On the numerical approximation of control problems for parabolic equations and systems

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joint work with F. HUBERT*, J. LE ROUSSEAU†, G. OLIVE*

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Outline

NOTATIONS

$$(S)\begin{cases}\partial_t y + \mathcal{A}y = \mathcal{B}v & \text{in }]0,T[,\\ y(0)=y^0,\end{cases}$$

- E and U two Hilbert spaces,
- $y \in L^\infty(]0,T[,E)$, $\mathcal{A} : D(\mathcal{A}) \subset E \mapsto E$ is some "elliptic" unbounded operator,
- $\mathcal{B} : U \mapsto E$ a bounded operator,
- $v \in L^2(]0,T[,U)$ is the control. Its cost is $\|v\|_{L^2(0,T;U)} = \left(\int_0^T \|v(t)\|_U^2\, dt\right)^{\frac{1}{2}}$.

APPROXIMATE CONTROL PROBLEM

For all $\beta > 0$, can we find $v \in L^2(0,T;U)$ s.t. the solution y satisfies $\|y(T)\|_E \leq \beta$?

NULL CONTROL PROBLEM

Can we find $v \in L^2(]0,T[;U)$ such that the solution y satisfies $y(T) = 0$?

(Fattorini-Russel, '71) (Lebeau-Robbiano, '95)

(Fursikov-Imanuvilov, '96) (Alessandrini-Escauriaza, '08)

(Ammar-Khodja, Benabdallah, González-Burgos, de Teresa, '11)

THE 1D HEAT EQUATION

$$(S)\begin{cases}\partial_t y - \partial_x(\gamma(x)\partial_x y) = 1_\omega v & \text{in } (0,T)\times\Omega,\\ y = 0 & \text{on } (0,T)\times\partial\Omega,\\ y(0) = y^0\end{cases}$$

that is $E = L^2(\Omega)$, $\mathcal{A} = -\partial_x(\gamma(x)\partial_x .)$, $U = L^2(\Omega)$, $\mathcal{B} = 1_\omega$ with $\omega \subset \Omega$.

1D PARABOLIC SYSTEMS

$$(S)\begin{cases}\partial_t y_i - \partial_x(\gamma_i(x)\partial_x y_i) + \sum_{j=1}^n \alpha_{ij}(x) y_j = 1_{\omega_i} B_i(x) v & \text{in } (0,T)\times\Omega,\\ y_i = 0 & \text{on } (0,T)\times\partial\Omega,\\ y_i(0) = y_i^0\end{cases}$$

$y = (y_1, ..., y_n)^t$, $E = (L^2(\Omega))^n$, $U = (L^2(\Omega))^p$, $B_i(x) \in \mathcal{M}_{1,p}(\mathbb{R})$,

$$\mathcal{B} = \begin{pmatrix} 1_{\omega_1} B_1(x) \\ \vdots \\ 1_{\omega_n} B_n(x) \end{pmatrix}.$$

Interesting (and much more difficult) case : $p < n$. Some components are controlled thanks to the coupling terms.

THE SAME IN MULTI-D

HUM-PENALTY IDEA : Given $\varepsilon > 0$, minimize the functional

$$F_\varepsilon : v \in L^2(]0,T[,U) \mapsto \frac{1}{2}\int_0^T \|v(t)\|_U^2\,dt + \frac{1}{2\varepsilon}\|y_v(T)\|_E^2.$$

DUAL PROBLEM : Find a minimizer of the dual functional

$$J_\varepsilon : q_F \in E \mapsto \frac{1}{2}\int_0^T \|\mathcal{B}^* q(t)\|_U^2\,dt + \frac{\varepsilon}{2}\|q_F\|_E^2 + (y_0, q(0))_E,$$

where $t \mapsto q(t)$ is the solution to the backward problem

$$-\partial_t q + \mathcal{A}^* q = 0, \quad q(T) = q_F. \tag{ADJ}$$

EXISTENCE AND DUALITY

For any $\varepsilon > 0$, F_ε has a unique minimizer v_ε, J_ε has a unique minimizer $q_{F,\varepsilon}$ and

$$v_\varepsilon(t) = \mathcal{B}^* q_\varepsilon(t), \ \forall t \in [0,T],$$

$$y_{v_\varepsilon}(T) = -\varepsilon q_{F,\varepsilon}.$$

$$\inf_v F_\varepsilon(v) = F_\varepsilon(v_\varepsilon) = -\inf_{q_F} J_\varepsilon(q_F) = -J_\varepsilon(q_{F,\varepsilon}).$$

Moreover $(\|y_{v_\varepsilon}(T)\|_E)_\varepsilon$ is bounded.

Proof

HUM-PENALTY IDEA : Given $\varepsilon > 0$, minimize the functional

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CONVERGENCE OF THE PENALTY METHOD - APPROXIMATE CONTROL PROBLEM

(S) is approximately controllable from y_0 at time $T \iff \|y_{v_\varepsilon}(T)\|_E \xrightarrow[\varepsilon\to 0]{} 0.$

- $\Leftarrow$ is straightforward.
- $\Rightarrow$ Assume that, up to a subsequence, $\|y_{v_\varepsilon}(T)\|_E^2 \geq \alpha > 0$, for any $\varepsilon > 0$. By assumption, there exists $\hat{v} \in L^2(]0,T[,U)$ such that $\|y_{\hat{v}}(T)\|_E^2 \leq \alpha/2$.

$$\frac{\alpha}{2\varepsilon} \leq \frac{1}{2\varepsilon}\|y_{v_\varepsilon}(T)\|_E^2 \leq F_\varepsilon(v_\varepsilon) \leq F_\varepsilon(\hat{v}) = \frac{1}{2}\|\hat{v}\|_{L^2(0,T,U)}^2 + \frac{1}{2\varepsilon}\|y_{\hat{v}}(T)\|_E^2.$$

It follows $\frac{\alpha}{4\varepsilon} \leq \frac{1}{2}\|\hat{v}\|_{L^2(0,T,U)}^2$, and we get a contradiction when $\varepsilon \to 0$.

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CONVERGENCE OF THE PENALTY METHOD - NULL CONTROL PROBLEM

$$(S) \text{ is null-controllable from } y_0 \text{ at time } T \iff \sup_\varepsilon \left(\inf_v F_\varepsilon(v)\right) < +\infty.$$

OBSERVABILITY

Null-controllability of (S) is equivalent to the observability inequality

$$\|q(0)\|_E^2 \leq C_{\text{obs}}^2 \int_0^T \|\mathcal{B}^* q(t)\|_U^2\,dt, \quad \forall q \text{ sol. of (ADJ)}, \tag{OBS}$$

and we have $\|y_{v_\varepsilon}(T)\|_E \leq C_{\text{obs}}\|y_0\|_E\sqrt{\varepsilon}$, $\quad v_\varepsilon \xrightarrow[\varepsilon\to 0]{} v_0$, in $L^2(0,T;U)$.

(Lebeau-Robbiano, '95)

$$\mathcal{A} = -\text{div}(\gamma\nabla.), \;\; \mathcal{B} = \mathcal{B}^* = 1_\omega,$$

Eigenfunctions : $\mathcal{A}\phi_k = \mu_k\phi_k$. Stable subspaces $E_\mu = \text{Span}\{\phi_k, \;\; \mu_k \leq \mu\}$.

NON UNIFORM PARTIAL OBSERVABILITY INEQUALITY

$$\|q(0)\|^2_{L^2(\Omega)} \leq C\frac{e^{C\sqrt{\mu}}}{T}\int_0^T\int_\omega |q|^2\,dt\,dx, \quad \forall q_F \in E_\mu.$$

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THIS USES A SPECTRAL INEQUALITY CONCERNING EIGENFUNCTIONS OF $\mathcal{A}$

$$\|\phi\|^2_{L^2(\Omega)} \leq Ce^{C\sqrt{\mu}}\|\phi\|^2_{L^2(\omega)}, \;\; \forall \phi \in E_\mu, \forall \mu > 0.$$

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GLOBAL CARLEMAN ESTIMATE FOR THE ELLIPTIC OP. $\mathcal{P} = -\partial_t^2 + \mathcal{A}$

There exists a $\varphi(t,x) > 0$ such that, $\forall s > 0$ large enough and $u(0,.) = 0$, we have

$$s^3\|e^{s\varphi}u\|^2_{L^2(Q)} + s\|e^{s\varphi}\nabla u\|^2_{L^2(Q)} + s^3e^{2s\varphi(T)}|u(T,.)|^2_{L^2(\Omega)}$$
$$\leq C\left(\|e^{s\varphi}\mathcal{P}u\|^2_{L^2(Q)} + se^{2s\varphi(T)}|\nabla_x u(T,.)|^2_{L^2(\Omega)} + s|e^{s\varphi(0,.)}\partial_t u(0,.)|^2_{L^2(\omega)}\right),$$

Apply this inequality to $u = \sum_{\mu_j\leq\mu}\alpha_j\frac{\sinh(\sqrt{\mu_j}t)}{\sqrt{\mu_j}}\phi_j$ and $s = C\sqrt{\mu}$.

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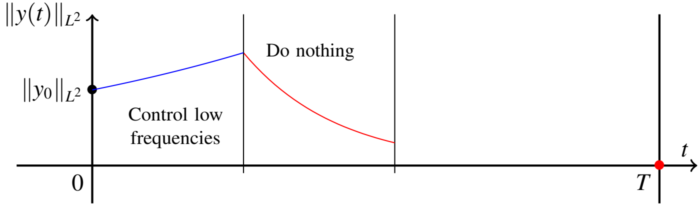

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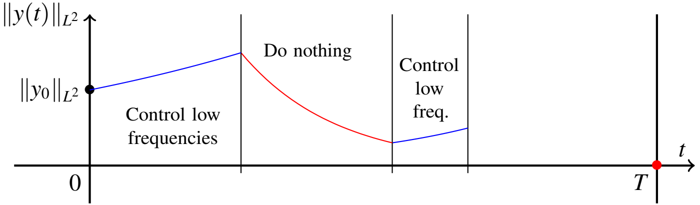

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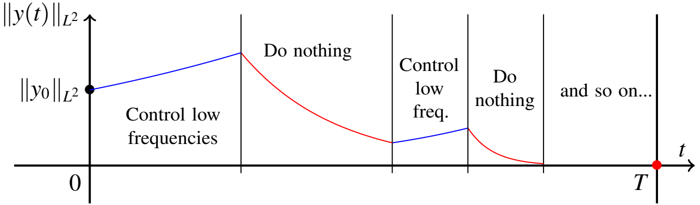

(Fursikov-Imanuvilov, '96)

GLOBAL CARLEMAN ESTIMATE FOR THE PARABOLIC OP. $\mathcal{P} = -\partial_t + \mathcal{A}^*$

Set $\theta(t) = (t(T-t))^{-1}$, and $Q =]0,T[\times\Omega$.

THEOREM

There exists (another) $x \mapsto \varphi(x) > 0$, such that for any $\tau > 0$ large enough and any q vanishing on $\partial\Omega$.

$$\|(\tau\theta)^{-\frac{1}{2}} e^{-\tau\theta\varphi} \partial_t q\|^2_{L^2(Q)} + \|(\tau\theta)^{\frac{1}{2}} e^{-\tau\theta\varphi} \nabla q\|^2_{L^2(Q)} + \|(\tau\theta)^{\frac{3}{2}} e^{-\tau\theta\varphi} q\|^2_{L^2(Q)}$$
$$\leq C\left(\|e^{-\tau\theta\varphi}\mathcal{P}q\|^2_{L^2(Q)} + \|(\tau\theta)^{\frac{3}{2}} e^{-\tau\theta\varphi} q\|^2_{L^2((0,T)\times\omega)}\right)$$

Writing that

$$\|q(0)\|^2_{L^2(\Omega)} \leq C\int_{T/4}^{3T/4} \|q(t)\|^2_{L^2(\Omega)}\,dt \leq C_\tau \|(\tau\theta)^{\frac{3}{2}} e^{-\tau\theta\varphi} q\|^2_{L^2(Q)},$$
$$\leq C'_\tau \|(\tau\theta)^{\frac{3}{2}} e^{-\tau\theta\varphi} q\|^2_{L^2((0,T)\times\omega)},$$

gives the observability inequality.

Goal of the talk

Derivation and analysis of methods to compute (an approximation of) the HUM control for (S).

Our approach : Combine the penalty idea with numerical approximation.

What happens when $\varepsilon \to 0$, $h \to 0$, $\delta t \to 0$?

GOAL OF THE TALK

Derivation and analysis of methods to compute (an approximation of) the HUM control for (S).

OUR APPROACH : Combine the penalty idea with numerical approximation.

What happens when $\varepsilon \to 0$, $h \to 0$, $\delta t \to 0$?

OTHER APPROACHES AND REFERENCES

- Seminal works **(Carthel-Glowinski-Lions, '94) (Glowinski-Lions, '94)**
- Uniform controllability results for 1D heat equation **(Lopez-Zuazua, '98) (Zuazua, '06)**
- Analysis of the problem by using the controllability properties of the continuous problem

  (Labbé-Trélat, '06)
- "Numerical Carleman" approach **(Fernández-Cara – Münch, '10,'11)**

$$\text{Minimize} \left\{ (y,v), \text{s.t. } y_v(T) = 0 \right\} \mapsto \int_0^T \int_\Omega e^{2\theta\varphi} |y|^2 \, dt\, dx + \int_0^T \int_\omega (T-t)^3 e^{2\theta\varphi} |v|^2 \, dt\, dx.$$

- Variational approach **(Münch–Pedregal, '11)**

(Labbé-Trélat, '06) (B.-Hubert-Le Rousseau, '10)

For any $h > 0$ (supposed to be some space discretization parameter) :

- $(E_h, (\cdot,\cdot)_h)$ euclidean space, with norm $|\cdot|_h$.
- $\mathcal{M}_h, \mathcal{A}_h \in L(E_h, E_h)$ which are SDP in $(E_h, (\cdot,\cdot)_h)$.
- Associated scalar products and norms

$$\langle x, y\rangle_h = (\mathcal{M}_h x, y)_h\,, \quad \|x\|_h = \langle x, x\rangle_h^{\frac{1}{2}} = \left|\mathcal{M}_h^{\frac{1}{2}} x\right|_h, \quad \forall x, y \in E_h.$$

- Another Euclidean space $(U_h, [\cdot,\cdot]_h)$, with norm $[\![\cdot]\!]_h$.
- A linear operator $\mathcal{B}_h : U_h \to E_h$, and $\mathcal{B}_h^\star : E_h \to U_h$ its adjoint.
- We shall assume that there exists $C > 0$ such that

$$[\![\mathcal{B}_h^\star x]\!]_h \leq C\, \|x\|_h\,, \quad \forall h > 0, \forall x \in E_h,$$

GENERAL PHILOSOPHY : Choose your favorite scheme !

EXAMPLES

- FD : cartesian meshes, $\mathcal{M}_h = \mathrm{Id}$, $\mathcal{A}_h$=the 5-point discrete Laplacian in 2D
- FV : orthogonal meshes, $\mathcal{M}_h = \mathrm{diag}(|\kappa|)_{\kappa\in\mathcal{T}}$, $\mathcal{A}_h$ =flux balance matrix
- Galerkin : $\mathcal{M}_h$ =mass matrix, $\mathcal{A}_h$ =rigidity matrix, $\mathcal{B}_h = \left(\int_\omega \phi_i \phi_j \, dx\right)$.

THE SEMI-DISCRETE PARABOLIC PROBLEM

$$(S)\begin{cases}\partial_t y + \mathcal{A}y = \mathcal{B}v,\\ y(0) = y_0,\end{cases} \qquad \Longrightarrow \qquad (S_h)\begin{cases}\mathcal{M}_h\partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h,\\ y_h(0) = y_{0,h},\end{cases}$$

SIMPLIFICATION IN THIS TALK : Mass matrix $\mathcal{M}_h = \mathrm{Id}$.

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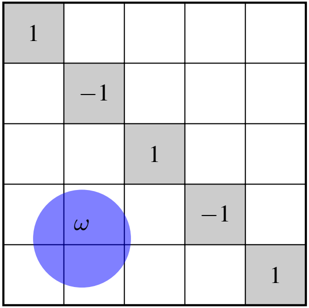
ISSUES

- The semi-discrete problem (S_h) can be non controllable even if (S) is.



(Kavian '01, Zuazua '03)

Indeed, it may exist eigenfunctions satisfiying

$$\mathcal{A}_h^* \psi_h = \mu_h \psi_h, \;\text{ and }\; \mathcal{B}_h^\star \psi_h = 0.$$

$\rightsquigarrow$ Non-controllability since for any v_h we have

$$\frac{d}{dt}\langle y, \psi_h\rangle_h + \mu_h \langle y, \psi_h\rangle_h = 0.$$

- It is certainly a **theoretical difficulty** : what can we do to deal with this issue ?
- Is it an actual difficulty **in practice** since $\mu_h \sim \frac{C}{h^2}$?

THE SEMI-DISCRETE PARABOLIC PROBLEM

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PENALTY + DISCRETIZATION

• Given $y_{0,h} \in E_h$ and $\varepsilon > 0$, let us consider (recall that $\dim E_h < +\infty$)

$$J_{\varepsilon,h} : q_F \in E_h \longmapsto \frac{1}{2}\int_0^T [\![\mathcal{B}_h^\star q_h(t)]\!]_h^2 \, dt + \frac{\varepsilon}{2}\|q_F\|_h^2 + \langle y_{0,h}, q_h(0)\rangle_h \, ,$$

where $t \mapsto q_h(t) \in E_h$ is the solution to $-\partial_t q_h + \mathcal{A}_h^* q_h = 0, \quad q_h(T) = q_F.$

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where $t \mapsto q_h(t) \in E_h$ is the solution to $-\partial_t q_h + \mathcal{A}_h^* q_h = 0, \quad q_h(T) = q_F$.

- We set $v_{\varepsilon,h} = \mathcal{B}_h^* q_{\varepsilon,h}(t)$ and $t \mapsto y_{\varepsilon,h}(t)$ the associated solution to (S_h).
  - For $h > 0$ fixed, we may have

$$\lim_{\varepsilon\to 0} \|y_{\varepsilon,h}(T)\|_h \neq 0.$$

  More precisely

$$\left\| y_{\varepsilon,h}(T) - \text{the non-controllable part of } e^{-T\mathcal{A}_h} y_{0,h} \right\|_h \leq C_h \sqrt{\varepsilon}.$$

THE SEMI-DISCRETE PARABOLIC PROBLEM

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- Given $y_{0,h} \in E_h$ and $\varepsilon > 0$, let us consider (recall that $\dim E_h < +\infty$)

$$J_{\varepsilon,h} : q_F \in E_h \longmapsto \frac{1}{2}\int_0^T [\![\mathcal{B}_h^\star q_h(t)]\!]_h^2 \, dt + \frac{\varepsilon}{2}\|q_F\|_h^2 + \langle y_{0,h}, q_h(0)\rangle_h ,$$

where $t \mapsto q_h(t) \in E_h$ is the solution to $-\partial_t q_h + \mathcal{A}_h^* q_h = 0, \quad q_h(T) = q_F$.

- We set $v_{\varepsilon,h} = \mathcal{B}_h^* q_{\varepsilon,h}(t)$ and $t \mapsto y_{\varepsilon,h}(t)$ the associated solution to (S_h).
  - For $h > 0$ fixed, we may have

$$\lim_{\varepsilon\to 0} \|y_{\varepsilon,h}(T)\|_h \neq 0.$$

  - If (S) is null-controllable, we can hope that for some $C > 0$ and any $\varepsilon > 0$, there exists $h_\varepsilon^* > 0$

$$\|y_{\varepsilon,h}(T)\|_h \leq C\sqrt{\varepsilon}\|y_{0,h}\|_h, \quad \text{for any } h < h_\varepsilon^*,$$

$$v_{\varepsilon,h} \xrightarrow[h\to 0]{} v_\varepsilon, \quad \text{in some sense,}$$

  as soon as $(y_{0,h})_h$ converges to y_0.

- For $h > 0$ fixed, we may have

$$\lim_{\varepsilon \to 0} \|y_{\varepsilon,h}(T)\|_h \neq 0.$$

- We can hope that for some $C > 0$ and any $\varepsilon > 0$, there exists $h^*_\varepsilon > 0$

$$\|y_{\varepsilon,h}(T)\|_h \leq C\sqrt{\varepsilon}\|y_{0,h}\|_h, \quad \text{for any } h < h^*_\varepsilon,$$

$$v_{\varepsilon,h} \xrightarrow[h\to 0]{} v_\varepsilon, \quad \text{in some sense},$$

as soon as $(y_{0,h})_h$ converges to y_0.

QUESTIONS :

1. **Approximate control problem** : Is it possible to give an estimate of h^*_ε ?
2. **Null control problem** : Is it possible to choose $\varepsilon > 0$ as a function of h : $\varepsilon = \phi(h)$ such that

$$\lim_{h\to 0} \|y_{\phi(h),h}(T)\|_h = 0, \quad \|v_{\phi(h),h}\|_{L^2(]0,T[,U_h)} \leq C,$$

and can we estimate those quantities ?

3. If many such $h \mapsto \phi(h)$ exist, how do I choose one ?

$$(\psi_{j,h}, \mu_{j,h})_j \text{ eigenelements of } \mathcal{A}_h,$$
$$E_{\mu,h} = \text{Span}(\psi_{j,h}, \ \ \mu_{j,h} \leq \mu).$$

ASSUMPTION (UNIFORM DISCRETE LEBEAU-ROBBIANO INEQUALITY)

There exists $h_0 > 0$, $\alpha \in [0,1)$, $\beta > 0$, and $\kappa, \ell > 0$ such that

$$\|\psi\|_h^2 \leq \kappa e^{\kappa\mu^\alpha} [\![\mathcal{B}_h^\star \psi]\!]_h^2, \quad \forall \psi \in E_{\mu,h}, \ \forall \mu < \frac{\ell}{h^\beta}, \ \forall h < h_0. \qquad (\mathcal{H}_{\alpha,\beta})$$

FUNDAMENTAL REMARK

For dimension reasons, such an inequality **can not** be true for any $\mu > 0$, that is for all ψ in the whole E_h.

(B.-Hubert-Le Rousseau '09,'10)

We proved that the uniform discrete Lebeau-Robbiano inequality $(\mathcal{H}_{\alpha,\beta})$ holds for

- Finite difference schemes on regular Cartesian meshes in any dimension.
- A scalar elliptic operator $\mathcal{A}$ with diagonal diffusion tensor.
- Distributed control problem $\mathcal{B}_h = 1_\omega$.
- We obtain :
  - $\alpha = 1/2$ (i.e. the constant is $\sim e^{\sqrt{\mu}}$).
  - $\beta = 2$ (related to α and to the order of the differential operator).

MAIN TOOL : **Global semi-discrete elliptic Carleman estimates**

THEOREM

There exists $C > 0$, $h_0 > 0$, $s_0 > 0$, $\varepsilon_0 > 0$ such that

$$s^3 \|e^{s\varphi} u_h\|^2_{L^2(]0,T_*[,E_h)} + s^3 e^{2s\varphi(T_*)} \|u_h(T_*,.)\|_h \leq C\|e^{s\varphi}(-\partial_t^2 + \mathcal{A}_h)u_h\|^2_{L^2(]0,T_*[,E_h)}$$
$$+ Cse^{2s\varphi(T_*)} \|\nabla_h u_h(T_*)\|_h^2 + Cse^{s\varphi(0,.)} \|\mathbf{1}_\omega \partial_t u_h(0)\|_h^2,$$

for all $s \geq s_0$, $0 < h \leq h_0$ and $\boldsymbol{sh \leq \varepsilon_0}$, *and $u_h \in \mathscr{C}^2([0,T_*],E_h)$.*

THEN CHOOSE $u_h(t) = \sum_{\mu_j \leq \mu} \alpha_j \frac{\sinh(\sqrt{\mu_j} t)}{\sqrt{\mu_j}} \psi_{j,h}$ and $s \sim \sqrt{\mu} \Leftarrow$ restriction on μ.

WHY FINITE DIFFERENCES ?

- The proof uses *discrete differential calculus* :
  - Conjugate the operator with weights
  - Compute the square of the new equation.
  - Integrate by parts a lot of times
  - 4th order operators appear
- This computations seem difficult to perform for
  - Finite Volume : flux balance formalism ...
  - Galerkin / Finite element : variational formulation ...

WHY *sh* HAS TO BE SMALL ENOUGH ?

- Continuous level

$$e^{-s\phi}\partial_x^2\left(e^{s\phi}u\right) = \partial_x^2 u + \text{l.o.t.}$$

- Discrete level

$$e^{-s\phi}\mathcal{A}_h\left(e^{s\phi}u\right) = \left(e^{-s\phi}\overline{\overline{e^{s\phi}}}\right)\mathcal{A}_h u + \text{l.o.t},$$

with

$$\left(\overline{\overline{f}}\right)_i = \frac{f_{i+1}+2f_i+f_{i-1}}{4} = f_i + \left(\frac{h^2}{4}\frac{f_{i+1}-2f_i+f_{i-1}}{h^2}\right).$$

Thus,

$$\left(e^{-s\phi}\overline{\overline{e^{s\phi}}}\right) = 1 + h^2\partial_x^2(e^{s\phi}) + \ldots = 1 + O\left((hs)^2\right).$$

(Carthel-Glowinski-Lions, '94) (Glowinski-Lions, '94)

$$(S_h)\begin{cases}\partial_t y + \mathcal{A}_h y = \mathcal{B}_h v,\\ y_h(0) = y_0.\end{cases}$$

Consider the approximate control problem for (S_h) by penalty introducing

$$q_F \in E_h \mapsto J_{\varepsilon,h}(q_F) = \frac{1}{2}\int_0^T [\![\mathcal{B}_h^\star q(t)]\!]_h^2\, dt + \frac{\varepsilon}{2}\|q_F\|_h^2 + \langle y_0, q(0)\rangle_h .$$

We denote by $q_{F,\varepsilon,h}$ its minimizer and $t \mapsto q_{\varepsilon,h}(t)$ the associated adjoint state.

THEOREM

Assume that the uniform discrete Lebeau-Robbiano inequality $(\mathcal{H}_{\alpha,\beta})$ holds, then there exists $h_0 > 0$ and constants $C, C_{\text{obs}} > 0$ such that :

- *For any $h < h_0$, and $\varepsilon > e^{-C/h^\beta}$, the control $v_{\varepsilon,h}(t) = \mathcal{B}_h^\star q_{\varepsilon,h}(t)$ is such that*

$$\|v_{\varepsilon,h}\|_{L^2(0,T;U_h)} \leq C_{\text{obs}}, \quad \textit{and} \quad \|y_{\varepsilon,h}(T)\|_h \leq C_{\text{obs}}\sqrt{\varepsilon}.$$

ASSOCIATED RELAXED OBSERVABILITY INEQUALITY

$$\left\{\begin{array}{c}\forall h < h_0, \forall \varepsilon > e^{-C/h^\beta}\\ \forall \varepsilon < \varepsilon_0, \forall h < \frac{C'}{|\log \varepsilon|^{1/\beta}}\end{array}\right\}, \forall q_F \in E_h,\ \|q(0)\|_h^2 \leq C_{\text{obs}}^2 \left(\int_0^T [\![\mathcal{B}_h^\star q(t)]\!]_h^2\, dt + \varepsilon \|q_F\|_h^2\right).$$

COMPUTATION OF AN APPROXIMATE CONTROL FOR $\varepsilon > 0$ FIXED :

The sequence $(v_{\varepsilon,h})_h$ converges towards the HUM-penalized control v_ε for (S).

COMPUTATION OF A NULL-CONTROL :

Choose a function $h \mapsto \phi(h)$ such that $\phi(h) > e^{-C/h^\beta}$ for any h.
The sequence $(v_{\phi(h),h})_h$ converges, at least weakly, towards a null-control for (S) and we have

$$\left\|y_{\phi(h),h}(T)\right\|_h \leq C_{\text{obs}}\sqrt{\phi(h)}, \quad \forall 0 < h < h_0.$$

- Recall that, in general, a null-control for (S_h) does not exist

  $\Rightarrow$ Taking $\varepsilon = 0$ is meaningless.
- Taking $\varepsilon = \phi(h)$ exponentially small is theoretically possible but

  this is not reasonable and in fact completely useless.
- In practice, choosing $\phi(h) = h^{2p}$ for some p related to the approximation order p of the scheme under study is sufficient.

See some numerical illustrations later

Outline

(B. - Le Rousseau, '12)

FINITE DIFFERENCES ON REGULAR (MULTI-D) CARTESIAN MESHES

We set $\theta(t) = (t + \alpha h)^{-1}(T - t + \alpha h)^{-1}$.

THEOREM

For any $\tau > 0$ large enough, there exists $\alpha > 0$ and $h_0 > 0$ such that for any function q and any $h < h_0$ we have

$$\|(\tau\theta)^{\frac{1}{2}} e^{-\tau\theta\varphi} D_h q\|^2_{L^2(Q)} + \|(\tau\theta)^{\frac{3}{2}} e^{-\tau\theta\varphi} q\|^2_{L^2(Q)}$$
$$\leq C\Big(\|e^{-\tau\theta\varphi}(-\partial_t + \mathcal{A}_h^*)q\|^2_{L^2(Q)} + \|(\tau\theta)^{\frac{3}{2}} e^{-\tau\theta\varphi} q\|^2_{L^2((0,T)\times\omega)}\Big)$$
$$+ Ch^{-2}\Big(|e^{-\tau\theta\varphi} q_{|t=0}|^2_{L^2(\Omega)} + |e^{-\tau\theta\varphi} q_{|t=T}|^2_{L^2(\Omega)}\Big)$$

THEOREM (RELAXED OBSERVABILITY INEQUALITY)

There exists $C > 0$ s.t. for any function $a_h \in L^\infty(]0, T[, E_h)$, and any $h \leq \min(h_0, h_1)$ with $h_1 \sim \|a_h\|_\infty^{-\frac{2}{3}}$, any solution of $-\partial_t q + \mathcal{A}_h^* q + a_h q = 0$ satisfies

$$|q(0)|^2_{L^2(\Omega)} \leq C_{\text{obs}} \|q\|^2_{L^2((0,T)\times\omega)} + e^{-\frac{C^{-1}}{h} + T\|a_h\|_\infty} |q(T)|^2_{L^2(\Omega)}.$$

with $C_{\text{obs}} = e^{C(1 + \frac{1}{T} + T\|a_h\|_\infty + \|a_h\|_\infty^{\frac{2}{3}})}$.

(Fernández-Cara – Zuazua, '00), (B. – Le Rousseau, '12)

$$\partial_t y_h + \mathcal{A}_h y_h + g(y_h) y_h = \mathbf{1}_\omega v_h, \quad y_h(0) = y_{0,h},$$

SUBLINEAR CASE : $|g(s)| \leq M$

There exists $C > 0$, such that for any initial data $y_{0,h} \in E_h$, and any $h < h_0$, there exists a semi-discrete control v_h such that

$$\|v_h\|_{L^2(]0,T[,U_h)} \leq C \|y_{0,h}\|_h, \quad \text{and} \quad \|y_h(T)\|_h \leq C e^{-\frac{C^{-1}}{h}} \|y_0\|_h .$$

(Fernández-Cara – Zuazua, '00), (B. – Le Rousseau, '12)

$$\partial_t y_h + \mathcal{A}_h y_h + g(y_h) y_h = \mathbf{1}_\omega v_h, \quad y_h(0) = y_{0,h},$$

SUPERLINEAR CASE : $|g(s)| \leq M \ln(1+|s|)^r$, $r < 3/2$

- In 1D : For any initial data $y_{0,h} \in E_h$ and $h < h_0$ there exists a v_h such that

$$\|v_h\|_{L^2(]0,T[,U_h)} \leq C_{\|y_{0,h}\|_h}, \quad \text{and} \quad \|y_h(T)\|_h \leq C_{\|y_{0,h}\|_h} e^{-\frac{C^{-1}}{h}}.$$

- In multi-D : same result but with a non-uniform bound of the control

$$\|v_h\|_{L^2(]0,T[,U_h)} \leq C_{\|y_{0,h}\|_h} h^{-\alpha}, \quad \text{and} \quad \|y_h(T)\|_h \leq C_{\|y_{0,h}\|_h} e^{-\frac{C^{-1}}{h}}.$$

N.B. : it is known that for $r > 2$ the problem is not null-controllable.

LINEARIZATION + FIXED-POINT PROCEDURE

$$(S_{z_h}) : \partial_t y_h + \mathcal{A}_h y_h + \underbrace{g(z_h)}_{=a_h} y_h = \mathbf{1}_\omega v_h,$$

$$\Lambda_h : z_h \in \text{Some space} \longmapsto v_h \text{the HUM-pen. control for } (S_{z_h})$$

$$\longmapsto y_h \in \text{the same space as } z_h.$$

OUTLINE

We have seen that some uniform approximate/null controllability properties hold for

$$(S_h)\begin{cases}\partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h,\\ y_h(0) = y_{0,h}.\end{cases}$$

WHAT ABOUT TIME DISCRETIZATION OF SUCH A SYSTEM ?

- We study **unconditionally stable schemes** : the implicit Euler scheme and the Crank-Nicolson scheme (in fact any θ-scheme with $\theta \in [1/2, 1]$).

$$(S_{h,\delta t})\begin{cases}y^0 = y_0,\\ \dfrac{y^{n+1}-y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta)y^n) = \mathcal{B}_h v^{n+1},\ \forall n \in [\![0, M-1]\!]\end{cases}$$

- We show that most of the results for the semi-discrete situation holds for fully-discrete systems uniformly in δt and h (provided δt is not too large with respect to h, *this will be made precise below*).
- Finally, we show that, for a fixed $h > 0$,

$$v_{h,\delta t} \xrightarrow[\delta t \to 0]{} v_h.$$

(Zheng, '08), (Ervedoza-Valein, '10)

THE PRIMAL OPTIMIZATION PROBLEM : Minimize the following functional

$$F_{\varepsilon,h,\delta t} : v \in U_h^M \longmapsto \frac{1}{2}\sum_{n=1}^{M} \delta t \, [\![v^n]\!]_h^2 + \frac{1}{2\varepsilon} \|\mathcal{L}(y_0, v)\|_h^2 ,$$

where $\mathcal{L}(y_0, v) = \mathcal{L}_0 y_0 + \mathcal{L}_v v$ is the value of y^M for the corresponding solution of

$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta) y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

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where $\mathcal{L}(y_0, v) = \mathcal{L}_0 y_0 + \mathcal{L}_v v$ is the value of y^M for the corresponding solution of

$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta)y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

DUAL OPTIMIZATION PROBLEM : General duality theory gives

$$J_{\varepsilon,h,\delta t} : q_F \in E_h \longmapsto \frac{1}{2}\sum_{n=1}^{M} \delta t \, \left[\!\left[(\mathcal{L}_v^* q_F)^n\right]\!\right]_h^2 + \frac{\varepsilon}{2} \|q_F\|_h^2 + \langle y_0, \mathcal{L}_0^* q_F \rangle_h .$$

$$\boxed{\text{Argmin}\, F_{\varepsilon,h,\delta t} = \mathcal{L}_v^* \left(\text{Argmin}\, J_{\varepsilon,h,\delta t}\right).}$$

ASSOCIATED (RELAXED) OBSERVABILITY INEQUALITY

$$\|\mathcal{L}_0^* q_F\|_h^2 \leq C_{\text{obs}}^2 \sum_{n=1}^{M} \delta t \, \left[\!\left[(\mathcal{L}_v^* q_F)^n\right]\!\right]_h^2 + ??? \, \|q_F\|_h^2 .$$

We defined $\mathcal{L}(y_0, v) = \mathcal{L}_0 y_0 + \mathcal{L}_v v = y^M$, where $(y^n)_n$ is given by

$$(S_{h,\delta t}) \begin{cases} y^0 = y_0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta)y^n) = \mathcal{B}_h v^{n+1}. \end{cases}$$

A STRAIGHTFORWARD, BUT NECESSARY, COMPUTATION LEADS TO :

- Given $q_F \in E_h$, we solve the following backward θ-scheme-like

$$(S^*_{h,\delta t}) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta)q^{n+1}) = 0, \quad \forall n \in [\![1, M-1]\!]. \end{cases}$$

then, we have

$$\begin{cases} \mathcal{L}_0^* q_F = q^1 - \delta t(1-\theta)\mathcal{A}_h q^1, \\ (\mathcal{L}_v^* q_F)^n = \mathcal{B}_h^\star q^n, \quad \forall n \in [\![1, M]\!]. \end{cases}$$

The dual functional that we will thus consider is the following

$$J_{\varepsilon,h,\delta t} : q_F \in E_h \longmapsto \frac{1}{2}\sum_{n=1}^{M} \delta t \, [\![\mathcal{B}_h^\star q^n]\!]_h^2 + \frac{\varepsilon}{2}\|q_F\|_h^2 - \left\langle y_0, q^1 - \delta t(1-\theta)\mathcal{A}_h q^1 \right\rangle_h,$$

where $(q^n)_n$ is defined by

$$(S^*_{h,\delta t}) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta\mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta)q^{n+1}) = 0, \ \ \forall n \in [\![1, M-1]\!]. \end{cases}$$

For $q_F \in E_h$ given, the adjoint problem associated with the time discretisation proposed is given by

$$(S^*_{h,\delta t}) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta) q^{n+1}) = 0, \quad \forall n \in [\![1, M-1]\!]. \end{cases}$$

THEOREM (THE CASE $\theta > 1/2$)

Assume that the uniform discrete L-R inequality $(\mathcal{H}_{\alpha,\beta})$ holds, choose $0 < \gamma \leq \beta$ and $C_T > 0$. For any $\delta t \leq C_T h^\gamma$ the following relaxed observability inequality holds

$$\left\| q^1 - \mathcal{A}_h q^1 \right\|_h^2 \leq C_{\text{obs}}^2 \sum_{n=1}^{M} \delta t \, [\![\mathcal{B}_h^\star q^n]\!]_h^2 + C e^{-C/h^\gamma} \|q_F\|_h^2 .$$

Thus, for any such δt, there exists a full-discrete control $v_{h,\delta t}$ s.t.

$$\sum_{n=1}^{M} \delta t \, [\![v^n]\!]_h^2 \leq C_{\text{obs}}^2 \|y_0\|_h^2, \quad \text{and} \quad \left\| y^M \right\|_h \leq C_{\text{obs}} e^{-C/h^\gamma} \|y_0\|_h .$$

For $q_F \in E_h$ given, the adjoint problem associated with the time discretisation proposed is given by

$$(S^*_{h,\delta t}) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta) q^{n+1}) = 0, \ \ \forall n \in [\![1, M-1]\!]. \end{cases}$$

THEOREM (THE CRANK-NICOLSON SCHEME - $\theta = 1/2$)

Assume that the uniform discrete L-R inequality ($\mathcal{H}_{\alpha,\beta}$) holds, choose $0 < \gamma \leq \beta$ and $C_T > 0$, and $\delta > 0$. For any $\delta t \leq C_T h^\gamma$ and $\delta t \rho(\mathcal{A}_h) \leq \delta$ the following relaxed observability inequality holds

$$\left\| q^1 - \delta t (1-\theta) \mathcal{A}_h q^1 \right\|_h^2 \leq C_{\text{obs}}^2 \sum_{n=1}^{M} \delta t \, [\![\mathcal{B}_h^\star q^n]\!]_h^2 + C e^{-C/h^\gamma} \| q_F \|_h^2 .$$

Thus, for any such δt, there exists a full-discrete control $v_{h,\delta t}$ s.t.

$$\sum_{n=1}^{M} \delta t \, [\![v^n]\!]_h^2 \leq C_{\text{obs}}^2 \| y_0 \|_h^2 , \quad \textit{and} \quad \left\| y^M \right\|_h \leq C_{\text{obs}} e^{-C/h^\gamma} \| y_0 \|_h .$$

For $q_F \in E_h$ given, the adjoint problem associated with the time discretisation proposed is given by

$$(S^*_{h,\delta t}) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta) q^{n+1}) = 0, \ \ \forall n \in [\![1, M-1]\!]. \end{cases}$$

THEOREM (THE CASE $\theta > 1/2$ - USEFUL STATEMENT)

Assume that the uniform discrete L-R inequality $(\mathcal{H}_{\alpha,\beta})$ holds and let $h \mapsto \phi(h)$ such that $\phi(h) \geq e^{-C/h^\beta}$. For any $\delta t \leq C_T |\log \phi(h)|$ the following relaxed observability inequality holds

$$\left\| q^1 - \delta t(1-\theta)\mathcal{A}_h q^1 \right\|_h^2 \leq C_{\text{obs}}^2 \sum_{n=1}^{M} \delta t \, [\![\mathcal{B}_h^\star q^n]\!]_h^2 + \phi(h) \, \|q_F\|_h^2 .$$

Thus, for any such δt, there exists a full-discrete control $v_{h,\delta t}$ s.t.

$$\sum_{n=1}^{M} \delta t \, [\![v^n]\!]_h^2 \leq C_{\text{obs}}^2 \, \|y_0\|_h^2 , \ \ \textit{and} \ \ \left\| y^M \right\|_h \leq C_{\text{obs}} \sqrt{\phi(h)} \, \|y_0\|_h .$$

For $q_F \in E_h$ given, the adjoint problem associated with the time discretisation proposed is given by

$$(S^*_{h,\delta t}) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta) q^{n+1}) = 0, \ \ \forall n \in [\![1, M-1]\!]. \end{cases}$$

Theorem (Crank-Nicolson - $\theta = 1/2$ - useful (?) statement)

Assume that the uniform discrete L-R inequality $(\mathcal{H}_{\alpha,\beta})$ holds and let $h \mapsto \phi(h)$ such that $\phi(h) \geq e^{-C/h^\beta}$. For any $\delta t \leq C_T |\log \phi(h)|$ and $\delta t \rho(\mathcal{A}_h) \leq \delta$ the following relaxed observability inequality holds

$$\left\| q^1 - \frac{\delta t}{2} \mathcal{A}_h q^1 \right\|_h^2 \leq C_{\text{obs}}^2 \sum_{n=1}^{M} \delta t \, [\![\mathcal{B}_h^\star q^n]\!]_h^2 + \phi(h) \, \|q_F\|_h^2 .$$

Thus, for any such δt, there exists a full-discrete control $v_{h,\delta t}$ s.t.

$$\sum_{n=1}^{M} \delta t \, [\![v^n]\!]_h^2 \leq C_{\text{obs}}^2 \, \|y_0\|_h^2 , \ \ \textit{and} \ \ \left\| y^M \right\|_h \leq C_{\text{obs}} \sqrt{\phi(h)} \, \|y_0\|_h .$$

OUTLINE

• We analyse the error induced by time discretisation in this problem.

THEOREM

Under the same assumptions as in previous results, the following error estimate holds

$$\left\| v_h - \sum_{n=1}^{M} 1_{(t^{n-1},t^n)} v^n \right\|_{L^2(]0,T[,U_h)} \leq C\delta t \frac{\rho(\mathcal{A}_h)}{\sqrt{\phi(h)}} \left(1 + \delta t^{\frac{3}{2}} \rho(\mathcal{A}_h)^{\frac{3}{2}}\right) \|y_0\|_h .$$

REMARKS

- The estimate is not uniform in h, even if we are interested in the approximate control problem where $\phi(h) = \varepsilon > 0$.
  $\rightsquigarrow$ The above result is probably not optimal.
- We have a similar **second order** estimate for CN provided a suitable time interpolation operator is used.

The functional we want to minimise reads (ε is fixed or $\varepsilon = \phi(h)$)

$$J_{\varepsilon,h,\delta t} : q_F \in E_h \longmapsto \frac{1}{2}\sum_{n=1}^{M} \delta t \left[\!\left[(\mathcal{L}_v^* q_F)^n \right]\!\right]_h^2 + \frac{\varepsilon}{2} \|q_F\|_h^2 + \langle y_0, \mathcal{L}_0^* q_F \rangle_h .$$

We solve this problem by a conjugate gradient (in $(E_h, \langle .,. \rangle_h)$). **(Glowinski-Lions, '94)**

COMPUTATION OF THE GRADIENT

$$\nabla J_{\varepsilon,h,\delta t}(q_F) = \mathcal{L}_v \mathcal{L}_v^* q_F + \varepsilon q_F + \mathcal{L}_0 y_0 = (\mathcal{L}_v \mathcal{L}_v^* + \varepsilon \mathrm{Id}) q_F + \mathcal{L}_0 y_0,$$

COMPUTATION OF THE HUM OPERATOR : for q_F given

$$(S^*_{h,\delta t}) \begin{cases} q^{M+1} = q_F, \\ \dfrac{q^M - q^{M+1}}{\delta t} + \theta \mathcal{A}_h q^M = 0, \\ \dfrac{q^n - q^{n+1}}{\delta t} + \mathcal{A}_h(\theta q^n + (1-\theta) q^{n+1}) = 0, \ \ \forall n \in [\![1, M-1]\!], \end{cases}$$

then by solving

$$(S_{h,\delta t}) \begin{cases} y^0 = 0, \\ \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h(\theta y^{n+1} + (1-\theta) y^n) = \mathcal{B}_h \underbrace{\mathcal{B}_h^\star q^{n+1}}_{=(\mathcal{L}_v^* q_F)^n}, \ \forall n \in [\![0, M-1]\!], \end{cases}$$

and we finally have $\mathcal{L}_v \mathcal{L}_v^* q_F = y^M$.

- Advantages
  - Many time stepping schemes can be adapted (higher order methods like BDF2 or RK3, RK4, etc ...).
  - Any reasonable space discretization method for any space dimension can be chosen, independently.
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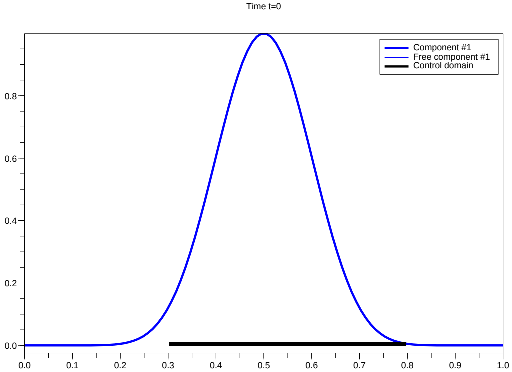
  - Comments
    - For $\varepsilon = \phi(h)$, the problem is not so ill-posed **but** preconditioning is a very important and chalenging issue.
    - Computational time of each CG iteration can be large and memory consuming : use of parareal algorithms can be useful. **(Lions-Maday-Turinici, ...)**

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$



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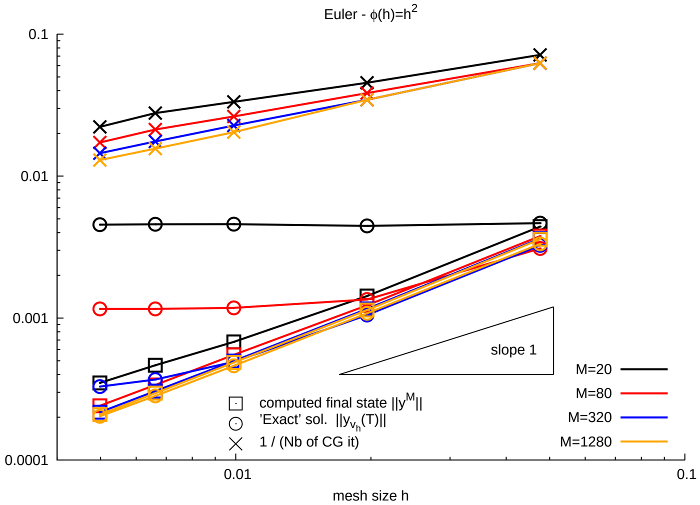

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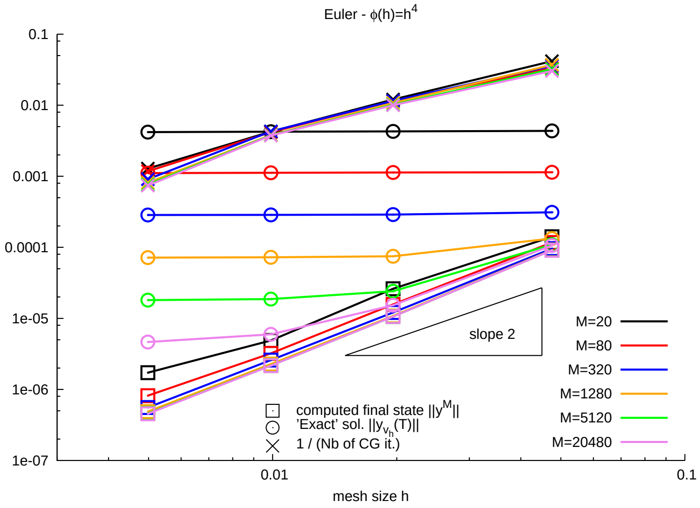

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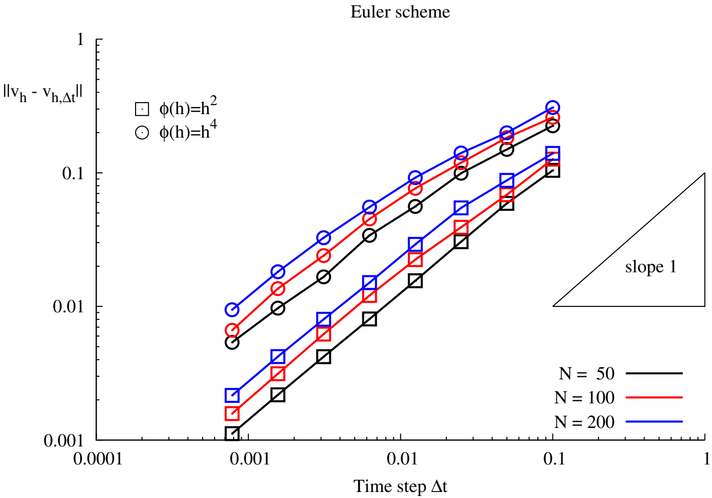

$$\partial_t y - 0.1\partial_x^2 y - 1.5y = 1_{]0.3,0.8[} v,$$

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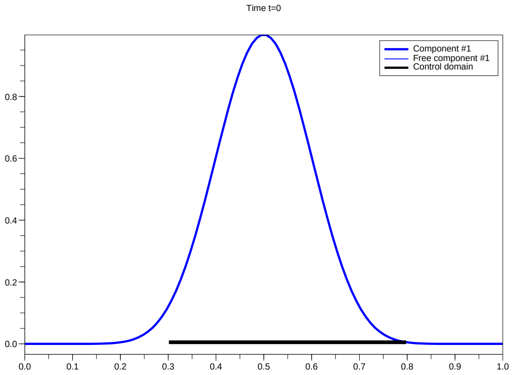

(Fernández-Cara – Münch, '11)

$$\partial_t y - 0.1\partial_x^2 y - 5y\log^{1.4}(1+|y|) = 1_{]0.2,0.8[}v,$$
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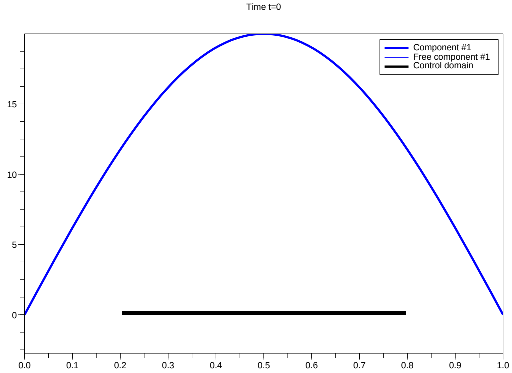

PICARD ITERATIONS WITH RELAXATION In order to solve $\Lambda y = y$, we use

$$y^{k+1} = \omega(\Lambda y^k) + (1-\omega)y^k, \quad \forall k \geq 0$$

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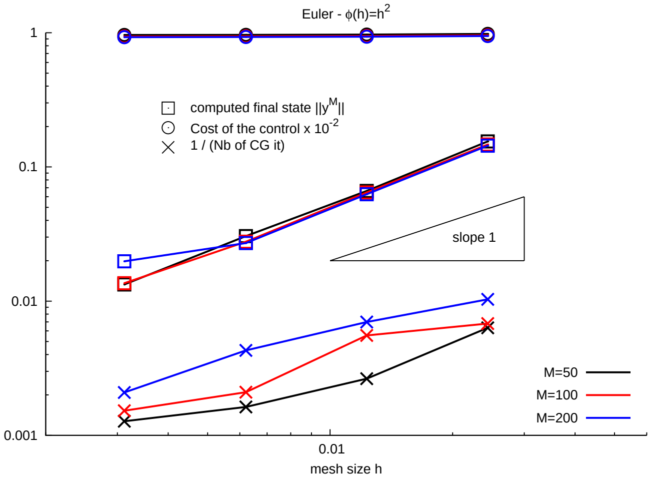

(De Teresa – González-Burgos, '08) (Ammar-Khodja – Benabdallah – Dupaix – González-Burgos, '09)

$$\partial_t y - \partial_x \left(\begin{pmatrix} 0.1 & 0 \\ 0 & 0.1(2+\sin(4x)) \end{pmatrix} \partial_x y \right) + \begin{pmatrix} 0 & 1_{]\mathbf{0.5},\mathbf{0.8}[}(x) \\ 0 & 0 \end{pmatrix} y = \begin{pmatrix} 0 \\ 1_{]\mathbf{0.2},\mathbf{0.8}[}(x) \end{pmatrix} v.$$

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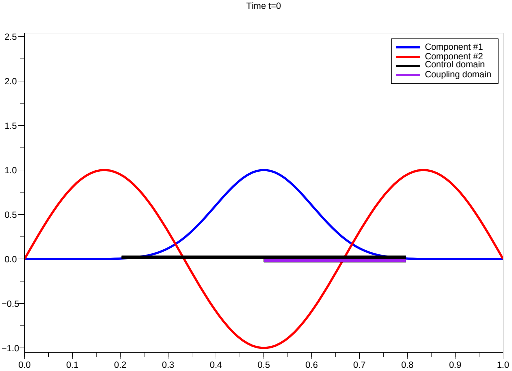

NUMERICAL PARAMETERS :

$N = 100$, uniform mesh, Euler scheme $M = 200$, $\phi(h) = h^4$.

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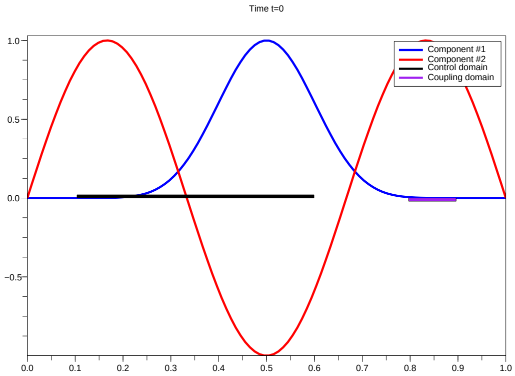

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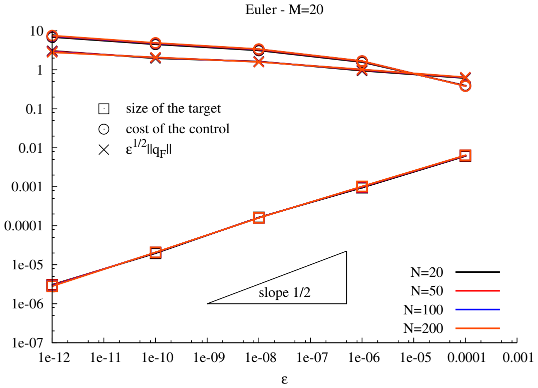

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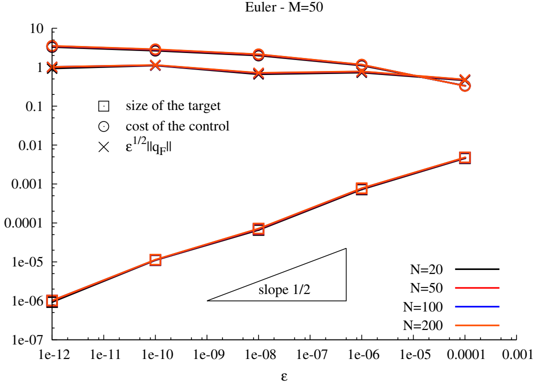

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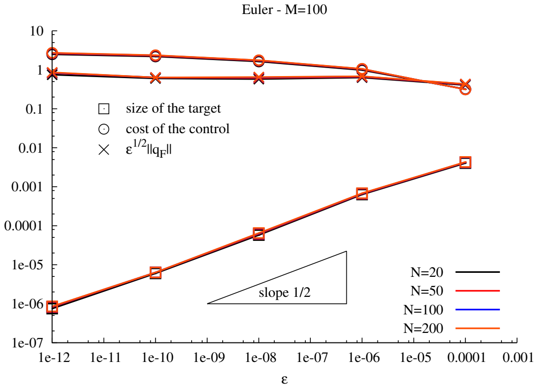

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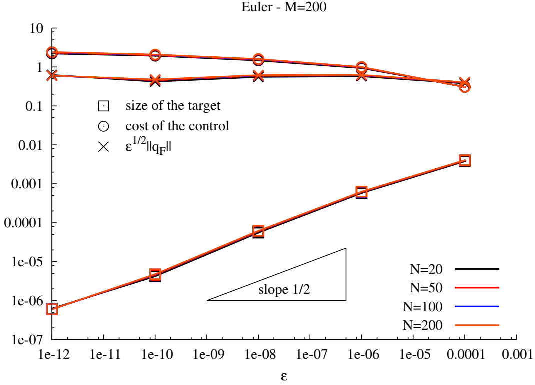

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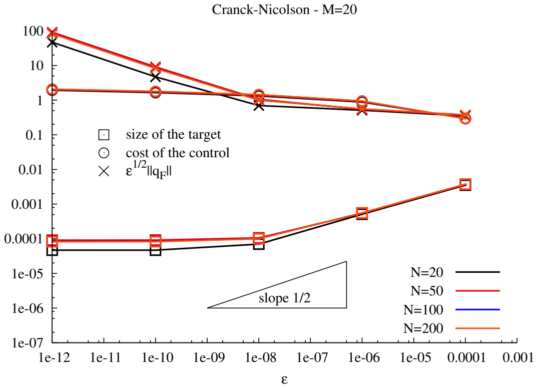

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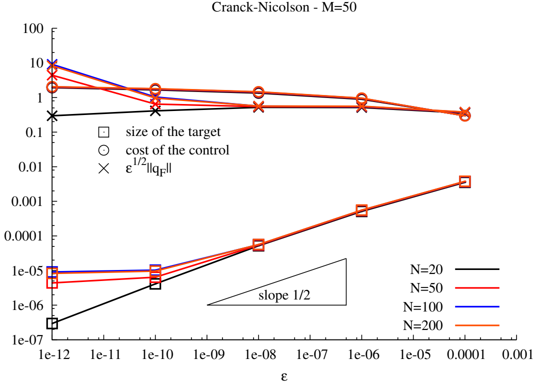

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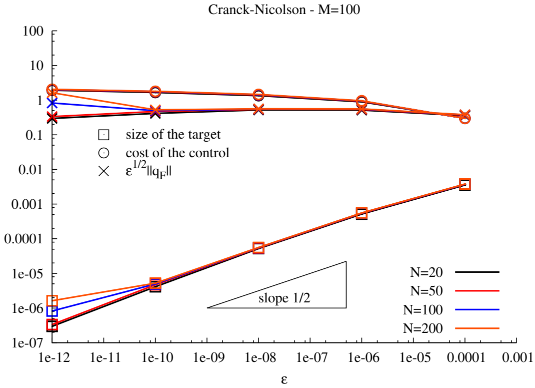

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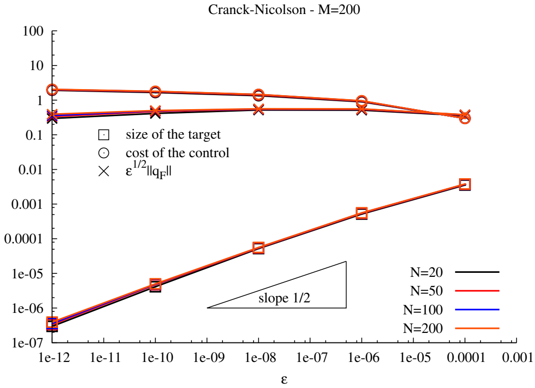

(Ammar-Khodja – Benabdallah – Dupaix – González-Burgos, '09)

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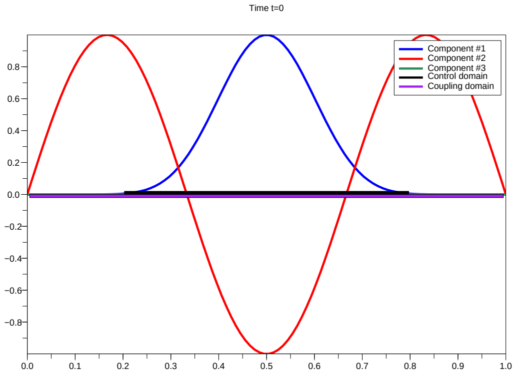

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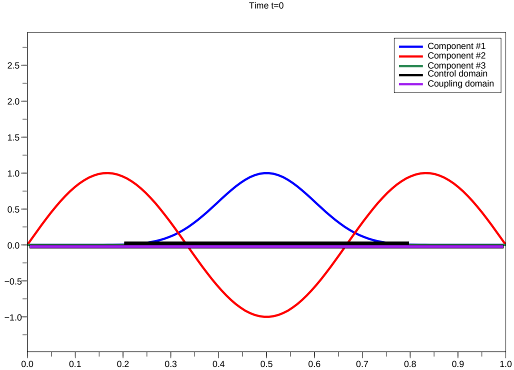

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(Benabdallah – Cristofol – De Teresa – Gaitan, '10)

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$\Omega =]0,1[$

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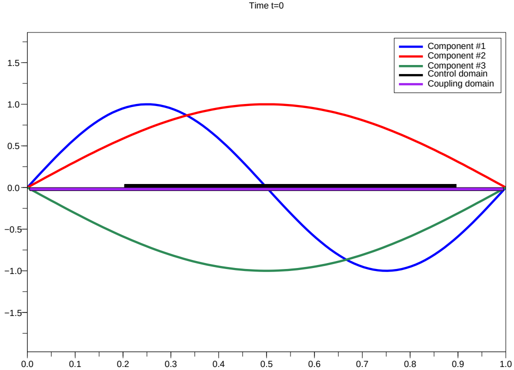
$$y_0(x) = \begin{pmatrix} \sin(2\pi x) \\ \sin(\pi x) \\ -\sin(\pi x) \end{pmatrix}.$$



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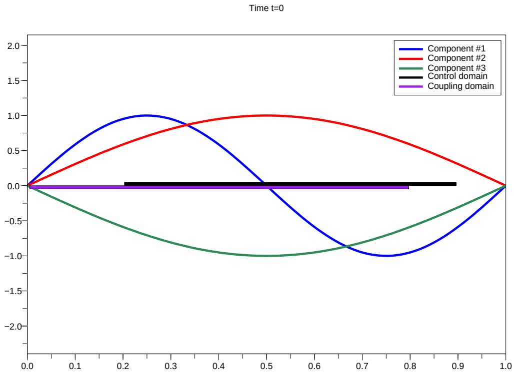

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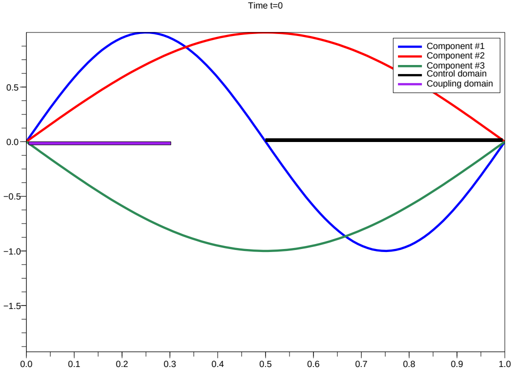

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ZOOM

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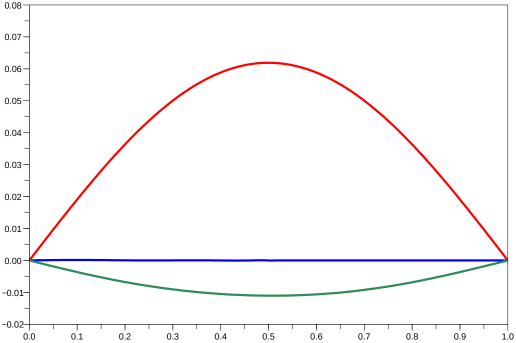

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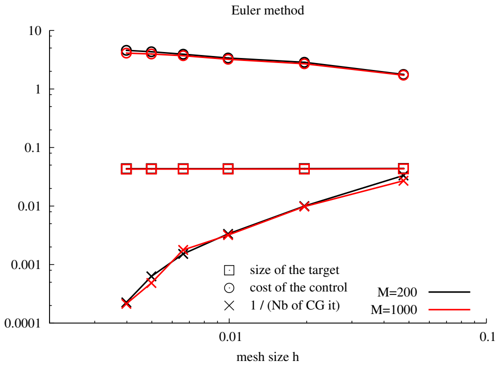

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$$\partial_t y - \frac{1}{\pi^2}\partial_x\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1.2 & 0\\ 0 & 0 & 3\end{pmatrix}\partial_x y\right) = 1_{]0.2,0.8[}\begin{pmatrix}1\\ 2\\ -1\end{pmatrix} v.$$

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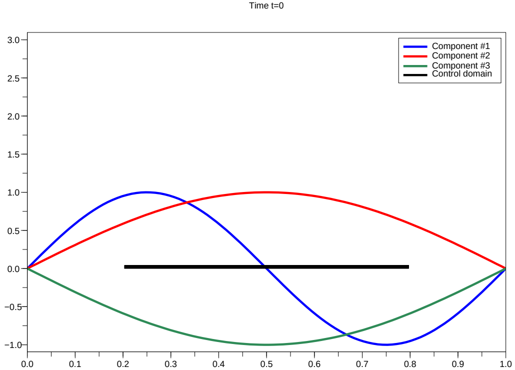

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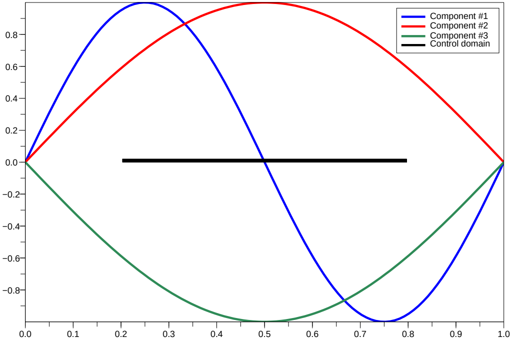
$$\partial_t y - \partial_x \left(\frac{1}{\pi^2} \begin{pmatrix} 2+5\times 1_{]0,0.2[} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & (2-1.8\times 1_{]0.8,1[}) \end{pmatrix} \partial_x y \right) = 1_{]0.2,0.8[} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} v.$$

$$\Omega =]0,1[$$

$$T = 2$$

$$y_0(x) = \begin{pmatrix} \sin(2\pi x) \\ \sin(\pi x) \\ -\sin(\pi x) \end{pmatrix}.$$

Time t=0



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$$\partial_t y - \partial_x \left(\frac{1}{\pi^2} \begin{pmatrix} 2 + 5 \times 1_{]0.2,0.6[} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & (2 - 1.8 \times 1_{]0.5,0.8[}) \end{pmatrix} \partial_x y \right) = 1_{]0.2,0.8[} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} v.$$

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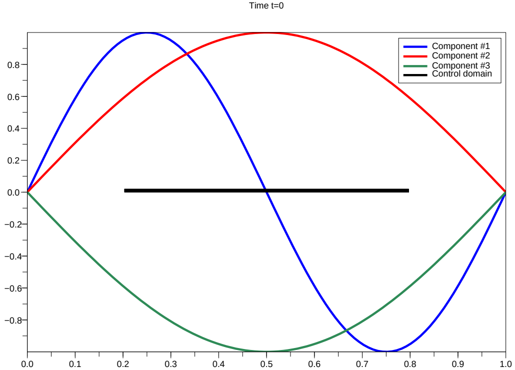
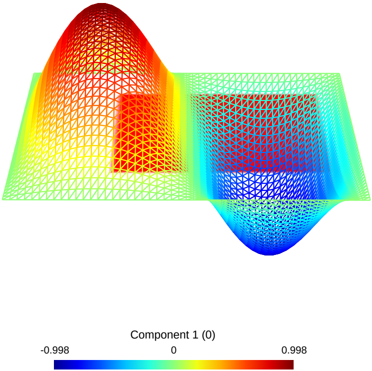
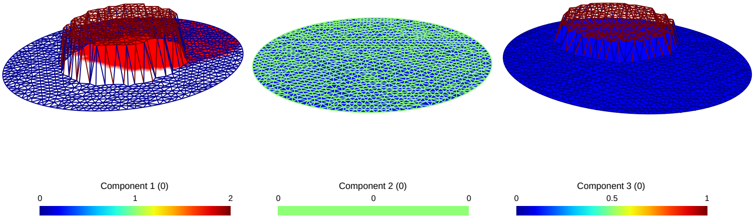


PARAMETERS : $N = 100$, uniform mesh, Euler scheme $M = 200$, $\phi(h) = h^4$.

$$\partial_t y - 0.05\Delta y = 1_{]0.3,0.9[\times]0.2,0.8[} v,$$
$$y(0,x) = \sin(2\pi x_1)\sin(\pi x_2), \quad \text{and} \quad y_F(x) = -0.4\sin(\pi x_1)\sin(2\pi x_2).$$



$$\partial_t y - \begin{pmatrix} 1/\pi^2 & 0 & 0 \\ 0 & 1/\pi^2 & 0 \\ 0 & 0 & 2/\pi^2 \end{pmatrix} \Delta y + \begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & -1 \\ 2 & -1 & -3 \end{pmatrix} y = \begin{pmatrix} 1_\omega \\ 0 \\ 0 \end{pmatrix} v,$$



SUMMARY

- Analysis of uniform controllability properties with respect to δt and/or h for semi/fully discrete problems.
  - Elliptic discrete Carleman estimates
  - Parabolic discrete Carleman estimates
  - Optimal relaxed observability inequalities.
  - Error analysis in time.
- We may use numerical simulations to investigate open problems.

PERSPECTIVES

- Extend our analysis to other cases
  - Non symmetric scalar operators.
  - Systems with few controls.
  - Boundary control problems.
  - The fully discrete problem for semilinear problems.
  - Analysis for other space discretizations (Finite Volume, Finite Element, ...)
- From a numerical point of view
  - A deeper understanding of the structure of the HUM operator should lead to reasonable preconditioning methods.
  - Is there more suitable solvers than standard Conjugate Gradient ?
  - How to compute efficiently the control for semi-linear problems ?

That's all folks !

$$F_\varepsilon : v \in L^2(]0,T[,U) \mapsto \frac{1}{2}\int_0^T \|v(t)\|_U^2\,dt + \frac{1}{2\varepsilon}\|y_v(T)\|_E^2.$$

$$J_\varepsilon : q_F \in E \mapsto \frac{1}{2}\int_0^T \|\mathcal{B}^* q(t)\|_U^2\,dt + \frac{\varepsilon}{2}\|q_F\|_E^2 + (y_0, q(0))_E.$$

- Existence and uniqueness of minimizers v_ε and $q_{F,\varepsilon}$ is standard (convexity).
- Euler-Lagrange equation for J_ε (notice that $\inf_{q_F} J_\varepsilon \leq 0$)

$$0 = \int_0^T (\mathcal{B}\mathcal{B}^* q_\varepsilon(t), \tilde{q}(t))_E\,dt + \varepsilon(q_{F,\varepsilon}, \tilde{q}_F)_E + (y_0, \tilde{q}(0))_E, \quad \forall \tilde{q} \text{ sol of (ADJ)}.$$

We set $v_\varepsilon = \mathcal{B}^* q_\varepsilon$, and we compute

$$\int_0^T (\partial_t y_{v_\varepsilon} + \mathcal{A} y_{v_\varepsilon}, \tilde{q})_E\,dt = \int_0^T (\mathcal{B}\mathcal{B}^* q_\varepsilon, \tilde{q})_E\,dt$$

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$$(y_{v_\varepsilon}(T), \tilde{q}_F)_E - (y_0, \tilde{q}(0))_E + \int_0^T \big(y_{v_\varepsilon}, \underbrace{-\partial_t \tilde{q} + \mathcal{A}^* \tilde{q}}_{=0}\big)_E\,dt = \int_0^T (\mathcal{B}\mathcal{B}^* q_\varepsilon, \tilde{q})_E\,dt$$

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Moreover, using the parabolic dissipation property, we have

$$\|y_{v_\varepsilon}(T)\|_E^2 = \varepsilon^2\|q_{F,\varepsilon}\|_E^2 \leq 2\varepsilon|(y_0, q_\varepsilon(0))_E| \leq 2\varepsilon\|y_0\|_E\|q_\varepsilon(0)\|_E$$
$$\leq 2\varepsilon\|y_0\|_E\|q_{F,\varepsilon}\|_E = 2\|y_0\|_E\|y_{v_\varepsilon}(T)\|_E,$$

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- Choose any $\tilde{v} \in L^2(]0,T[,U)$ and $y_{\tilde{v}}$ the associated solution

$$\partial_t y_{\tilde{v}} + \mathcal{A} y_{\tilde{v}} = \mathcal{B}\tilde{v},$$

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$$\int_0^T (q_\varepsilon, \partial_t y_{\tilde{v}} + \mathcal{A} y_{\tilde{v}})_E\,dt = \int_0^T (\mathcal{B}\tilde{v}, q_\varepsilon)_E\,dt = \int_0^T (\tilde{v}, \underbrace{\mathcal{B}^* q_\varepsilon}_{=v_\varepsilon})_U\,dt$$

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$$-\frac{1}{\varepsilon}\left(y_{v_\varepsilon}(T), y_{\tilde{v}}(T) - e^{-T\mathcal{A}}y_0\right)_E = \int_0^T (\tilde{v}, v_\varepsilon)_U\,dt.$$

This the Euler-Lagrange equation for $F_\varepsilon \Rightarrow v_\varepsilon$ is the minimizer of F_ε.

Back