Full discretization of distributed control problems for parabolic equations

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NOTATIONS

$$(S)\begin{cases}\partial_t y + \mathcal{A}y = \mathcal{B}v & \text{in } (0,T),\\ y(0) = y^0\end{cases}$$

with $T > 0$, E and U two Hilbert spaces, $y \in L^2(]0,T[,E)$, $\mathcal{A} : D(\mathcal{A}) \subset E \mapsto E$ is some "elliptic" unbounded operator, $\mathcal{B} : U \mapsto E$ a **bounded** operator, and $v \in L^2(]0,T[,U)$ is the control.

$$\text{Cost of the control : } \|v\|_{L^2(0,T;U)} = \left(\int_0^T \|v(t)\|_U^2\,dt\right)^{\frac{1}{2}}.$$

APPROXIMATE CONTROL PROBLEM

For $y_T \in E$ and $\beta > 0$ given, can we find $v \in L^2(0,T;U)$ such that the solution y to (S) satisfies $\|y(T) - y_T\|_E \leq \beta$?

NULL CONTROL PROBLEM

Can we find $v \in L^2(]0,T[;U)$ such that the solution y to (S) satisfies $y(T) = 0$?

(Lebeau-Robbiano, '95) (Fursikov-Imanuvilov, '96)

THE 1D HEAT EQUATION

$$(S)\begin{cases}\partial_t y - \partial_x(\gamma(x)\partial_x y) = 1_\omega v & \text{in } (0,T)\times\Omega,\\ y = 0 & \text{on } (0,T)\times\partial\Omega,\\ y(0) = y^0 \end{cases}$$

that is $E = L^2(\Omega)$, $\mathcal{A} = -\partial_x(\gamma(x)\partial_x .)$, $U = L^2(\Omega)$, $\mathcal{B} = 1_\omega$ with $\omega \subset \Omega$.

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1D PARABOLIC SYSTEMS

$$(S)\begin{cases}\partial_t y_i - \partial_x(\gamma_i(x)\partial_x y_i) + \sum_{j=1}^n \alpha_{ij}(x) y_j = 1_{\omega_i} B_i(x) v & \text{in } (0,T)\times\Omega,\\ y_i = 0 & \text{on } (0,T)\times\partial\Omega,\\ y_i(0) = y_i^0\end{cases}$$

$y = (y_1, \dots, y_n)^t$, $E = (L^2(\Omega))^n$,

$$\mathcal{A} = \begin{pmatrix} -\partial_x(\gamma_1(x)\partial_x .) & 0 & \dots & \\ 0 & -\partial_x(\gamma_2(x)\partial_x .) & 0 & \dots \\ & & \ddots & \\ & \dots & 0 & -\partial_x(\gamma_n(x)\partial_x .)\end{pmatrix} + \begin{pmatrix}\alpha_{ij}(x)\end{pmatrix}_{ij},$$

$$U = (L^2(\Omega))^p,\ B_i(x) \in \mathcal{M}_{1,p}(\mathbb{R}),\ \mathcal{B} = \begin{pmatrix} 1_{\omega_1} B_1(x) \\ \vdots \\ 1_{\omega_n} B_n(x)\end{pmatrix}.$$

THE SAME IN MULTI-D

From now on, let us only consider the target $y_T = 0$.

HUM idea : Given $\varepsilon > 0$, minimize the functional

$$F_\varepsilon : v \in L^2(]0,T[,U) \mapsto \frac{1}{2}\int_0^T \|v(t)\|_U^2 + \frac{1}{2\varepsilon}\|y_v(T)\|_E^2.$$

Dual problem : Find a minimizer of the dual functional

$$J_\varepsilon : q_F \in E \mapsto \frac{1}{2}\int_0^T \|\mathcal{B}^* q(t)\|_U^2 + \frac{\varepsilon}{2}\|q_F\|_E^2 + (y_0, q(0))_E,$$

where $t \mapsto q(t)$ is the solution to the backward problem

$$-\partial_t q + \mathcal{A}^* q = 0, \quad q(T) = q_F.$$

General statement - convergence of the penalty method

For any $\varepsilon > 0$, J_ε has a unique minimizer $q_{F,\varepsilon}$.

If we assume that the problem (S) is approximately controllable, then the control $v_\varepsilon = \mathcal{B}^* q_\varepsilon$ for our parabolic problem leads to a solution such that $\|y_{v_\varepsilon}(T)\|_E \to 0$ when $\varepsilon \to 0$.

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DUAL PROBLEM : Find a minimizer of the dual functional

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NULL-CONTROLLABLE SYSTEMS

If we assume that the following observability inequality for the adjoint pb holds :

$$\|q(0)\|_E^2 \leq C_{\text{obs}}^2 \int_0^T \|\mathcal{B}^* q(t)\|_U^2 \, dt,$$

then we have $\|y_{v_\varepsilon}(T)\|_E \leq C_{\text{obs}}\sqrt{\varepsilon}, \quad v_\varepsilon \xrightarrow[\varepsilon\to 0]{} v,$ in $L^2(0,T;U)$, where v is the (unique) null-control of minimal $L^2(0,T;U)$ norm (the so-called HUM control).

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(Labbé-Trélat, '06) (B.-Hubert-Le Rousseau, '10)

For any $h > 0$ (supposed to be some space discretization parameter) :

- $(E_h, (\cdot,\cdot)_h)$ euclidean space, with norm $|\cdot|_h$.
- $\mathcal{M}_h, \mathcal{A}_h \in L(E_h, E_h)$ which are SDP in $(E_h, (\cdot,\cdot)_h)$.
  In the FE framework : $\mathcal{M}_h$ is the mass matrix, $\mathcal{A}_h$ the rigidity matrix.

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- Associated scalar products and norms

$$\forall x, y \in E_h, \ \ \langle x, y\rangle_h = (\mathcal{M}_h x, y)_h\,, \ \ \|x\|_h = \langle x, x\rangle_h^{\frac{1}{2}} = \left|\mathcal{M}_h^{\frac{1}{2}} x\right|_h.$$

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  In the FE framework : $\|.\|_h$ is the L^2-norm .
- Another Euclidean space $(U_h, [\cdot,\cdot]_h)$, with norm $[\![\cdot]\!]_h$.
- A linear operator $\mathcal{B}_h : U_h \to E_h$, and $\mathcal{B}_h^\star$ its adjoint :

$$\forall v \in U_h, \forall x \in E_h, \ \ (\mathcal{B}_h v, x)_h = [\mathcal{B}_h^\star x, v]_h\,.$$

- We shall assume that there exists $C > 0$ such that

$$[\![\mathcal{B}_h^\star x]\!]_h \leq C\,\|x\|_h\,, \ \ \forall h > 0, \forall x \in E_h,$$

  $\rightsquigarrow$ our analysis does not include boundary controls !

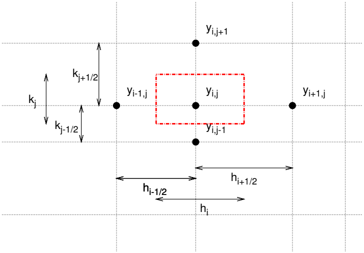
Examples for the heat equation 1/3

$$\Omega=(0,1)^2,\ E=L^2(\Omega),\ \mathcal{A}=-\mathrm{div}\left(\begin{pmatrix}\gamma^1(x) & 0\\ 0 & \gamma^2(x)\end{pmatrix}\nabla.\right),\ \omega\subset\Omega,\ U=L^2(\omega)$$

- $E_h=\mathbb{R}^N$, $N=n_1\times n_2$ the total number of discretization points

$$(x,y)_h=\sum_{i,j} h_i k_j x_{i,j} y_{i,j},$$

$$h_i=(h_{i+\frac12}+h_{i-\frac12})/2,$$
$$k_j=(k_{j+\frac12}+k_{j-\frac12})/2.$$



- $U_h=\mathbb{R}^k$, k being the number of discretization cells which intersect the control domain ω equipped with the same inner product as E_h.
- $\mathcal{A}_h\in M_N(\mathbb{R})$ is the classical 5-diagonal matrix given by

$$(\mathcal{A}_h y)_{i,j}=-\frac{\gamma^1_{i+\frac12,j}\dfrac{y_{i+1,j}-y_{i,j}}{h_{i+\frac12}}-\gamma^1_{i-\frac12,j}\dfrac{y_{i,j}-y_{i-1,j}}{h_{i-\frac12}}}{h_i}-\frac{\gamma^2_{i,j+\frac12}\dfrac{y_{i,j+1}-y_{i,j}}{k_{j+\frac12}}-\gamma^2_{i,j-\frac12}\dfrac{y_{i,j}-y_{i,j-1}}{k_{j-\frac12}}}{k_j},$$

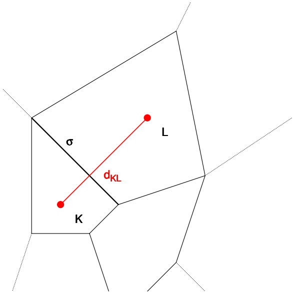
- $\mathcal{M}_h\in M_N(\mathbb{R})$ is the identity matrix : No mass matrix in FD schemes
- $\mathcal{B}_h\in M_{N,k}(\mathbb{R})$ is the rectangle matrix corresponding to the natural embedding of ω in Ω.

$\Omega \subset \mathbb{R}^2$ polygonal, $E = L^2(\Omega)$, $\mathcal{A} = -\gamma\Delta$, $\omega \subset \Omega$, $U = L^2(\omega)$

- $E_h = \mathbb{R}^{\mathcal{T}}$, where $\mathcal{T} = (\kappa)_{\kappa\in\mathcal{T}}$ is an admissible set of polygonal cells, $N = |\mathcal{T}|$.

$$(x, y)_h = \sum_{\mathcal{K}\in\mathcal{T}} x_{\mathcal{K}} y_{\mathcal{K}}.$$

- The mass matrix $\mathcal{M}_h \in M_N(\mathbb{R})$ is diagonal ; its entries are the volumes $|\kappa|$ of each $\kappa \in \mathcal{T}$.



- $\mathcal{A}_h \in M_N(\mathbb{R})$ is the finite volume matrix defined with

$$(\mathcal{A}_h y)_{\mathcal{K}} = \gamma \sum_{\mathcal{L}\in\mathcal{N}_{\mathcal{K}}} |\sigma| \frac{y_{\mathcal{K}} - y_{\mathcal{L}}}{d_{\mathcal{K}\mathcal{L}}}.$$

- $U_h = \mathbb{R}^{\mathcal{T}_\omega}$, $\mathcal{T}_\omega$ is the subset of $\mathcal{T}$ form with the cells which intersect the control domain ω, equipped with the inner product defined by $\mathcal{M}_h$.
- $\mathcal{B}_h \in M_{N,k}(\mathbb{R})$ is the rectangle matrix corresponding to the natural embedding of $\mathcal{T}_\omega$ into $\mathcal{T}$.

Let $X_h \subset H_0^1(\Omega)$, and $Y_h \subset L^2(\Omega)$ be finite dimensional spaces and $(\phi_i^h)_i \subset X_h$, $(\psi_j^h)_j \subset Y_h$ two basis of these spaces.

- $E_h = \mathbb{R}^{\dim X_h}$, the elements in E_h being the coordinates in the basis, $(\cdot,\cdot)_h$ is the usual Euclidean inner product.
- $U_h = \mathbb{R}^{\dim Y_h}$, the elements in U_h representing the coordinates of elements in Y_h in the basis, $[\cdot,\cdot]_h$ is the usual Euclidean inner product.
- The matrix $\mathcal{M}_h \in M_N(\mathbb{R})$ is the mass matrix associated with $(\phi_i^h)_i$. Its entries are $\displaystyle\int_\Omega \phi_i^h \phi_j^h \, dx$.
- The matrix $\mathcal{B}_h \in M_{N,k}(\mathbb{R})$ is the matrix whose entries are $\displaystyle\int_\omega \phi_i^h \psi_j^h \, dx$.
- The matrix $\mathcal{A}_h \in M_N(\mathbb{R})$ is the rigidity matrix associated with the diffusion operator. Its entries are $\displaystyle\int_\Omega \gamma(x)\nabla\phi_i^h \cdot \nabla\phi_j^h \, dx$.

Mass lumping technique :

The scheme can be slightly modified by replacing $\mathcal{M}_h$ by a diagonal matrix containing the sum of the entries in each row of $\mathcal{M}_h$

$$\Rightarrow \text{avoids the computation of } \mathcal{M}_h^{-1}.$$

THE SEMI-DISCRETE PARABOLIC PROBLEM

$$(S)\begin{cases}\partial_t y + \mathcal{A}y = \mathcal{B}v,\\ y(0)=y_0,\end{cases} \quad \Longrightarrow (S_h)\begin{cases}\mathcal{M}_h\partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h,\\ y_h(0)=y_{0,h},\end{cases}$$

SIMPLIFICATION IN THIS TALK : Mass matrix $\mathcal{M}_h = \mathrm{Id}$.

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FACTS AND QUESTIONS

- The semi-discrete problem (S_h) can be non controllable even if (S) is. Indeed, it may exists some eigenfunctions ψ_h of $\mathcal{A}_h$ such that

  $$\mathcal{B}_h^\star \psi_h = 0.$$

  Such an initial data can not be controlled. **(Kavian '01, Zuazua '03)**
- It is certainly a **theoretical difficulty** : what can we do to overcome the problem ?
- Is it an actual difficulty in practice ?

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SOME PRECISIONS

- Given $y_{0,h} \in E_h$, $\varepsilon > 0$, let us minimize (recall that $\dim E_h < +\infty$)

$$J_{\varepsilon,h} : q_F \in E_h \longmapsto \frac{1}{2}\int_0^T [\![\mathcal{B}_h^\star q_h(t)]\!]_h^2 + \frac{\varepsilon}{2}\|q_F\|_h^2 + \langle y_{0,h}, q_h(0)\rangle_h ,$$

where $t \mapsto q_h(t) \in E_h$ is the solution to $-\partial_t q_h + \mathcal{A}_h^* q_h = 0, \quad q_h(T) = q_F.$

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$$(S)\begin{cases}\partial_t y + \mathcal{A}y = \mathcal{B}v,\\ y(0) = y_0,\end{cases} \quad\Longrightarrow (S_h)\begin{cases}\partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h,\\ y_h(0) = y_{0,h},\end{cases}$$

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where $t \mapsto q_h(t) \in E_h$ is the solution to $-\partial_t q_h + \mathcal{A}_h^* q_h = 0, \quad q_h(T) = q_F$.

- We let $v_{\varepsilon,h} = \mathcal{B}_h^* q_{\varepsilon,h}(t)$ and $t \mapsto y_{h,\varepsilon}(t)$ the associated solution to (S_h).
  - For $h > 0$ fixed, we may have

$$\lim_{\varepsilon\to 0} \|y_{h,\varepsilon}(T)\|_h = +\infty, \quad \text{and} \quad \lim_{\varepsilon\to 0} \|v_{h,\varepsilon}\|_{L^2(0,T;U_h)} = +\infty.$$

  - We can hope that for some $C > 0$ and any $\varepsilon > 0$, there exists $h_\varepsilon^* > 0$

$$\text{for any } h < h_\varepsilon^*, \quad \|y_{h,\varepsilon}(T)\|_h \leq C\sqrt{\varepsilon}\|y_{0,h}\|_h,$$

    and that $(v_{h,\varepsilon})_h$ converges (in some sense) towards v_ε if $(y_{0,h})_h$ converges in some sense towards y_0.

- For $h > 0$ fixed, we may have (for suitable $y_{0,h}$)

$$\lim_{\varepsilon \to 0} \|y_{h,\varepsilon}(T)\|_h = +\infty, \text{ and } \lim_{\varepsilon \to 0} \|v_{h,\varepsilon}\|_{L^2(0,T;U_h)} = +\infty.$$

- We can hope that for some $C > 0$ and any $\varepsilon > 0$, there exists $h^*_\varepsilon > 0$

$$\text{for any } h < h^*_\varepsilon, \;\; \|y_{h,\varepsilon}(T)\|_h \leq C\sqrt{\varepsilon}\|y_{0,h}\|_h,$$

  and that $(v_{h,\varepsilon})_h$ converges (in some sense) towards v_ε if $(y_{0,h})_h$ converges in some sense towards some y_0.

FIRST SERIES OF QUESTIONS

1. If we are interested in the **approximate control problem** : Is it possible to give an estimate of h^*_ε ?
2. If we are interested in the **null control problem** : Is it possible to choose $\varepsilon > 0$ as a function of h : $\varepsilon = \phi(h)$ such that

$$\lim_{h \to 0} \|y_{h,\phi(h)}(T)\|_h = 0, \;\; \|v_{h,\phi(h)}\|_{L^2(0,T;U_h)} \leq C,$$

   and can we estimate those quantities ?
3. If many such $h \mapsto \phi(h)$ exist, how do I choose one in practice ?

In the sequel of the talk I will assume the following

ASSUMPTION (UNIFORM DISCRETE LEBEAU-ROBBIANO INEQUALITY)

There exists $h_0 > 0$, $\alpha \in [0,1)$, $\beta > 0$, and $\kappa, \ell > 0$ such that for any $h < h_0$ and for any $(a_j)_j \in \mathbb{R}^{\mathbb{N}}$, we have

$$\Big\| \sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \Big\|_h^2 \leq \kappa e^{\kappa \mu^\alpha} \left[\!\left[\mathcal{B}_h^\star \Big(\sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \Big) \right]\!\right]_h^2, \quad \forall \mu < \frac{\ell}{h^\beta}, \qquad (\mathcal{H}_{\alpha,\beta})$$

where $(\mu_{j,h})_j$ are the eigenvalues of $\mathcal{A}_h$ and $(\psi_{j,h})_j$ the corresponding orthonormal eigenvectors.

FUNDAMENTAL REMARK

For dimension reasons, such an inequality **can not** be true for any $\mu > 0$.

SOME RESTRICTIONS OF THE METHOD UP TO NOW

- Only the symmetric case $\mathcal{A}_h = \mathcal{A}_h^*$.
- No boundary control.
- Time-independent coefficients.

NOTATION SIMPLIFICATION : $y_h \to y$, $v_h \to v$, $\mu_{j,h} \to \mu_j$, ...

(B.-Hubert-Le Rousseau '09,'10)

We proved that the uniform discrete Lebeau-Robbiano inequality $(\mathcal{H}_{\alpha,\beta})$ holds for

- Finite difference schemes on regular Cartesian meshes in any dimension.
- A scalar elliptic operator $\mathcal{A}$ with diagonal diffusion tensor (possibly depending smoothly on x).
- Distributed control problem $\mathcal{B}_h = 1_\omega$.
- We obtain :
  - $\alpha = 1/2$ (i.e. the constant is $\sim e^{\sqrt{\mu}}$).
  - $\beta = 2$ (related to α and to the order of the differential operator).

MAIN TOOL : **Global discrete elliptic Carleman estimates** with precise dependence of the large Carleman parameters with respect to the discretization parameter h.

PERSPECTIVES :

The same kind of property *should be true* in more general situations :

- For non-symmetric $\mathcal{A}_h$ (heat equation with first order terms, parabolic systems with non symmetric coupling, etc ...).
- Finite volume schemes.
- Galerkin discretizations.

To our knowledge, these are still open problems.

(Carthel-Glowinski-Lions, '94) (Glowinski-Lions, '94)

$$(S_h) \begin{cases} \partial_t y + \mathcal{A}_h y = \mathcal{B}_h v, \\ y_h(0) = y_0. \end{cases}$$

Consider the approximate control problem for (S_h) by penalty introducing

$$q_F \in E_h \mapsto J_{\varepsilon,h}(q_F) = \frac{1}{2}\int_0^T [\![\mathcal{B}_h^\star q(t)]\!]_h^2 \, dt + \frac{\varepsilon}{2}\|q_F\|_h^2 + \langle y_0, q(0)\rangle_h .$$

We denote by $q_{F,\varepsilon,h}$ its minimizer and $t \mapsto q_{\varepsilon,h}(t)$ the associated adjoint state.

THEOREM

Assume that the uniform discrete Lebeau-Robbiano inequality $(\mathcal{H}_{\alpha,\beta})$ holds, then there exists $h_0 > 0$ and constants $C, C_{\text{obs}} > 0$ such that :

- *For any $h < h_0$, and $\varepsilon > e^{-C/h^\beta}$, the control $v_{h,\varepsilon}(t) = \mathcal{B}_h^\star q_{h,\varepsilon}(t)$ is such that*

$$\|v_{h,\varepsilon}\|_{L^2(0,T;U_h)} \leq C_{\text{obs}}, \quad \textit{and} \quad \|y_{h,\varepsilon}(T)\|_h \leq C_{\text{obs}}\sqrt{\varepsilon}.$$

ASSOCIATED RELAXED OBSERVABILITY INEQUALITY

$$\left\{ \begin{array}{c} \forall h < h_0, \forall \varepsilon > e^{-C/h^\beta} \\ \forall \varepsilon < \varepsilon_0, \forall h < \frac{C'}{|\log \varepsilon|^{1/\beta}} \end{array} \right\}, \forall q_F \in E_h, \ \|q(0)\|_h^2 \leq C_{\text{obs}}^2 \left(\int_0^T [\![\mathcal{B}_h^\star q(t)]\!]_h^2 \, dt + \varepsilon \|q_F\|_h^2 \right).$$

COMPUTATION OF AN APPROXIMATE CONTROL FOR $\varepsilon > 0$ FIXED :

The sequence $(v_{h,\varepsilon})_h$ converges towards the control v_ε solution of the approximate penalized control problem for the initial PDE (S).

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COMPUTATION OF A NULL-CONTROL :

If we choose a function $h \mapsto \phi(h)$ such that $\phi(h) > e^{-C/h^\beta}$ for any h, then the sequence $(v_{h,\phi(h)})_h$ converges, at least weakly, towards a null-control of the initial PDE (S) and we have

$$\left\| y_{h,\phi(h)}(T) \right\|_h \leq C_{\text{obs}} \sqrt{\phi(h)}, \quad \forall 0 < h < h_0.$$

- Recall that, in general, a null-control for (S_h) does not exist

  $\Rightarrow \varepsilon = 0$ is meaningless.
- Taking $\varepsilon = \phi(h)$ exponentially small is theoretically possible but

  this is not reasonable and in fact completely useless.
- In practice, choosing $\phi(h) = h^{2p}$ for some integer p related to the approximation order p of the scheme under study is sufficient.

  See some numerical evidences above

We have seen that some uniform approximate/null controllability properties hold for

$$(S_h)\begin{cases}\partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h,\\ y_h(0) = y_{0,h}.\end{cases}$$

WHAT ABOUT TIME DISCRETIZATION OF SUCH A SYSTEM ?

- We study **unconditionally stable schemes** : the implicit Euler scheme and the Crank-Nicolson scheme (in fact any θ-scheme with $\theta \in [1/2, 1]$).

$$(S_{h,\delta t})\begin{cases} y^0 = y_0,\\ \dfrac{y^{n+1}-y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta)y^n) = \mathcal{B}_h v^{n+1},\ \forall n \in [\![0, M-1]\!]\end{cases}$$

- We show that most of the results of the semi-discrete situation holds for fully-discrete systems uniformly in δt and h (provided δt is not too large with respect to h, *this will be made precise below*).
- Finally, we show that, $h > 0$ being fixed, the full discrete control $v_{h,\delta t}$ we will construct converges towards the semi-discrete control v_h at first or second order in time.

(Zheng, '08), (Ervedoza-Valein, '10)

THE PRIMAL OPTIMIZATION PROBLEM : Minimize the following functional

$$F_{\varepsilon,h,\delta t} : v \in U_h^M \longmapsto \frac{1}{2}\sum_{n=1}^{M} \delta t \, [\![v^n]\!]_h^2 + \frac{1}{2\varepsilon} \|\mathcal{L}(y_0, v)\|_h^2 ,$$

where $\mathcal{L}(y_0, v) = \mathcal{L}_0 y_0 + \mathcal{L}_v v$ is the value of y^M for the corresponding solution of

$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta) y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

REMARKS

- The definition of $\mathcal{L}(y_0, v)$ has to be adapted to the time discretisation scheme.
  Example : BDF2 method **(Glowinski-Lions,'94)**

$$\begin{cases} y^0 = y_0, \\ \dfrac{y^1 - y^0}{\delta t} + \dfrac{2}{3}\mathcal{A}_h y^1 + \dfrac{1}{3}\mathcal{A}_h y^0 = \dfrac{2}{3}\mathcal{B}_h v^1, \\ \dfrac{3y^{n+1} - 2y^n + \frac{1}{2}y^{n-1}}{\delta t} + \mathcal{A}_h y^{n+1} = \mathcal{B}_h v^{n+1}, \ \forall n \in [\![1, M-2]\!], \\ y^M = 2y^{M-1} - y^{M-2}. \end{cases}$$

- Other choices for the full-discrete $L^2(]0,T[, U_h)$ norm could be more suitable.

THE PRIMAL OPTIMIZATION PROBLEM : Minimize the following functional

$$F_{\varepsilon,h,\delta t} : v \in U_h^M \longmapsto \frac{1}{2}\sum_{n=1}^{M} \delta t \, [\![v^n]\!]_h^2 + \frac{1}{2\varepsilon} \, \|\mathcal{L}(y_0, v)\|_h^2 ,$$

where $\mathcal{L}(y_0, v) = \mathcal{L}_0 y_0 + \mathcal{L}_v v$ is the value of y^M for the corresponding solution of

$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta)y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

DUAL OPTIMIZATION PROBLEM : General duality theory gives

$$J_{\varepsilon,h,\delta t} : q_F \in E_h \longmapsto \frac{1}{2}\sum_{n=1}^{M} \delta t \, \left[\!\left[(\mathcal{L}_v^* q_F)^n\right]\!\right]_h^2 + \frac{\varepsilon}{2} \, \|q_F\|_h^2 + \langle y_0, \mathcal{L}_0^* q_F \rangle_h .$$

$$\text{Argmin}\, F_{\varepsilon,h,\delta t} = \mathcal{L}_v^* \left(\text{Argmin}\, J_{\varepsilon,h,\delta t}\right).$$

ASSOCIATED OBSERVABILITY INEQUALITY

$$\|\mathcal{L}_0^* q_F\|_h^2 \leq C_{\text{obs}}^2 \sum_{n=1}^{M} \delta t \, \left[\!\left[(\mathcal{L}_v^* q_F)^n\right]\!\right]_h^2 .$$

We defined $\mathcal{L}(y_0, v) = \mathcal{L}_0 y_0 + \mathcal{L}_v v = y^M$, where $(y^n)_n$ is given by

$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta) y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

• For any $q_F \in E_h, y_0 \in E_h, v \in (U_h)^M$ we must have

$$\langle \mathcal{L}(y_0, v), q_F \rangle_h = \langle \mathcal{L}_0 y_0, q_F \rangle_h + \langle \mathcal{L}_v v, q_F \rangle_h = \langle y_0, \mathcal{L}_0^* q_F \rangle_h + \sum_{n=1}^{M} \delta t \left[(\mathcal{L}_v^* q_F)^n, v^n \right]_h .$$

- By adding any element of $\ker \mathcal{B}_h$ to any v^n, you do not change $\mathcal{L}(y_0, v)$:

$$\Longrightarrow (\mathcal{L}_v^* q_F)^n \in (\ker \mathcal{B}_h)^{\perp} = \operatorname{Im} \mathcal{B}_h^\star, \quad \forall n \in [\![1, M]\!].$$

- We thus write $(\mathcal{L}_v^* q_F)^n = \mathcal{B}_h^\star q^n, \forall n \in [\![1, M]\!]$.

We defined $\mathcal{L}(y_0, v) = \mathcal{L}_0 y_0 + \mathcal{L}_v v = y^M$, where $(y^n)_n$ is given by

$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta) y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

- For any $q_F \in E_h, y_0 \in E_h, v \in (U_h)^M$ we look for $(\mathcal{L}_v^* q_F)^n = \mathcal{B}_h^\star q^n$ satisfying

$$\langle \mathcal{L}(y_0, v), q_F \rangle_h = \langle y_0, \mathcal{L}_0^* q_F \rangle_h + \sum_{n=1}^{M} \delta t \, [\mathcal{B}_h^\star q^n, v^n]_h .$$

We defined $\mathcal{L}(y_0, v) = \mathcal{L}_0 y_0 + \mathcal{L}_v v = y^M$, where $(y^n)_n$ is given by

$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta)y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

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$$\langle \mathcal{L}(y_0, v), q_F \rangle_h = \langle y_0, \mathcal{L}_0^* q_F \rangle_h + \sum_{n=1}^{M} \delta t \, \langle q^n, \mathcal{B}_h v^n \rangle_h \, .$$

We defined $\mathcal{L}(y_0, v) = \mathcal{L}_0 y_0 + \mathcal{L}_v v = y^M$, where $(y^n)_n$ is given by

$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta) y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

- For any $q_F \in E_h, y_0 \in E_h, v \in (U_h)^M$ we look for $(\mathcal{L}_v^* q_F)^n = \mathcal{B}_h^\star q^n$ satisfying

$$\langle \mathcal{L}(y_0, v), q_F \rangle_h = \langle y_0, \mathcal{L}_0^* q_F \rangle_h + \sum_{n=1}^{M} \delta t \, \langle q^n, \mathcal{B}_h v^n \rangle_h \,.$$

By introducing the solution $(y^n)_n$ to $(S_{h,\delta t})$ we get

$$\left\langle y^M, q_F \right\rangle_h = \langle y_0, \mathcal{L}_0^* q_F \rangle_h + \sum_{n=1}^{M} \left\langle q^n, y^n - y^{n-1} \right\rangle_h$$
$$+ \sum_{n=1}^{M} \delta t \left\langle q^n, \mathcal{A}_h(\theta y^n + (1-\theta) y^{n-1}) \right\rangle_h .$$

We defined $\mathcal{L}(y_0, v) = \mathcal{L}_0 y_0 + \mathcal{L}_v v = y^M$, where $(y^n)_n$ is given by

$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta)y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

- For any $q_F \in E_h, y_0 \in E_h, v \in (U_h)^M$ we look for $(\mathcal{L}_v^* q_F)^n = \mathcal{B}_h^\star q^n$ satisfying

$$\langle \mathcal{L}(y_0, v), q_F \rangle_h = \langle y_0, \mathcal{L}_0^* q_F \rangle_h + \sum_{n=1}^{M} \delta t \, \langle q^n, \mathcal{B}_h v^n \rangle_h .$$

By introducing the solution $(y^n)_n$ to $(S_{h,\delta t})$ we get

$$\left\langle y^M, q_F \right\rangle_h = \left\langle y_0, \mathcal{L}_0^* q_F - q^1 \right\rangle_h + \sum_{n=1}^{M-1} \left\langle q^n - q^{n+1}, y^n \right\rangle_h + \left\langle y^M, q^M \right\rangle_h$$
$$+ \sum_{n=1}^{M} \delta t \left\langle q^n, \mathcal{A}_h(\theta y^n + (1-\theta) y^{n-1}) \right\rangle_h .$$

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$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta)y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

- For any $q_F \in E_h, y_0 \in E_h, v \in (U_h)^M$ we look for $(\mathcal{L}_v^* q_F)^n = \mathcal{B}_h^\star q^n$ satisfying

$$\langle \mathcal{L}(y_0, v), q_F \rangle_h = \langle y_0, \mathcal{L}_0^* q_F \rangle_h + \sum_{n=1}^{M} \delta t \, \langle q^n, \mathcal{B}_h v^n \rangle_h .$$

By introducing the solution $(y^n)_n$ to $(S_{h,\delta t})$ we get

$$\left\langle y^M, q_F \right\rangle_h = \left\langle y_0, \mathcal{L}_0^* q_F - q^1 \right\rangle_h + \sum_{n=1}^{M-1} \left\langle q^n - q^{n+1}, y^n \right\rangle_h + \left\langle y^M, q^M \right\rangle_h$$

$$+ \sum_{n=1}^{M} \delta t \left\langle q^n, \mathcal{A}_h(\theta y^n + (1-\theta) y^{n-1}) \right\rangle_h .$$

We defined $\mathcal{L}(y_0, v) = \mathcal{L}_0 y_0 + \mathcal{L}_v v = y^M$, where $(y^n)_n$ is given by

$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta)y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

- For any $q_F \in E_h, y_0 \in E_h, v \in (U_h)^M$ we look for $(\mathcal{L}_v^* q_F)^n = \mathcal{B}_h^\star q^n$ satisfying

$$\langle \mathcal{L}(y_0, v), q_F \rangle_h = \langle y_0, \mathcal{L}_0^* q_F \rangle_h + \sum_{n=1}^{M} \delta t \, \langle q^n, \mathcal{B}_h v^n \rangle_h .$$

By introducing the solution $(y^n)_n$ to $(S_{h,\delta t})$ we get

$$\left\langle y^M, q_F \right\rangle_h = \left\langle y_0, \mathcal{L}_0^* q_F - q^1 \right\rangle_h + \sum_{n=1}^{M-1} \left\langle q^n - q^{n+1}, y^n \right\rangle_h + \left\langle y^M, q^M \right\rangle_h$$
$$+ \sum_{n=1}^{M-1} \delta t \left\langle y^n, \mathcal{A}_h(\theta q^n + (1-\theta) q^{n+1}) \right\rangle_h + \delta t \theta \left\langle y^M, \mathcal{A}_h q^M \right\rangle_h + \delta t (1-\theta) \left\langle y^0, \mathcal{A}_h q^1 \right\rangle_h .$$

We defined $\mathcal{L}(y_0, v) = \mathcal{L}_0 y_0 + \mathcal{L}_v v = y^M$, where $(y^n)_n$ is given by

$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta)y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

- For any $q_F \in E_h, y_0 \in E_h, v \in (U_h)^M$ we look for $(\mathcal{L}_v^* q_F)^n = \mathcal{B}_h^\star q^n$ satisfying

$$\langle \mathcal{L}(y_0, v), q_F \rangle_h = \langle y_0, \mathcal{L}_0^* q_F \rangle_h + \sum_{n=1}^{M} \delta t \, \langle q^n, \mathcal{B}_h v^n \rangle_h .$$

By introducing the solution $(y^n)_n$ to $(S_{h,\delta t})$ we get

$$\left\langle y^M, q_F \right\rangle_h = \left\langle y_0, \mathcal{L}_0^* q_F - q^1 + \delta t(1-\theta)\mathcal{A}_h q^1 \right\rangle_h + \left\langle y^M, q^M + \delta t \theta \mathcal{A}_h q^M \right\rangle_h$$
$$+ \sum_{n=1}^{M-1} \left\langle y^n, q^n - q^{n+1} + \delta t \mathcal{A}_h(\theta q^n + (1-\theta) q^{n+1}) \right\rangle_h .$$

We defined $\mathcal{L}(y_0, v) = \mathcal{L}_0 y_0 + \mathcal{L}_v v = y^M$, where $(y^n)_n$ is given by

$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta) y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

- For any $q_F \in E_h, y_0 \in E_h, v \in (U_h)^M$ we look for $(\mathcal{L}_v^* q_F)^n = \mathcal{B}_h^\star q^n$ satisfying

$$\langle \mathcal{L}(y_0, v), q_F \rangle_h = \langle y_0, \mathcal{L}_0^* q_F \rangle_h + \sum_{n=1}^{M} \delta t \, \langle q^n, \mathcal{B}_h v^n \rangle_h .$$

By introducing the solution $(y^n)_n$ to $(S_{h,\delta t})$ we get

$$\left\langle y^M, q_F \right\rangle_h = \left\langle y_0, \underbrace{\mathcal{L}_0^* q_F - q^1 + \delta t (1-\theta) \mathcal{A}_h q^1}_{=0} \right\rangle_h + \left\langle y^M, \underbrace{q^M + \delta t \theta \mathcal{A}_h q^M}_{=q_F} \right\rangle_h$$
$$+ \sum_{n=1}^{M-1} \left\langle y^n, \underbrace{q^n - q^{n+1} + \delta t \mathcal{A}_h (\theta q^n + (1-\theta) q^{n+1})}_{=0} \right\rangle_h .$$

We defined $\mathcal{L}(y_0, v) = \mathcal{L}_0 y_0 + \mathcal{L}_v v = y^M$, where $(y^n)_n$ is given by

$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta) y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

- For any $q_F \in E_h, y_0 \in E_h, v \in (U_h)^M$ we look for $(\mathcal{L}_v^* q_F)^n = \mathcal{B}_h^\star q^n$ satisfying

$$\langle \mathcal{L}(y_0, v), q_F \rangle_h = \langle y_0, \mathcal{L}_0^* q_F \rangle_h + \sum_{n=1}^{M} \delta t \, \langle q^n, \mathcal{B}_h v^n \rangle_h .$$

CONCLUSION : given $q_F \in E_h$, we solve the following backward θ-scheme-like

$$(S_{h,\delta t}^*) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta) q^{n+1}) = 0, \ \ \forall n \in [\![1, M-1]\!]. \end{cases}$$

then, we have

$$\begin{cases} \mathcal{L}_0^* q_F = q^1 - \delta t (1-\theta) \mathcal{A}_h q^1, \\ (\mathcal{L}_v^* q_F)^n = \mathcal{B}_h^\star q^n, \ \ \forall n \in [\![1, M]\!]. \end{cases}$$

Remark : q_F itself does not directly appears in $\mathcal{L}_v^* q_F$.

The dual functional that we will thus consider is the following

$$J_{\varepsilon,h,\delta t} : q_F \in E_h \longmapsto \frac{1}{2}\sum_{n=1}^{M} \delta t \, [\![\mathcal{B}_h^\star q^n]\!]_h^2 + \frac{\varepsilon}{2}\|q_F\|_h^2 - \Big\langle y_0, q^1 - \delta t(1-\theta)\mathcal{A}_h q^1 \Big\rangle_h,$$

where $(q^n)_n$ is defined by

$$(S^*_{h,\delta t}) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta)q^{n+1}) = 0, \ \ \forall n \in [\![1, M-1]\!]. \end{cases}$$

For $q_F \in E_h$ given, the adjoint problem associated with the time discretisation proposed is given by

$$(S^*_{h,\delta t}) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta) q^{n+1}) = 0, \ \ \forall n \in [\![1, M-1]\!]. \end{cases}$$

THEOREM (USELESS ...)

The fully discrete system $(S_{h,\delta t})$ is controllable if and only if any solution of the adjoint system $(S^*_{h,\delta t})$ satisfies the following observability inequality

$$\underbrace{\left\| q^1 - \delta t (1-\theta) \mathcal{A}_h q^1 \right\|_h^2}_{= \left\| \mathcal{L}_0^* q_F \right\|_h^2} \leq C_{\text{obs}}^2 \sum_{n=1}^{M} \delta t \underbrace{[\![\mathcal{B}_h^\star q^n]\!]_h^2}_{= [\![(\mathcal{L}_v^* q_F)^n]\!]_h^2} .$$

Unfortunately, as we have seen, this does not hold in general.

For $q_F \in E_h$ given, the adjoint problem associated with the time discretisation proposed is given by

$$(S^*_{h,\delta t}) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta) q^{n+1}) = 0, \quad \forall n \in [\![1, M-1]\!]. \end{cases}$$

THEOREM (THE CASE $\theta > 1/2$)

Assume that the uniform discrete L-R inequality $(\mathcal{H}_{\alpha,\beta})$ holds, choose $0 < \gamma \leq \beta$ and $C_T > 0$. For any $\delta t \leq C_T h^\gamma$ the following relaxed observability inequality holds

$$\left\| q^1 - \delta t (1-\theta) \mathcal{A}_h q^1 \right\|_h^2 \leq C_{\text{obs}}^2 \sum_{n=1}^{M} \delta t \, [\![\mathcal{B}_h^\star q^n]\!]_h^2 + C e^{-C/h^\gamma} \|q_F\|_h^2 .$$

Thus, for any such δt, there exists a full-discrete control $v_{h,\delta t}$ s.t.

$$\sum_{n=1}^{M} \delta t \, [\![v^n]\!]_h^2 \leq C_{\text{obs}}^2 \|y_0\|_h^2 , \quad \textit{and} \quad \left\| y^M \right\|_h \leq C_{\text{obs}} e^{-C/h^\gamma} \|y_0\|_h .$$

For $q_F \in E_h$ given, the adjoint problem associated with the time discretisation proposed is given by

$$(S^*_{h,\delta t}) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta) q^{n+1}) = 0, \;\; \forall n \in [\![1, M-1]\!]. \end{cases}$$

THEOREM (THE CRANK-NICOLSON SCHEME - $\theta = 1/2$)

Assume that the uniform discrete L-R inequality $(\mathcal{H}_{\alpha,\beta})$ holds, choose $0 < \gamma \leq \beta$ and $C_T > 0$, and $\delta > 0$. For any $\delta t \leq C_T h^\gamma$ and $\delta t \rho(\mathcal{A}_h) \leq \delta$ the following relaxed observability inequality holds

$$\left\| q^1 - \frac{\delta t}{2} \mathcal{A}_h q^1 \right\|_h^2 \leq C_{\text{obs}}^2 \sum_{n=1}^{M} \delta t \, [\![\mathcal{B}_h^\star q^n]\!]_h^2 + C e^{-C/h^\gamma} \, \|q_F\|_h^2 .$$

Thus, for any such δt, there exists a full-discrete control $v_{h,\delta t}$ s.t.

$$\sum_{n=1}^{M} \delta t \, [\![v^n]\!]_h^2 \leq C_{\text{obs}}^2 \, \|y_0\|_h^2 , \;\; \textit{and} \;\; \left\| y^M \right\|_h \leq C_{\text{obs}} e^{-C/h^\gamma} \, \|y_0\|_h .$$

For $q_F \in E_h$ given, the adjoint problem associated with the time discretisation proposed is given by

$$(S^*_{h,\delta t}) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta) q^{n+1}) = 0, \quad \forall n \in [\![1, M-1]\!]. \end{cases}$$

THEOREM (THE CASE $\theta > 1/2$ - USEFUL STATEMENT)

Assume that the uniform discrete L-R inequality $(\mathcal{H}_{\alpha,\beta})$ holds and let $h \mapsto \phi(h)$ such that $\phi(h) \geq e^{-C/h^\beta}$. For any $\delta t \leq C_T |\log \phi(h)|$ the following relaxed observability inequality holds

$$\left\| q^1 - \delta t(1-\theta)\mathcal{A}_h q^1 \right\|_h^2 \leq C_{\text{obs}}^2 \sum_{n=1}^{M} \delta t \, [\![\mathcal{B}_h^\star q^n]\!]_h^2 + \phi(h) \, \|q_F\|_h^2 .$$

Thus, for any such δt, there exists a full-discrete control $v_{h,\delta t}$ s.t.

$$\sum_{n=1}^{M} \delta t \, [\![v^n]\!]_h^2 \leq C_{\text{obs}}^2 \, \|y_0\|_h^2 , \quad \textit{and} \quad \left\| y^M \right\|_h \leq C_{\text{obs}} \sqrt{\phi(h)} \, \|y_0\|_h .$$

For $q_F \in E_h$ given, the adjoint problem associated with the time discretisation proposed is given by

$$(S^*_{h,\delta t}) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta) q^{n+1}) = 0, \quad \forall n \in [\![1, M-1]\!]. \end{cases}$$

THEOREM (CRANK-NICOLSON - $\theta = 1/2$ - USEFUL (?) STATEMENT)

Assume that the uniform discrete L-R inequality $(\mathcal{H}_{\alpha,\beta})$ holds and let $h \mapsto \phi(h)$ such that $\phi(h) \geq e^{-C/h^\beta}$. For any $\delta t \leq C_T |\log \phi(h)|$ and $\delta t \rho(\mathcal{A}_h) \leq \delta$ the following relaxed observability inequality holds

$$\left\| q^1 - \frac{\delta t}{2} \mathcal{A}_h q^1 \right\|_h^2 \leq C_{\text{obs}}^2 \sum_{n=1}^{M} \delta t \, [\![\mathcal{B}_h^\star q^n]\!]_h^2 + \phi(h) \, \|q_F\|_h^2 .$$

Thus, for any such δt, there exists a full-discrete control $v_{h,\delta t}$ s.t.

$$\sum_{n=1}^{M} \delta t \, [\![v^n]\!]_h^2 \leq C_{\text{obs}}^2 \, \|y_0\|_h^2 , \quad \textit{and} \quad \left\| y^M \right\|_h \leq C_{\text{obs}} \sqrt{\phi(h)} \, \|y_0\|_h .$$

Main idea : adapt the Lebeau-Robbiano original strategy

Step 1 : Use the discrete L.R. inequality to prove controllability of frequency modes less than μ with cost $e^{C\mu^{\alpha}}\|y_0\|_h$.

Step 2 : Construct a suitable full discrete control by a discrete **finite** time slicing procedure :

$$\{0,\ldots,M\} = \bigsqcup_{j=1}^{J}\{M_j',\ldots,M_j'+2M_j\}. \qquad (\star)$$

- Between discrete times M_j' and $M_j'+M_j$:

  Use a control for frequencies less than $2^{j/\alpha}$ (Step 1).
- Between discrete times $M_j'+M_j+1$ and $M_j'+2M_j$:

  Let the system evolve without control and take advantage of the parabolic dissipation since the solution only contains frequencies greater than $2^{j/\alpha}$.

New difficulties

- δt has to be small enough (i.e. M large enough) in order to construct a suitable slicing $(\star)$.
- The full-discrete heat semi-group

$$(\mathrm{Id}+\theta\delta t\mathcal{A}_h)^{-1}(\mathrm{Id}+(1-\theta)\delta t\mathcal{A}_h)$$

do not have the same dissipation properties than the semi-discrete semi-group

$$e^{-\delta t\mathcal{A}_h}.$$

THE θ-SCHEME FOR $\theta > 1/2$

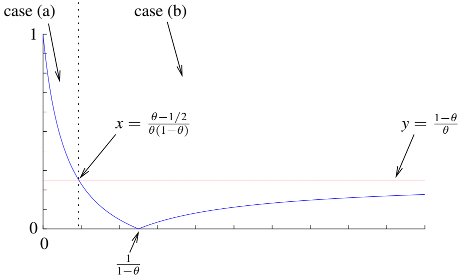
The iteration matrix for the system is

$$\mathcal{C}_{h,\delta t} = (\mathrm{Id} + \theta \delta t \mathcal{A}_h)^{-1} (\mathrm{Id} - (1-\theta)\delta t \mathcal{A}_h).$$

- Let us analyse $\mathrm{Sp}(\mathcal{C}_{h,\delta t})$:

Image of $\mathrm{Sp}(\delta t \mathcal{A}_h)$ through

$$x \mapsto \frac{1-(1-\theta)x}{1+\theta x}$$



- In practice, $\rho(\delta t \mathcal{A}_h) \sim C\frac{\delta t}{h^p}$, for some p (e.g. $p = 2$ for classical FD)

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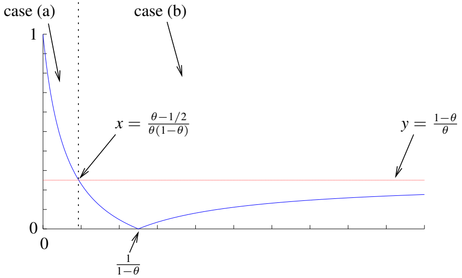

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$\rightsquigarrow$ Case (a) : For $\delta t \times \mu_{i,h}$ less than $\frac{\theta-1/2}{\theta(1-\theta)}$: we have exponential damping.

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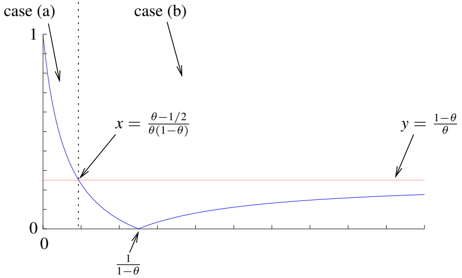


- In practice, $\rho(\delta t\mathcal{A}_h) \sim C\frac{\delta t}{h^p}$, for some p (e.g. $p=2$ for classical FD)

$\rightsquigarrow$ Case (b) : For $\delta t \times \mu_{i,h}$ greater than $\frac{\theta-1/2}{\theta(1-\theta)}$ (possibly $\to +\infty$) the damping factor can be $\sim (1-\theta)/\theta < 1$ but we assumed $\delta t \leq C_T h^\gamma$:

$$\left(\frac{1-\theta}{\theta}\right)^M \leq \left(\frac{1-\theta}{\theta}\right)^{\frac{M\delta t}{C_T h^\gamma}} = e^{-\xi\frac{M\delta t}{h^\gamma}} \sim e^{-\xi\frac{T}{h^\gamma}}\,.$$

THE CRANK-NICOLSON SCHEME ($\theta = 1/2$)

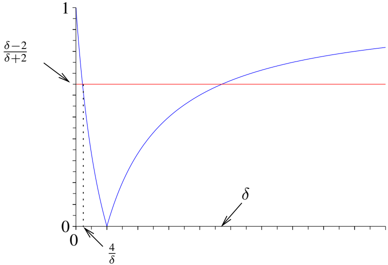
The iteration matrix for the system is

$$\mathcal{C}_{h,\delta t} = \left(\mathrm{Id} - \frac{\delta t}{2}\mathcal{A}_h\right)^{-1}\left(\mathrm{Id} + \frac{\delta t}{2}\mathcal{A}_h\right).$$

- Let us analyse $\mathrm{Sp}(\mathcal{C}_{h,\delta t})$:

Image of $\mathrm{Sp}(\delta t \mathcal{A}_h)$ through

$$x \mapsto \frac{1 - x/2}{1 + x/2}$$



- For large $\delta t \times \mu_{i,h}$, the damping factor can be ~ 1. Here we use

$$\delta t \rho(\mathcal{A}_h) \leq \delta.$$

We thus split the analysis into two cases :

- The case $\delta t \times \mu_{i,h}$ less than $4/\delta$: natural exponential damping
- The case $\delta t \times \mu_{i,h}$ greater than $4/\delta$: damping bounded by $\frac{\delta-2}{\delta+2} < 1$.

We consider a free trajectory of the semi-discrete problem

$$\hat{y}_F = e^{-T\mathcal{A}_h}\hat{y}_0.$$

Problem 1

Starting from any $y_0 \in E_h$, can we drive the solution of the semi-discrete system

$$\partial_t y + \mathcal{A}_h y = \mathcal{B}_h v, \quad y(0) = y_0,$$

to $\hat{y}_F$ at time T ?

$\rightsquigarrow$ equivalent to the null-controllability problem with initial data $\hat{y}_0 - y_0$

We consider a free trajectory of the semi-discrete problem

$$\hat{y}_F = e^{-T\mathcal{A}_h}\hat{y}_0.$$

PROBLEM 2

Starting from any $y_0 \in E_h$, can we drive the solution of the full-discrete system

$$\frac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta)y^n) = \mathcal{B}_h v^{n+1}, \quad y^0 = y_0,$$

to $\hat{y}_F$ at discrete time M ?

$\rightsquigarrow$ This is not equivalent to the null-controllability problem with initial data $\hat{y}_0 - y_0$

INDEED : The full-discrete free trajectory starting at $\hat{y}_0$ is not equal to $\hat{y}_F$ at time M.

- In most cases (but not always) $\hat{y}_F$ belongs to some full discrete trajectory

$$\hat{y}_F = \Big((\mathrm{Id} + \theta\mathcal{A}_h)^{-1}(\mathrm{Id} + (1-\theta)\mathcal{A}_h)\Big)^M \tilde{y}_0^{\delta t}.$$

- We do not want the estimates to depend on $\tilde{y}_0^{\delta t}$ since :
  - In general we do not want to compute $\tilde{y}_0^{\delta t}$.
  - Its norm can be large with respect to that of $\hat{y}_F$.

We consider a free trajectory of the semi-discrete problem

$$\hat{y}_F = e^{-T\mathcal{A}_h}\hat{y}_0.$$

OUR RESULT : Under suitable assumptions, by minimizing the functional

$$J^{h,\delta t}(q_F) = \frac{1}{2}\sum_{n=1}^{M}\delta t\,[\![\mathcal{B}_h^\star q^n]\!]_h^2 + \frac{\phi(h)}{2}\,\|q_F\|_h^2 - \langle \hat{y}_F, q_F\rangle_h + \left\langle y_0, q^1 - \delta t(1-\theta)\mathcal{A}_h q^1\right\rangle_h,$$

we produce a full discrete control $v_{h,\delta t} = (\mathcal{B}_h^\star q^n)_n$ such that

- The cost of the control satisfies

$$\sum_{n=1}^{M}\delta t\,[\![v^n]\!]_h^2 \leq C_{\text{obs}}^2\left(\|y_0 - \hat{y}_0\|_h + C_s\delta t^{\zeta_1}\left\|\mathcal{A}_h^{\frac{1}{2}}\hat{y}_0\right\|_h\right)^2 + e^{-C/\delta t^{\zeta_2}}\|\hat{y}_0\|_h^2,$$

  for some $\zeta_1, \zeta_2 > 0$.
- The controlled solution $(y^n)_n$ associated with $v_{h,\delta t}$ and y_0 is such that

$$\left\|y^M - \hat{y}_F\right\|_h \leq \sqrt{\phi(h)}C_{\text{obs}}\left(\|y_0 - \hat{y}_0\|_h + C\delta t^{\zeta_1}\left\|\mathcal{A}_h^{\frac{1}{2}}\hat{y}_0\right\|_h\right) + e^{-C/\delta t^{\zeta_2}}\|\hat{y}_0\|_h.$$

MAIN TOOL : Estimate of the difference between the two initial data

$$\left\|\tilde{y}_0^{\delta t} - \hat{y}_0\right\|_h \leq C\delta t^{\zeta}\left\|\mathcal{A}_h^{\frac{1}{2}}\hat{y}_0\right\|_h.$$

Outline

- The error analysis in space is intricate (low regularity expected for the HUM null-control, ...).
- We try here to analyse the error induced by time discretisation in this problem.

FRAMEWORK

- We assume the uniform discrete Lebeau-Robbiano ($\mathcal{H}_{\alpha,\beta}$) to hold.
- We suppose that $h > 0$ is fixed, that $y_0 \in E_h$ is given and that $h \mapsto \phi(h)$ is given.
- The minimization of the functional

$$J^h(q_F) = \frac{1}{2}\int_0^T [\![\mathcal{B}_h^\star q(t)]\!]_h^2 \, dt + \frac{\phi(h)}{2}\|q_F\|_h^2 + \langle y_0, q(0)\rangle_h ,$$

  leads to a semi-discrete control $t \mapsto v_h(t) \in L^2(]0,T[, U_h)$.
- For simplicity, we consider the implicit Euler scheme (similar results hold for $\theta \in [1/2, 1[$). The minimization of the functional

$$J^{h,\delta t}(q_F) = \frac{1}{2}\sum_{n=1}^{M} \delta t \, [\![\mathcal{B}_h^\star q^n]\!]_h^2 + \frac{\phi(h)}{2}\|q_F\|_h^2 + \left\langle y_0, q^1 \right\rangle_h ,$$

  leads to a full discrete control $v_{h,\delta t} = (v^n)_n \in (U_h)^M$.

GOAL : Prove an error estimate between $v_{h,\delta t}$ and v_h.

THEOREM

Under the same assumptions than previous results (in particular $\delta t \leq Ch^\gamma$), the following error estimate holds

$$\Bigg\| v_h - \underbrace{\sum_{n=1}^{M} 1_{(t^{n-1},t^n)} v^n}_{\overset{\text{def}}{=} \mathcal{F}_0[v_{h,\delta t}]} \Bigg\|_{L^2(]0,T[,U_h)} \leq C\delta t \frac{\rho(\mathcal{A}_h)}{\sqrt{\phi(h)}} \left(1 + \delta t^{\frac{3}{2}} \rho(\mathcal{A}_h)^{\frac{3}{2}}\right) \|y_0\|_h .$$

REMARKS

- First order in time estimate (second order for CN provided a suitable time interpolation operator is used in place of $\mathcal{F}_0[.]$).
- The estimate is not uniform in h, even if we are interested in the approximate control problem where $\phi(h) = \varepsilon > 0$. The result is probably not optimal.

SKETCH OF PROOF

- Write the Euler-Lagrange equations corresponding to the two minimization problems we consider (the semi-discrete and the full-discrete).
- Compare the two Euler-Lagrange equations by using error estimates in time for the adjoint problem.

▶ Proof details

The functional we want to minimise reads (ε is fixed or $\varepsilon = \phi(h)$)

$$J_{\varepsilon,h,\delta t} : q_F \in E_h \longmapsto \frac{1}{2}\sum_{n=1}^{M} \delta t \left[\!\left[(\mathcal{L}_v^* q_F)^n \right]\!\right]_h^2 + \frac{\varepsilon}{2} \|q_F\|_h^2 + \langle y_0, \mathcal{L}_0^* q_F \rangle_h .$$

We solve this problem by conjugate gradient (in $(E_h, \langle ., . \rangle_h)$). **(Glowinski-Lions, '94)**

COMPUTATION OF THE GRADIENT

$$\nabla J_{\varepsilon,h,\delta t}(q_F) = \mathcal{L}_v \mathcal{L}_v^* q_F + \varepsilon q_F + \mathcal{L}_0 y_0,$$

and we have seen that $\mathcal{L}_v \mathcal{L}_v^* q_F$ is computed by solving first

$$(S_{h,\delta t}^*) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta) q^{n+1}) = 0, \quad \forall n \in [\![1, M-1]\!], \end{cases}$$

then by solving

$$(S_{h,\delta t}) \begin{cases} y^0 = 0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta) y^n) = \mathcal{B}_h \underbrace{\mathcal{B}_h^\star q^{n+1}}_{=(\mathcal{L}_v^* q_F)^n}, \ \forall n \in [\![0, M-1]\!], \end{cases}$$

and we finally have $\mathcal{L}_v \mathcal{L}_v^* q_F = y^M$.

- Advantages compared to other approaches **(Münch et al, '09,'10,'11)**
  - Many time stepping schemes can be adapted (higher order methods like BDF2 or RK3, RK4, etc ...).
  - Any reasonable space discretization method for any space dimension can be chosen, independently.
  - You can use some **black-box** direct and adjoint solver $\Rightarrow$ very easy implementation.

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  - You can use some **black-box** direct and adjoint solver $\Rightarrow$ very easy implementation.
- Performance issues :
  - Condition number for $\varepsilon > 0$ (almost independent of δt) :
  $$\|\mathcal{L}_v\mathcal{L}_v^* + \varepsilon \text{Id}\| \leq C + \varepsilon,$$
  $$\|(\mathcal{L}_v\mathcal{L}_v^* + \varepsilon \text{Id})^{-1}\| \sim \frac{C}{\varepsilon}.$$
  For instance, for $\varepsilon = \phi(h) = h^2$ we have the same condition number as for the discrete Laplace matrix ...
  Recall that : Nb of iterations of CG $\sim \sqrt{\text{condition number}} \sim 1/\sqrt{\varepsilon}$.
  - Condition number for $\varepsilon = 0$:
    - We have seen that $\mathcal{L}_v\mathcal{L}_v^*$ could be not invertible ! !
    - Even if we assume that it is invertible and that the uniform observability inequality holds
    $$\left\|\mathcal{L}_0^* q_F\right\|_h \leq C\|\mathcal{L}_v^* q_F\|,$$
    the very bad condition number comes from
    $$\|q_F\|_h^2 \leq Ce^{C/h^p} \left\|\mathcal{L}_0^* q_F\right\|_h^2 \leq C' e^{C/h^p} \left\langle \mathcal{L}_v\mathcal{L}_v^* q_F, q_F\right\rangle_h .$$
  - Summary :
    - System is not so ill-posed **but** preconditioning is a very important and difficult issue.
    - Computational time of each CG iteration can be large and memory consuming : use of parareal algorithms can be useful **(Lions-Maday-Turinici, ...)**

For the null-control problem, we recall that we choose $\varepsilon = \phi(h)$ and the computed control $v_{h,\delta t} = (v^n)_n$ and the **computed approximated solution** $y_{h,\delta t} = (y^n)_n$ satisfy

$$\left\| y^M \right\|_h \leq C_{\text{obs}} \sqrt{\phi(h)} \, \|y_{0,h}\|_h \,, \quad \text{and} \quad \sum_{n=1}^{M} \delta t \, [\![v^n]\!]_h^2 \leq C_{\text{obs}}^2 \, \|y_{0,h}\|_h^2 \,.$$

EXAMPLE : FD app. of the 1D heat equation $\partial_t y - \gamma \partial_x^2 y = 1_\omega v$ in $]0,1[$.
We build the piecewise constant function (K_i=cells associated to discretisation points)

$$\tilde{v}_{h,\delta t} = \sum_{n=1}^{M} \sum_i \delta t 1_{]t^{n-1},t^n[\times K_i} v_i^n \ \in L^2(]0,T[\times\Omega),$$

that we introduce into the original PDE :

$$\partial_t \tilde{y}_{h,\delta t} - \gamma \partial_x^2 \tilde{y}_{h,\delta t} = 1_\omega \tilde{v}_{h,\delta t}, \ \text{ with } \ \tilde{y}_{h,\delta t}(0) = \sum_i y_{0,h,i} 1_{K_i}.$$

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CONTINUITY OF SOLUTIONS WITH RESPECT TO DATA

$$\| \underbrace{y(T)}_{=0} - \tilde{y}_{h,\delta t}(T) \|_{L^2} \leq C \|y_0 - \tilde{y}_{h,\delta t}(0)\|_{L^2} + C \|v - \tilde{v}_{h,\delta t}\|_{L^2(]0,T[\times\omega)}.$$

STANDARD A PRIORI ERROR ESTIMATE : $\|\tilde{y}_{h,\delta t}(T) - y^M\|_{L^2} \sim C_{v,y_0}(\delta t^p + h^q)$.

CONCLUSION : $\|v - \tilde{v}_{h,\delta t}\|_{L^2(]0,T[\times\omega)} \sim C_{v,y_0}(\delta t^p + h^q) + C_{\text{obs}} \sqrt{\phi(h)} \|y_0\|_{L^2}$.

$$\|\tilde{y}_{h,\delta t}(T)\|_h \sim \|v - \tilde{v}_{h,\delta t}\|_{L^2(]0,T[\times\omega)} \sim C_{v,y_0}(\delta t^p + h^q) + C_{\text{obs}}\sqrt{\phi(h)}\|y_0\|_{L^2}.$$

CONCLUSION

- The choice of $\phi(h)$ has to be related to the rate of convergence $\delta t^p + h^q$ of the approximation scheme used.
- Even for large time steps we may compute very small targets y^M, **but** they are meaningless since the actual control and controlled solution are very poorly approximated.
- For the results to be meaningful, the time step has to be chosen small enough : the same choice as the one done for computing the free solution is OK.

EXACT COMPUTATION OF THE RECONSTITUTED FINAL STATE $\tilde{y}_{h,\delta t}(T)$

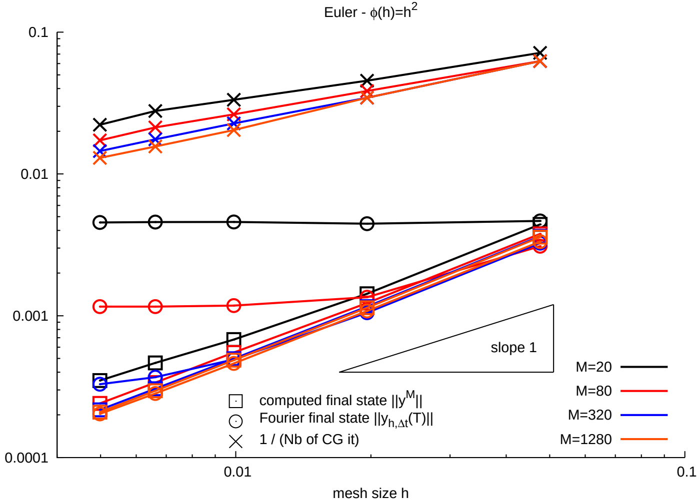
In Fourier variable $\left(\mathcal{F}_k z = \int_0^1 z(x)\sin(k\pi x)\,dx\right)$ we have the ODE

$$\frac{d}{dt}\mathcal{F}_k(\tilde{y}_{h,\delta t}) + \gamma k^2\pi^2\mathcal{F}_k(\tilde{y}_{h,\delta t}) = \sum_{n=1}^{M}\delta t 1_{]t^{n-1},t^n[}\sum_i v_i^n\mathcal{F}_k(1_{\omega\cap K_i}),$$

which can be solved explicitly and then

$$\|\tilde{y}_{h,\delta t}(T)\|_{L^2}^2 \sim \sum_k |\mathcal{F}_k(\tilde{y}_{h,\delta t})(T)|^2.$$

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$



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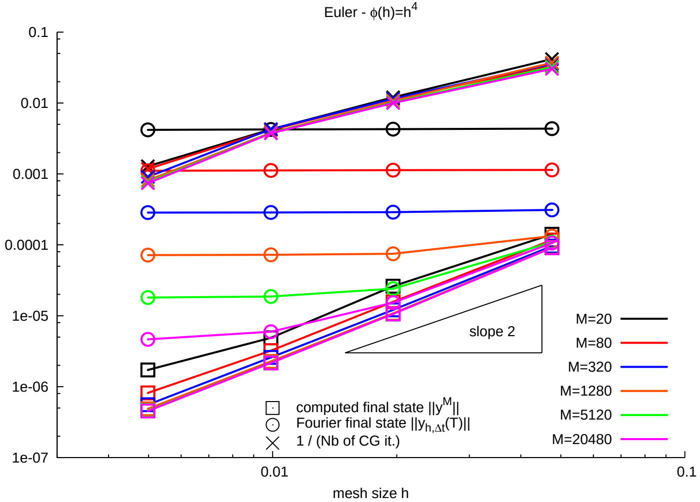

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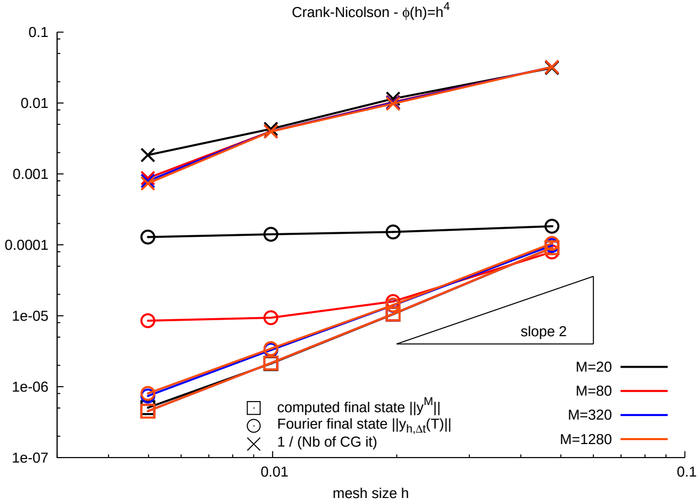

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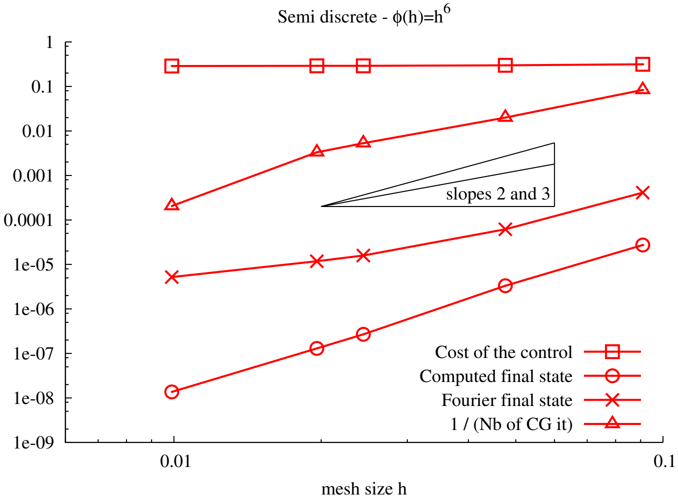

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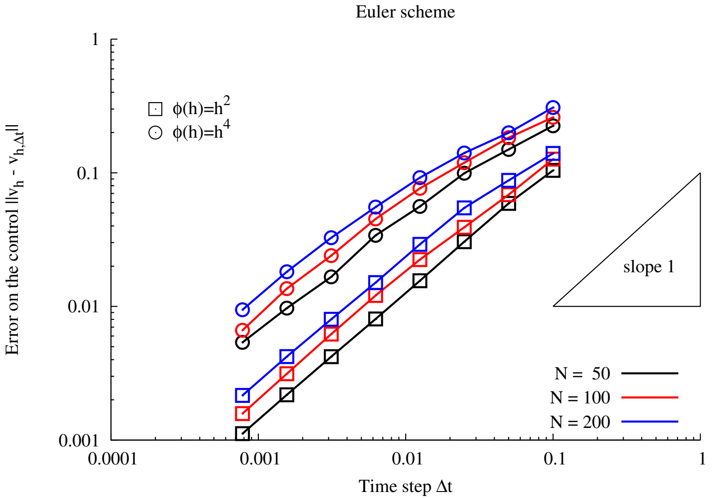

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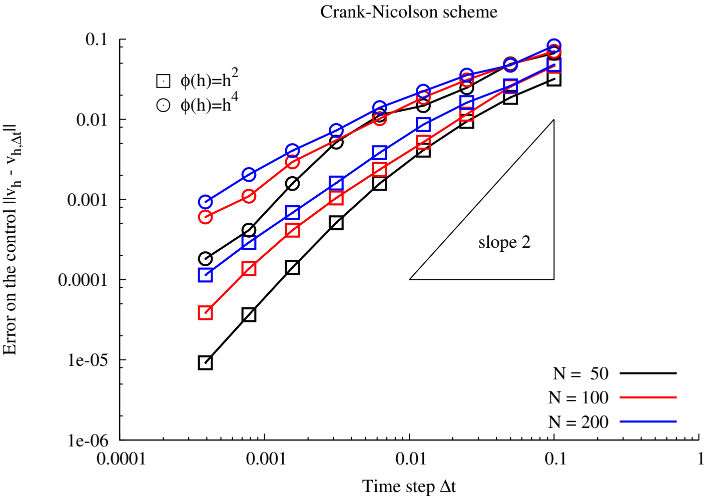
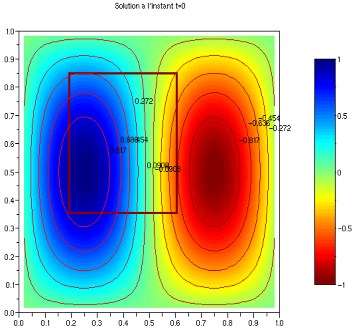
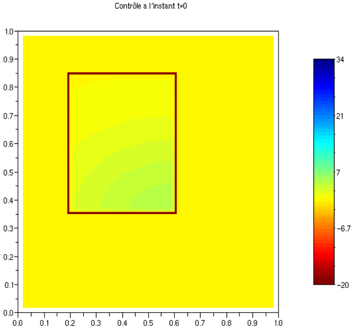


$$\partial_t y - 0.1\Delta y = 1_\omega v,$$
$$y(0,x) = \sin(2\pi x_1)\sin(\pi x_2), \quad \text{and} \quad y_F(x) = 0.1\sin(\pi x_1)\sin(2\pi x_2).$$





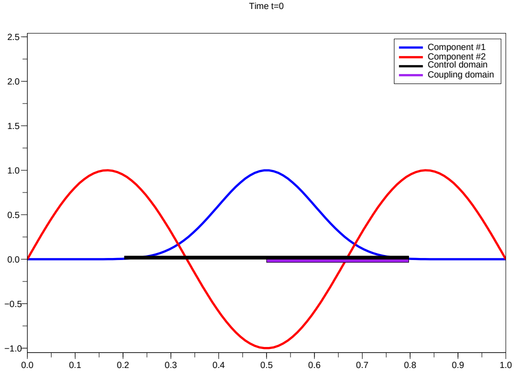
(De Teresa – González-Burgos, '08) (Ammar-Khodja – Benabdallah – Dupaix – González-Burgos, '09)

$$\partial_t y - \partial_x \left(\begin{pmatrix} 0.1 & 0 \\ 0 & 0.1(2+\sin(4x)) \end{pmatrix} \partial_x y \right) + \begin{pmatrix} 0 & 1_{]0.5,0.8[}(x) \\ 0 & 0 \end{pmatrix} y = \begin{pmatrix} 0 \\ 1_{]0.2,0.8[}(x) \end{pmatrix} v.$$

$$\Omega =]0,1[$$

$$T = 1$$

$$y_0(x) = \begin{pmatrix} \sin(\pi x)^{10} \\ \sin(3\pi x) \end{pmatrix}.$$



NUMERICAL PARAMETERS :

$$N = 100, \text{ uniform mesh, Euler scheme } M = 200, \ \phi(h) = h^4.$$

(De Teresa-Kavian '09, De Teresa-Rosier '10 ?)

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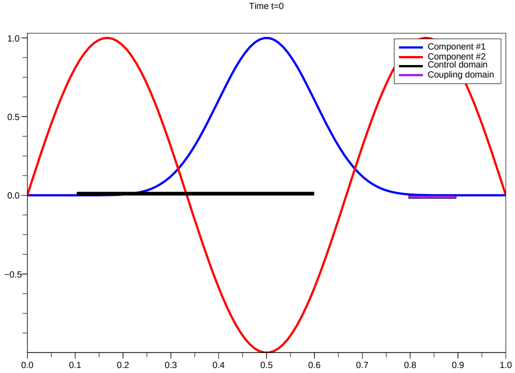

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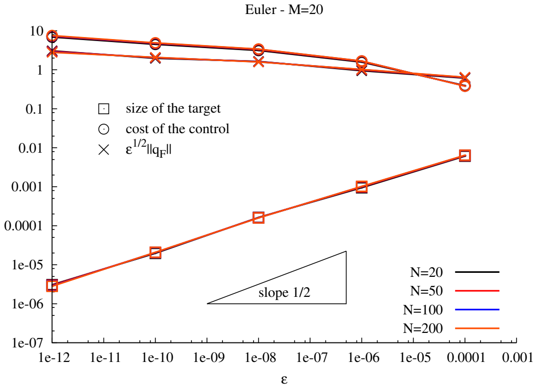

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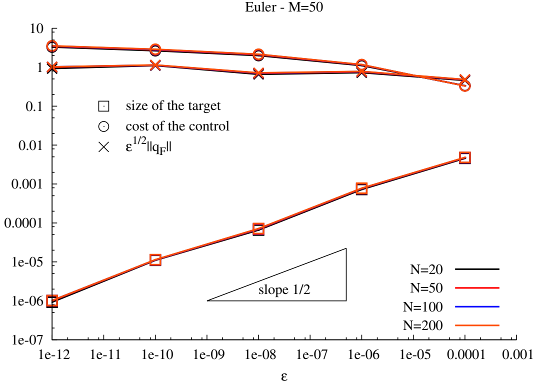
$$T = 4$$

$$y_0(x) = \begin{pmatrix} \sin(\pi x)^{10} \\ \sin(3\pi x) \end{pmatrix}.$$



NUMERICAL PARAMETERS :

$N = 100$, uniform mesh, Euler scheme $M = 200$, $\phi(h) = h^4$.

(De Teresa-Kavian '09, De Teresa-Rosier '10 ?)

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 1_{]0.8,0.9[}(x) \\ 0 & 0 \end{pmatrix} y = \begin{pmatrix} 0 \\ 1_{]0.1,0.6[}(x) \end{pmatrix} v.$$

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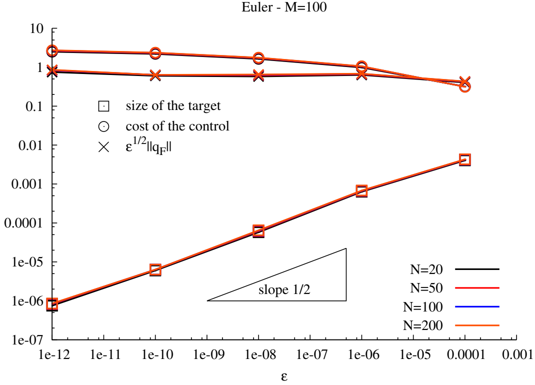

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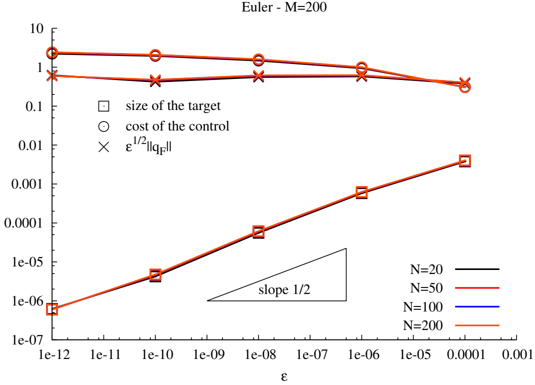

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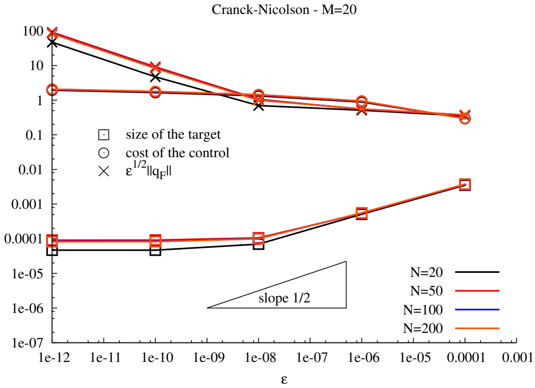

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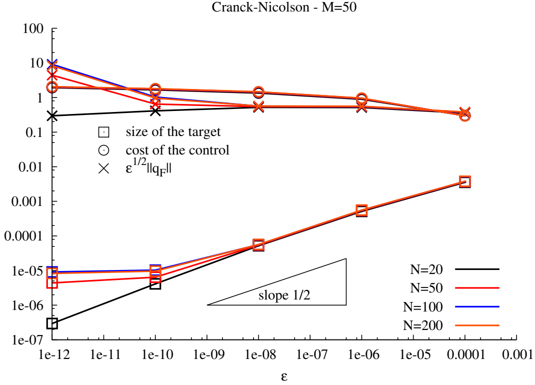

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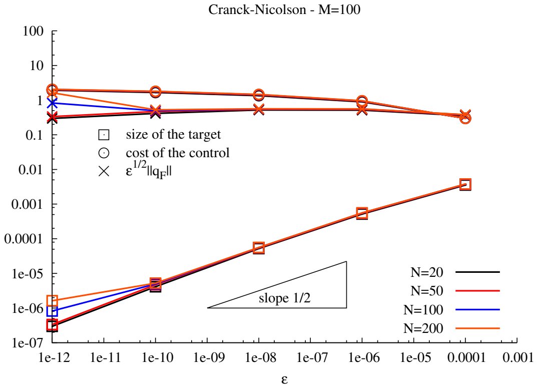

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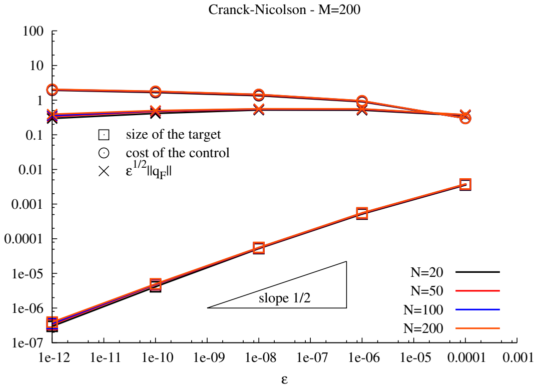


NUMERICAL PARAMETERS :

$N = 100$, uniform mesh, Euler scheme $M = 200$, $\phi(h) = h^4$.

(Ammar-Khodja – Benabdallah – Dupaix – González-Burgos, '09)

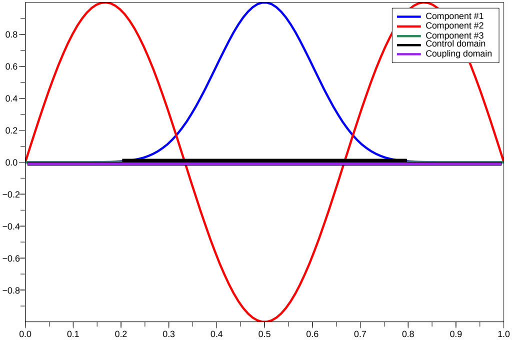
$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \\ 1_{]\mathbf{0.2},\mathbf{0.8}[}(x) \end{pmatrix} v.$$

$$\Omega =]0,1[$$

$$T = 1$$

$$y_0(x) = \begin{pmatrix} \sin(\pi x)^{10} \\ \sin(3\pi x) \\ 0 \end{pmatrix}.$$

Time t=0



PARAMETERS : $N = 100$, uniform mesh, Euler scheme $M = 200$, $\phi(h) = h^4$.

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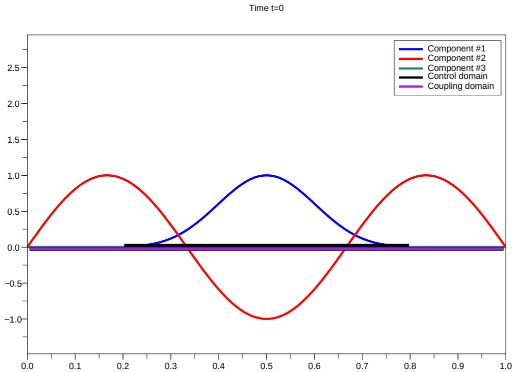


PARAMETERS : $N = 100$, uniform mesh, Euler scheme $M = 200$, $\phi(h) = h^4$.

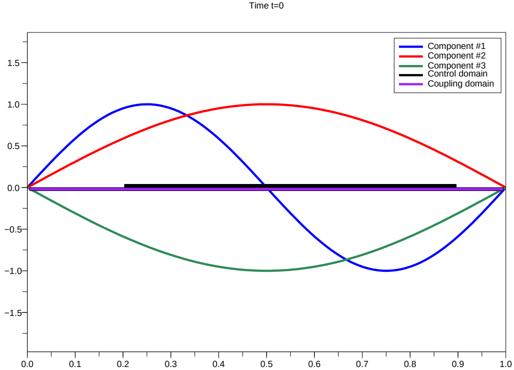
(Benabdallah – Cristofol – De Teresa – Gaitan, '10)

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ x+1 & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1_{]0.2,0.9[} \\ 0 \\ 0 \end{pmatrix} v.$$

$$\Omega =]0,1[$$

$$T = 3$$

$$y_0(x) = \begin{pmatrix} \sin(2\pi x) \\ \sin(\pi x) \\ -\sin(\pi x) \end{pmatrix}.$$



PARAMETERS : $N = 100$, uniform mesh, Euler scheme $M = 200$, $\phi(h) = h^4$.

(Benabdallah – Cristofol – De Teresa – Gaitan, '10)

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ x 1_{]0,0.8[}(x) & 0 & 0 \\ (x+1)1_{]0,0.8[}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1_{]0.2,0.9[} \\ 0 \\ 0 \end{pmatrix} v.$$

$$\Omega =]0,1[$$

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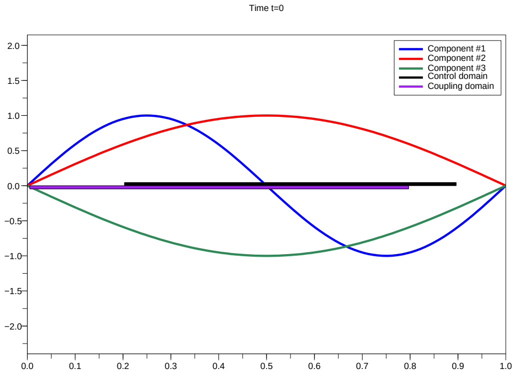

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$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ x1_{]0,0.3[}(x) & 0 & 0 \\ (x+1)1_{]0,0.3[}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1_{]0.5,1.0[} \\ 0 \\ 0 \end{pmatrix} v.$$

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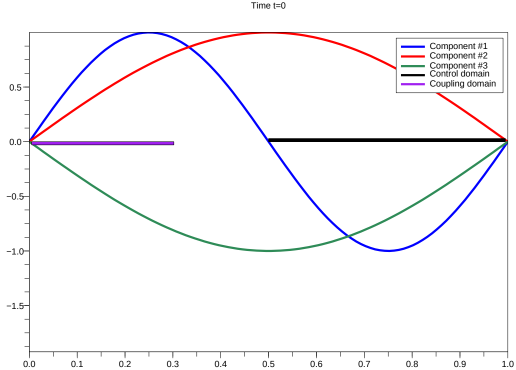

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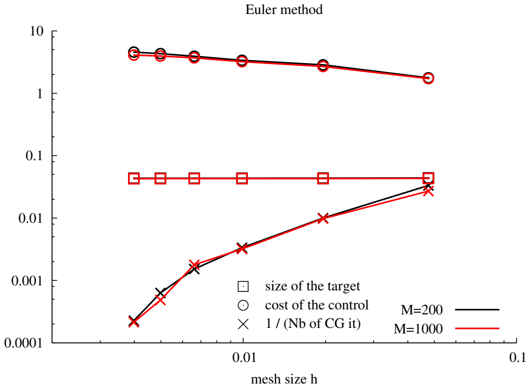


PARAMETERS : $N = 100$, uniform mesh, Euler scheme $M = 200$, $\phi(h) = h^4$.

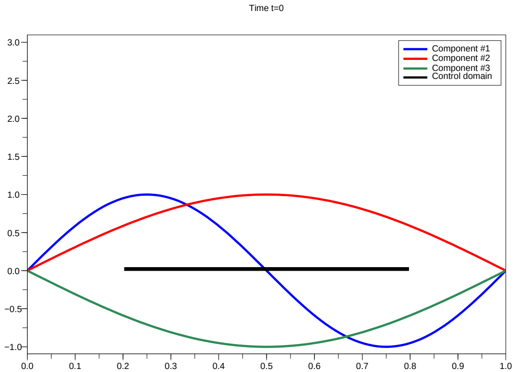
(Ammar-Khodja – Benabdallah – Dupaix – González-Burgos, '09)

$$\partial_t y - \frac{1}{\pi^2}\partial_x\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1.2 & 0\\ 0 & 0 & 3\end{pmatrix}\partial_x y\right) = 1_{]\mathbf{0.2},\mathbf{0.8}[}\begin{pmatrix}1\\ 2\\ -1\end{pmatrix} v.$$

$$\Omega =]0,1[$$

$$T = 2$$

$$y_0(x) = \begin{pmatrix}\sin(2\pi x)\\ \sin(\pi x)\\ -\sin(\pi x)\end{pmatrix}.$$



PARAMETERS : $N = 100$, uniform mesh, Euler scheme $M = 200$, $\phi(h) = h^4$.

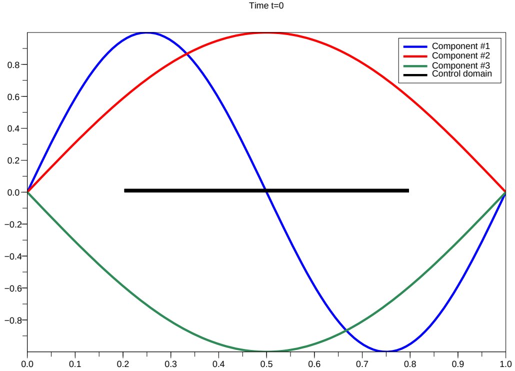
(Ammar-Khodja – Benabdallah – Dupaix – González-Burgos, '09)

$$\partial_t y - \partial_x \left(\frac{1}{\pi^2} \begin{pmatrix} 2 + 5 \times 1_{]0,0.2[} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & (2 - 1.8 \times 1_{]0.8,1[}) \end{pmatrix} \partial_x y \right) = 1_{]0.2,0.8[} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} v.$$

$$\Omega =]0,1[$$

$$T = 2$$

$$y_0(x) = \begin{pmatrix} \sin(2\pi x) \\ \sin(\pi x) \\ -\sin(\pi x) \end{pmatrix}.$$



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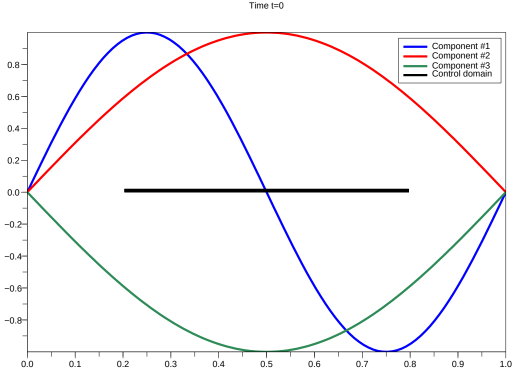

PARAMETERS : $N = 100$, uniform mesh, Euler scheme $M = 200$, $\phi(h) = h^4$.

$$\partial_t y - \partial_x \left(\begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{pmatrix} \partial_x y \right) + \begin{pmatrix} 2 & -3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix} y = \begin{pmatrix} 1_{]\mathbf{0.7},\mathbf{1.0}[} & 0 \\ 0 & 1_{]\mathbf{0.1},\mathbf{0.5}[} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

$$\Omega =]0,1[$$

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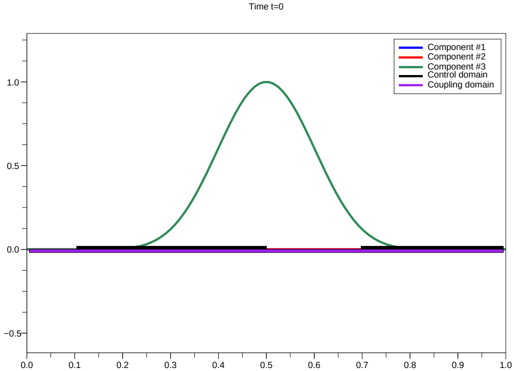
$$y_0(x) = \begin{pmatrix} 0 \\ 0 \\ \sin(\pi x)^{10} \end{pmatrix}.$$



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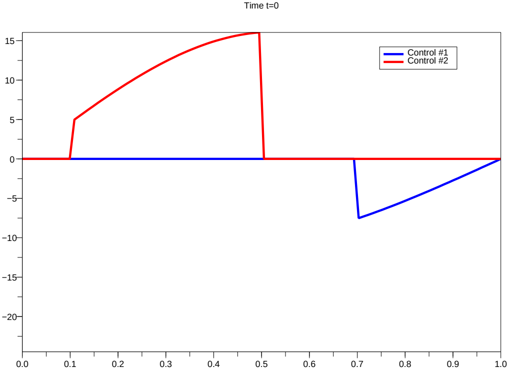

PARAMETERS : $N = 100$, uniform mesh, Euler scheme $M = 200$, $\phi(h) = h^4$.

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} y = \begin{pmatrix} 1_{]\mathbf{0.7},\mathbf{1.0}[} & 0 \\ 0 & 1_{]\mathbf{0.1},\mathbf{0.5}[} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

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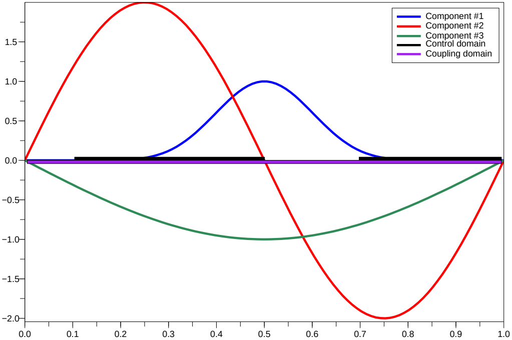
Time t=0



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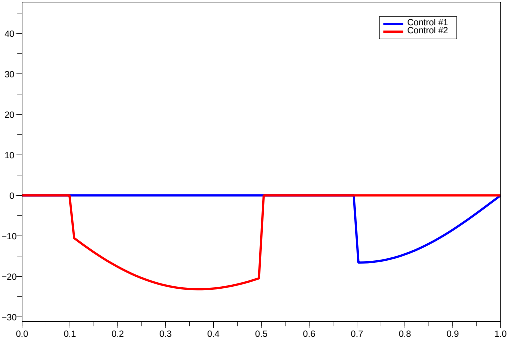

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SUMMARY

- Analysis of uniform controllability properties with respect to δt and h for full discrete problems.
  - Elliptic Discrete Carleman estimates
  - Optimal relaxed observability inequalities.
  - Error analysis in time.
- We may use numerical simulations to investigate open problems.

PERSPECTIVES

- Extend our analysis to other cases
  - Time variable coefficients.
  - Non symmetric scalar operators.
  - Systems.
  - Semi-linear problems.
  - Boundary control problems.
  - **Main tool :** Semi-discrete parabolic Carleman estimates.
- From a numerical point of view
  - Analysis for other numerical schemes (Finite Volumes, Finite Elements, ...)
  - A deeper understanding of the structure of the HUM operator should lead to reasonable preconditioning methods.
  - Is there more suitable solvers than standard Conjugate Gradient ?

EULER-LAGRANGE EQUATION FOR J^h

We denote the minimizer by q^F_{opt} and $t \mapsto q_{opt}(t)$ the corresponding solution to the semi-discrete adjoint problem :

$$0 = \int_0^T [\mathcal{B}_h^\star q_{opt}(t), \mathcal{B}_h^\star \tilde{q}(t)]_h \, dt + \phi(h) \left\langle q^F_{opt}, \tilde{q}^F \right\rangle_h + \langle y_0, \tilde{q}(0) \rangle_h ,$$

for any $\tilde{q}^F \in E_h$.

EULER-LAGRANGE EQUATION FOR $J^{h,\delta t}$

We denote the minimizer by $q^F_{opt,\delta t}$ and by $(q^n_{opt,\delta t})_n$ the corresponding solution to the full-discrete adjoint problem

$$0 = \sum_{n=1}^{M} \delta t \left[\mathcal{B}_h^\star q^n_{opt,\delta t}, \mathcal{B}_h^\star \tilde{q}^n\right]_h + \phi(h) \left\langle q^F_{opt,\delta t}, \tilde{q}^F \right\rangle_h + \left\langle y_0, \tilde{q}^1 \right\rangle_h ,$$

for any $\tilde{q}^F \in E_h$.

$$\int_0^T \left[\mathcal{B}_h^\star q_{opt}(t), \mathcal{B}_h^\star \tilde{q}(t)\right]_h \, dt + \phi(h) \left\langle q_{opt}^F, \tilde{q}^F \right\rangle_h + \left\langle y_0, \tilde{q}(0) \right\rangle_h = 0,$$

$$\sum_{n=1}^{M} \delta t \left[\mathcal{B}_h^\star q_{opt,\delta t}^n, \mathcal{B}_h^\star \tilde{q}^n\right]_h + \phi(h) \left\langle q_{opt,\delta t}^F, \tilde{q}^F \right\rangle_h + \left\langle y_0, \tilde{q}^1 \right\rangle_h$$

$$= 0,$$

$$\int_0^T \left[\mathcal{B}_h^\star q_{opt}(t), \mathcal{B}_h^\star \tilde{q}(t)\right]_h \, dt + \phi(h) \left\langle q_{opt}^F, \tilde{q}^F \right\rangle_h + \left\langle y_0, \tilde{q}(0) \right\rangle_h = 0,$$

$$\boxed{\sum_{n=1}^{M} \delta t \left[\mathcal{B}_h^\star q_{opt,\delta t}^n, \mathcal{B}_h^\star \tilde{q}^n\right]_h} + \phi(h) \left\langle q_{opt,\delta t}^F, \tilde{q}^F \right\rangle_h + \left\langle y_0, \tilde{q}^1 \right\rangle_h = 0,$$

TRANSFORMATION OF THESE EQUATIONS

$$\delta t \left[\mathcal{B}_h^\star q_{opt,\delta t}^n, \mathcal{B}_h^\star \tilde{q}^n\right]_h = \int_{t^{n-1}}^{t^n} \left[\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star (\mathcal{F}_0[\tilde{q}_{\delta t}](t))\right]_h \, dt$$

$$= \int_{t^{n-1}}^{t^n} \left[\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star \tilde{q}(t)\right]_h \, dt + \int_{t^{n-1}}^{t^n} \left[\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star \left(\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t)\right)\right]_h \, dt.$$

Proof details

$$\int_0^T \left[\mathcal{B}_h^\star q_{opt}(t), \mathcal{B}_h^\star \tilde{q}(t)\right]_h \, dt + \phi(h) \left\langle q_{opt}^F, \tilde{q}^F \right\rangle_h + \left\langle y_0, \tilde{q}(0) \right\rangle_h = 0,$$

$$\int_0^T \left[\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star \tilde{q}(t)\right]_h \, dt + \phi(h) \left\langle q_{opt,\delta t}^F, \tilde{q}^F \right\rangle_h + \left\langle y_0, \tilde{q}^1 \right\rangle_h$$

$$= - \int_0^T \left[\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star \left(\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t)\right)\right]_h \, dt,$$

$$\int_0^T \left[\mathcal{B}_h^\star q_{opt}(t), \mathcal{B}_h^\star \tilde{q}(t)\right]_h \, dt + \phi(h) \left\langle q_{opt}^F, \tilde{q}^F \right\rangle_h + \langle y_0, \tilde{q}(0)\rangle_h = 0,$$

$$\int_0^T \left[\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star \tilde{q}(t)\right]_h \, dt + \phi(h) \left\langle q_{opt,\delta t}^F, \tilde{q}^F \right\rangle_h + \boxed{\left\langle y_0, \tilde{q}^1 \right\rangle_h}$$

$$= - \int_0^T \left[\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star \left(\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t)\right)\right]_h \, dt,$$

TRANSFORMATION OF THESE EQUATIONS

$$\tilde{q}^1 = \tilde{q}(0) + \left(\tilde{q}^1 - \tilde{q}(0)\right)$$

$$\int_0^T \left[\mathcal{B}_h^\star q_{opt}(t), \mathcal{B}_h^\star \tilde{q}(t)\right]_h dt + \phi(h) \left\langle q_{opt}^F, \tilde{q}^F \right\rangle_h + \langle y_0, \tilde{q}(0)\rangle_h = 0,$$

$$\int_0^T \left[\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star \tilde{q}(t)\right]_h dt + \phi(h) \left\langle q_{opt,\delta t}^F, \tilde{q}^F \right\rangle_h + \langle y_0, \tilde{q}(0)\rangle_h$$
$$= - \int_0^T \left[\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star \left(\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t)\right)\right]_h dt - \left\langle y_0, \tilde{q}^1 - \tilde{q}(0) \right\rangle_h,$$

$$\int_0^T \left[\mathcal{B}_h^\star q_{opt}(t), \mathcal{B}_h^\star \tilde{q}(t)\right]_h \, dt + \phi(h) \left\langle q_{opt}^F, \tilde{q}^F \right\rangle_h + \langle y_0, \tilde{q}(0)\rangle_h = 0,$$

$$\int_0^T \left[\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star \tilde{q}(t)\right]_h \, dt + \phi(h) \left\langle q_{opt,\delta t}^F, \tilde{q}^F \right\rangle_h + \langle y_0, \tilde{q}(0)\rangle_h$$

$$= -\int_0^T \left[\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star \left(\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t)\right)\right]_h \, dt - \left\langle y_0, \tilde{q}^1 - \tilde{q}(0) \right\rangle_h,$$

SUBTRACTION OF THE EQUATIONS

$$\int_0^T \left[\mathcal{B}_h^\star q_{opt}(t) - \mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star \tilde{q}(t)\right]_h \, dt + \phi(h) \left\langle q_{opt}^F - q_{opt,\delta t}^F, \tilde{q}^F \right\rangle_h$$

$$= \int_0^T \left[\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star \left(\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t)\right)\right]_h \, dt + \left\langle y_0, \tilde{q}^1 - \tilde{q}(0) \right\rangle_h,$$

$\rightsquigarrow$ Now we choose $\tilde{q}^F = q_{opt}^F - q_{opt,\delta t}^F$, so that $\tilde{q}(t) = q_{opt}(t) - \underline{q}(t)$ and then

$$\mathcal{B}_h^\star \tilde{q}(t) = \left(\mathcal{B}_h^\star q_{opt}(t) - \mathcal{F}_0[v_{h,\delta t}](t)\right) + \mathcal{B}_h^\star \left(\mathcal{F}_0[q_{opt,\delta t}](t) - \underline{q}(t)\right).$$

$$\int_0^T [\![v(t) - \mathcal{F}_0[v_{h,\delta t}](t)]\!]_h^2 \, dt + \phi(h) \left\| q_{opt}^F - q_{opt,\delta t}^F \right\|_h^2$$

$$= \int_0^T \left[\mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star \left(\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t)\right)\right]_h dt + \left\langle y_0, \tilde{q}^1 - \tilde{q}(0) \right\rangle_h$$

$$- \int_0^T \left[v(t) - \mathcal{F}_0[v_{h,\delta t}](t), \mathcal{B}_h^\star \left(\mathcal{F}_0[q_{opt,\delta t}](t) - \underline{q}(t)\right)\right]_h dt.$$

- The error terms are estimated as follows by usual parabolic techniques :

$$\int_0^T [\![\mathcal{B}_h^\star \left(\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t)\right)]\!]_h^2 \, dt \leq C \int_0^T \|\mathcal{F}_0[\tilde{q}_{\delta t}](t) - \tilde{q}(t)\|_h^2 \, dt$$

$$\leq C \left(\delta t^2 \left\| \mathcal{A}_h^{\frac{1}{2}} \tilde{q}^F \right\|_h^2 + \delta t^5 \left\| \mathcal{A}_h^2 \tilde{q}^F \right\|_h^2 \right) \leq C \delta t^2 \left\| \tilde{q}^F \right\|_h^2 \rho_h (1 + \rho_h^3)$$

- We conclude by using Cauchy-Schwarz inequality.

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