}$, the damping factor can be ~ 1 . Here we use

$$\delta t \rho(\mathcal{A}_h) \leq \delta.$$

We thus split the analysis into two cases :

- The case $\delta t \times \mu_{i,h}$ less than $4/\delta$: natural exponential damping
- The case $\delta t \times \mu_{i,h}$ greater than $4/\delta$: damping bounded by $\frac{\delta-2}{\delta+2} < 1$.

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We consider a free trajectory of the semi-discrete problem

$$\hat{y}_F = e^{-T\mathcal{A}_h}\hat{y}_0.$$

PROBLEM 1

Starting from any $y_0 \in E_h$, can we drive the solution of the semi-discrete system

$$\partial_t y + \mathcal{A}_h y = \mathcal{B}_h v, \quad y(0) = y_0,$$

to \hat{y}_F at time T ?

\rightsquigarrow equivalent to the null-controllability problem with initial data $\hat{y}_0 - y_0$

We consider a free trajectory of the semi-discrete problem

$$\hat{y}_F = e^{-T\mathcal{A}_h}\hat{y}_0.$$

PROBLEM 2

Starting from any $y_0 \in E_h$, can we drive the solution of the full-discrete system

$$\frac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1 - \theta)y^n) = \mathcal{B}_h v^{n+1}, \quad y^0 = y_0,$$

to \hat{y}_F at discrete time M ?

\rightsquigarrow This is **not** equivalent to the null-controllability problem with initial data $\hat{y}_0 - y_0$

INDEED : The full-discrete free trajectory starting at \hat{y}_0 is not equal to \hat{y}_F at time M .

- In most cases (but not always) \hat{y}_F belongs to some full discrete trajectory

$$\hat{y}_F = \left((\text{Id} + \theta \mathcal{A}_h)^{-1} (\text{Id} + (1 - \theta) \mathcal{A}_h) \right)^M \tilde{y}_0^{\delta t}.$$

- We do not want the estimates to depend on $\tilde{y}_0^{\delta t}$ since :
 - In general we do not want to compute $\tilde{y}_0^{\delta t}$.
 - Its norm can be large with respect to that of \hat{y}_F .

We consider a free trajectory of the semi-discrete problem

$$\hat{y}_F = e^{-T\mathcal{A}_h} \hat{y}_0.$$

OUR RESULT : Under suitable assumptions, by minimizing the functional

$$J^{h,\delta t}(q^F) = \frac{1}{2} \sum_{n=1}^M \delta t \left[\|\mathcal{B}_h^* q^n\|_h^2 + \frac{\phi(h)}{2} \|q^F\|_h^2 - \left\langle \hat{y}_F, q^F \right\rangle_h + \left\langle y_0, q^1 - \delta t(1-\theta)\mathcal{A}_h q^1 \right\rangle_h \right],$$

we produce a full discrete control $v_{h,\delta t} = (\mathcal{B}_h^* q^n)_n$ such that

- The cost of the control satisfies

$$\sum_{n=1}^M \delta t \|v^n\|_h^2 \leq C_{\text{obs}}^2 \left(\|y_0 - \hat{y}_0\|_h + C_s \delta t^{\zeta_1} \left\| \mathcal{A}_h^{\frac{1}{2}} \hat{y}_0 \right\|_h \right)^2 + e^{-C/\delta t^{\zeta_2}} \|\hat{y}_0\|_h^2,$$

for some $\zeta_1, \zeta_2 > 0$.

- The controlled solution $(y^n)_n$ associated with $v_{h,\delta t}$ and y_0 is such that

$$\|y^M - \hat{y}_F\|_h \leq \sqrt{\phi(h)} C_{\text{obs}} \left(\|y_0 - \hat{y}_0\|_h + C \delta t^{\zeta_1} \left\| \mathcal{A}_h^{\frac{1}{2}} \hat{y}_0 \right\|_h \right) + e^{-C/\delta t^{\zeta_2}} \|\hat{y}_0\|_h.$$

MAIN TOOL : Estimate of the difference between the two initial data

$$\|\tilde{y}_0^{\delta t} - \hat{y}_0\|_h \leq C \delta t^{\zeta} \left\| \mathcal{A}_h^{\frac{1}{2}} \hat{y}_0 \right\|_h.$$

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- The error analysis in space is intricate (low regularity expected for the HUM null-control, ...).
- We try here to analyse the error induced by time discretisation in this problem.

FRAMEWORK

- We assume the uniform discrete Lebeau-Robbiano ($\mathcal{H}_{\alpha,\beta}$) to hold.
- We suppose that $h > 0$ is fixed, that $y_0 \in E_h$ is given and that $h \mapsto \phi(h)$ is given.
- The minimization of the functional

$$J^h(q^F) = \frac{1}{2} \int_0^T \|\mathcal{B}_h^* q(t)\|_h^2 dt + \frac{\phi(h)}{2} \|q^F\|_h^2 + \langle y_0, q(0) \rangle_h,$$

leads to a semi-discrete control $t \mapsto v_h(t) \in L^2([0, T[, U_h)$.

- For simplicity, we consider the implicit Euler scheme (similar results hold for $\theta \in [1/2, 1[)$. The minimization of the functional

$$J^{h,\delta t}(q^F) = \frac{1}{2} \sum_{n=1}^M \delta t \|\mathcal{B}_h^* q^n\|_h^2 + \frac{\phi(h)}{2} \|q^F\|_h^2 + \langle y_0, q^1 \rangle_h,$$

leads to a full discrete control $v_{h,\delta t} = (v^n)_n \in (U_h)^M$.

GOAL : Prove an error estimate between $v_{h,\delta t}$ and v_h .

THEOREM

Under the same assumptions than previous results (in particular $\delta t \leq Ch^\gamma$), the following error estimate holds

$$\left\| v_h - \underbrace{\sum_{n=1}^M 1_{(t^{n-1}, t^n)} v^n}_{\stackrel{\text{def}}{=} \mathcal{F}_0[v_h, \delta t]} \right\|_{L^2([0, T[, U_h])} \leq C \delta t \frac{\rho(\mathcal{A}_h)}{\sqrt{\phi(h)}} \left(1 + \delta t^{\frac{3}{2}} \rho(\mathcal{A}_h)^{\frac{3}{2}} \right) \|y_0\|_h.$$

REMARKS

- First order in time estimate (second order for CN provided a suitable time interpolation operator is used in place of $\mathcal{F}_0[.]$).
- The estimate is not uniform in h , even if we are interested in the approximate control problem where $\phi(h) = \varepsilon > 0$. The result is probably not optimal.

SKETCH OF PROOF

- Write the Euler-Lagrange equations corresponding to the two minimization problems we consider (the semi-discrete and the full-discrete).
- Compare the two Euler-Lagrange equations by using error estimates in time for the adjoint problem.

EULER-LAGRANGE EQUATION FOR J^h

We denote the minimizer by q_{opt}^F and $t \mapsto q_{opt}(t)$ the corresponding solution to the semi-discrete adjoint problem :

$$0 = \int_0^T [\mathcal{B}_h^* q_{opt}(t), \mathcal{B}_h^* \tilde{q}(t)]_h dt + \phi(h) \left\langle q_{opt}^F, \tilde{q}^F \right\rangle_h + \langle y_0, \tilde{q}(0) \rangle_h ,$$

for any $\tilde{q}^F \in E_h$.

EULER-LAGRANGE EQUATION FOR $J^{h,\delta t}$

We denote the minimizer by $q_{opt,\delta t}^F$ and by $(q_{opt,\delta t}^n)_n$ the corresponding solution to the full-discrete adjoint problem

$$0 = \sum_{n=1}^M \delta t [\mathcal{B}_h^* q_{opt,\delta t}^n, \mathcal{B}_h^* \tilde{q}^n]_h + \phi(h) \left\langle q_{opt,\delta t}^F, \tilde{q}^F \right\rangle_h + \left\langle y_0, \tilde{q}^1 \right\rangle_h ,$$

for any $\tilde{q}^F \in E_h$.

$$\int_0^T [\mathcal{B}_h^* q_{opt}(t), \mathcal{B}_h^* \tilde{q}(t)]_h dt + \phi(h) \langle q_{opt}^F, \tilde{q}^F \rangle_h + \langle y_0, \tilde{q}(0) \rangle_h = 0,$$

$$\sum_{n=1}^M \delta t \left[\mathcal{B}_h^* q_{opt, \delta t}^n, \mathcal{B}_h^* \tilde{q}^n \right]_h + \phi(h) \langle q_{opt, \delta t}^F, \tilde{q}^F \rangle_h + \langle y_0, \tilde{q}^1 \rangle_h$$

$$= 0,$$

$$\int_0^T [\mathcal{B}_h^* q_{opt}(t), \mathcal{B}_h^* \tilde{q}(t)]_h dt + \phi(h) \langle q_{opt}^F, \tilde{q}^F \rangle_h + \langle y_0, \tilde{q}(0) \rangle_h = 0,$$

$$\sum_{n=1}^M \delta t \left[\mathcal{B}_h^* q_{opt, \delta t}^n, \mathcal{B}_h^* \tilde{q}^n \right]_h + \phi(h) \langle q_{opt, \delta t}^F, \tilde{q}^F \rangle_h + \langle y_0, \tilde{q}^1 \rangle_h = 0,$$

TRANSFORMATION OF THESE EQUATIONS

$$\begin{aligned} \delta t \left[\mathcal{B}_h^* q_{opt, \delta t}^n, \mathcal{B}_h^* \tilde{q}^n \right]_h &= \int_{t^{n-1}}^{t^n} [\mathcal{F}_0[v_{h, \delta t}](t), \mathcal{B}_h^*(\mathcal{F}_0[\tilde{q}_{\delta t}](t))]_h dt \\ &= \int_{t^{n-1}}^{t^n} [\mathcal{F}_0[v_{h, \delta t}](t), \mathcal{B}_h^* \tilde{q}(t)]_h dt + \int_{t^{n-1}}^{t^n} [\mathcal{F}_0[v_{h, \delta t}](t), \mathcal{B}_h^* (\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t))]_h dt. \end{aligned}$$

$$\int_0^T [\mathcal{B}_h^* q_{opt}(t), \mathcal{B}_h^* \tilde{q}(t)]_h dt + \phi(h) \langle q_{opt}^F, \tilde{q}^F \rangle_h + \langle y_0, \tilde{q}(0) \rangle_h = 0,$$

$$\begin{aligned} & \int_0^T [\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^* \tilde{q}(t)]_h dt + \phi(h) \langle q_{opt,\delta t}^F, \tilde{q}^F \rangle_h + \langle y_0, \tilde{q}^1 \rangle_h \\ &= - \int_0^T [\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^* (\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t))]_h dt, \end{aligned}$$

$$\int_0^T [\mathcal{B}_h^* q_{opt}(t), \mathcal{B}_h^* \tilde{q}(t)]_h dt + \phi(h) \langle q_{opt}^F, \tilde{q}^F \rangle_h + \langle y_0, \tilde{q}(0) \rangle_h = 0,$$

$$\begin{aligned} \int_0^T [\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^* \tilde{q}(t)]_h dt + \phi(h) \langle q_{opt,\delta t}^F, \tilde{q}^F \rangle_h + \boxed{\langle y_0, \tilde{q}^1 \rangle_h} \\ = - \int_0^T [\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^* (\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t))]_h dt, \end{aligned}$$

TRANSFORMATION OF THESE EQUATIONS

$$\tilde{q}^1 = \tilde{q}(0) + (\tilde{q}^1 - \tilde{q}(0))$$

$$\int_0^T [\mathcal{B}_h^* q_{opt}(t), \mathcal{B}_h^* \tilde{q}(t)]_h dt + \phi(h) \langle q_{opt}^F, \tilde{q}^F \rangle_h + \langle y_0, \tilde{q}(0) \rangle_h = 0,$$

$$\begin{aligned} & \int_0^T [\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^* \tilde{q}(t)]_h dt + \phi(h) \langle q_{opt,\delta t}^F, \tilde{q}^F \rangle_h + \langle y_0, \tilde{q}(0) \rangle_h \\ &= - \int_0^T [\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^* (\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t))]_h dt - \langle y_0, \tilde{q}^1 - \tilde{q}(0) \rangle_h, \end{aligned}$$

$$\int_0^T [\mathcal{B}_h^* q_{opt}(t), \mathcal{B}_h^* \tilde{q}(t)]_h dt + \phi(h) \langle q_{opt}^F, \tilde{q}^F \rangle_h + \langle y_0, \tilde{q}(0) \rangle_h = 0,$$

$$\begin{aligned} & \int_0^T [\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^* \tilde{q}(t)]_h dt + \phi(h) \langle q_{opt,\delta t}^F, \tilde{q}^F \rangle_h + \langle y_0, \tilde{q}(0) \rangle_h \\ &= - \int_0^T [\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^* (\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t))]_h dt - \langle y_0, \tilde{q}^1 - \tilde{q}(0) \rangle_h, \end{aligned}$$

SUBTRACTION OF THE EQUATIONS

$$\begin{aligned} & \int_0^T [\mathcal{B}_h^* q_{opt}(t) - \mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^* \tilde{q}(t)]_h dt + \phi(h) \langle q_{opt}^F - q_{opt,\delta t}^F, \tilde{q}^F \rangle_h \\ &= \int_0^T [\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^* (\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t))]_h dt + \langle y_0, \tilde{q}^1 - \tilde{q}(0) \rangle_h, \end{aligned}$$

\rightsquigarrow Now we choose $\tilde{q}^F = q_{opt}^F - q_{opt,\delta t}^F$, so that $\tilde{q}(t) = q_{opt}(t) - \underline{q}(t)$ and then

$$\mathcal{B}_h^* \tilde{q}(t) = \left(\mathcal{B}_h^* q_{opt}(t) - \mathcal{F}_0[v_{h,\delta t}](t) \right) + \mathcal{B}_h^* \left(\mathcal{F}_0[q_{opt,\delta t}](t) - \underline{q}(t) \right).$$

$$\begin{aligned}
& \int_0^T \left\| v(t) - \mathcal{F}_0[v_{h,\delta t}](t) \right\|_h^2 dt + \phi(h) \left\| q_{opt}^F - q_{opt,\delta t}^F \right\|_h^2 \\
&= \int_0^T \left[\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^* (\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t)) \right]_h dt + \left\langle y_0, \tilde{q}^1 - \tilde{q}(0) \right\rangle_h \\
&\quad - \int_0^T \left[v(t) - \mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^* (\mathcal{F}_0[q_{opt,\delta t}](t) - \underline{q}(t)) \right]_h dt.
\end{aligned}$$

- The **error** terms are estimated as follows by usual parabolic techniques :

$$\begin{aligned}
& \int_0^T \left\| \mathcal{B}_h^* (\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t)) \right\|_h^2 dt \leq C \int_0^T \left\| \mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t) \right\|_h^2 dt \\
&\leq C \left(\delta t^2 \left\| \mathcal{A}_h^{\frac{1}{2}} \tilde{q}^F \right\|_h^2 + \delta t^5 \left\| \mathcal{A}_h^2 \tilde{q}^F \right\|_h^2 \right) \leq C \delta t^2 \left\| \tilde{q}^F \right\|_h^2 \rho_h (1 + \rho_h^3)
\end{aligned}$$

- We conclude by using Cauchy-Schwarz inequality.

Assume that \mathcal{A}_h is self-adjoint.

DISCRETE H^s NORMS $\|u_h\|_{s,h}^2 = \langle (\mathcal{A}_h)^s u_h, u_h \rangle_h$.

PROPOSITION

Let $s \in \mathbb{R}$, $q^F \in \mathbb{R}^{\mathfrak{M}}$, $t \mapsto q(t) \in \mathbb{R}^{\mathfrak{M}}$ the solution of the semi-discrete adjoint problem $-\partial_t q(t) + \mathcal{A}_h q(t) = 0$ with $q(T) = q^F$ and $(q^n)_n$ the solution of the fully discrete (backward) problem

$$\frac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h q^n = 0, \quad \forall n \in \{1, \dots, M\},$$

*associated with the same data $q^{M+1} = q^F$. Let the error be $E^n = q^n - q(t^{n-1})$. There exists $C_s > 0$ **independent of h** such that*

$$\sup_{1 \leq n \leq M} \|E^n\|_{s,h}^2 + \left(\sum_{n=1}^M \delta t \|E^n\|_{s+1,h}^2 \right)^{\frac{1}{2}} \leq C_s \delta t \|q^F\|_{s+2,h},$$

$$\sup_{1 \leq n \leq M} \|(T - t^{n-1})E^n\|_{s,h} \leq C_s \delta t \|q^F\|_{s,h} + C_s \delta t^2 \|q^F\|_{s+2,h}.$$

We apply those estimates for $s = 0$, $s = -1$ and with

$$q^F = q_{opt,\delta t}^F, \quad \text{and} \quad q^F = q_{opt,\delta t}^F - q_{opt}^F.$$

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$$\begin{aligned}\partial_t y - 0.1 \partial_x^2 y &= 1_{]0.3, 0.8[} v, \\ T = 1, y_0(x) &= \sin(\pi x)^{10}.\end{aligned}$$

$$\partial_t y - 0.1 \partial_x^2 y = 1_{]0.3, 0.8[} v,$$

$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

N	M				
	20	80	320	1280	$+\infty$
20	14	16	16	16	16
50	22	26	29	29	31
100	30	38	44	49	48
200	45	58	69	77	82

(A) Case $\phi(h) = h^2$

N	M				
	20	80	320	1280	$+\infty$
20	24	30	28	27	32
50	83	87	87	93	106
100	235	240	233	262	265
200	778	850	1098	1230	1374

(B) Case $\phi(h) = h^4$ TABLE : Conjugate gradient iterates; $\omega =]0.3, 0.8[$

$$\partial_t y - 0.1 \partial_x^2 y = 1_{]0.3, 0.8[} v,$$

$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

N	M				
	20	80	320	1280	$+\infty$
20	$7.17 \cdot 10^{-2}$	$6.54 \cdot 10^{-2}$	$6.38 \cdot 10^{-2}$	$6.34 \cdot 10^{-2}$	$6.33 \cdot 10^{-2}$
50	$7.98 \cdot 10^{-2}$	$7.08 \cdot 10^{-2}$	$6.85 \cdot 10^{-2}$	$6.79 \cdot 10^{-2}$	$6.78 \cdot 10^{-2}$
100	$8.5 \cdot 10^{-2}$	$7.44 \cdot 10^{-2}$	$7.15 \cdot 10^{-2}$	$7.07 \cdot 10^{-2}$	$7.05 \cdot 10^{-2}$
200	$9.1 \cdot 10^{-2}$	$7.75 \cdot 10^{-2}$	$7.39 \cdot 10^{-2}$	$7.3 \cdot 10^{-2}$	$7.27 \cdot 10^{-2}$

TABLE : Optimal energy; $\phi(h) = h^2$; $\omega =]0.3, 0.8[$

$$\partial_t y - 0.1 \partial_x^2 y = 1_{]0.3, 0.8[} v,$$

$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

N	M				
	20	80	320	1280	$+\infty$
20	0.11	$8.92 \cdot 10^{-2}$	$8.43 \cdot 10^{-2}$	$8.3 \cdot 10^{-2}$	$8.26 \cdot 10^{-2}$
50	0.12	$8.94 \cdot 10^{-2}$	$8.29 \cdot 10^{-2}$	$8.12 \cdot 10^{-2}$	$8.07 \cdot 10^{-2}$
100	0.12	$9.1 \cdot 10^{-2}$	$8.33 \cdot 10^{-2}$	$8.13 \cdot 10^{-2}$	$8.06 \cdot 10^{-2}$
200	0.13	$9.33 \cdot 10^{-2}$	$8.41 \cdot 10^{-2}$	$8.17 \cdot 10^{-2}$	$8.09 \cdot 10^{-2}$

TABLE : Optimal energy; $\phi(h) = h^4$; $\omega =]0.3, 0.8[$

$$\partial_t y - 0.1 \partial_x^2 y = 1_{]0.3, 0.8[} v,$$

$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

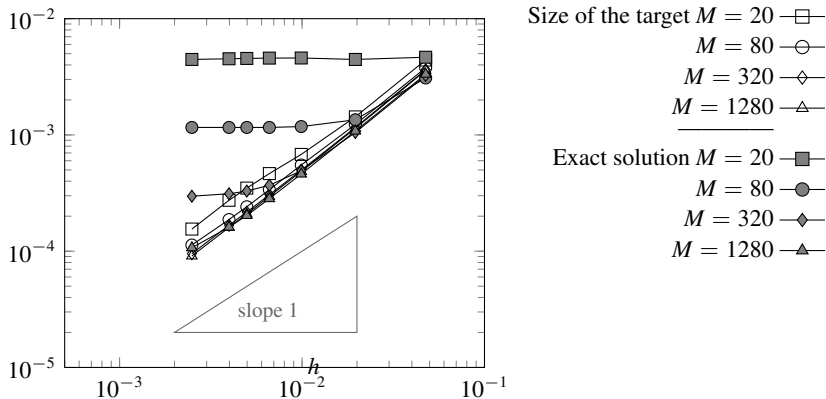


FIGURE : Convergence analysis with $\phi(h) = h^2$; $\omega =]0.3, 0.8[$

$$\partial_t y - 0.1 \partial_x^2 y = 1]_{0.3, 0.8}[v,$$

$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

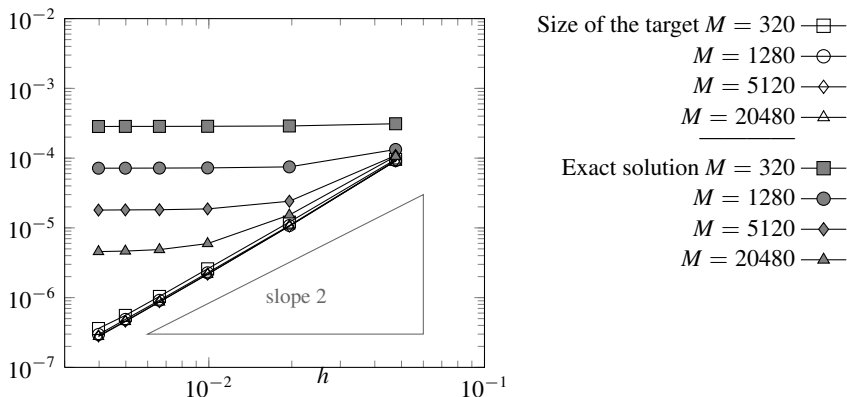


FIGURE : Convergence analysis with $\phi(h) = h^4$; $\omega =]0.3, 0.8[$

$$\partial_t y - 0.1 \partial_x^2 y = 1_{]0.3, 0.8[} v,$$

$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

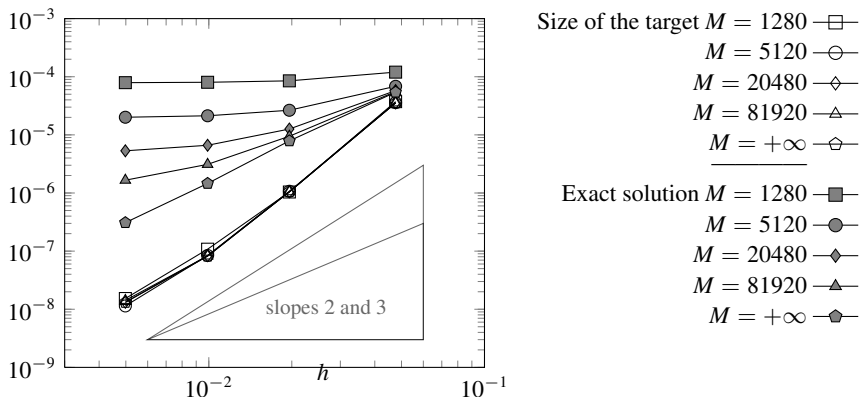


FIGURE : Convergence analysis with $\phi(h) = 1000h^6$; $\omega =]0.3, 0.8[$

$$\partial_t y - 0.1 \partial_x^2 y = \mathbf{1}_\Omega v,$$

$$T = 0.5, y_0(x) = \sin(\pi x)^{10}.$$

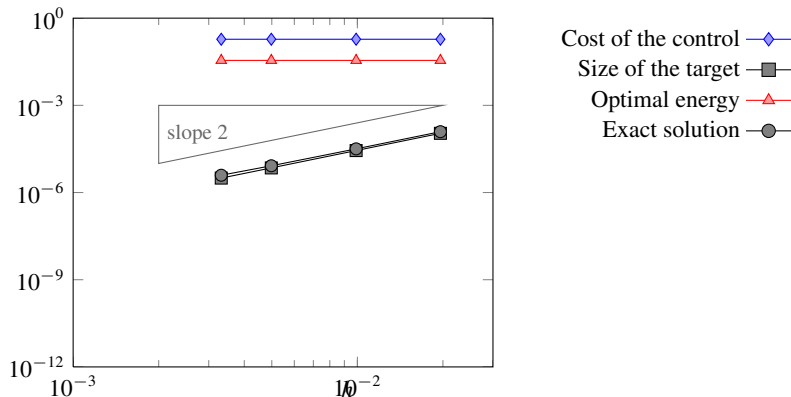


FIGURE : $\phi(h) = h^2$; Semi-discrete scheme

$$\partial_t y - 0.1 \partial_x^2 y = \mathbf{1}_\Omega v,$$

$$T = 0.5, y_0(x) = \sin(\pi x)^{10}.$$

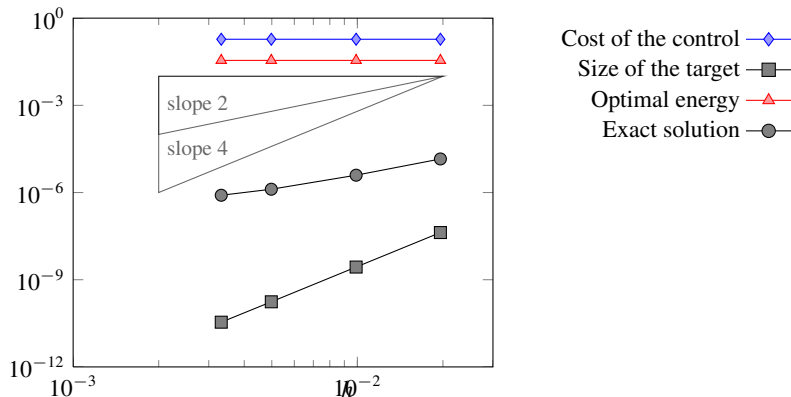


FIGURE : $\phi(h) = h^4$; Semi-discrete scheme

$$\partial_t y - 0.1 \partial_x^2 y - 1.5 y = 1_{]0.3, 0.8[} \mathcal{V},$$

$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

(Fernández-Cara – Münch, '11)

(B.-Le Rousseau, '13)

$$\begin{aligned}\partial_t y - 0.1 \partial_x^2 y - 5y \log^{1.4}(1 + |y|) &= 1_{]0.2, 0.8[} v, \\ T = 0.5, y_0(x) &= 20 \sin(\pi x).\end{aligned}$$

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$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

SHORT REVIEW OF KNOWN RESULTS

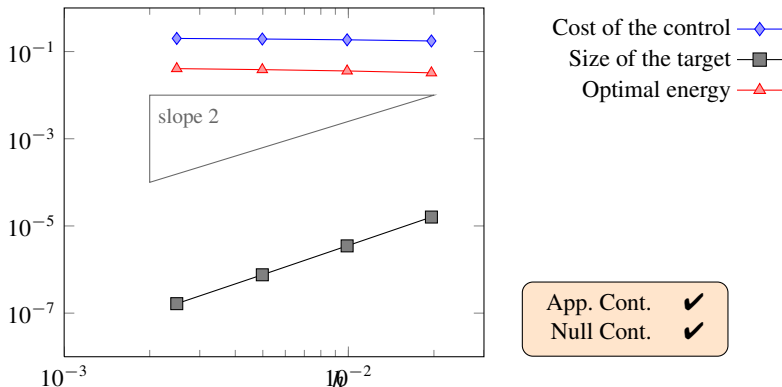
- In the case $a_{21} = \text{cte}$ the system is null-controllable if and only if $a_{21} \neq 0$
(Kalman-like condition)
(Ammar-Khodja-Benabdallah-Dupaix-González-Burgos, '09)
- In the case where $\text{Supp}(a_{21}) \cap \omega \neq \emptyset$, the system is null-controllable
(González-Burgos-de Teresa, '10)
- In the case where $\text{Supp}(a_{21}) \cap \omega = \emptyset$ and a_{21} has a constant sign, the system is null-controllable
(Rosier-de Teresa, '11)
- In the case where $\text{Supp}(a_{21}) \cap \omega = \emptyset$ and a_{21} changes its sign :
 - There are structural conditions for the system to be even approximatively controllable.
(B.- Olive, '13)
 - A minimal time condition for the null-controllability can occur
(Ammar-Khodja-Benabdallah-González-Burgos-de Teresa, '14)

$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 1 : $a_{21}(x) = \mathbf{1}_{]0.2, 0.9[}(x)$, $\omega =]0.1, 0.5[$, $y_0(x) = (\sin(3\pi x), \sin(\pi x)^{10})^t$.

$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 1 : $a_{21}(x) = \mathbf{1}_{]0.2, 0.9[}(x)$, $\omega =]0.1, 0.5[$, $y_0(x) = (\sin(3\pi x), \sin(\pi x)^{10})^t$.

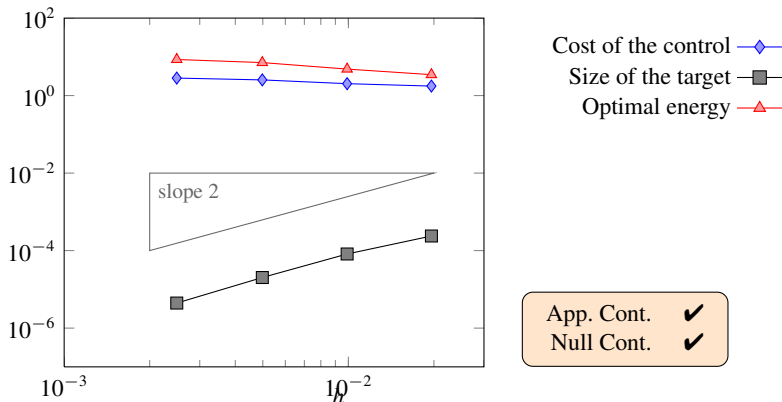


$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 2 : $a_{21}(x) = \mathbf{1}_{]0.7, 0.9[}(x)$, $\omega =]0.1, 0.5[$, $y_0(x) = (\sin(3\pi x), \sin(\pi x)^{10})^t$.

$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 2 : $a_{21}(x) = \mathbf{1}_{]0.7, 0.9[}(x)$, $\omega =]0.1, 0.5[$, $y_0(x) = (\sin(3\pi x), \sin(\pi x)^{10})^t$.



$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_{\omega} v.$$

CASE 3 : $a_{21}(x) = (x - \alpha) \mathbf{1}_{]0, 0.5[}(x)$, $\omega =]0.5, 1[$, $y_0(x) = (\sin(2\pi x), 3 \sin(2\pi x))^t$.

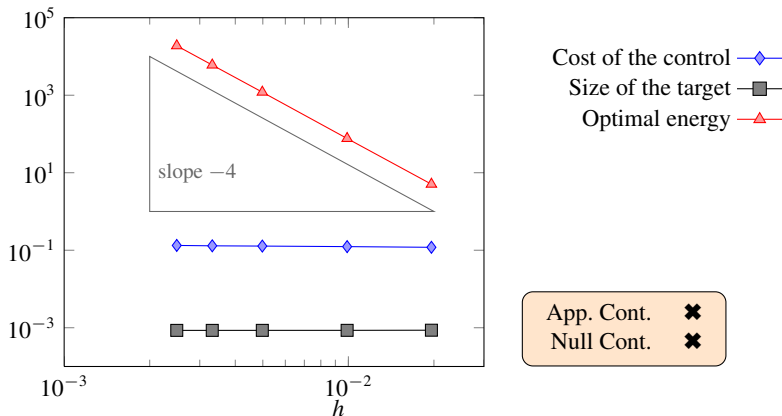


FIGURE : $\alpha = 1/4$

$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_{\omega} v.$$

CASE 3 : $a_{21}(x) = (x - \alpha) \mathbf{1}_{]0,0.5[}(x)$, $\omega =]0.5, 1[$, $y_0(x) = (\sin(2\pi x), 3 \sin(2\pi x))^t$.

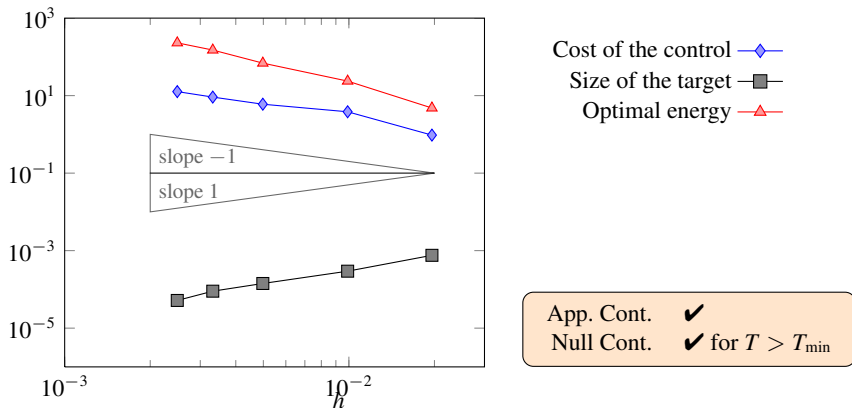


FIGURE : $\alpha = 1/8$

$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 4 : $a_{21}(x) = \mathbf{1}_{[\alpha-d, \alpha]} - \mathbf{1}_{[\alpha, \alpha+d]}$, $y_0(x) = (\sin(\pi x)^{10}, -2 \sin(2\pi x)^{10})^t$.

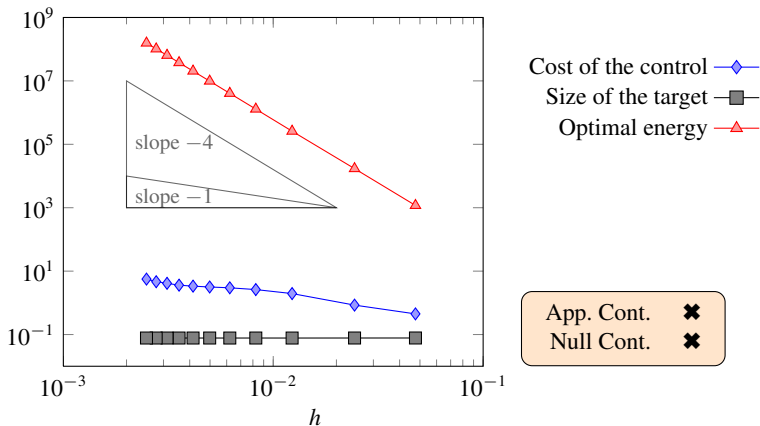


FIGURE : $\alpha = 1/2$, $d = \frac{1}{2\sqrt{3}}$, $\omega =]0.8, 1.0[$.

$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 4 : $a_{21}(x) = \mathbf{1}_{[\alpha-d, \alpha]} - \mathbf{1}_{[\alpha, \alpha+d]}$, $y_0(x) = (\sin(\pi x)^{10}, -2 \sin(2\pi x)^{10})^t$.

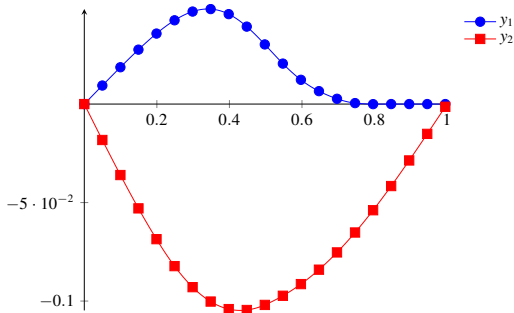


FIGURE : $\alpha = 1/2, d = \frac{1}{2\sqrt{3}}, \omega =]0.8, 1.0[$.

$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 4 : $a_{21}(x) = \mathbf{1}_{[\alpha-d, \alpha]} - \mathbf{1}_{[\alpha, \alpha+d]}$, $y_0(x) = (\sin(\pi x)^{10}, -2 \sin(2\pi x)^{10})^t$.

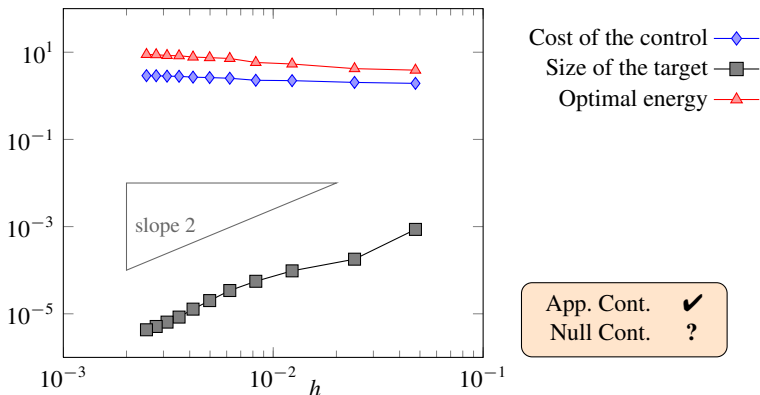


FIGURE : $\alpha = 1/2, d = \frac{1}{2\sqrt{3}}, \omega =]0, 0.2[\cup]0.8, 1.0[.$

$$\partial_t y - 0.1 \partial_x \left(\begin{pmatrix} 1 & 0 \\ 0 & \gamma(x) \end{pmatrix} \partial_x y \right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathbf{1}_\omega v.$$

$$\gamma(x) = \mathbf{1}_\omega + \alpha \mathbf{1}_{\omega^c}, y_0(x) = (\sin(\pi x)^{10}, -2 \sin(2\pi x)^{10})^t, \omega =]0.5, 1.0[$$

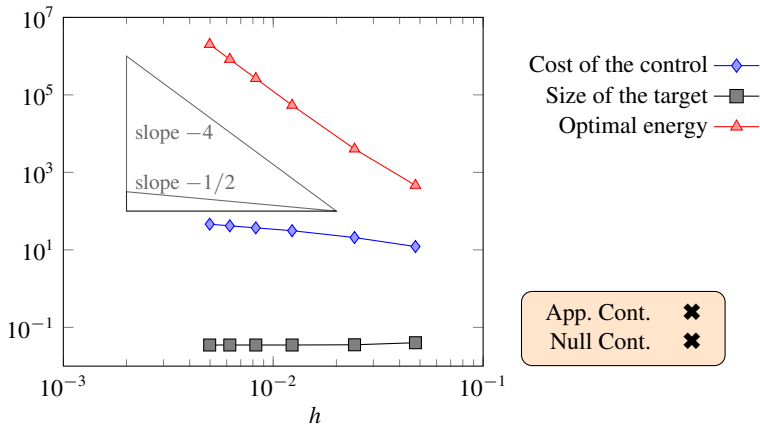
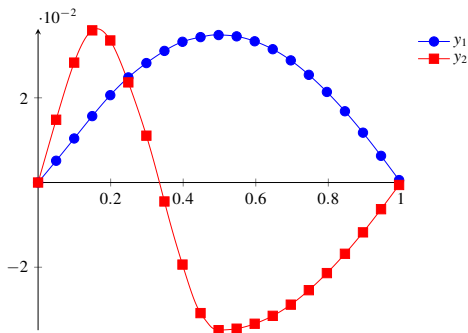


FIGURE : $\alpha = 1/9$

$$\partial_t y - 0.1 \partial_x \left(\begin{pmatrix} 1 & 0 \\ 0 & \gamma(x) \end{pmatrix} \partial_x y \right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathbf{1}_\omega v.$$

$$\gamma(x) = \mathbf{1}_\omega + \alpha \mathbf{1}_{\omega^c}, y_0(x) = (\sin(\pi x)^{10}, -2 \sin(2\pi x)^{10})^t, \omega =]0.5, 1.0[$$

FIGURE : $\alpha = 1/9$

$$\partial_t y - 0.1 \partial_x \left(\begin{pmatrix} 1 & 0 \\ 0 & \gamma(x) \end{pmatrix} \partial_x y \right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathbf{1}_\omega v.$$

$$\gamma(x) = \mathbf{1}_\omega + \alpha \mathbf{1}_{\omega^c}, y_0(x) = (\sin(\pi x)^{10}, -2 \sin(2\pi x)^{10})^t, \omega =]0.5, 1.0[$$

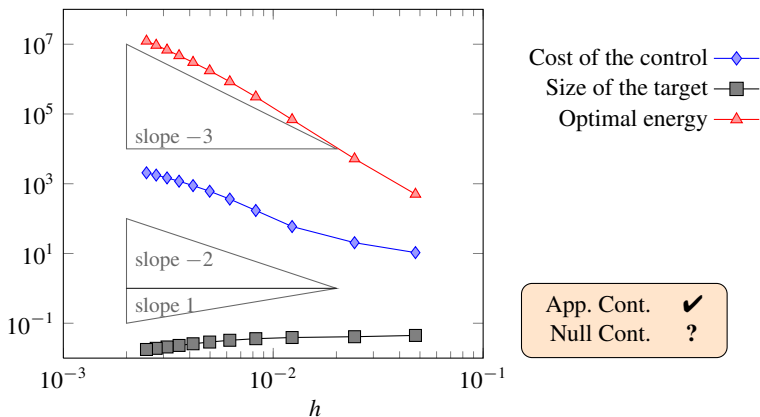


FIGURE : $\alpha = 1/8$

$$\partial_t y - 0.1 \partial_x \left(\begin{pmatrix} 1 & 0 \\ 0 & \gamma(x) \end{pmatrix} \partial_x y \right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathbf{1}_\omega v.$$

$$\gamma(x) = \mathbf{1}_\omega + \alpha \mathbf{1}_{\omega^c}, y_0(x) = (\sin(\pi x)^{10}, -2 \sin(2\pi x)^{10})^t, \omega =]0.5, 1.0[$$

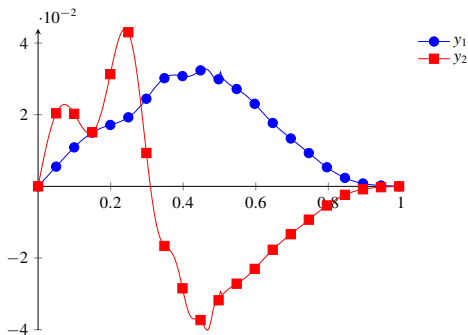


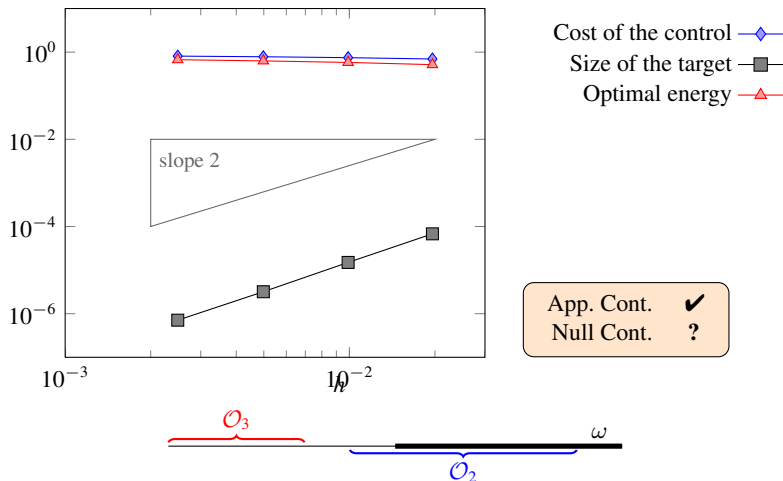
FIGURE : $\alpha = 1/8$

$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

Here also necessary and sufficient conditions for approximate controllability are known

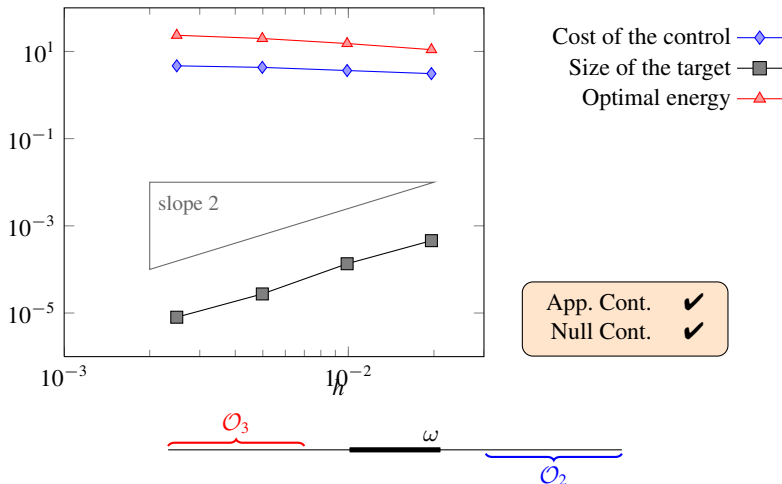
$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 1 : $\mathcal{O}_2 \cap \omega \neq \emptyset$



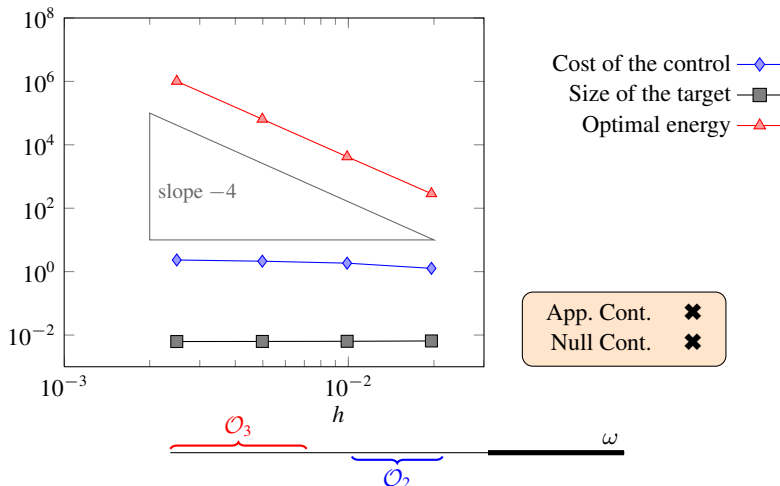
$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 2 : \mathcal{O}_2 and \mathcal{O}_3 are located in different connected components of $\Omega \setminus \omega$



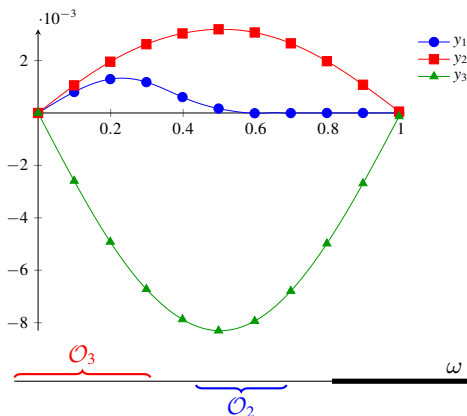
$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 3 : \mathcal{O}_2 and \mathcal{O}_3 are located in the same connected component of $\Omega \setminus \omega$



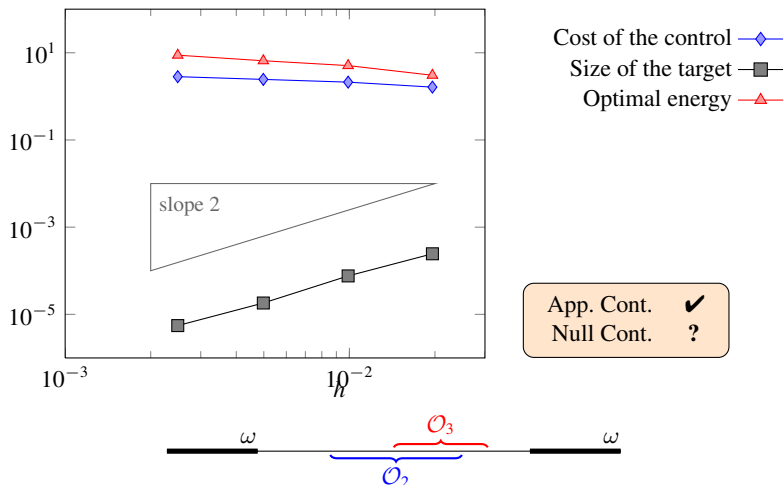
$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 3 : \mathcal{O}_2 and \mathcal{O}_3 are located in the same connected component of $\Omega \setminus \omega$



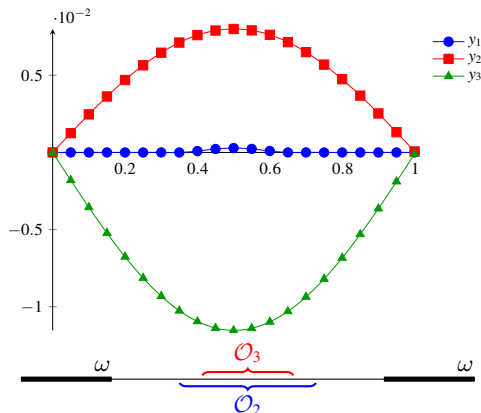
$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 4.1 : $\mathcal{O}_2 =]0.35, 0.65[$, $\mathcal{O}_3 =]0.5, 1/\sqrt{2}[$, $\omega =]0, 0.2[\cup]0.8, 1.0[$



$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 4.2 : $\mathcal{O}_2 =]0.35, 0.65[$, $\mathcal{O}_3 =]0.4, 0.6[$, $\omega =]0, 0.2[\cup]0.8, 1[$



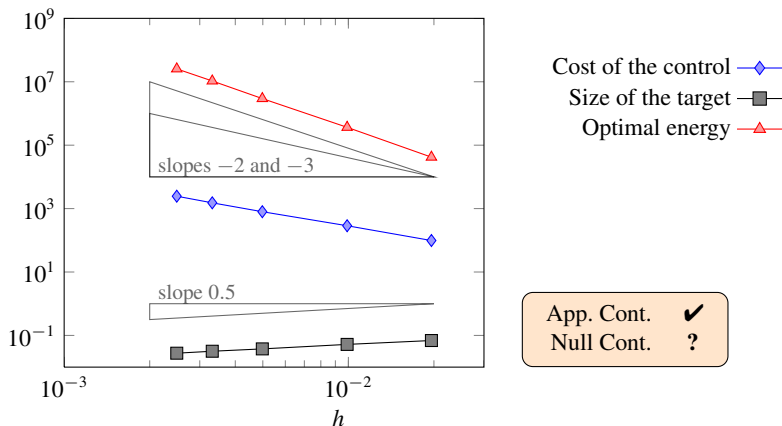
$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ 0 & a_{32}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

VERY SHORT REVIEW :

- If the supports of all the a_{ij} intersect the control domain ω and keeps a constant sign on a part of ω , then the system is null-controllable.
- Necessary and sufficient conditions for approximate controllability are known in the general case.

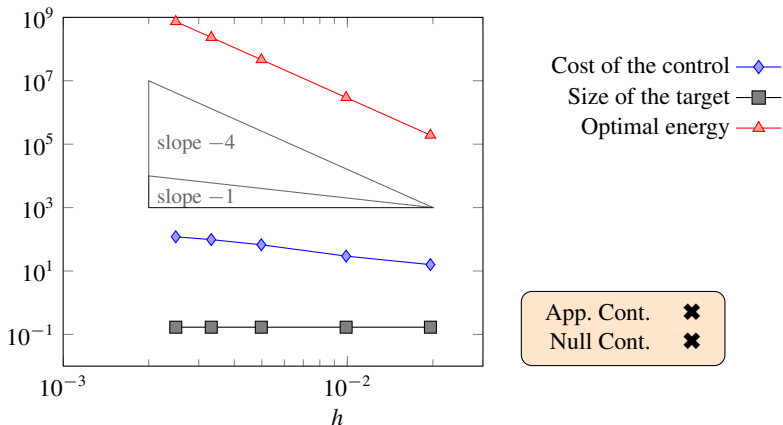
$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ 0 & a_{32}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 1 : $a_{21} = \mathbf{1}_{]0,0.5[}$, $a_{32} = 1$, $\omega =]0.5, 1[$.



$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ 0 & a_{32}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 2 : $a_{21} = \mathbf{1}_{]0,0.5[}$, $a_{32}(x) = x - 1/2$, $\omega =]0.5, 1[$.

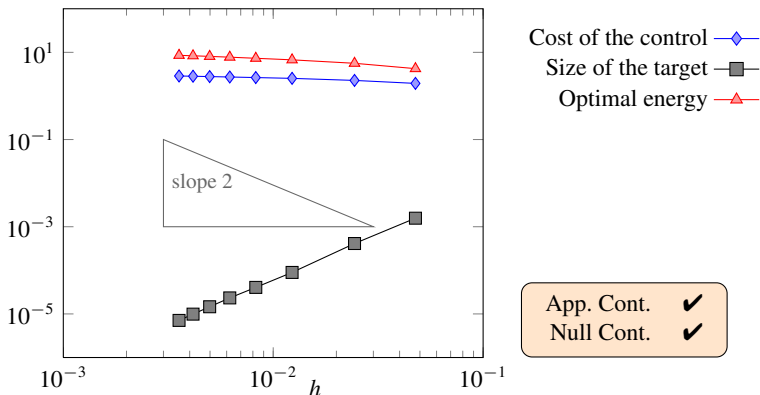


$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} y = \begin{pmatrix} \mathbf{1}_{\omega_1}(x) & 0 \\ 0 & \mathbf{1}_{\omega_2}(x) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

$$y_0(x) = \left(0, 0, \sin(\pi x)^{10}\right)^t, \quad \omega_1 =]0.7, 1.0[, \quad \omega_2 =]0.1, 0.5[$$

$$\partial_t y - 0.1 \partial_x^2 y + \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} y = \begin{pmatrix} \mathbf{1}_{\omega_1}(x) & 0 \\ 0 & \mathbf{1}_{\omega_2}(x) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

$$y_0(x) = \left(0, 0, \sin(\pi x)^{10}\right)^t, \quad \omega_1 =]0.7, 1.0[, \quad \omega_2 =]0.1, 0.5[$$



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$$\begin{aligned}\partial_t y - 0.05 \Delta y &= 1_{]0.3, 0.9[\times]0.2, 0.8[} \mathcal{V}, \\ y(0, x) &= \sin(2\pi x_1) \sin(\pi x_2), \quad \text{and} \quad y_F(x) = -0.4 \sin(\pi x_1) \sin(2\pi x_2).\end{aligned}$$

(Beauchard, Cannarsa, Guglielmi, '14), (Beauchard, Miller, Morancey, '14)

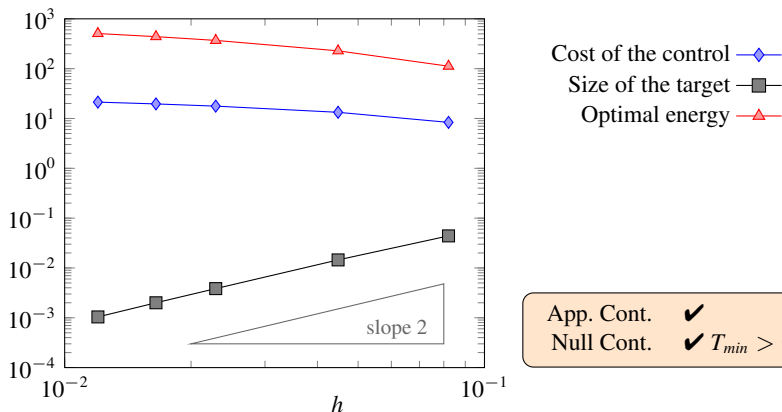
$$\Omega =]-1, 1[\times]0, 1[, \quad \omega =]\underbrace{0.75}_{=a}, 1[\times]0, 1[, \quad T = 0.5 \in]\frac{a^2}{2}, \frac{1+a^2}{2}[.$$

$$\partial_t y - \partial_{x_1}^2 y - x_1^2 \partial_{x_2}^2 y = 1_\omega v,$$

(Beauchard, Cannarsa, Guglielmi, '14), (Beauchard, Miller, Morancey, '14)

$$\Omega =]-1, 1[\times]0, 1[, \quad \omega =]\underbrace{0.75}_{=a}, 1[\times]0, 1[, \quad T = 0.5 \in \left] \frac{a^2}{2}, \frac{1+a^2}{2} \right[.$$

$$\partial_t y - \partial_{x_1}^2 y - x_1^2 \partial_{x_2}^2 y = 1_\omega v,$$



(Beauchard, Cannarsa, Guglielmi, '14), (Beauchard, Miller, Morancey, '14)

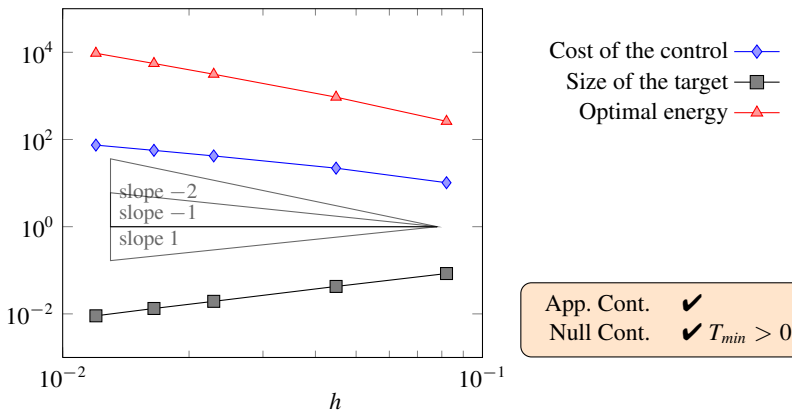
$$\Omega =]-1, 1[\times]0, 1[, \quad \omega =]\underbrace{0.75}_{=a}, 1[\times]0.6, 1[, \quad T = 0.5 > \frac{a^2}{2}.$$

$$\partial_t y - \partial_{x_1}^2 y - \underbrace{x_1^2}_{=a} \partial_{x_2}^2 y = 1_\omega v,$$

(Beauchard, Cannarsa, Guglielmi, '14), (Beauchard, Miller, Morancey, '14)

$$\Omega =]-1, 1[\times]0, 1[, \quad \omega =]\underbrace{0.75}_{=a}, 1[\times]0.6, 1[, \quad T = 0.5 > \frac{a^2}{2}.$$

$$\partial_t y - \partial_{x_1}^2 y - x_1^2 \partial_{x_2}^2 y = 1_\omega v,$$



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SUMMARY

- In the PDE world
 - Many **standard** results in controllability theory can be deduced from the analysis of the penalized HUM approach.
 - The penalized HUM approach always converge towards *something* as the penalization parameter tends to 0.
- In the discrete world
 - Necessity to relate the penalization parameter to discretisation parameters in a clever way.
 - Analysis of uniform null-controllability properties with respect to δt and/or h for semi/fully discrete problems.
 - Associated relaxed observability inequalities.
 - We may use numerical simulations to investigate open problems.
 - Even for non controllable problems, the numerical method applies and gives interesting results.

PERSPECTIVES

- Extend our analysis in the discrete setting to other cases
 - Non symmetric scalar operators.
 - Parabolic systems with few controls.
 - Boundary control problems.
 - Analysis for other space discretizations (Finite Volume, Finite Element, ...)
- From a computational point of view
 - A deeper understanding of HUM operators \rightsquigarrow preconditioning methods.
 - More suitable solvers than standard Conjugate Gradient ?