About the HUM method and its application
in particular to the numerical approximation of controls of PDEs

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- Two Hilbert spaces : the state space $(E, \langle .,. \rangle)$ and the control space $(U, [.,.])$.
- $\mathcal{A} : D(\mathcal{A}) \subset E \mapsto E$ is some *elliptic* operator such that $-\mathcal{A}$ generates an analytic semigroup in E.
- $\mathcal{B} : U \mapsto D(\mathcal{A}^\star)'$ the control (bounded) operator, $\mathcal{B}^\star$ its adjoint.
- COMPATIBILITY ASSUMPTION : we assume that

$$\left(t \mapsto \mathcal{B}^\star e^{-t\mathcal{A}^\star}\psi\right) \in L^2(0,T;U), \text{ and } \left[\!\left[\mathcal{B}^\star e^{-\cdot\mathcal{A}^\star}\psi\right]\!\right]_{L^2(0,T;U)} \leq C\|\psi\|, \ \forall \psi \in E.$$

Our controlled parabolic problem is

$$(S) \quad \begin{cases} \partial_t y + \mathcal{A}y = \mathcal{B}v & \text{in }]0,T[, \\ y(0) = y_0, \end{cases}$$

Here, $y_0 \in E$ is the initial data, $v \in L^2(]0,T[,U)$ is the control we are looking for.

THEOREM (WELL-POSEDNESS OF (S) IN A DUAL SENSE)

For any $y_0 \in E$ and $v \in L^2(0,T;U)$, there exists a unique $y = y_{v,y_0} \in \mathcal{C}^0([0,T],E)$ such that

$$\langle y(t),\psi\rangle - \left\langle y_0, e^{-t\mathcal{A}^\star}\psi\right\rangle = \int_0^t \left[v(s), \mathcal{B}^\star e^{-(t-s)\mathcal{A}^\star}\psi\right] ds, \ \forall t \in [0,T], \forall \psi \in E.$$

NOTATION : $\mathcal{L}_T(v|y_0) \stackrel{\text{def}}{=} y_{v,y_0}(T).$

$$(S)\begin{cases}\partial_t y + \mathcal{A}y = \mathcal{B}v & \text{in }]0,T[,\\ y(0) = y_0.\end{cases}$$

For a given (fixed) control time $T > 0$ and any $\delta \geq 0$, we set

$$\mathrm{Adm}(y_0,\delta) \stackrel{\text{def}}{=} \left\{ v \in L^2(0,T;U), \text{ s.t. } \left\| \mathcal{L}_T\left(v \middle| y_0\right) \right\| \leq \delta \right\}.$$

APPROXIMATE CONTROL PROBLEM FROM THE INITIAL DATA y_0

Do we have

$$\mathrm{Adm}(y_0,\delta) \neq \emptyset, \quad \forall \delta > 0\ ?$$

NULL-CONTROL PROBLEM FROM THE INITIAL DATA y_0

Do we have

$$\mathrm{Adm}(y_0,0) \neq \emptyset\ ?$$

(Fattorini-Russel, '71) (Lebeau-Robbiano, '95)
(Fursikov-Imanuvilov, '96) (Alessandrini-Escauriaza, '08)
(Ammar-Khodja, Benabdallah, González-Burgos, de Teresa, '11)

(Lions, '88) (Glowinski–Lions, '90s)

IDEAS

- To formulate control problems as constrainted optimisation problems.
- To write the associated **unconstrainted** dual optimisation problem.
- To find conditions for the solvability of the dual problem and prove that there are satisfied.

COST OF THE CONTROL

We set

$$F(v) \stackrel{\text{def}}{=} \frac{1}{2}\int_0^T [\![v(t)]\!]^2 \, dt, \ \ \forall v \in L^2(0,T;U),$$

and for any $\delta \geq 0$, we define (it it exists !), v^δ to be the unique minimiser

$$F(v^\delta) = \inf_{v \in \mathrm{Adm}(y_0,\delta)} F(v). \qquad (P^\delta)$$

DUAL PROBLEMS

- The dual pb of (P^0) is not coercive in the natural space E. We need to introduce a **big** abstract space obtained as the completion of E with respect to a suitable norm.
- The dual pb of (P^δ), $\delta > 0$ is coercive in E but is not smooth.

PRIMAL PROBLEM

$$F_\varepsilon(v) \stackrel{\text{def}}{=} \frac{1}{2}\int_0^T [\![v(t)]\!]^2\,dt + \frac{1}{2\varepsilon}\left\|\mathcal{L}_T(v|y_0)\right\|^2, \quad \forall v \in L^2(0,T;U),$$

we consider the following problem : to find $v_\varepsilon \in L^2(0,T;U)$ such that

$$F_\varepsilon(v_\varepsilon) = \inf_{v\in L^2(0,T;U)} F_\varepsilon(v). \qquad (P_\varepsilon)$$

PROPOSITION

For any $\varepsilon > 0$, the functional F_ε is strictly convex, continuous and coercive. Therefore, it admits a unique minimiser $v_\varepsilon \in L^2(0,T;U)$.

DUAL PROBLEM

(Fenchel-Rockafellar duality theorem)

$$J_\varepsilon(q^F) \stackrel{\text{def}}{=} \frac{1}{2}\int_0^T \left[\!\!\left[\mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q^F\right]\!\!\right]^2 dt + \frac{\varepsilon}{2}\left\|q^F\right\|^2 + \left\langle y_0, e^{-T\mathcal{A}^\star} q^F\right\rangle, \quad \forall q^F \in E.$$

PROPOSITION

For any $\varepsilon > 0$, the functional J_ε is strictly convex, continuous and coercive. Therefore, it admits a unique minimiser $q^F_\varepsilon \in E$.

REMARK

We do not require any particular assumption on the operators $\mathcal{A}$ and $\mathcal{B}$.
In particular **we do not assume** that the PDE (S) is (or is not) controllable.

PROPOSITION (DUALITY PROPERTIES PRECISED)

For any $\varepsilon > 0$, the minimisers v_ε and q_ε^F of the functionals F_ε and J_ε respectively, are related through the formulas

$$v_\varepsilon(t) = \mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q_\varepsilon^F, \text{ for a.e. } t \in]0,T[,$$

and

$$\mathcal{L}_T\left(v_\varepsilon \middle| y_0\right) = y_{v_\varepsilon, y_0}(T) = -\varepsilon q_\varepsilon^F.$$

As a consequence, we have

$$\inf_{L^2(0,T;U)} F_\varepsilon = F_\varepsilon(v_\varepsilon) = -J_\varepsilon(q_\varepsilon^F) = -\inf_E J_\varepsilon.$$

(B., '13)

THEOREM

1. *Problem (S) is approximately controllable from the initial data y_0 if and only if*

$$\mathcal{L}_T\left(v_\varepsilon \middle| y_0\right) = y_{v_\varepsilon, y_0}(T) \xrightarrow[\varepsilon \to 0]{} 0.$$

2. *Problem (S) is null-controllable from the initial data y_0 if and only if*

$$M_{y_0}^2 \stackrel{\text{def}}{=} 2 \sup_{\varepsilon > 0} \left(\inf_{L^2(0,T;U)} F_\varepsilon \right) = 2 \sup_{\varepsilon > 0} F_\varepsilon(v_\varepsilon) < +\infty.$$

IN THE NULL-CONTROLLABLE CASE

$$[\![v_\varepsilon]\!]_{L^2(0,T;U)} \leq M_{y_0}, \quad \text{and} \quad \left\| \mathcal{L}_T\left(v_\varepsilon \middle| y_0\right) \right\| \leq M_{y_0} \sqrt{\varepsilon}.$$

Moreover we have $[\![v^0]\!]_{L^2(0,T;U)} = M_{y_0}$ and

$$\boxed{v_\varepsilon \xrightarrow[\varepsilon \to 0]{} v^0, \quad \text{strongly in } L^2(0,T;U)}, \quad \text{and} \quad \boxed{\frac{\mathcal{L}_T\left(v_\varepsilon \middle| y_0\right)}{\sqrt{\varepsilon}} \xrightarrow[\varepsilon \to 0]{} 0},$$

where v^0 is the unique HUM null-control (that is the one of minimal L^2-norm).

Non observable adjoint states : $Q_F \stackrel{\text{def}}{=} \left\{ q^F \in E, \text{ s.t. } \mathcal{B}^\star e^{-t\mathcal{A}^\star} q^F = 0, \forall t \geq 0 \right\}.$

THEOREM (CONVERGENCE OF THE PENALISED HUM FINAL STATE)

For any $y_0 \in E$, the penalised-HUM sequence of controls $(v_\varepsilon)_\varepsilon$ satisfies

$$\mathcal{L}_T(v_\varepsilon | y_0) \xrightarrow[\varepsilon \to 0]{} \mathbb{P}_{Q_F}\left(e^{-T\mathcal{A}} y_0\right).$$

PROPOSITION (SELFADJOINT CASE)

Assume that $\mathcal{A}$ is selfadjoint, and set $Y_T \stackrel{\text{def}}{=} e^{-T\mathcal{A}^\star} Q_F)e^{-T\mathcal{A}} Q_F$ then

$$\mathbb{P}_{Q_F}\left(e^{-T\mathcal{A}} y_0\right) = e^{-T\mathcal{A}}\left(\mathbb{P}_{\overline{Y_T}} y_0\right).$$

Therefore, the system is approximately controllable from y_0 if and only if $\mathbb{P}_{\overline{Y_T}} y_0 = 0$.

- The set of (approximately) controllable initial data is $Y_T^\perp$.
- For any $y_0 \in Y_T$ we have

$$v_\varepsilon = 0, \quad \forall \varepsilon > 0,$$
$$\text{Adm}(y_0, \delta) \neq \emptyset \quad \Leftrightarrow \quad \delta \geq \left\| e^{-T\mathcal{A}} y_0 \right\|.$$

Non observable adjoint states : $Q_F \stackrel{\text{def}}{=} \left\{ q^F \in E, \text{ s.t. } \mathcal{B}^\star e^{-t\mathcal{A}^\star} q^F = 0, \forall t \geq 0 \right\}.$

THEOREM (CONVERGENCE OF THE PENALISED HUM FINAL STATE)

For any $y_0 \in E$, the penalised-HUM sequence of controls $(v_\varepsilon)_\varepsilon$ satisfies

$$\mathcal{L}_T(v_\varepsilon | y_0) \xrightarrow[\varepsilon \to 0]{} \mathbb{P}_{Q_F}\left(e^{-T\mathcal{A}} y_0\right).$$

COROLLARY (APP. CONTROLLABILITY AND UNIQUE CONTINUATION)

The system (S) is approximately controllable from the initial data y_0 if and only if

$$\left[\mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q^F = 0, \quad \forall t \in [0,T]\right] \Longrightarrow \left\langle y_0, e^{-T\mathcal{A}^\star} q^F \right\rangle = 0. \qquad \text{(UC)}$$

PROPOSITION (APP. CONTROLLABILITY AND WEAK OBSERVABILITY)

The property (UC) *is equivalent to the following* **weak observability inequality**

$$\left|\left\langle y_0, e^{-T\mathcal{A}^\star} q^F \right\rangle\right|^2 \leq C_{\varepsilon,y_0}^2 \left[\!\left[\mathcal{B}^\star e^{-(T-.)\mathcal{A}^\star} q^F\right]\!\right]_{L^2(0,T;U)}^2 + \varepsilon \left\| q^F \right\|^2, \quad \forall q^F \in E, \forall \varepsilon > 0.$$

THEOREM (NULL-CONTROLLABILITY AND OBSERVABILITY)

Problem (S) is null-controllable from y_0 if and only if, there exists $\widetilde{M}_{y_0} \geq 0$ such that

$$\left|\left\langle y_0, e^{-T\mathcal{A}^\star} q^F \right\rangle\right|^2 \leq \widetilde{M}_{y_0}^2 \left[\!\left[\mathcal{B}^\star e^{-(T-.)\mathcal{A}^\star} q^F\right]\!\right]_{L^2(0,T;U)}^2, \quad \forall q^F \in E.$$

Moreover, the best constant $\widetilde{M}_{y_0}$ is equal to the cost of the HUM control $[\![v^0]\!]_{L^2(0,T;U)}$.

For each $\varepsilon > 0$, let $y_{0,\varepsilon} \in E$ such that $(y_{0,\varepsilon})_\varepsilon$ is bounded in E and

$$e^{-T\mathcal{A}} y_{0,\varepsilon} \xrightarrow[\varepsilon \to 0]{} e^{-TA} y_0.$$

ASSOCIATED HUM FUNCTIONALS

$$\tilde{F}_\varepsilon(v) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T [\![v(t)]\!]^2 \, dt + \frac{1}{2\varepsilon} \left\| \mathcal{L}_T\left(v \middle| y_{0,\varepsilon}\right) \right\|^2, \quad \forall v \in L^2(0,T;U),$$

$$\tilde{J}_\varepsilon(q^F) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T \left[\!\left[\mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q^F \right]\!\right]^2 dt + \frac{\varepsilon}{2} \left\| q^F \right\|^2 + \left\langle y_{0,\varepsilon}, e^{-T\mathcal{A}^\star} q^F \right\rangle, \quad \forall q^F \in E.$$

We denote by $\tilde{v}_\varepsilon$ the unique minimiser of $\tilde{F}_\varepsilon$.

CONTROLLABILITY CONDITIONS

(S) is app. cont. from $y_0 \iff \mathcal{L}_T\left(\tilde{v}_\varepsilon \middle| y_{0,\varepsilon}\right) \xrightarrow[\varepsilon \to 0]{} 0.$

$\sup_{\varepsilon > 0} \left(\inf_{L^2(0,T;U)} \tilde{F}_\varepsilon \right) < +\infty \implies$ (S) is null-controllable from y_0.

$$\left. \begin{array}{c} \text{(S) is null-controllable from } y_0 \\ \sup_{\varepsilon>0} \frac{1}{\varepsilon} \left\| e^{-T\mathcal{A}}(y_0 - y_{0,\varepsilon}) \right\|^2 < +\infty \end{array} \right\} \implies \sup_{\varepsilon > 0} \left(\inf_{L^2(0,T;U)} \tilde{F}_\varepsilon \right) < +\infty.$$

For each $\varepsilon > 0$, let $y_{0,\varepsilon} \in E$ such that $(y_{0,\varepsilon})_\varepsilon$ is bounded in E and

$$e^{-T\mathcal{A}} y_{0,\varepsilon} \xrightarrow[\varepsilon\to 0]{} e^{-TA} y_0.$$

ASSOCIATED HUM FUNCTIONALS

$$\tilde{F}_\varepsilon(v) \stackrel{\text{def}}{=} \frac{1}{2}\int_0^T [\![v(t)]\!]^2\, dt + \frac{1}{2\varepsilon}\left\|\mathcal{L}_T\left(v\middle|y_{0,\varepsilon}\right)\right\|^2, \quad \forall v \in L^2(0,T;U),$$

$$\tilde{J}_\varepsilon(q^F) \stackrel{\text{def}}{=} \frac{1}{2}\int_0^T \left[\!\!\left[\mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q^F\right]\!\!\right]^2 dt + \frac{\varepsilon}{2}\left\|q^F\right\|^2 + \left\langle y_{0,\varepsilon}, e^{-T\mathcal{A}^\star} q^F\right\rangle, \quad \forall q^F \in E.$$

We denote by $\tilde{v}_\varepsilon$ the unique minimiser of $\tilde{F}_\varepsilon$.

CONTROLLABILITY CONDITIONS

$$\left.\begin{array}{r} \text{(S) is null-controllable from } y_0 \\ \sup_{\varepsilon>0} \frac{1}{\varepsilon}\left\|e^{-T\mathcal{A}}(y_0 - y_{0,\varepsilon})\right\|^2 < +\infty \end{array}\right\} \Longrightarrow \sup_{\varepsilon>0}\left(\inf_{L^2(0,T;U)} \tilde{F}_\varepsilon\right) < +\infty.$$

DISCUSSION : Assume $\mathcal{A} = \mathcal{A}^\star$ and $Q_F \neq \{0\}$, then take $y_{0,\varepsilon} = \varepsilon^\alpha z$, $z \in e^{-T\mathcal{A}^\star} Q_F$

$$\inf_{L^2(0,T;U)} \tilde{F}_\varepsilon = \frac{\varepsilon^{2\alpha-1}}{2}\left\|e^{-T\mathcal{A}} z\right\|^2 \xrightarrow[\varepsilon\to 0]{} +\infty, \quad \text{as soon as } \alpha < 1/2.$$

$$y_{0,\varepsilon} \xrightarrow[\varepsilon\to 0]{} \boxed{0} \Longleftarrow \text{this initial data is indeed null-controllable !!.}$$

For each $\varepsilon > 0$, let $y_{0,\varepsilon} \in E$ such that $(y_{0,\varepsilon})_\varepsilon$ is bounded in E and

$$e^{-T\mathcal{A}} y_{0,\varepsilon} \xrightarrow[\varepsilon \to 0]{} e^{-TA} y_0.$$

ASSOCIATED HUM FUNCTIONALS

$$\tilde{F}_\varepsilon(v) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T [\![v(t)]\!]^2 \, dt + \frac{1}{2\varepsilon} \left\| \mathcal{L}_T\left(v \middle| y_{0,\varepsilon}\right) \right\|^2, \quad \forall v \in L^2(0,T;U),$$

$$\tilde{J}_\varepsilon(q^F) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T \left[\!\!\left[\mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q^F \right]\!\!\right]^2 dt + \frac{\varepsilon}{2} \left\| q^F \right\|^2 + \left\langle y_{0,\varepsilon}, e^{-T\mathcal{A}^\star} q^F \right\rangle, \quad \forall q^F \in E.$$

We denote by $\tilde{v}_\varepsilon$ the unique minimiser of $\tilde{F}_\varepsilon$.

PROPOSITION (RELAXED OBSERVABILITY INEQUALITY)

Assume that

$$\sup_{\varepsilon > 0} \frac{1}{\varepsilon} \left\| e^{-T\mathcal{A}} (y_0 - y_{0,\varepsilon}) \right\|^2 < +\infty.$$

The system (S) is null-controllable from the initial data y_0 if and only if

$$\left| \left\langle y_{0,\varepsilon}, e^{-T\mathcal{A}^\star} q^F \right\rangle \right|^2 \leq M \left(\left[\!\!\left[\mathcal{B}^\star e^{-(T-.)\mathcal{A}^\star} q^F \right]\!\!\right]^2_{L^2(0,T;U)} + \varepsilon \left\| q^F \right\|^2 \right), \quad \forall q^F \in E.$$

We do not require the system to be null-controllable from any of the $(y_{0,\varepsilon})_\varepsilon$.

FRAMEWORK

For any $h > 0$, we are given

- A discrete state space $(E_h, \langle\cdot,\cdot\rangle_h)$.
- An approximate operator $\mathcal{A}_h$ on E_h.
- A discrete control space $(U_h, [\cdot,\cdot]_h)$.
- A linear operator $\mathcal{B}_h : U_h \to E_h$, $\mathcal{B}_h^\star$ being its adjoint $\langle \mathcal{B}_h u, x\rangle_h = [\mathcal{B}_h^\star x, u]_h$.

The semi-discrete control problem is (S_h) $\begin{cases} \partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h, \\ y_h(0) = y_{0,h}. \end{cases}$

Its solution is refered to as $t \mapsto y_{v_h,y_{0,h}}(t) \in E_h$ and we set

$$\mathcal{L}_T^h\left(v_h \middle| y_{0,h}\right) \stackrel{\text{def}}{=} y_{v_h,y_{0,h}}(T).$$

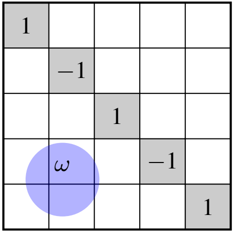
QUESTIONS

Assume that $(y_{0,h})_h$ are, in some sense, approximations of a $y_0 \in E$.

1. Can we relate the controllability properties of (S) starting from y_0 to the ones of (S_h) starting from $y_{0,h}$?
2. Can we obtain uniform bounds (w.r.t. h) for the associated controls v_h ?

MAIN ISSUES RELATED TO DISCRETISATION

1. It may happen that (S_h) is not controllable even if (S) is.
   EXAMPLE : the 2D 5-point discrete Laplace operator $\mathcal{A}_h$. **(Kavian, Zuazua)**



There exists a non trivial $\psi_h \in E_h$ such that

$$\mathcal{A}_h^\star \psi_h = \mu_h \psi_h, \text{ and } \mathcal{B}_h^\star \psi_h = 0.$$

$$\Longrightarrow \psi_h \in Q_{F,h} \stackrel{\text{def}}{=} \{q_h^F \in E_h, \text{ s.t. } \mathcal{B}_h^\star e^{-t\mathcal{A}_h^\star} q_h^F = 0, \ \forall t \geq 0\},$$

For any control $v_h \in L^2(0,T;U_h)$, $\dfrac{d}{dt} \langle y_h(t), \psi_h \rangle_h + \mu_h \langle y_h(t), \psi_h \rangle = 0,$

and thus

$$\left\langle \mathcal{L}_T^h(v_h | y_{0,h}), \psi_h \right\rangle_h = \langle y_h(T), \psi_h \rangle_h = e^{-\mu_h T} \langle y_{0,h}, \psi_h \rangle_h . \tag{1}$$

REMARK : The eigenvalue μ_h is very large $\sim \frac{C}{h^2}$ thus $\left\langle \mathcal{L}_T^h(v_h | y_{0,h}), \psi_h \right\rangle_h$ is exponentially small.

2. Even if (S) and (S_h) are both controllable, it is not necessarily desirable to compute a null-control v_h of (S_h) to obtain a suitable approximation of a null-control of (S).

$$F_{\varepsilon,h}(v_h) \stackrel{\text{def}}{=} \frac{1}{2}\int_0^T [\![v_h(t)]\!]_h^2\,dt + \frac{1}{2\varepsilon}\left\|\mathcal{L}_T^h(v_h|y_{0,h})\right\|_h^2, \quad \forall v_h \in L^2(0,T;U_h),$$

$$J_{\varepsilon,h}(q_h^F) \stackrel{\text{def}}{=} \frac{1}{2}\int_0^T \left[\!\!\left[\mathcal{B}_h^\star e^{-(T-t)\mathcal{A}_h^\star} q_h^F\right]\!\!\right]_h^2 dt + \frac{\varepsilon}{2}\left\|q_h^F\right\|_h^2 + \left\langle y_{0,h}, e^{-T\mathcal{A}_h^\star} q_h^F\right\rangle_h, \quad \forall q_h^F \in E_h.$$

- For each value of $h>0$, all the previous results apply.
- We denote by $v_{\varepsilon,h}$ the unique minimiser of $F_{\varepsilon,h}$.

GOAL

One would like to let $(\varepsilon,h)\to(0,0)$ but this should be done with some care.

COMMENTS

1. Even if (S) is controllable from y_0, in the cases where $Q_{F,h}\neq\{0\}$ we may have

$$\lim_{\varepsilon\to 0}\left\|\mathcal{L}_T^h(v_{\varepsilon,h}|y_{0,h})\right\|_h \neq 0, \quad \forall h>0$$

2. One can prove that for any $h>0$

$$\sup_{\varepsilon>0} [\![v_{\varepsilon,h}]\!]_{L^2(0,T;U_h)} < +\infty.$$

Let $h \in]0,+\infty[\mapsto \phi(h) \in]0,+\infty[$ be given such that $\lim_{h\to 0}\phi(h) = 0$.

DEFINITION

For a given family of initial data $Y_0 = (y_{0,h})_h \in \prod_{h>0} E_h$, we say that the family of problems (S_h) is $\phi(h)$-null controllable from Y_0, if there exists a $h_0 > 0$ such that

$$M_{Y_0}^2 \stackrel{\text{def}}{=} 2 \sup_{0<h<h_0} \left(\inf_{L^2(0,T;U_h)} F_{\phi(h),h} \right) < +\infty,$$

where $F_{\phi(h),h}$ is built upon $y_{0,h}$.

THEOREM (RELAXED OBSERVABILITY)

For a given $Y_0 \in E_{\text{init}}$, the problems (S_h) are $\phi(h)$-null-controllable from Y_0 if and only if there exists $h_0 > 0$ and $\widetilde{M}_{Y_0} > 0$, such that, for any $0 < h < h_0$

$$\left| \left\langle y_{0,h}, e^{-T\mathcal{A}_h^\star} q_h^F \right\rangle_h \right|^2 \leq \widetilde{M}_{Y_0}^2 \left(\left[\!\left[\mathcal{B}_h^\star e^{-(T-.)\mathcal{A}_h^\star} q_h^F \right]\!\right]^2_{L^2(0,T;U_h)} + \phi(h) \left\| q_h^F \right\|_h^2 \right), \quad \forall q_h^F \in E_h.$$

In such case, the best constant $\widetilde{M}_{Y_0}$ is equal to M_{Y_0} and

$$\left[\!\left[v_{\phi(h),h} \right]\!\right]_{L^2(0,T;U_h)} \leq M_{Y_0}, \text{ and } \left\| \mathcal{L}_T^h\left(v_{\phi(h),h} | y_{0,h}\right) \right\|_h \leq M_{Y_0}\sqrt{\phi(h)}, \ \forall 0 < h < h_0.$$

Let $h \in]0,+\infty[\mapsto \phi(h) \in]0,+\infty[$ be given such that $\lim_{h\to 0}\phi(h) = 0$.

DEFINITION

For a given family of initial data $Y_0 = (y_{0,h})_h \in \prod_{h>0} E_h$, we say that the family of problems (S_h) is $\phi(h)$-null controllable from Y_0, if there exists a $h_0 > 0$ such that

$$M_{Y_0}^2 \stackrel{\text{def}}{=} 2 \sup_{0<h<h_0} \left(\inf_{L^2(0,T;U_h)} F_{\phi(h),h} \right) < +\infty,$$

where $F_{\phi(h),h}$ is built upon $y_{0,h}$.

PROPOSITION

Assume that, for some $C_{\text{obs}} > 0$, the following relaxed observability inequality holds

$$\left\| e^{-T\mathcal{A}_h^\star} q_h^F \right\|_h^2 \leq C_{\text{obs}}^2 \left(\left[\!\left[\mathcal{B}_h^\star e^{-(T-.)\mathcal{A}_h^\star} q_h^F \right]\!\right]_{L^2(0,T;U_h)}^2 + \phi(h) \left\| q_h^F \right\|_h^2 \right), \quad \left| \begin{array}{r} \forall q_h^F \in E_h, \\ \forall 0 < h < h_0 \end{array} \right.$$

then for any **bounded** *family Y_0, the problems (S_h) are $\phi(h)$-null-controllable from Y_0 and we have*

$$M_{Y_0} \leq C_{\text{obs}} \left(\sup_{0<h<h_0} \|y_{0,h}\|_h \right).$$

(Lasiecka-Triggiani, '00) (Labbé-Trélat, '06)

- We suppose given $\widetilde{P}_h : E_h \to D((\mathcal{A}^\star)^{\frac{1}{2}})$ and $\widetilde{Q}_h : U_h \to U$ such that

$$\|y_h\|_h = \left\|\widetilde{P}_h y_h\right\|, \forall y_h \in E_h, \quad \text{and} \quad [\![u_h]\!]_h = \left[\!\left[\widetilde{Q}_h u_h\right]\!\right].$$

- We set $P_h = (\widetilde{P}_h)^* : D((\mathcal{A}^\star)^{\frac{1}{2}})' \to E_h$ and $Q_h = (\widetilde{Q}_h)^* : U \to U_h$ and we assume that

$$P_h\widetilde{P}_h = \text{Id}_{E_h}, \text{ and } Q_h\widetilde{Q}_h = \text{Id}_{U_h}.$$

- We define now $\mathcal{A}_h$ and $\mathcal{B}_h$ through their adjoints by the formulas

$$\mathcal{A}_h^\star = P_h\mathcal{A}^\star\widetilde{P}_h, \quad \mathcal{B}_h^\star = Q_h\mathcal{B}^\star\widetilde{P}_h.$$

- + Standard approximation properties ...

EXAMPLE : Finite element Galerkin approximation.

(Labbé-Trélat, '06)

THEOREM

Assume that (S) *is null-controllable at time* T.
There exists a $\beta > 0$, **depending on the approximation properties of E_h and U_h** *such that the relaxed-observability inequality holds as soon as*

$$\boxed{\liminf_{h\to 0} \frac{\phi(h)}{h^\beta} > 0.}$$

In that case, for any $y_0 \in E$*, we can define* $y_{0,h} = P_h y_0$ *and build the associated penalised HUM discrete controls* $v_{\phi(h),h}$.
Then, there is a null-control $v \in \mathrm{Adm}(y_0, 0)$ *such that, up to a subsequence, we have*

$$\widetilde{Q}_h v_{\phi(h),h} \xrightarrow[h\to 0]{} v, \quad \text{in } L^2(0,T;U), \quad \text{and} \quad \widetilde{P}_h y_h \xrightarrow[h\to 0]{} y_{v,y_0}, \quad \text{in } L^2(0,T;E).$$

- The limit control v may not be the HUM control.
- Proving **strong convergence** of the discrete control is very difficult.
- In practive, the power β is low : for the 1D heat equation, Neumann boundary control, $\mathbb{P}^1$ finite element, we get $\beta = 0.45$. It means that

$$\|y_h(T)\|_h \approx_0 \sqrt{\phi(h)} = h^{0.225} \quad \Longleftarrow \text{Very poor convergence.}$$

(B.-Hubert-Le Rousseau, '09-...)

We assume that $\mathcal{A}_h$ is SPD and let $(\psi_{j,h}, \mu_{j,h})_j$ its eigenelements.

ASSUMPTION : DISCRETE LEBEAU-ROBBIANO SPECTRAL INEQUALITY

There exists $h_0 > 0$, $\alpha \in [0,1)$, $\beta > 0$, and $\kappa, \ell > 0$ such that, for any $h < h_0$ and for any $(a_j)_j \in \mathbb{R}^{\mathbb{N}}$, we have

$$\Big\| \sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \Big\|_h^2 \leq \kappa e^{\kappa \mu^\alpha} \Big[\!\Big[\mathcal{B}_h^\star \Big(\sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \Big) \Big]\!\Big]_h^2, \quad \forall \mu < \frac{\ell}{h^\beta}. \qquad (\mathcal{H}_{\alpha,\beta})$$

THEOREM

Assume that assumption $(\mathcal{H}_{\alpha,\beta})$ holds, then there exists $h_0 > 0$, $C > 0$ such that, the relaxed observability inequality holds as soon as the function ϕ satisfies

$$\liminf_{h \to 0} \frac{\phi(h)}{e^{-C/h^\beta}} > 0.$$

Thus, for any bounded family of initial data $Y_0 \in E_{\text{init}}$, and for any $0 < h < h_0$ we have

$$[\![v_{\phi(h),h}]\!]_{L^2(0,T;U_h)} \leq C_{\text{obs}} \|y_{0,h}\|_h, \text{ and } \left\| \mathcal{L}_T^h \left(v_{\phi(h),h} \middle| y_{0,h} \right) \right\|_h \leq C_{\text{obs}} \|y_{0,h}\|_h \sqrt{\phi(h)}.$$

DISCRETE LEBEAU-ROBBIANO SPECTRAL INEQUALITY

There exists $h_0 > 0$, $\alpha \in [0,1)$, $\beta > 0$, and $\kappa, \ell > 0$ such that, for any $h < h_0$ and for any $(a_j)_j \in \mathbb{R}^{\mathbb{N}}$, we have

$$\Big\| \sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \Big\|_h^2 \leq \kappa e^{\kappa \mu^\alpha} \Big[\!\Big[\mathcal{B}_h^\star \Big(\sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \Big) \Big]\!\Big]_h^2, \qquad \boxed{\forall \mu < \frac{\ell}{h^\beta}}. \qquad (\mathcal{H}_{\alpha,\beta})$$

IMPORTANT OBSERVATION

Excepted in very particular cases, the assumption $(\mathcal{H}_{\alpha,\beta})$ has no chance to hold true without restriction on μ, see the counter-example of Kavian.

DISCRETE LEBEAU-ROBBIANO SPECTRAL INEQUALITY

There exists $h_0 > 0$, $\alpha \in [0,1)$, $\beta > 0$, and $\kappa, \ell > 0$ such that, for any $h < h_0$ and for any $(a_j)_j \in \mathbb{R}^{\mathbb{N}}$, we have

$$\Big\| \sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \Big\|_h^2 \leq \kappa e^{\kappa \mu^\alpha} \Big[\!\Big[\mathcal{B}_h^\star \Big(\sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \Big) \Big]\!\Big]_h^2, \quad \forall \mu < \frac{\ell}{h^\beta}. \qquad (\mathcal{H}_{\alpha,\beta})$$

THEOREM

We assume that $\mathcal{A}_h$ is the usual finite difference approximation of $-\mathrm{div}(\gamma\nabla .)$ for a smooth γ on a regular Cartesian mesh and that $\mathcal{B}_h = 1_\omega$. Then,

Assumption $(\mathcal{H}_{\alpha,\beta})$ holds for $\alpha = 1/2$ and $\beta = 2$.

MAIN TOOL OF THE PROOF : Uniform discrete elliptic Carleman estimates for an augmented semi-discrete elliptic operator $-\partial_s^2 + \mathcal{A}_h$.

OPTIMALITY : The maximal eigenvalue of $\mathcal{A}_h$ is $\sim \frac{C}{h^2}$ thus $(\mathcal{H}_{\alpha,\beta})$ gives a bound for a constant portion of the spectrum of $\mathcal{A}_h$. Moreover, $\alpha = 1/2$ is the exponent of the usual Lebeau-Robbiano inequality.

CONSEQUENCE : The $\phi(h)$-null-controllability holds for any $\phi(h) \geq e^{-C/h^2}$.

We have introduced and analyzed the $\phi(h)$-null-controllability hold for

$$(S_h)\ \begin{cases} \partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h, \\ y_h(0) = y_{0,h}. \end{cases}$$

WHAT ABOUT TIME DISCRETIZATION OF SUCH A SYSTEM ?

We study **unconditionally stable schemes** : the θ-scheme with $\theta \in [1/2, 1]$

$$(S_{h,\delta t})\ \begin{cases} \dfrac{y_h^{n+1} - y_h^n}{\delta t} + \mathcal{A}_h(\theta y_h^{n+1} + (1-\theta) y_h^n) = \mathcal{B}_h v_h^{n+1},\ \forall n \in [\![0, M-1]\!], \\ y_h^0 = y_{0,h} \in E_h, \end{cases}$$

where, $\delta t = T/M$, $v_{h,\delta t} = (v_h^n)_{1 \leq n \leq M} \in (U_h)^M$ is a fully-discrete control function whose cost is defined by

$$[\![v_{h,\delta t}]\!]_{L^2_{\delta t}(0,T;U_h)} \stackrel{\text{def}}{=} \left(\sum_{n=1}^{M} \delta t \, [\![v_h^n]\!]_h^2 \right)^{\frac{1}{2}}.$$

The value at the final time iteration of the controlled solution of $(S_{h,\delta t})$ is denoted by

$$\mathcal{L}_T^{h,\delta t}\left(v_{h,\delta t} \middle| y_{0,h}\right) \stackrel{\text{def}}{=} y_h^M.$$

THE PENALISED HUM PRIMAL FUNCTIONAL

$$F_{\varepsilon,h,\delta t}(v_{h,\delta t}) \stackrel{\text{def}}{=} \frac{1}{2} [\![v_{h,\delta t}]\!]^2_{L^2_{\delta t}(0,T;U_h)} + \frac{1}{2\varepsilon} \left\| \mathcal{L}_T^{h,\delta t}\left(v_{h,\delta t} \middle| y_{0,h}\right) \right\|_h^2 .$$

DEFINITION (DUAL FUNCTIONAL)

We define the functional

$$J_{\varepsilon,h,\delta t}(q_h^F) \stackrel{\text{def}}{=} \frac{1}{2} \left[\!\!\left[\mathcal{B}_h^\star \mathcal{L}_T^{*,h,\delta t}\left(q_h^F\right) \right]\!\!\right]^2_{L^2_{\delta t}(0,T;U_h)} + \frac{\varepsilon}{2} \left\| q_h^F \right\|_h^2 - \left\langle y_{0,h}, q_h^1 - \delta t(1-\theta)\mathcal{A}_h q_h^1 \right\rangle_h, \quad \forall q_h^F \in E_h,$$

where $\mathcal{L}_T^{*,h,\delta t}\left(q_h^F\right) = (q_h^n)_{1\leq n\leq M}$ *is the solution of the following adjoint problem*

$$\begin{cases} q_h^{M+1} = q_h^F, \\ \dfrac{q_h^M - q_h^{M+1}}{\delta t} + \theta \mathcal{A}_h q_h^M = 0, \\ \dfrac{q_h^n - q_h^{n+1}}{\delta t} + \mathcal{A}_h(\theta q_h^n + (1-\theta) q_h^{n+1}) = 0, \ \forall n \in [\![1, M-1]\!]. \end{cases} \qquad (S^*_{h,\delta t})$$

THE PENALISED HUM PRIMAL FUNCTIONAL

$$F_{\varepsilon,h,\delta t}(v_{h,\delta t}) \stackrel{\text{def}}{=} \frac{1}{2} [\![v_{h,\delta t}]\!]^2_{L^2_{\delta t}(0,T;U_h)} + \frac{1}{2\varepsilon} \left\| \mathcal{L}_T^{h,\delta t}\left(v_{h,\delta t} \middle| y_{0,h}\right) \right\|_h^2 .$$

$$J_{\varepsilon,h,\delta t}(q_h^F) \stackrel{\text{def}}{=} \frac{1}{2} \left[\!\!\left[\mathcal{B}_h^\star \mathcal{L}_T^{*,h,\delta t}\left(q_h^F\right) \right]\!\!\right]^2_{L^2_{\delta t}(0,T;U_h)} + \frac{\varepsilon}{2} \left\| q_h^F \right\|_h^2 - \left\langle y_{0,h}, q_h^1 - \delta t(1-\theta)\mathcal{A}_h q_h^1 \right\rangle_h, \quad \forall q_h^F \in E_h,$$

THEOREM (DUALITY)

The functionals $F_{\varepsilon,h,\delta t}$ and $J_{\varepsilon,h,\delta t}$ are in duality, in the sense that their respective minimisers $v_{\varepsilon,h,\delta t} \in L^2(0,T;U_h)$ and $q^F_{\varepsilon,h,\delta t} \in E_h$ satisfy

$$\inf_{L^2_{\delta t}(0,T;U_h)} F_{\varepsilon,h,\delta t} = F_{\varepsilon,h,\delta t}(v_{\varepsilon,h,\delta t}) = -J_{\varepsilon,h,\delta t}(q^F_{\varepsilon,h,\delta t}) = -\inf_{E_h} J_{\varepsilon,h,\delta t},$$

and moreover

$$v_{\varepsilon,h} = \mathcal{B}_h^\star \mathcal{L}_T^{*,h,\delta t}\left(q^F_{\varepsilon,h,\delta t}\right).$$

THEOREM (CASE $\theta \in]1/2, 1]$)

Assume that the discrete Lebeau-Robbiano inequality $(\mathcal{H}_{\alpha,\beta})$ *holds and let* ϕ *be such that*

$$\liminf_{h\to 0} \frac{\phi(h)}{e^{-C/h^\beta}} > 0.$$

Then, there exists $h_0 > 0, C_T > 0, C_{\text{obs}} > 0$ *such that for any* $0 < h < h_0$ *and any* $\delta t \leq C_T |\log \phi(h)|^{-1}$, *the following relaxed observability inequality holds*

$$\left\| q_h^1 - \delta t(1-\theta)\mathcal{A}_h q_h^1 \right\|_h^2 \leq C_{\text{obs}}^2 \left([\![\mathcal{B}_h^\star q_h^n]\!]^2_{L^2_{\delta t}(0,T;U_h)} + \phi(h) \left\| q_h^F \right\|_h^2 \right), \ \ \forall q_h^F \in E_h.$$

Thus, for any such δt *and* h *and any initial data* $y_{0,h} \in E_h$, *the full-discrete control* $v_{\phi(h),h,\delta t}$, *obtained by minimising* $F_{\phi(h),h,\delta t}$ *(or equivalently* $J_{\phi(h),h,\delta t}$*) satisfies*

$$[\![v_{\phi(h),h,\delta t}]\!]_{L^2_{\delta t}(0,T;U_h)} \leq C_{\text{obs}} \left\| y_{0,h} \right\|_h,$$

$$\left\| \mathcal{L}_T^{h,\delta t}(v_{\phi(h),h,\delta t} | y_{0,h}) \right\|_h \leq C_{\text{obs}} \sqrt{\phi(h)} \left\| y_{0,h} \right\|_h.$$

CASE $\theta = 1/2$: An additional condition on δt is required.

GENERAL PRINCIPLE : Minimise dual functionals $J_{\varepsilon,h}$ or $J_{\varepsilon,h,\delta t}$ (with $\varepsilon = \phi(h)$).

PROPOSITION (GRADIENTS AND GRAMIAM OPERATORS)

For any $h > 0, \delta t > 0, \varepsilon > 0$ and any $q_h^F \in E_h$, we have

$$\nabla J_{\varepsilon,h}(q_h^F) = \underbrace{\mathcal{L}_T^h\left(\mathcal{B}_h^\star e^{-(T-.)\mathcal{A}_h^\star} q_h^F \big| 0\right)}_{\stackrel{\text{def}}{=}\Lambda^h q_h^F} + \varepsilon q_h^F + \mathcal{L}_T^h\left(0\big|y_{0,h}\right),$$

$$\nabla J_{\varepsilon,h,\delta t}(q_h^F) = \underbrace{\mathcal{L}_T^{h,\delta t}\left(\mathcal{B}_h^\star \mathcal{L}_T^{*,h,\delta t}\left(q_h^F\right) \big| 0\right)}_{\stackrel{\text{def}}{=}\Lambda^{h,\delta t} q_h^F} + \varepsilon q_h^F + \mathcal{L}_T^{h,\delta t}\left(0\big|y_{0,h}\right),$$

where $\mathcal{L}_T^{*,h,\delta t}\left(q_h^F\right)$ is the solution of the adjoint fully-discrete pb associated with q_h^F.

COMPUTATION OF GRAMIAN OPERATORS

The computation of $\Lambda_\bullet q_h^F$ amounts to

1. Solve a backward parabolic problem.
2. Apply $\mathcal{B}_h^\star$
3. Solve a forward parabolic problem with the control previously computed.

GENERAL PRINCIPLE : Minimise dual functionals $J_{\varepsilon,h}$ or $J_{\varepsilon,h,\delta t}$ (with $\varepsilon = \phi(h)$).

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For any $h > 0, \delta t > 0, \varepsilon > 0$ and any $q_h^F \in E_h$, we have

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$$\nabla J_{\varepsilon,h,\delta t}(q_h^F) = \underbrace{\mathcal{L}_T^{h,\delta t}\left(\mathcal{B}_h^\star \mathcal{L}_T^{*,h,\delta t}\left(q_h^F\right) \middle| 0\right)}_{\stackrel{\text{def}}{=}\Lambda^{h,\delta t} q_h^F} + \varepsilon q_h^F + \mathcal{L}_T^{h,\delta t}\left(0\middle|y_{0,h}\right),$$

where $\mathcal{L}_T^{*,h,\delta t}\left(q_h^F\right)$ is the solution of the adjoint fully-discrete pb associated with q_h^F.

EQUATIONS TO SOLVE

The semi/fully-discrete controls ar computed by solving the equations

$$(\Lambda^h + \varepsilon \mathrm{Id}) q_h^F = -\mathcal{L}_T^h\left(0\middle|y_{0,h}\right),$$

$$(\Lambda^{h,\delta t} + \varepsilon \mathrm{Id}) q_h^F = -\mathcal{L}_T^{h,\delta t}\left(0\middle|y_{0,h}\right).$$

In practice, we use a conjugate gradient algorithm.

GENERAL PRINCIPLE : Minimise dual functionals $J_{\varepsilon,h}$ or $J_{\varepsilon,h,\delta t}$ (with $\varepsilon = \phi(h)$).

PROPOSITION (GRADIENTS AND GRAMIAM OPERATORS)

For any $h>0, \delta t>0, \varepsilon>0$ and any $q_h^F \in E_h$, we have

$$\nabla J_{\varepsilon,h}(q_h^F) = \underbrace{\mathcal{L}_T^h\left(\mathcal{B}_h^\star e^{-(T-.)\mathcal{A}_h^\star} q_h^F \middle| 0\right)}_{\stackrel{\text{def}}{=}\Lambda^h q_h^F} + \varepsilon q_h^F + \mathcal{L}_T^h\left(0 \middle| y_{0,h}\right),$$

$$\nabla J_{\varepsilon,h,\delta t}(q_h^F) = \underbrace{\mathcal{L}_T^{h,\delta t}\left(\mathcal{B}_h^\star \mathcal{L}_T^{*,h,\delta t}\left(q_h^F\right) \middle| 0\right)}_{\stackrel{\text{def}}{=}\Lambda^{h,\delta t} q_h^F} + \varepsilon q_h^F + \mathcal{L}_T^{h,\delta t}\left(0 \middle| y_{0,h}\right),$$

where $\mathcal{L}_T^{*,h,\delta t}\left(q_h^F\right)$ is the solution of the adjoint fully-discrete pb associated with q_h^F.

CONDITION NUMBER

$$\text{Basic estimate : } \varepsilon \left\|q_h^F\right\|_h \leq \left\|(\Lambda^\bullet + \varepsilon \mathrm{Id}) q_h^F\right\|_h \leq (C+\varepsilon)\left\|q_h^F\right\|_h.$$

$$\boxed{\mathrm{Cond}(\Lambda^\bullet + \varepsilon \mathrm{Id}) \sim \frac{1}{\varepsilon}.}$$

Two main principles

1. $\varepsilon = \phi(h)$ should not be too small in order to maintain a reasonable condition number (i.e. computational cost)

$$\mathrm{Cond}(\Lambda^{\bullet} + \phi(h)\mathrm{Id}) \sim \frac{1}{\phi(h)}.$$

2. The size of the computed solution at time T is

$$\|y_h(T)\|_h \approx C_{\mathrm{obs}}\sqrt{\phi(h)}.$$

It seems reasonnable to choose

$$\phi(h) \sim_{h\to 0} h^{2p},$$

where p is the order of accuracy of the numerical method under study.

Remarks

- Computing a null-control for (S_h), i.e. taking $\varepsilon = \phi(h) = 0$, is not possible in general.
- Choosing $\phi(h)$ much smaller than h^{2p} (like e^{-C/h^2}) is a useless computational effort.

We set $E = E_h = \mathbb{R}$, $\mathcal{A} = \lambda > 0$, $\mathcal{A}_h = (\lambda + \delta_h) \in \mathbb{R}$ with $\delta_h \xrightarrow[h \to 0]{} 0$, $\mathcal{B} = \mathcal{B}_h = 1$.

$$(S) \begin{cases} y' + \lambda y = v, \\ y(0) = 1, \end{cases} \text{ and } (S_h) \begin{cases} y_h' + (\lambda + \delta_h) y_h = v_h, \\ y_h(0) = 1. \end{cases}$$

Uncontrolled solution $e^{-T\mathcal{A}_h} y_{0,h} = e^{-(\lambda+\delta_h)T}$.

Gramiam "operators"

$$\Lambda_h q^F = \frac{1 - e^{-2(\lambda+\delta_h)T}}{2(\lambda+\delta_h)} q^F, \text{ and } \Lambda q^F = \frac{1 - e^{-2\lambda T}}{2\lambda} q^F, \ \forall q^F \in \mathbb{R},$$

Proposition

The corresponding semi-discrete penalised and exact HUM controls are

$$v_{\varepsilon,h}(t) = -e^{-(T-t)(\lambda+\delta_h)} \frac{2(\lambda+\delta_h) e^{-(\lambda+\delta_h)T}}{1 - e^{-2(\lambda+\delta_h)T} + \boxed{2\varepsilon(\lambda+\delta_h)}},$$

$$v(t) = -e^{-(T-t)\lambda} \frac{2\lambda e^{-\lambda T}}{1 - e^{-2\lambda T}}.$$

$$(S)\begin{cases} y' + \lambda y = v, \\ y(0) = 1, \end{cases} \quad \text{and} \quad (S_h)\begin{cases} y_h' + (\lambda + \delta_h) y_h = v_h, \\ y_h(0) = 1. \end{cases}$$

Proposition

The corresponding semi-discrete penalised and exact HUM controls are

$$v_{\varepsilon,h}(t) = -e^{-(T-t)(\lambda+\delta_h)} \frac{2(\lambda+\delta_h)e^{-(\lambda+\delta_h)T}}{1 - e^{-2(\lambda+\delta_h)T} + \boxed{2\varepsilon(\lambda+\delta_h)}},$$

$$v(t) = -e^{-(T-t)\lambda} \frac{2\lambda e^{-\lambda T}}{1 - e^{-2\lambda T}}.$$

Error estimates

$$[\![v - v_{\varepsilon,h}]\!]_{L^2(0,T;U)} \leq C(\lambda, T)(|\delta_h| + \varepsilon), \quad \text{for } \delta_h \text{ and } \varepsilon \text{ small},$$

$$\mathcal{L}_T\left(v_{\varepsilon,h} \middle| 1\right) = C_1(\lambda,T)\delta_h + C_2(\lambda,T)\varepsilon + O(\varepsilon^2 + \delta_h^2),$$

with $C_i(\lambda, T) > 0$.

Conclusion : The *optimal* choice is to take $\varepsilon = \phi(h) \sim \delta_h$.

$$\partial_t y - \partial_x^2 y = 1_\Omega v, \quad \text{in } \Omega =]0,1[,$$

in the particular case where $\omega = \Omega$.

STANDARD FINITE DIFFERENCE APPROXIMATION ON A UNIFORM GRID

$$\partial_t y_i - \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = v_i, \quad \forall i \in \{1, ..., N\}.$$

EIGENFUNCTIONS OF $\mathcal{A}$

$$\phi_k(x) = \sin(k\pi x), \quad \lambda_k = k^2\pi^2, \quad \forall k \geq 1.$$

EIGENFUNCTIONS OF $\mathcal{A}_h$

$$\phi_{k,h} = (\sin(k\pi x_i))_i, \quad \lambda_{k,h} = \frac{4\sin^2\left(\frac{k\pi h}{2}\right)}{h^2}, \quad \forall 1 \leq k \leq 1/h.$$

EQUATIONS FOR THE k-TH EIGENMODE

$$y' + \lambda_k y = v, \quad y_h' + \lambda_{k,h} y_h = v_h.$$

Here

$$\delta_{k,h} = \lambda_{k,h} - \lambda_k \underset{h\to 0}{\sim} -\frac{k^4\pi^4}{12} h^2.$$

Outline

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[} v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

N	M: 20	80	320	1280	$+\infty$
20	14	16	16	16	16
50	22	26	29	29	31
100	30	38	44	49	48
200	45	58	69	77	82

(A) Case $\phi(h) = h^2$

N	M: 20	80	320	1280	$+\infty$
20	24	30	28	27	32
50	83	87	87	93	106
100	235	240	233	262	265
200	778	850	1098	1230	1374

(B) Case $\phi(h) = h^4$

TABLE : Conjugate gradient iterates ; $\omega =]0.3, 0.8[$

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

N	M 20	80	320	1280	$+\infty$
20	$7.17 \cdot 10^{-2}$	$6.54 \cdot 10^{-2}$	$6.38 \cdot 10^{-2}$	$6.34 \cdot 10^{-2}$	$6.33 \cdot 10^{-2}$
50	$7.98 \cdot 10^{-2}$	$7.08 \cdot 10^{-2}$	$6.85 \cdot 10^{-2}$	$6.79 \cdot 10^{-2}$	$6.78 \cdot 10^{-2}$
100	$8.5 \cdot 10^{-2}$	$7.44 \cdot 10^{-2}$	$7.15 \cdot 10^{-2}$	$7.07 \cdot 10^{-2}$	$7.05 \cdot 10^{-2}$
200	$9.1 \cdot 10^{-2}$	$7.75 \cdot 10^{-2}$	$7.39 \cdot 10^{-2}$	$7.3 \cdot 10^{-2}$	$7.27 \cdot 10^{-2}$

TABLE : Optimal energy ; $\phi(h) = h^2$; $\omega =]0.3, 0.8[$

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

N	M 20	80	320	1280	$+\infty$
20	0.11	$8.92 \cdot 10^{-2}$	$8.43 \cdot 10^{-2}$	$8.3 \cdot 10^{-2}$	$8.26 \cdot 10^{-2}$
50	0.12	$8.94 \cdot 10^{-2}$	$8.29 \cdot 10^{-2}$	$8.12 \cdot 10^{-2}$	$8.07 \cdot 10^{-2}$
100	0.12	$9.1 \cdot 10^{-2}$	$8.33 \cdot 10^{-2}$	$8.13 \cdot 10^{-2}$	$8.06 \cdot 10^{-2}$
200	0.13	$9.33 \cdot 10^{-2}$	$8.41 \cdot 10^{-2}$	$8.17 \cdot 10^{-2}$	$8.09 \cdot 10^{-2}$

TABLE : Optimal energy ; $\phi(h) = h^4$; $\omega =]0.3, 0.8[$

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

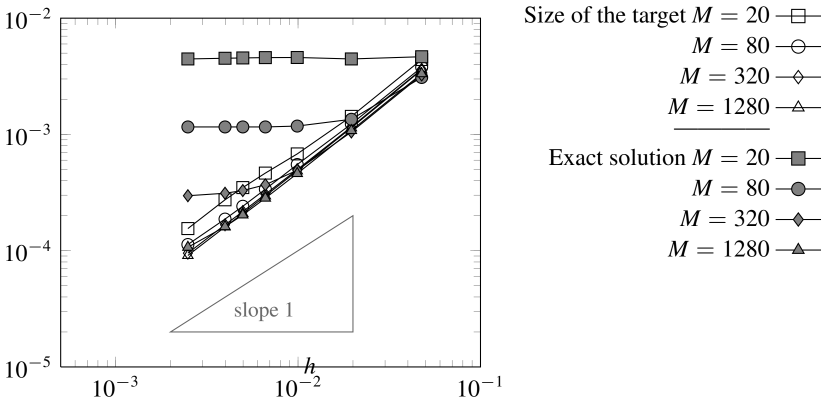


FIGURE : Convergence analysis with $\phi(h) = h^2$; $\omega =]0.3, 0.8[$

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[} v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

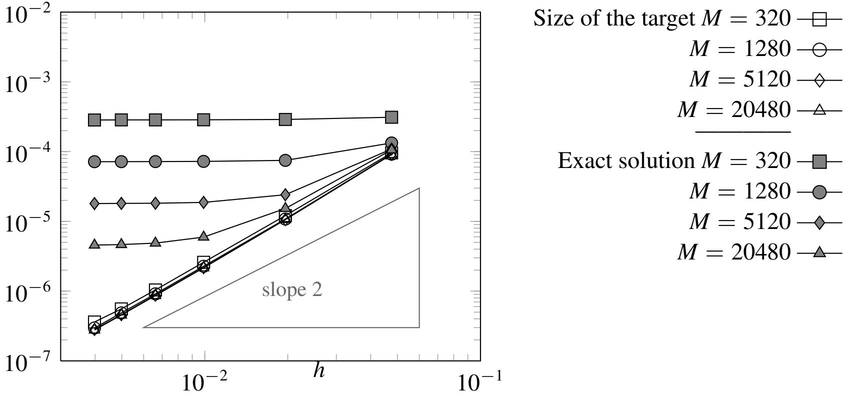


FIGURE : Convergence analysis with $\phi(h) = h^4$; $\omega =]0.3, 0.8[$

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

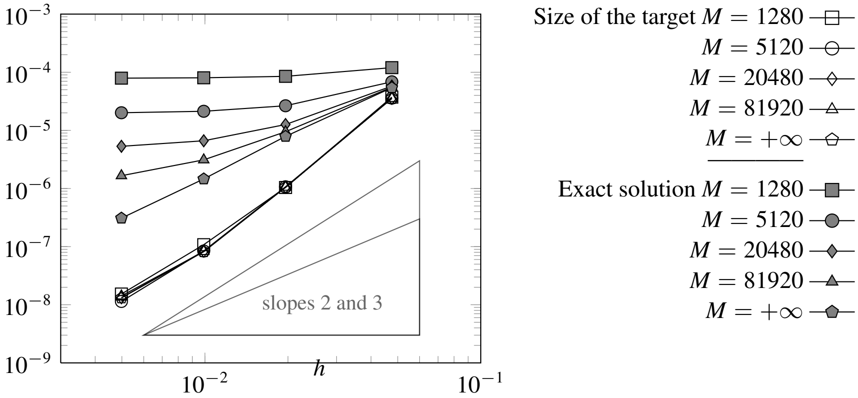


FIGURE : Convergence analysis with $\phi(h) = 1000h^6$; $\omega =]0.3, 0.8[$

$$\partial_t y - 0.1\partial_x^2 y = 1_\Omega v,$$

$$T = 0.5, y_0(x) = \sin(\pi x)^{10}.$$

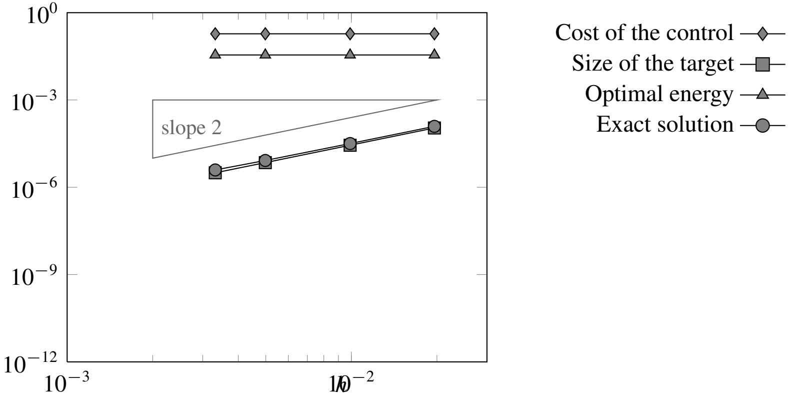


FIGURE : $\phi(h) = h^2$; Semi-discrete scheme

$$\partial_t y - 0.1\partial_x^2 y = 1_\Omega v,$$
$$T = 0.5, y_0(x) = \sin(\pi x)^{10}.$$

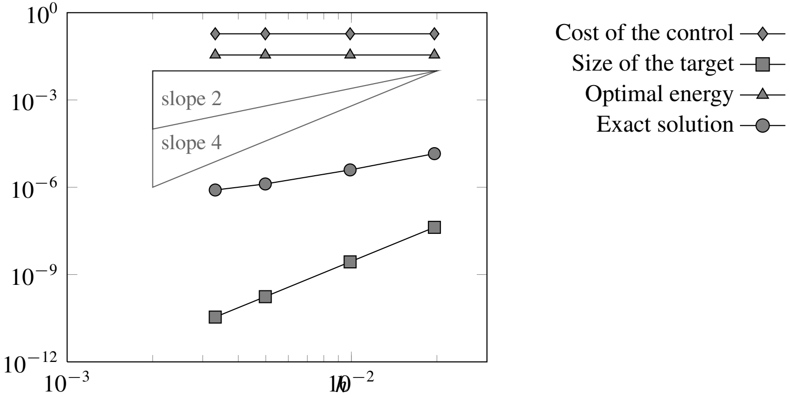


FIGURE : $\phi(h) = h^4$; Semi-discrete scheme

$$\partial_t y - 0.1\partial_x^2 y - 1.5y = 1_{]0.3,0.8[}v,$$

$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

(Fernández-Cara – Münch, '11)

(B.–Le Rousseau, '13)

$$\partial_t y - 0.1\partial_x^2 y - 5y\log^{1.4}(1+|y|) = 1_{]0.2,0.8[}v,$$
$$T = 0.5, y_0(x) = 20\sin(\pi x).$$

OUTLINE

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

SHORT REVIEW OF KNOWN RESULTS

- In the case $a_{21} = \text{cte}$ the system is null-controllable if and only if $a_{21} \neq 0$ (Kalman-like condition)

  (Ammar-Khodja–Benabdallah–Dupaix–González-Burgos, '09)
- In the case where $\text{Supp}(a_{21}) \cap \omega \neq \emptyset$, the system is null-controllable

  (González-Burgos–de Teresa, '10)
- In the case where $\text{Supp}(a_{21}) \cap \omega = \emptyset$ and a_{21} has a constant sign, the system is null-controllable

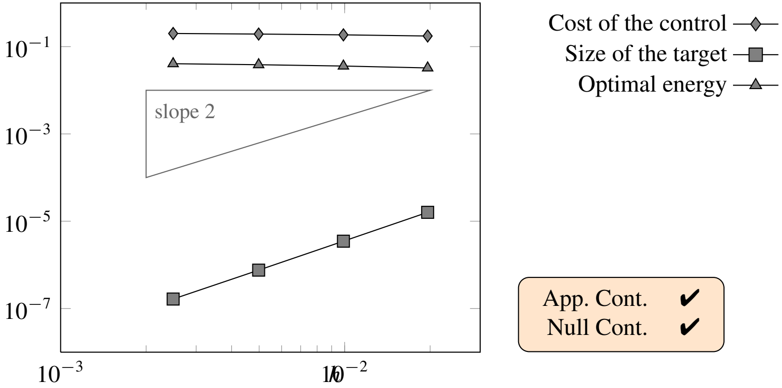
  (Rosier–de Teresa, '11)
- In the case where $\text{Supp}(a_{21}) \cap \omega = \emptyset$ and a_{21} changes it sign :
  - There are structural conditions for the system to be even approximatively controllable **(B.– Olive, '13)**
  - A minimal time condition for the null-controllability can occur **(Ammar-Khodja–Benabdallah–González-Burgos–de Teresa, '14)**

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 1 : $a_{21}(x) = \mathbf{1}_{]0.2,0.9[}(x)$, $\omega =]0.1, 0.5[$, $y_0(x) = \left(\sin(3\pi x), \sin(\pi x)^{10}\right)^t$.

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 1 : $a_{21}(x) = \mathbf{1}_{]0.2,0.9[}(x)$, $\omega =]0.1, 0.5[$, $y_0(x) = \left(\sin(3\pi x), \sin(\pi x)^{10}\right)^t$.



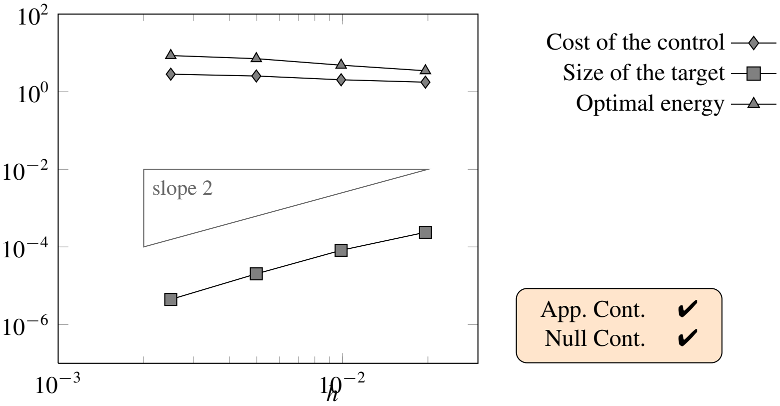
App. Cont. ✔
Null Cont. ✔

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 2 : $a_{21}(x) = \mathbf{1}_{]0.7,0.9[}(x)$, $\omega =]0.1, 0.5[$, $y_0(x) = \left(\sin(3\pi x), \sin(\pi x)^{10}\right)^t$.

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

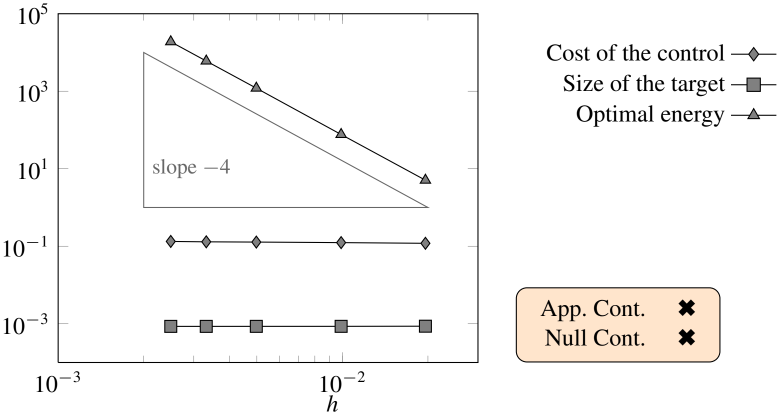
CASE 2 : $a_{21}(x) = \mathbf{1}_{]0.7,0.9[}(x)$, $\omega =]0.1, 0.5[$, $y_0(x) = \left(\sin(3\pi x), \sin(\pi x)^{10}\right)^t$.



App. Cont. ✔
Null Cont. ✔

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 3 : $a_{21}(x) = (x-\alpha)\mathbf{1}_{]0,0.5[}(x)$, $\omega =]0.5, 1[$, $y_0(x) = (\sin(2\pi x), 3\sin(2\pi x))^t$.

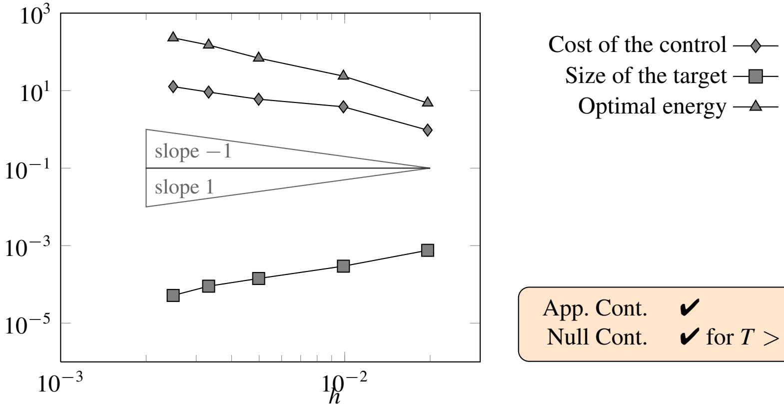


App. Cont.	✖
Null Cont.	✖

FIGURE : $\alpha = 1/4$

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 3 : $a_{21}(x) = (x-\alpha)\mathbf{1}_{]0,0.5[}(x)$, $\omega =]0.5, 1[$, $y_0(x) = (\sin(2\pi x), 3\sin(2\pi x))^t$.



App. Cont.	✔
Null Cont.	✔ for $T > T_{\min}$

FIGURE : $\alpha = 1/8$

A THREE EQUATION CASCADE SYSTEM

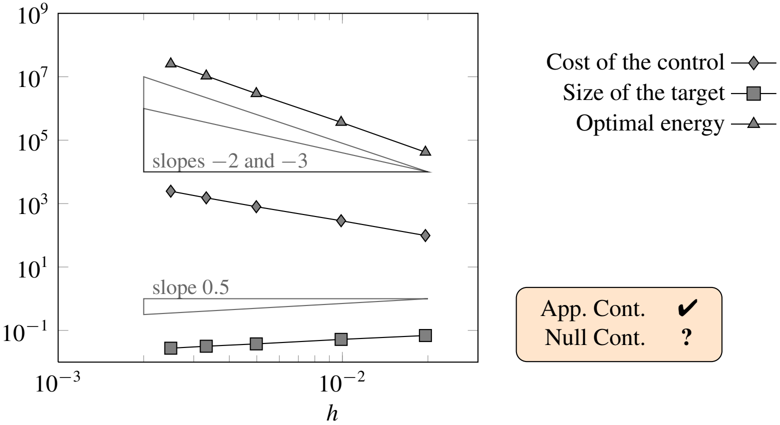
$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ 0 & a_{32}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

VERY SHORT REVIEW :

- If the supports of all the a_{ij} intersect the control domain ω and keeps a constant sign on a part of ω, then the system is null-controllable.
- Necessary and sufficient conditions for approximate controllability are known in the general case.

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ 0 & a_{32}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

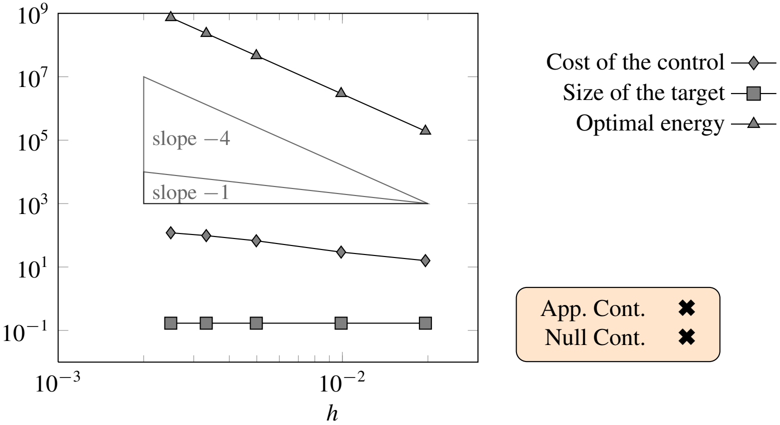
CASE 1 : $a_{21} = \mathbf{1}_{]0,0.5[}, \ a_{32} = 1, \ \omega =]0.5, 1[$.



App. Cont. ✔
Null Cont. ?

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ 0 & a_{32}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 2 : $a_{21} = \mathbf{1}_{]0,0.5[},\ a_{32}(x) = x - 1/2,\ \omega =]0.5, 1[.$



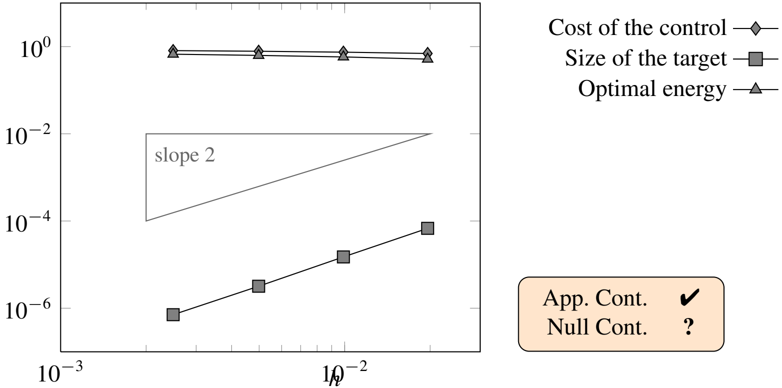
App. Cont. ✖
Null Cont. ✖

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

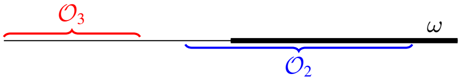
Here also necessary and sufficient conditions for approximate controllability are known

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 1 : $\mathcal{O}_2 \cap \omega \neq \emptyset$

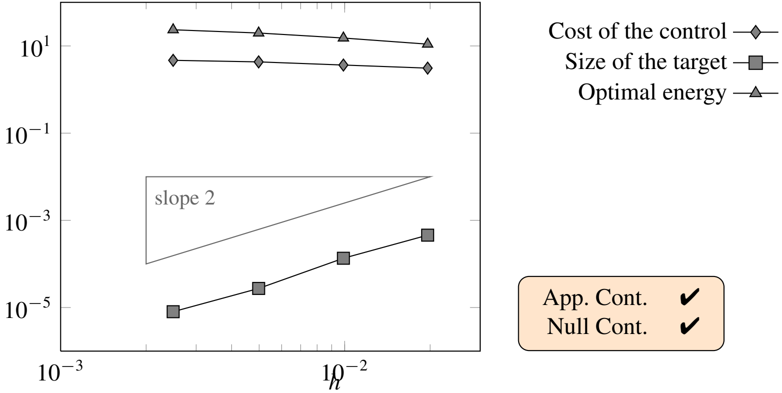


App. Cont. ✔
Null Cont. ?

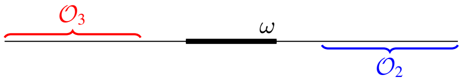


$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 2 : $\mathcal{O}_2$ and $\mathcal{O}_3$ are located in different connected components of $\Omega \setminus \omega$

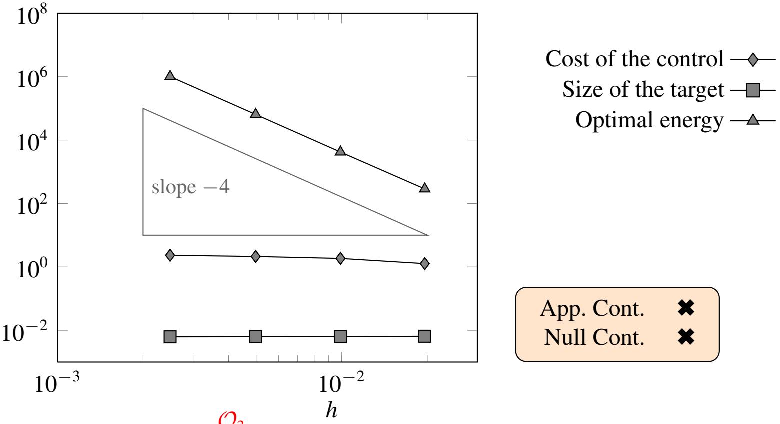


App. Cont. ✔
Null Cont. ✔

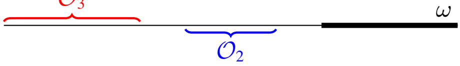


$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 3 : $\mathcal{O}_2$ and $\mathcal{O}_3$ are located in the same connected component of $\Omega \setminus \omega$

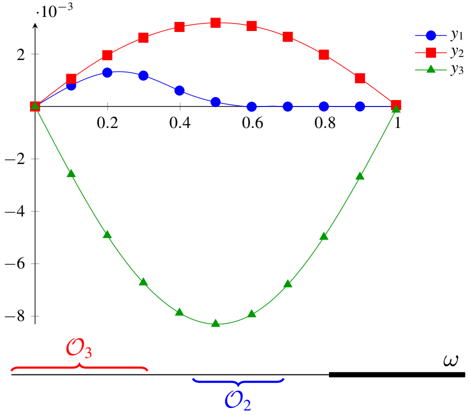


App. Cont. ✖
Null Cont. ✖



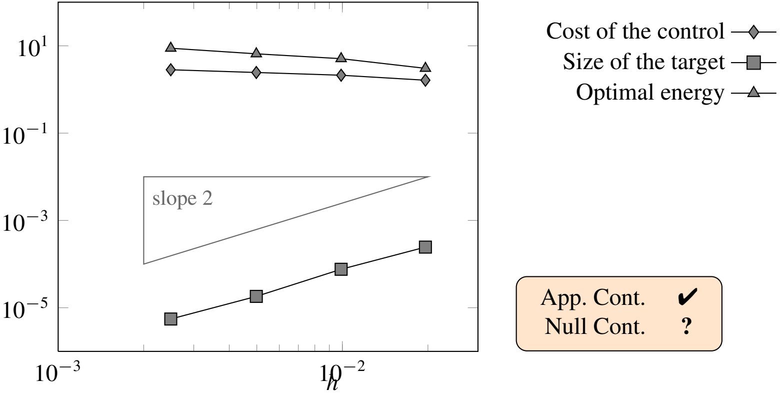
$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 3 : $\mathcal{O}_2$ and $\mathcal{O}_3$ are located in the same connected component of $\Omega \setminus \omega$

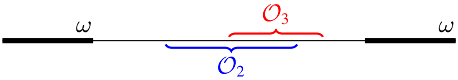


$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 4.1 : $\mathcal{O}_2 =]0.35, 0.65[, \mathcal{O}_3 =]0.5, 1/\sqrt{2}[, \omega =]0, 0.2[\cup]0.8, 1.0[$

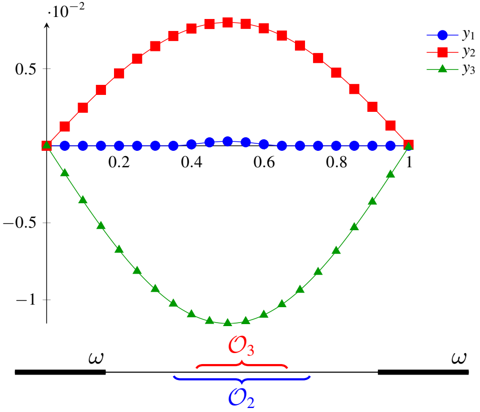


App. Cont. ✔
Null Cont. ?



$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 4.2 : $\mathcal{O}_2 =]0.35, 0.65[$, $\mathcal{O}_3 =]0.4, 0.6[$, $\omega =]0, 0.2[\cup]0.8, 1[$



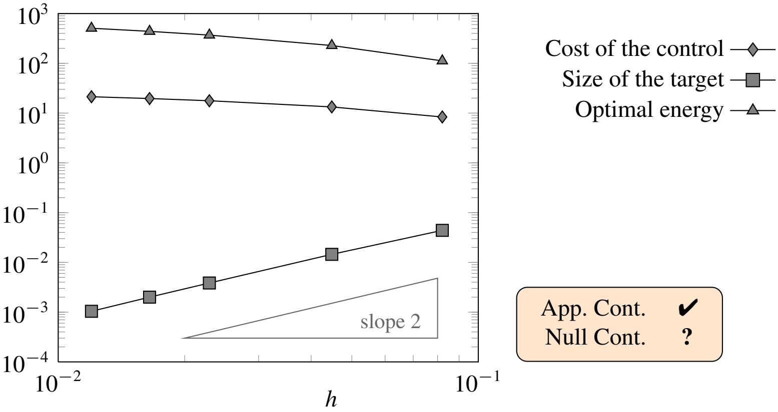
$$\partial_t y - 0.05\Delta y = 1_{]0.3,0.9[\times]0.2,0.8[}v,$$
$$y(0,x) = \sin(2\pi x_1)\sin(\pi x_2), \quad \text{and} \quad y_F(x) = -0.4\sin(\pi x_1)\sin(2\pi x_2).$$

$$\Omega =]-1,1[\times]0,1[, \quad \omega =]0.75,1[\times]0,1[.$$

$$\partial_t y - \partial_{x_1}^2 y - x_1^2 \partial_{x_2}^2 y = 1_\omega v,$$

$$\Omega =]-1, 1[\times]0, 1[, \quad \omega =]0.75, 1[\times]0, 1[.$$

$$\partial_t y - \partial^2_{x_1} y - x_1^2 \partial^2_{x_2} y = 1_\omega v,$$



App. Cont.	✔
Null Cont.	?

$$\Omega =]-1,1[\times]0,1[, \quad \omega =]0.75,1[\times]0.6,1[.$$

$$\partial_t y - \partial^2_{x_1} y - x_1^2 \partial^2_{x_2} y = 1_\omega v,$$

SUMMARY

- In the PDE world
  - Many **standard** results in controllability theory can be deduced from the analysis of the penalized HUM approach.
  - The penalized HUM approach always converge towards *something* as the penalization parameter tends to 0.
- In the discrete world
  - Necessity to relate the penalization parameter to discretisation parameters in a clever way.
  - Analysis of uniform null-controllability properties with respect to δt and/or h for semi/fully discrete problems.
  - Associated relaxed observability inequalities.
  - We may use numerical simulations to investigate open problems.
  - Even for non controllable problems, the numerical method applies and gives interesting results.

PERSPECTIVES

- Extend our analysis in the discrete setting to other cases
  - Non symmetric scalar operators.
  - Parabolic systems with few controls.
  - Boundary control problems.
  - Analysis for other space discretizations (Finite Volume, Finite Element, ...)
- From a computational point of view
  - A deeper understanding of HUM operators $\rightsquigarrow$ preconditioning methods.
  - More suitable solvers than standard Conjugate Gradient ?