An introduction to finite volume methods for diffusion problems

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The main points that I will not discuss

- The 3D case : many things can be done ... with some efforts.
- Parabolic equation.
- Non-linear problems.

Flow of an incompressible fluid in a porous medium

$$\operatorname{div} v = f, \quad \text{mass conservation, } f \text{ represents sinks/wells,}$$

$$v = -\varphi(x, \nabla p), \quad \text{filtration velocity constitutive law.}$$

Linear regime

- Darcy law :

$$v = -\frac{K(x)}{\mu} \nabla p,$$

the tensor $K(x)$ is the permeability, μ the viscosity.

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Non-linear regimes

- Darcy-Forchheimer law : in case of high pressure gradients

$$-\nabla p = \frac{1}{k} v + \beta |v| v, \iff v = \frac{-2k\nabla p}{1 + \sqrt{1 + 4\beta k^2 |\nabla p|}}.$$

- Power law : Non-newtonian effects

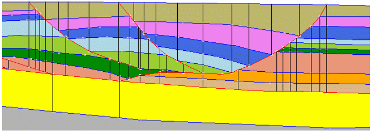
$$|v|^{n-1} v = -k\nabla p, \iff v = -|k\nabla p|^{\frac{1}{n}-1}(k\nabla p).$$

Monotonicity

Observe that in each case, $\nabla p \mapsto v = -\varphi(x, \nabla p)$ is monotone.

Heterogeneities, Discontinuities, Anisotropy

Example of the underground structure



Each color represents a different medium

$$-\mathrm{div}\,(\varphi(x, \nabla p)) = f.$$

- $\varphi(x, \cdot)$ can be linear in some areas.
- $\varphi(x, \cdot)$ can be non-linear in other areas.
- Some rocks are very permeable, other are almost impermeable.
- Some rocks have an isotropic structure, other are very anisotropic due to the particular structure at the pore scale.

Transmission conditions

- Pressure is continuous at interfaces.
- Mass flux $\varphi(x, \nabla p) \cdot \boldsymbol{\nu}$ is continuous across interfaces.

MULTIPHASE (WATER/OIL) FLOWS

- Coupling of a Darcy-like equation ($\rightsquigarrow$ velocity) with an hyperbolic problem ($\rightsquigarrow$ oil concentration=saturation).
- In practice, a time splitting scheme can be used :
  1) We solve a Darcy problem whose permeability is a function of saturation
  $$-\text{div}\,(K(u^n)\nabla p) = f, \quad + \text{B.C.}$$
  2) We solve the nonlinear conservation law by using the previously computed velocity $v^n = -K(u^n)\nabla p$
  $$\frac{(\theta u)^{n+1} - (\theta u)^n}{\delta t} + \text{div}\,(f(u^n)v^n) = 0.$$

REMARKS

1. In this course, I will only concentrate on the first step **but** it is worth mentioning that FV methods have been initially introduced in the hyperbolic framework.
2. We need a good approximation of the mass fluxes $(v \cdot \boldsymbol{\nu})$ at interfaces of the mesh in the second step, **with explicit formulas** if possible.

ADDITIONAL EQUATION FOR A POLLUTANT

$$\partial_t(\theta c) + \text{div}\,(cv) - \text{div}\,(D(c,v)\nabla c) = 0,$$

here $D(c,v)$ is a diffusion/dispersion full tensor depending on the concentration and the Darcy velocity v, typically through the tensor product $v \otimes v$.

Electrocardiology

(Coudière–Pierre–Turpault '09)

$$\begin{cases} u = u_i - u_e, & \\ C(\partial_t u + f(u)) = -\mathrm{div}\,(G_e \nabla u_e), & \text{in the heart,} \\ \mathrm{div}\,((G_i + G_e)\nabla u_e) = -\mathrm{div}\,(G_i \nabla u), & \text{in the heart,} \\ \mathrm{div}\,(G_T \nabla u_T) = 0, & \text{in the torso,} \\ (G_i \nabla u_e)\cdot \boldsymbol{\nu} = -(G_i \nabla u)\cdot \boldsymbol{\nu}, & \text{at the interface heart/torso,} \\ (G_e \nabla u_e)\cdot \boldsymbol{\nu} = -(G_T \nabla u_T)\cdot \boldsymbol{\nu}, & \text{at the interface heart/torso.} \end{cases}$$

Drift-diffusion models for semi-conductors

(Chainais–Hillairet - Peng '03,'04)

$$\begin{cases} \partial_t N - \mathrm{div}\,(\nabla N - N\nabla\psi) = 0, \\ \partial_t P - \mathrm{div}\,(\nabla P + P\nabla\psi) = 0, \\ \lambda^2 \Delta\psi = N - P. \end{cases}$$

Maxwell, Stokes, Elasticity ...

1. Introduction
   - Complex flows in porous media
   - Very short battle : FV / FE /FD
2. 1D Finite Volume method for the Poisson problem
   - Notations. Construction
   - Analysis of the scheme in the FD spirit
   - Analysis of the scheme in the FV spirit
   - Extensions
3. The basic FV scheme for the 2D Laplace problem
   - Notations. Construction
   - Analysis of the TPFA scheme
   - Extensions of the TPFA scheme
   - TPFA drawbacks

Finite differences methods

- Mostly based on Taylor expansions of (smooth) solutions.
- Cartesian geometry only (at least without any additional tools).
- "Replace" derivatives by differential quotients

$$\frac{\partial u}{\partial x} \rightsquigarrow \frac{u_{i+1}-u_i}{\Delta x}, \quad \frac{\partial^2 u}{\partial x^2} \rightsquigarrow \frac{u_{i+1}-2u_i+u_{i-1}}{\Delta x^2}.$$

Galerkin methods

- Based on a variational formulation of the PDE.
- Solve the formulation on a suitable finite dimensional subspace of the energy space
  - Piecewise polynomials : Finite Elements
  - Fourier-like basis : Spectral Methods

Finite volume methods

- Based on the conservation form of the PDE :

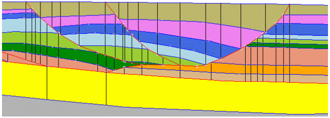
$$\mathrm{div}\,(\mathbf{something}) = \mathrm{source}.$$

- Integrate the balance equation on each cell $\mathcal{K}$ and apply Stokes formula

$$\int_{\mathcal{K}} \mathrm{source} = \sum_{\text{edges of } \mathcal{K}} \text{Outward flux of } \mathbf{something} \text{ across the edge}$$

- Approximate each flux and write the discrete balance equation obtained from this approximation.

Geometry : FE : ✔, FV : ✔, FD : ✖



Weak constraints on the meshes FE : ✔/✖, FV : ✔, FD : ✖

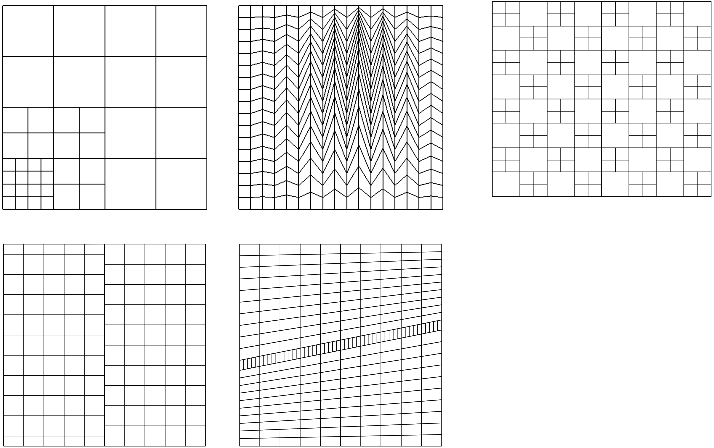
- Non conforming meshes.
- Local refinement.
- Very stretched cells.

Expected properties of the scheme

- Local mass conservativity, and mass flux consistency.
  FE : ✖, FV : ✔
- Preservation of basic properties of the PDEs (well-posedness,...).
  FE : ✔, FV : ✔
- Preservation of physical bounds on solutions.
  FE : ✖/✔, FV : ✖/✔
- Accuracy on coarse meshes with high anisotropies and heterogeneities.
  FE : ✖, FV : ✖/✔

Here, we restrict ourselves to low order schemes

For higher order methods $\rightsquigarrow$ Discontinuous Galerkin



Notations

The PDE under study

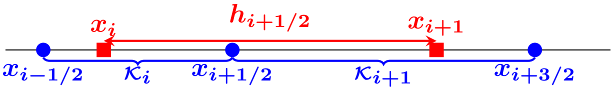
$$\begin{cases} -\partial_x(k(x)\partial_x u) = f(x), & \text{in } \Omega =]0,1[, \\ u(0) = u(1) = 0. \end{cases}$$

- f is an integrable function (let say continuous ...).
- k is bounded, smooth, and $\inf k > 0$.

A FV mesh $\mathcal{T}$ of $]0,1[$

- **Control volumes :** Compact ($\sim$ disjoint) intervals $(\kappa_i)_{1\leq i\leq N}$ that cover $[0,1] = \overline{\Omega}$

$$\kappa_i = [x_{i-1/2}, x_{i+1/2}],$$



- **Centers :** A set of points $x_i \in \kappa_i$, $\forall i$. Not necessarily mass centers
- **Boundary :** To ease presentation, we set $x_0 = 0$ and $x_{N+1} = 1$.
- We set $h_i = |\kappa_i| = x_{i+1/2} - x_{i-1/2}$ the measure (=length) of κ_i.
- We set $h_{i+1/2} = x_{i+1} - x_i$ the distance between neighboring centers.

Cell-centered Finite Volume philosophy

A cell-centered scheme

- Concerns one single unknown u_i per control volume, *supposed* to be an approximation of the exact solution at the center x_i.
- Consists in writing a (discrete) flux balance equation on each control volume.

Assume the solution is smooth

$$\int_{\mathcal{K}_i} \partial_x(-k\partial_x u)\,dx = \int_{\mathcal{K}_i} f(x)\,dx, \quad \forall i = 1, ..., N.$$

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$$\Big[-(k\partial_x u)(x_{i+1/2})\Big] - \Big[-(k\partial_x u)(x_{i-1/2})\Big] = \int_{\mathcal{K}_i} f(x)\,dx, \quad \forall i = 1, ..., N.$$

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Flux approximation

$$-(k\partial_x u)(x_{i+1/2}) \boxed{\approx} - k(x_{i+1/2})\frac{u(x_{i+1}) - u(x_i)}{h_{i+1/2}}.$$

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- Consists in writing a (discrete) flux balance equation on each control volume.

Assume the solution is smooth

$$\left[-k(x_{i+1/2})\frac{u(x_{i+1})-u(x_i)}{h_{i+1/2}}\right]-\left[-k(x_{i-1/2})\frac{u(x_i)-u(x_{i-1})}{h_{i-1/2}}\right] \boxed{\approx} \int_{\mathcal{K}_i} f(x)\,dx = h_i f_i,$$

where f_i is the mean-value of f on $\mathcal{K}_i$.

The FV scheme - version #1

We look for $u^{\mathcal{T}} = (u_i)_{1\leq i\leq N} \in \mathbb{R}^{\mathcal{T}}$ satisfying

$$\boxed{-k_{i+1/2}\frac{u_{i+1}-u_i}{h_{i+1/2}} + k_{i-1/2}\frac{u_i-u_{i-1}}{h_{i-1/2}} = h_i f_i, \quad \forall i \in \{1,\dots,N\},}$$

with $u_0 = u_{N+1} = 0$ (**homogeneous Dirichlet BC**).

THE FV SCHEME - VERSION #1

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with $u_0 = u_{N+1} = 0$ (**homogeneous Dirichlet BC**).

THE FV SCHEME - VERSION #2

We look for $u^{\mathcal{T}} = (u_i)_{1\leq i\leq N} \in \mathbb{R}^{\mathcal{T}}$ satisfying

$$\boxed{F_{i+1/2}(u^{\mathcal{T}}) - F_{i-1/2}(u^{\mathcal{T}}) = h_i f_i, \quad \forall i \in \{1,\dots,N\},}$$

with

$$F_{i+1/2}(u^{\mathcal{T}}) \stackrel{\text{def}}{=} -k_{i+1/2}\frac{u_{i+1}-u_i}{h_{i+1/2}}, \quad i = 1,\dots,N-1,$$

$$F_{1/2}(u^{\mathcal{T}}) \stackrel{\text{def}}{=} -k_{1/2}\frac{u_1-0}{h_{1/2}},$$

$$F_{N+1/2}(u^{\mathcal{T}}) \stackrel{\text{def}}{=} -k_{N+1/2}\frac{0-u_N}{h_{N+1/2}}.$$

Matrix form of the scheme $Au^{\mathcal{T}} = B$ with

$$u^{\mathcal{T}} = (u_i)_{1\leq i\leq N}, \quad B = (h_i f_i)_{1\leq i\leq N},$$

where A is **tridiagonal** and given by

$$a_{i,i} = \frac{k_{i+1/2}}{h_{i+1/2}} + \frac{k_{i-1/2}}{h_{i-1/2}}, \quad a_{i,i+1} = -\frac{k_{i+1/2}}{h_{i+1/2}}, \quad a_{i,i-1} = -\frac{k_{i-1/2}}{h_{i-1/2}}.$$

First step of the analysis

Matrix form of the scheme $Au^{\mathcal{T}} = B$ with

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- Observe that A is **symmetric and positive definite**

$$\begin{aligned}
(Au^{\mathcal{T}}, v^{\mathcal{T}}) &= \sum_{i=1}^{N} (F_{i+1/2}(u^{\mathcal{T}}) - F_{i-1/2}(u^{\mathcal{T}}))v_i \\
&= -F_{1/2}(u^{\mathcal{T}})v_1 + F_{N+1/2}(u^{\mathcal{T}})v_N + \sum_{i=1}^{N-1} F_{i+1/2}(u^{\mathcal{T}})(v_i - v_{i+1}) \\
&= -\sum_{i=0}^{N} F_{i+1/2}(u^{\mathcal{T}})(v_{i+1} - v_i) \\
&= \sum_{i=0}^{N} k_{i+1/2}\frac{u_{i+1} - u_i}{h_{i+1/2}}(v_{i+1} - v_i) \\
&= \sum_{i=0}^{N} h_{i+1/2}k_{i+1/2}\left(\frac{u_{i+1} - u_i}{h_{i+1/2}}\right)\left(\frac{v_{i+1} - v_i}{h_{i+1/2}}\right)
\end{aligned}$$

Matrix form of the scheme $Au^{\mathcal{T}} = B$ with

$$u^{\mathcal{T}} = (u_i)_{1\leq i\leq N}, \quad B = (h_i f_i)_{1\leq i\leq N},$$

where A is **tridiagonal** and given by

$$a_{i,i} = \frac{k_{i+1/2}}{h_{i+1/2}} + \frac{k_{i-1/2}}{h_{i-1/2}}, \quad a_{i,i+1} = -\frac{k_{i+1/2}}{h_{i+1/2}}, \quad a_{i,i-1} = -\frac{k_{i-1/2}}{h_{i-1/2}}.$$

"Variationnal" formulation of the FV scheme

To find $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ such that

$$\underbrace{\sum_{i=0}^{N} h_{i+1/2}k_{i+1/2}\left(\frac{u_{i+1}-u_i}{h_{i+1/2}}\right)\left(\frac{v_{i+1}-v_i}{h_{i+1/2}}\right)}_{\text{sum on the interfaces}} = \underbrace{\sum_{i=1}^{N} h_i f_i v_i,}_{\text{sum on the control volumes}} \qquad \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

$$\approx \int_0^1 k\,\partial_x u\,\partial_x v\,dx \qquad\qquad \approx \int_0^1 f\,v\,dx$$

Comments

- Matrix assembly for FV schemes is done **by interfaces/edges**.
- Variationnal formulation is useful in the analysis.

Let $u^{\mathcal{T}}$ be the solution to

$$Au^{\mathcal{T}} = (h_i f_i)_{1\leq i\leq N}.$$

THEOREM (DISCRETE MAXIMUM PRINCIPLE)

$$\Big(f_i \geq 0, \forall i\Big) \Longrightarrow \Big(u_i \geq 0, \forall i\Big).$$

Proof

THEOREM (L^∞-STABILITY)

There exists $C>0$ depending only on k such that

$$\|u^{\mathcal{T}}\|_\infty \leq C\|f\|_\infty.$$

Proof

Lemma (Discrete Poincaré inequality)

For any $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, we have

$$\|u^{\mathcal{T}}\|_2 \stackrel{\text{def}}{=} \left(\sum_{i=1}^{N} h_i |u_i|^2\right)^{\frac{1}{2}} \leq \|u^{\mathcal{T}}\|_\infty \leq \underbrace{\left(\sum_{i=0}^{N} h_{i+1/2} \left|\frac{u_{i+1}-u_i}{h_{i+1/2}}\right|^2\right)^{\frac{1}{2}}}_{\textit{discrete } H_0^1 \textit{ norm}}.$$

Proof : Telescoping summation + Cauchy-Schwarz

$$u_i = (u_i - u_{i-1}) + (u_{i-1} - u_{i-2}) + \cdots + (u_1 - 0).$$

Theorem (L^2-stability)

For any $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ solution of $Au^{\mathcal{T}} = (h_i f_i)_{1\leq i\leq N}$, we have the estimate

$$\|u^{\mathcal{T}}\|_2 \leq \frac{\|f\|_2}{\inf k}.$$

In fact, we have a much better estimate : a discrete H^1 estimate.

We assume $k(x) = 1$; the PDE is then $-\partial_x^2 u = f$.
We choose mass centers $x_i = \frac{x_{i-1/2}+x_{i+1/2}}{2}$.

- Inside the domain :

$$FV \Leftrightarrow FD \Leftrightarrow \frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} = f_i,$$

  the schemes are identical (except possibly the source term discretisation).

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  the schemes are identical (except possibly the source term discretisation).

- On the boundary :

$$FV \Leftrightarrow \frac{u_2 - u_1}{h} - \frac{u_1 - 0}{h/2} = h f_1.$$

$$FD \Leftrightarrow \frac{\frac{u_2-u_1}{h} - \frac{u_1-0}{h/2}}{3h/4} = f_1.$$

  Here the two schemes are not exactly the same.

The FV scheme inside the domain writes

$$-\frac{\frac{u_{i+1}-u_i}{h_{i+1/2}} - \frac{u_i-u_{i-1}}{h_{i-1/2}}}{h_i} = f_i \qquad \left(= \frac{1}{h_i}\int_{K_i} (-\partial_x^2 u)\, dx = -\partial_x^2 u(x_i) + O(h) \right).$$

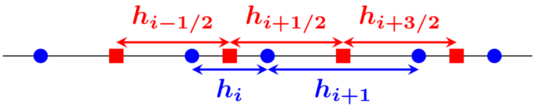
Is the scheme consistent in the usual FD sense ?

Let $u \in \mathcal{C}^2$

$$\partial_x^2 u(x_i) - \frac{\frac{u(x_{i+1})-u(x_i)}{h_{i+1/2}} - \frac{u(x_i)-u(x_{i-1})}{h_{i-1/2}}}{h_i}$$

$$= \underbrace{\left(1 - \frac{h_{i-1/2}+h_{i+1/2}}{2h_i}\right)}_{=r_i} \partial_x^2 u(x_i) + O(h)$$

Example :



$$\boxed{\text{We find that : } r_i = 1/4, \text{or } r_i = -1/2.}$$

The FV scheme is not consistent in the FD sense

... however the scheme is L^∞ stable and convergent !

The FV scheme inside the domain reads

$$-\frac{\frac{u_{i+1}-u_i}{h_{i+1/2}} - \frac{u_i-u_{i-1}}{h_{i-1/2}}}{h_i} = f_i.$$

An alternate consistency property

Assume that $u \in \mathcal{C}^3$. We set

$$\bar{u}_i \overset{\text{def}}{=} u(x_i) + \frac{h_i^2}{8}\partial_x^2 u(x_i) = u(x_i) + O(h^2).$$

We find that

$$\partial_x^2 u(x_i) - \frac{\frac{\bar{u}_{i+1}-\bar{u}_i}{h_{i+1/2}} - \frac{\bar{u}_i-\bar{u}_{i-1}}{h_{i-1/2}}}{h_i} = O(h) \quad !!$$

$\Rightarrow$ This proves (in a quite non-natural way) the first order convergence of the scheme

$$\boxed{\sup_i |u(x_i) - u_i| \leq \sup_i |u(x_i) - \bar{u}_i| + \sup_i |\bar{u}_i - u_i| \leq Ch.}$$

- The **flux** notion is crucial in the Finite Volume framework but it does not enter the game in the previous proof.
- Consistency in the FD sense is not adapted to FV.
- The previous convergence proof is not natural and does not have a simple generalisation to more complex problems.

Conclusion

We need new tools and proofs.

Recall the scheme under flux balance form :

$$F_{i+1/2}(u^{\mathcal{T}}) - F_{i-1/2}(u^{\mathcal{T}}) = h_i f_i, \quad \forall i \in \{1, \ldots, N\},$$

$$F_{i+1/2}(u^{\mathcal{T}}) = -k_{i+1/2} \frac{u_{i+1} - u_i}{h_{i+1/2}}, \quad 0 = 1, \ldots, N.$$

RECALL THE SCHEME UNDER FLUX BALANCE FORM :

$$F_{i+1/2}(u^{\mathcal{T}}) - F_{i-1/2}(u^{\mathcal{T}}) = h_i f_i, \quad \forall i \in \{1, \dots, N\},$$

$$F_{i+1/2}(u^{\mathcal{T}}) = -k_{i+1/2} \frac{u_{i+1} - u_i}{h_{i+1/2}}, \quad 0 = 1, \dots, N.$$

THE EXACT SOLUTION u SATISFIES :

$$\overline{F}_{i+1/2}(u) - \overline{F}_{i-1/2}(u) = h_i f_i, \quad \forall i \in \{1, \dots, N\},$$

where the **exact fluxes** are defined by

$$\overline{F}_{i+1/2}(u) = -k_{i+1/2}(\partial_x u)(x_{i+1/2}).$$

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The exact solution u satisfies :

$$\overline{F}_{i+1/2}(u) - \overline{F}_{i-1/2}(u) = h_i f_i, \quad \forall i \in \{1, \dots, N\},$$

where the **exact fluxes** are defined by

$$\overline{F}_{i+1/2}(u) = -k_{i+1/2}(\partial_x u)(x_{i+1/2}).$$

"Projection" of the exact solution on $\mathcal{T}$:

$$\mathbb{P}^{\mathcal{T}} u = (u(x_i))_{1 \leq i \leq N} \in \mathbb{R}^{\mathcal{T}}.$$

Flux consistency error terms

$$R_{i+1/2}(u) \stackrel{\text{def}}{=} \underbrace{\left(-k_{i+1/2} \frac{u(x_{i+1}) - u(x_i)}{h_{i+1/2}} \right)}_{= F_{i+1/2}(\mathbb{P}^{\mathcal{T}} u)} - \underbrace{\left(- k_{i+1/2}(\partial_x u)(x_{i+1/2}) \right)}_{= \overline{F}_{i+1/2}(u)}.$$

We define the error $e^{\mathcal{T}} = u^{\mathcal{T}} - \mathbb{P}^{\mathcal{T}} u$.

Recall

$$F_{i+1/2}(u^{\mathcal{T}}) - F_{i-1/2}(u^{\mathcal{T}}) = h_i f_i, \quad \forall i \in \{1, \ldots, N\}, \tag{1}$$

$$\overline{F}_{i+1/2}(u) - \overline{F}_{i-1/2}(u) = h_i f_i, \quad \forall i \in \{1, \ldots, N\}, \tag{2}$$

$$R_{i+1/2}(u) = F_{i+1/2}(\mathbb{P}^{\mathcal{T}} u) - \overline{F}_{i+1/2}(u), \quad \forall i \in \{0, \ldots, N\}. \tag{3}$$

We define the error $e^{\mathcal{T}} = u^{\mathcal{T}} - \mathbb{P}^{\mathcal{T}} u$.

Recall

$$F_{i+1/2}(u^{\mathcal{T}}) - F_{i-1/2}(u^{\mathcal{T}}) = h_i f_i, \quad \forall i \in \{1, \ldots, N\}, \tag{1}$$

$$\overline{F}_{i+1/2}(u) - \overline{F}_{i-1/2}(u) = h_i f_i, \quad \forall i \in \{1, \ldots, N\}, \tag{2}$$

$$R_{i+1/2}(u) = F_{i+1/2}(\mathbb{P}^{\mathcal{T}} u) - \overline{F}_{i+1/2}(u), \quad \forall i \in \{0, \ldots, N\}. \tag{3}$$

The Finite Volume computation

- We subtract (1) and (2), and we use (3)

$$F_{i+1/2}(u^{\mathcal{T}} - \mathbb{P}^{\mathcal{T}} u) - F_{i-1/2}(u^{\mathcal{T}} - \mathbb{P}^{\mathcal{T}} u) = -R_{i+1/2}(u) + R_{i-1/2}(u).$$

- We multiply the equations by e_i and make the sum

$$\sum_{i=0}^{N} h_{i+1/2} k_{i+1/2} \left(\frac{e_{i+1} - e_i}{h_{i+1/2}} \right)^2 = - \sum_{i=0}^{N} h_{i+1/2} R_{i+1/2}(u) \left(\frac{e_{i+1} - e_i}{h_{i+1/2}} \right).$$

- We use Cauchy-Schwarz inequality

$$\|e^{\mathcal{T}}\|_\infty \leq \left(\sum_{i=0}^{N} h_{i+1/2} \left(\frac{e_{i+1} - e_i}{h_{i+1/2}} \right)^2 \right)^{\frac{1}{2}} \leq \frac{1}{C} \left(\sum_{i=0}^{N} h_{i+1/2} |R_{i+1/2}(u)|^2 \right)^{\frac{1}{2}}.$$

It remains to estimate the terms $R_{i+1/2}(u)$

Assume that $u \in \mathcal{C}^2$, then Taylor formulas lead to

$$\begin{aligned} \left|R_{i+1/2}(u)\right| &= \left| \left(-k_{i+1/2} \frac{u(x_{i+1}) - u(x_i)}{h_{i+1/2}} \right) - \left(-k_{i+1/2} (\partial_x u)(x_{i+1/2}) \right) \right|. \\ &\leq \|k\|_\infty \|\partial_x^2 u\|_\infty h. \end{aligned}$$

We conclude

$$\boxed{\sup_i |u_i - u(x_i)| = \|e^{\mathcal{T}}\|_\infty \leq \|k\|_\infty \|\partial_x^2 u\|_\infty h}$$

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We conclude

$$\boxed{\sup_i |u_i - u(x_i)| = \|e^{\mathcal{T}}\|_\infty \leq \|k\|_\infty \|\partial_x^2 u\|_\infty h}$$

Remarks :

- If we have $x_{i+1/2} = \frac{x_i + x_{i+1}}{2}$, then the scheme is second order. However, in general, **this is not true**, since we have more naturally
$$x_i = \frac{x_{i-1/2} + x_{i+1/2}}{2}.$$
- Observe that, if there **one single** interface such that $R_{i+1/2}(u) = O(1)$, then the scheme is only of order $1/2$.

ASSUMPTION : the diffusion coefficient k is piecewise constant and the mesh is assumed to be *compatible* with the jumps of k (i.e. k is constant on each control volume).

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REMARKS

- The diffusion coefficient $k_{i+1/2} = k(x_{i+1/2})$ is not defined anymore.
- The solution u has no chance to be $\mathcal{C}^2$! Indeed, if f is $\mathcal{C}^0$, we only have

$$k(\partial_x u) \in \mathcal{C}^1.$$

- Therefore, it is the **total flux** which is smooth

$$\text{Exact total flux :} \quad \overline{F}_{i+1/2}(u) = -\Big(k(\partial_x u)\Big)(x_{i+1/2}).$$

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Definition of the numerical flux

We still consider the two-point formula

$$F_{i+1/2}(u^{\mathcal{T}}) = -k_{i+1/2}\frac{u_{i+1} - u_i}{h_{i+1/2}},$$

but how do we choose the diffusion coefficient $k_{i+1/2}$?

$$k_{i+1/2} = k_i? \quad k_{i+1/2} = k_{i+1}? \quad k_{i+1/2} = (k_i + k_{i+1})/2?$$

In the control volume κ_i

- $x \mapsto k(x)$ is a constant $= k_i$ thus $u \in \mathcal{C}^2$ in κ_i.

$$\Rightarrow \quad (k\partial_x u)(x_{i+1/2}^-) \approx k_i \frac{u(x_{i+1/2}) - u(x_i)}{x_{i+1/2} - x_i}.$$

In the control volume κ_{i+1}

- $x \mapsto k(x)$ is a constant $= k_{i+1}$ thus $u \in \mathcal{C}^2$ in κ_{i+1}.

$$\Rightarrow \quad (k\partial_x u)(x_{i+1/2}^+) \approx k_{i+1} \frac{u(x_{i+1}) - u(x_{i+1/2})}{x_{i+1} - x_{i+1/2}}.$$

In the control volume κ_i

- $x \mapsto k(x)$ is a constant $= k_i$ thus $u \in \mathcal{C}^2$ in κ_i.

$$\Rightarrow \quad (k\partial_x u)(x^-_{i+1/2}) \approx k_i \frac{u(x_{i+1/2}) - u(x_i)}{x_{i+1/2} - x_i}.$$

In the control volume κ_{i+1}

- $x \mapsto k(x)$ is a constant $= k_{i+1}$ thus $u \in \mathcal{C}^2$ in κ_{i+1}.

$$\Rightarrow \quad (k\partial_x u)(x^+_{i+1/2}) \approx k_{i+1} \frac{u(x_{i+1}) - u(x_{i+1/2})}{x_{i+1} - x_{i+1/2}}.$$

We write the total flux continuity at $x_{i+1/2}$

$$\overline{F}_{i+1/2}(u) = -(k\partial_x u)(x^+_{i+1/2}) = -(k\partial_x u)(x^-_{i+1/2}).$$

In the control volume κ_i

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$$\overline{F}_{i+1/2}(u) = -(k\partial_x u)(x_{i+1/2}^+) = -(k\partial_x u)(x_{i+1/2}^-).$$

We try to eliminate the interface value

$$\overline{F}_{i+1/2}(u) = -\frac{\frac{x_{i+1}-x_{i+1/2}}{k_{i+1}}(k\partial_x u)(x_{i+1/2}^+) + \frac{x_{i+1/2}-x_i}{k_i}(k\partial_x u)(x_{i+1/2}^-)}{\frac{x_{i+1}-x_{i+1/2}}{k_{i+1}} + \frac{x_{i+1/2}-x_i}{k_i}}$$

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- $x \mapsto k(x)$ is a constant $= k_i$ thus $u \in \mathcal{C}^2$ in κ_i.

$$\Rightarrow \quad (k\partial_x u)(x_{i+1/2}^-) \approx k_i \frac{u(x_{i+1/2}) - u(x_i)}{x_{i+1/2} - x_i}.$$

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$$\overline{F}_{i+1/2}(u) = -(k\partial_x u)(x_{i+1/2}^+) = -(k\partial_x u)(x_{i+1/2}^-).$$

We try to eliminate the interface value

$$\overline{F}_{i+1/2}(u) \approx -\underbrace{\frac{h_{i+1/2}}{\frac{x_{i+1}-x_{i+1/2}}{k_{i+1}} + \frac{x_{i+1/2}-x_i}{k_i}}}_{\stackrel{\text{def}}{=} k_{i+1/2},\ \text{harmonic mean}} \frac{u(x_{i+1}) - u(x_i)}{h_{i+1/2}}$$

The FV scheme is thus the following : To find $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ such that

$$F_{i+1/2}(u^{\mathcal{T}}) - F_{i-1/2}(u^{\mathcal{T}}) = h_i f_i, \quad \forall 1 \leq i \leq N,$$

with

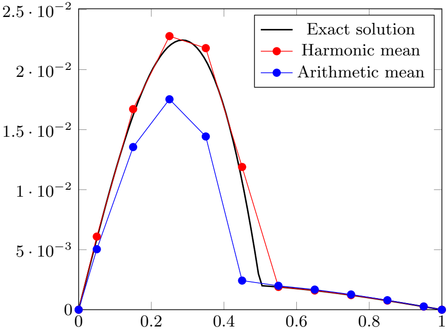
$$F_{i+1/2}(u^{\mathcal{T}}) = -k_{i+1/2} \frac{u_{i+1} - u_i}{h_{i+1/2}},$$

$$\boxed{k_{i+1/2} = \frac{h_{i+1/2} k_i k_{i+1}}{(x_{i+1} - x_{i+1/2}) k_i + (x_{i+1/2} - x_i) k_{i+1}}.}$$

Theoretical results

- Existence and uniqueness of the solution.
- L^∞ and L^2 stability.
- First order H^1-convergence for any source term in $f \in L^2(\Omega)$.
- We observe (without proof) a second order convergence in L^∞.

The other choices of $k_{i+1/2}$ lead to a poor convergence rate.



- $k_{i+1/2}$ = arithmetic mean $\rightsquigarrow$ convergence rate $\frac{1}{2}$.
- $k_{i+1/2}$ = harmonic mean $\rightsquigarrow$ convergence rate 1.

(Eymard, Gallouët, Herbin, '00→ '09)

Definition in 2D

- Ω a connected bounded polygonal domain in $\mathbb{R}^2$.
- An admissible orthogonal mesh $\mathcal{T}$ is made of
  - a finite set of non empty compact **convex polygonal** subdomains of Ω refered to as $\mathcal{K}$, called **control volumes** such that
    - If $\mathcal{K} \neq \mathcal{L}$, then $\mathring{\mathcal{K}} \cap \mathring{\mathcal{L}} = \emptyset$.
    - $\overline{\Omega} = \bigcup_{\mathcal{K} \in \mathcal{T}} \mathcal{K}$.
  - A set of points, called centers, $(x_{\mathcal{K}})_{\mathcal{K} \in \mathcal{T}}$ such that
    - For any $\mathcal{K} \in \mathcal{T}$, $x_{\mathcal{K}} \in \mathring{\mathcal{K}}$.
    - For any $\mathcal{K}, \mathcal{L} \in \mathcal{T}$, $\mathcal{K} \neq \mathcal{L}$ such that $\mathcal{K} \cap \mathcal{L}$ is a segment, then it is an edge of $\mathcal{K}$ and an edge of $\mathcal{L}$ is denoted $\mathcal{K}|\mathcal{L}$ and satisfies the **orthogonality condition**
$$[x_{\mathcal{K}}, x_{\mathcal{L}}] \perp \mathcal{K}|\mathcal{L}.$$

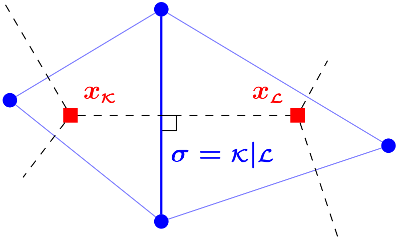
Notations

- Mesh size : $\text{size}(\mathcal{T}) = \max_{\mathcal{K} \in \mathcal{T}} \big(\text{diam}(\mathcal{K})\big)$.
- Set of edges : $\mathcal{E}$, $\mathcal{E}_{ext}$, $\mathcal{E}_{int}$, $\mathcal{E}_{\mathcal{K}}$
- Unit normals : $\boldsymbol{\nu}_{\mathcal{K}}$, $\boldsymbol{\nu}_{\mathcal{K}\sigma}$, $\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}$
- Volumes/Areas/Measures : $|\mathcal{K}|$, $|\sigma|$
- Distances : $d_{\mathcal{K}\sigma}$, $d_{\mathcal{L}\sigma}$, $d_{\mathcal{K}\mathcal{L}}$, d_σ

Consider the following problem

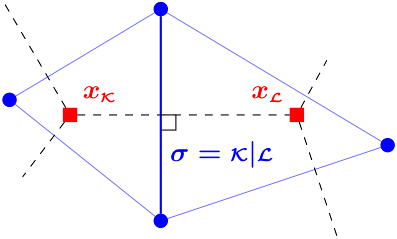
$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

and an admissible orthogonal mesh $\mathcal{T}$



FLUX BALANCE EQUATION ON THE CONTROL VOLUME $\mathcal{K}$

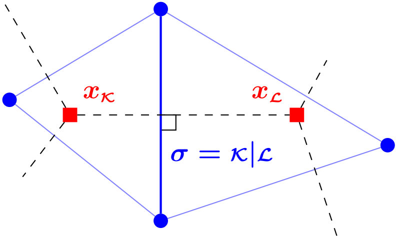
$$|\mathcal{K}|f_\mathcal{K} \stackrel{\text{def}}{=} \int_\mathcal{K} f = \int_\mathcal{K} -\Delta u = \sum_{\sigma\in\mathcal{E}_\mathcal{K}} \underbrace{-\int_\sigma \nabla u \cdot \boldsymbol{\nu}_{\mathcal{K}\sigma}}_{\stackrel{\text{def}}{=} \overline{F}_{\mathcal{K},\sigma}(u)}$$



$$|\mathcal{K}| f_\mathcal{K} = \sum_{\sigma \in \mathcal{E}_\mathcal{K}} \overline{F}_{\mathcal{K},\sigma}(u).$$

LOCAL CONSERVATIVITY PROPERTY FOR THE INITIAL PROBLEM

$$\overline{F}_{\mathcal{K},\sigma}(u) = -\overline{F}_{\mathcal{L},\sigma}(u), \quad \text{for } \sigma = \mathcal{K}|\mathcal{L}.$$



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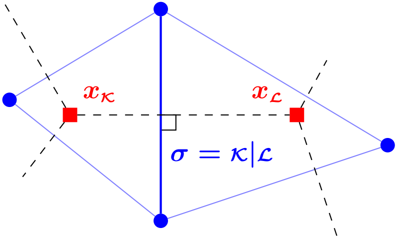
CELL-CENTERED UNKNOWNS — We are looking for $u_\mathcal{K} \sim u(x_\mathcal{K})$

Notation : $u^\mathcal{T} = (u_\mathcal{K})_{\mathcal{K}\in\mathcal{T}} \in \mathbb{R}^\mathcal{T}$.



$$|\mathcal{K}| f_{\mathcal{K}} = \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} \overline{F}_{\mathcal{K},\sigma}(u).$$

Local conservativity property for the initial problem

$$\overline{F}_{\mathcal{K},\sigma}(u) = -\overline{F}_{\mathcal{L},\sigma}(u), \quad \text{for } \sigma = \mathcal{K}|\mathcal{L}.$$

Cell-centered unknowns

We are looking for $u_{\mathcal{K}} \sim u(x_{\mathcal{K}})$

Notation : $u^{\mathcal{T}} = (u_{\mathcal{K}})_{\mathcal{K} \in \mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$.

Numerical fluxes

A family of maps $u^{\mathcal{T}} \mapsto F_{\mathcal{K},\sigma}(u^{\mathcal{T}})$ in order to approximate $\overline{F}_{\mathcal{K},\sigma}(u)$

Numerical scheme

We look for $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ such that $|\mathcal{K}| f_{\mathcal{K}} = \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} F_{\mathcal{K},\sigma}(u^{\mathcal{T}})$ for any $\mathcal{K} \in \mathcal{T}$.

Case of an interior edge

$\sigma \in \mathcal{E}_{int}$, $\sigma = \kappa|\mathcal{L}$.

$$x_{\mathcal{L}} - x_{\mathcal{K}} = d_{\mathcal{K}\mathcal{L}} \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}.$$

$$\text{For } x \in \sigma, \qquad (\nabla u(x)) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} = \frac{u(x_{\mathcal{L}}) - u(x_{\mathcal{K}})}{d_{\mathcal{K}\mathcal{L}}} + O(\text{size}(\mathcal{T}))$$

$$\Longrightarrow \overline{F}_{\mathcal{K},\sigma}(u) = -|\sigma| \frac{u(x_{\mathcal{L}}) - u(x_{\mathcal{K}})}{d_{\mathcal{K}\mathcal{L}}} + O(\text{size}(\mathcal{T})^2)$$

CASE OF AN INTERIOR EDGE

$\sigma \in \mathcal{E}_{int}$, $\sigma = \mathcal{K}|\mathcal{L}$.

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$$\text{For } x \in \sigma, \qquad (\nabla u(x)) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} = \frac{u(x_{\mathcal{L}}) - u(x_{\mathcal{K}})}{d_{\mathcal{K}\mathcal{L}}} + O(\text{size}(\mathcal{T}))$$

$$\Longrightarrow \overline{F}_{\mathcal{K},\sigma}(u) = -|\sigma| \frac{u(x_{\mathcal{L}}) - u(x_{\mathcal{K}})}{d_{\mathcal{K}\mathcal{L}}} + O(\text{size}(\mathcal{T})^2)$$

Thus, we define

$$\boxed{F_{\mathcal{K},\sigma}(u^{\mathcal{T}}) \stackrel{\text{def}}{=} -|\sigma| \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}}}.}$$

REMARK AND DEFINITION

The scheme is built so as to be conservative

$$F_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = -F_{\mathcal{L},\sigma}(u^{\mathcal{T}})$$

We set

$$F_{\mathcal{K},\mathcal{L}}(u^{\mathcal{T}}) \stackrel{\text{def}}{=} F_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = -F_{\mathcal{L},\sigma}(u^{\mathcal{T}}).$$

Case of a boundary edge $\sigma \in \mathcal{E}_{ext}$.

$$x_\sigma - x_\mathcal{K} = d_{\mathcal{K}\sigma} \boldsymbol{\nu}_{\mathcal{K}\sigma}.$$

$$(\nabla u(x)) \cdot \boldsymbol{\nu}_{\mathcal{K}\sigma} \sim \frac{u(x_\sigma) - u(x_\mathcal{K})}{d_{\mathcal{K}\sigma}} = \frac{\mathbf{0} - u(x_\mathcal{K})}{d_{\mathcal{K}\sigma}} \Leftarrow \text{Boundary data}$$

$$\Longrightarrow \overline{F}_{\mathcal{K},\sigma}(u) = -|\sigma| \frac{-u(x_\mathcal{K})}{d_{\mathcal{K}\sigma}} + O(\text{size}(\mathcal{T})^2)$$

Thus we define

$$\boxed{F_{\mathcal{K},\sigma}(u^\mathcal{T}) \stackrel{\text{def}}{=} -|\sigma| \frac{-u_\mathcal{K}}{d_{\mathcal{K}\sigma}}.}$$

Definition of the TPFA scheme

We look for $u^{\mathcal{T}} = (u_{\mathcal{K}})_{\mathcal{K}\in\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ such that

$$\begin{cases} \displaystyle\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} F_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = |\kappa| f_{\mathcal{K}}, & \forall \kappa \in \mathcal{T}, \\ F_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = -|\sigma| \dfrac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}}}, & \text{for } \sigma = \kappa|\mathcal{L} \in \mathcal{E}_{int}, \\ F_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = -|\sigma| \dfrac{-u_{\mathcal{K}}}{d_{\mathcal{K}\sigma}}, & \text{for } \sigma \in \mathcal{E}_{ext}. \end{cases} \tag{TPFA}$$

- It is a linear system of N equations with N unknowns (N=nb of control volumes in $\mathcal{T}$).
- The scheme is also known as VF4/FV4 : 4-point stencil for a triangle 2D mesh.
- On a 2D uniform Cartesian mesh : we recover the usual 5-point scheme.

Definition of the TPFA scheme

We look for $u^{\mathcal{T}} = (u_{\mathcal{K}})_{\mathcal{K}\in\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ such that

$$\begin{cases} \displaystyle\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} F_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = |\mathcal{K}| f_{\mathcal{K}}, & \forall \mathcal{K}\in\mathcal{T}, \\ F_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = -|\sigma|\dfrac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}}}, & \text{for } \sigma = \mathcal{K}|\mathcal{L} \in \mathcal{E}_{int}, \\ F_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = -|\sigma|\dfrac{-u_{\mathcal{K}}}{d_{\mathcal{K}\sigma}}, & \text{for } \sigma \in \mathcal{E}_{ext}. \end{cases} \tag{TPFA}$$

Notations - Piecewise constant approximation

- We define $f^{\mathcal{T}} = (f_{\mathcal{K}})_{\mathcal{K}\in\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$.
- With each set of unknowns $v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, we associate the **piecewise constant** function
$$v^{\mathcal{T}}(x) = \sum_{\mathcal{K}\in\mathcal{T}} v_{\mathcal{K}} \mathbf{1}_{\mathcal{K}}(x).$$
- Natural norms $\|v^{\mathcal{T}}\|_{L^\infty} = \sup_{\mathcal{K}\in\mathcal{T}} |v_{\mathcal{K}}|$, $\|v^{\mathcal{T}}\|_{L^2} = \left(\sum_{\mathcal{K}\in\mathcal{T}} |\mathcal{K}||v_{\mathcal{K}}|^2\right)^{\frac{1}{2}}$.

FV methods are non-conforming methods

NOTATIONS : Oriented difference quotients

- For any couple of neighboring control volumes $(\mathcal{K}, \mathcal{L})$ we set

$$D_{\mathcal{K}\mathcal{L}}(u^{\mathcal{T}}) \stackrel{\text{def}}{=} \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}}} \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}.$$

- For any interior edge $\sigma \in \mathcal{E}_{int}$ we set

$$D_\sigma(u^{\mathcal{T}}) \stackrel{\text{def}}{=} D_{\mathcal{K}\mathcal{L}}(u^{\mathcal{T}}) = D_{\mathcal{L}\mathcal{K}}(u^{\mathcal{T}}).$$

- For any exterior edge $\sigma \in \mathcal{E}_{ext}$ we set $D_\sigma(u^{\mathcal{T}}) \stackrel{\text{def}}{=} \frac{0-u_{\mathcal{K}}}{d_{\mathcal{K}\sigma}} \boldsymbol{\nu}_{\mathcal{K}\sigma}$.

LEMMA (DISCRETE INTEGRATION BY PARTS)

Let $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ be a solution of (TPFA) **if it exists**, *then for any $v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$*

$$\underbrace{\sum_{\sigma \in \mathcal{E}} d_\sigma |\sigma| D_\sigma(u^{\mathcal{T}}) \cdot D_\sigma(v^{\mathcal{T}})}_{\stackrel{\text{def}}{=} [u^{\mathcal{T}}, v^{\mathcal{T}}]_{1,\mathcal{T}}} = \sum_{\mathcal{K} \in \mathcal{T}} |\mathcal{K}| v_{\mathcal{K}} f_{\mathcal{K}} = (v^{\mathcal{T}}, f^{\mathcal{T}})_{L^2}.$$

$\rightsquigarrow$ **Local conservativity of the scheme is crucial here.**

Lemma (Discrete integration by parts)

Let $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ be a solution of (TPFA) **if it exists**, *then for any $v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$*

$$\underbrace{\sum_{\sigma \in \mathcal{E}} d_\sigma |\sigma| D_\sigma(u^{\mathcal{T}}) \cdot D_\sigma(v^{\mathcal{T}})}_{\stackrel{\text{def}}{=} [u^{\mathcal{T}}, v^{\mathcal{T}}]_{1,\mathcal{T}}} = \sum_{\mathcal{K} \in \mathcal{T}} |\mathcal{K}| v_{\mathcal{K}} f_{\mathcal{K}} = (v^{\mathcal{T}}, f^{\mathcal{T}})_{L^2}.$$

$\rightsquigarrow$ **Local conservativity of the scheme is crucial here.**

Proposition

The bilinear form

$$(u^{\mathcal{T}}, v^{\mathcal{T}}) \in \mathbb{R}^{\mathcal{T}} \times \mathbb{R}^{\mathcal{T}} \mapsto [u^{\mathcal{T}}, v^{\mathcal{T}}]_{1,\mathcal{T}},$$

is an inner product in $\mathbb{R}^{\mathcal{T}}$ that we call **discrete H_0^1 inner product**.
The associated norm $\|\cdot\|_{1,\mathcal{T}}$ is called **discrete H_0^1 norm**.

Theorem

For any source term $f \in L^2(\Omega)$, the scheme (TPFA) *has a unique solution $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ and we have*

$$\|u^{\mathcal{T}}\|_{1,\mathcal{T}}^2 \leq \|u^{\mathcal{T}}\|_{L^2}\|f^{\mathcal{T}}\|_{L^2} \leq \|u^{\mathcal{T}}\|_{L^2}\|f\|_{L^2}.$$

In order to get a useful discrete-H^1 estimate, we need

Theorem (Discrete Poincaré inequality)

For any orthogonal admissible mesh $\mathcal{T}$, we have

$$\|v^{\mathcal{T}}\|_{L^2} \leq \mathrm{diam}(\Omega)\|v^{\mathcal{T}}\|_{1,\mathcal{T}}, \quad \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

▸ Proof

Matrix of the system :

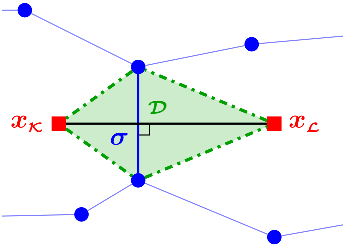
- A is symmetric definite positive (See : Discrete integration by parts).
- A is a M-matrix $\Rightarrow$ Discrete maximum principle

$$f^{\mathcal{T}} \geq 0 \Longrightarrow u^{\mathcal{T}} \geq 0.$$

Indeed, the line of the system $Au^{\mathcal{T}} = f^{\mathcal{T}}$ corresponding to the control volume κ reads

$$\sum_{\mathcal{L} \in V_{\mathcal{K}}} \underbrace{\tau_{\mathcal{K}\mathcal{L}}}_{\geq 0}(u_{\mathcal{K}} - u_{\mathcal{L}}) = |\kappa| f_{\mathcal{K}}.$$

Diamond cells



Discrete gradient

For any $v^\mathcal{T} \in \mathbb{R}^\mathcal{T}$, and any $\mathcal{D} \in \mathfrak{D}$, we set

$$\nabla_\mathcal{D}^\mathcal{T} v^\mathcal{T} \stackrel{\text{def}}{=} \begin{cases} d\dfrac{u_\mathcal{L}-u_\mathcal{K}}{d_{\mathcal{K}\mathcal{L}}}\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} = dD_\sigma(u^\mathcal{T}), & \text{for } \sigma \in \mathcal{E}_{int}, \\ d\dfrac{0-u_\mathcal{K}}{d_{\mathcal{K}\mathcal{L}}}\boldsymbol{\nu}_{\mathcal{K}\sigma} = dD_\sigma(u^\mathcal{T}), & \text{for } \sigma \in \mathcal{E}_{ext}, \end{cases}$$

$$\nabla^\mathcal{T} v^\mathcal{T} \stackrel{\text{def}}{=} \sum_{\mathcal{D}\in\mathfrak{D}} \mathbf{1}_\mathcal{D} \nabla_\mathcal{D}^\mathcal{T} v^\mathcal{T} \in (L^2(\Omega))^2.$$

Link with the discrete H_0^1 norm

$$\|v^\mathcal{T}\|_{1,\mathcal{T}}^2 = \frac{1}{d}\|\nabla^\mathcal{T} v^\mathcal{T}\|_{L^2}^2.$$

Theorem (Weak compactness)

Let $(\mathcal{T}_n)_n$ be a sequence of admissible orthogonal meshes such that $\text{size}(\mathcal{T}_n) \to 0$ *and* $(u^{\mathcal{T}_n})_n$ *a familly of discrete functions defined on each of these meshes and such that*

$$\boxed{\sup_n \|u^{\mathcal{T}_n}\|_{1,\mathcal{T}_n} < +\infty.}$$

Then

- *There exists a function $u \in L^2(\Omega)$ and a subsequence $(u^{\mathcal{T}_{\varphi(n)}})_n$ that* **strongly** *converges towards u in $L^2(\Omega)$.*

THEOREM (WEAK COMPACTNESS)

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Then

- *There exists a function* $u \in L^2(\Omega)$ *and a subsequence* $(u^{\mathcal{T}_{\varphi(n)}})_n$ *that* **strongly** *converges towards* u *in* $L^2(\Omega)$*.*

Moreover,

- *The function* u *belongs to* $H^1_0(\Omega)$*.*
- *The sequence of discrete gradients* $(\nabla^{\mathcal{T}_{\varphi(n)}} u^{\mathcal{T}_{\varphi(n)}})_n$ **weakly** *converges towards* ∇u *in* $(L^2(\Omega))^d$*.*

▸ Proof

THEOREM (CONVERGENCE OF THE TPFA SCHEME)

Let $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ be the unique solution to the PDE.

Let $(\mathcal{T}_n)_n$ be a family of admissible orthogonal meshes such that $\mathrm{size}(\mathcal{T}_n) \to 0$.

For any n, let $u^{\mathcal{T}_n} \in \mathbb{R}^{\mathcal{T}_n}$ be the unique solution of the TPFA scheme on the mesh $\mathcal{T}_n$ associated with the source term f.

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Then, we have

1. *The sequence $(u^{\mathcal{T}_n})_n$* **strongly** *converges towards u in $L^2(\Omega)$.*
2. *The sequence $(\nabla^{\mathcal{T}_n} u^{\mathcal{T}_n})_n$* **weakly** *converges towards ∇u in $(L^2(\Omega))^d$.*
3. **Strong** *convergence of the gradients* **DOES NOT HOLD** *(excepted for $f = u = 0$).*

▸ Proof

First remarks

- Convergence of the scheme : no need of any regularity assumption on u.
- For error estimates we will assume that $u \in H^2(\Omega)$.

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Principle of the analysis

- We want to compare $u^{\mathcal{T}}$ with the projection $\mathbb{P}^{\mathcal{T}} u = (u(x_{\mathcal{K}}))_{\mathcal{K}}$ of the exact solution on the mesh. The error is thus defined by

$$e^{\mathcal{T}} \stackrel{\text{def}}{=} \mathbb{P}^{\mathcal{T}} u - u^{\mathcal{T}}.$$

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$$e^{\mathcal{T}} \stackrel{\text{def}}{=} \mathbb{P}^{\mathcal{T}} u - u^{\mathcal{T}}.$$

- We compare the numerical fluxes computed on $\mathbb{P}^{\mathcal{T}} u$ with exact fluxes

$$|\sigma| R_{\mathcal{K},\sigma}(u) \stackrel{\text{def}}{=} F_{\mathcal{K},\sigma}(\mathbb{P}^{\mathcal{T}} u) - \overline{F}_{\mathcal{K},\sigma}(u),$$

that is

$$R_{\mathcal{K},\sigma}(u) = \frac{u(x_{\mathcal{L}}) - u(x_{\mathcal{K}})}{d_{\mathcal{K}\mathcal{L}}} - \frac{1}{|\sigma|} \int_\sigma \nabla u \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} \, dx, \quad \forall \sigma \in \mathcal{E}_{int}.$$

$$|\sigma| R_{\mathcal{K},\sigma}(u) \stackrel{\text{def}}{=} F_{\mathcal{K},\sigma}(\mathbb{P}^{\mathcal{T}} u) - \overline{F}_{\mathcal{K},\sigma}(u).$$

- We subtract the exact fluxes balance equation (that is the PDE integrated on κ)

$$|\kappa| f_{\mathcal{K}} = \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} \overline{F}_{\mathcal{K},\sigma} = \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} F_{\mathcal{K},\sigma}(\mathbb{P}^{\mathcal{T}} u) - \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} |\sigma| R_{\mathcal{K},\sigma}(u),$$

  and the numerical scheme

$$|\kappa| f_{\mathcal{K}} = \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} F_{\mathcal{K},\sigma}(u^{\mathcal{T}}).$$

  We get

$$\sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} F_{\mathcal{K},\sigma}(e^{\mathcal{T}}) = \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} |\sigma| R_{\mathcal{K},\sigma}(u), \quad \forall \kappa \in \mathcal{T}. \tag{$\star$}$$

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- We get
$$\sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} F_{\mathcal{K},\sigma}(e^{\mathcal{T}}) = \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} |\sigma| R_{\mathcal{K},\sigma}(u), \quad \forall \mathcal{K} \in \mathcal{T}. \qquad (\star)$$

- We multiply $(\star)$ by $e_{\mathcal{K}}$ and we sum over $\mathcal{K}$.
- We Notice that the flux consistency error terms are conservative $R_{\mathcal{K},\sigma}(u) = -R_{\mathcal{L},\sigma}(u)$, thus we get
$$\|e^{\mathcal{T}}\|_{1,\mathcal{T}}^2 = [e^{\mathcal{T}}, e^{\mathcal{T}}]_{1,\mathcal{T}} = \sum_{\sigma \in \mathcal{E}} d_\sigma |\sigma| D_\sigma(e^{\mathcal{T}})^2 = \sum_{\sigma \in \mathcal{E}} d_\sigma |\sigma| R_\sigma(u) D_\sigma(e^{\mathcal{T}}).$$

- We use the Cauchy-Schwarz inequality
$$\|e^{\mathcal{T}}\|_{1,\mathcal{T}} \leq \left(\sum_{\sigma \in \mathcal{E}} d_\sigma |\sigma| |R_\sigma(u)|^2 \right)^{\frac{1}{2}}.$$

Recall

$$\|e^{\mathcal{T}}\|_{1,\mathcal{T}} \leq \left(\sum_{\sigma\in\mathcal{E}} d_\sigma |\sigma| |R_\sigma(u)|^2\right)^{\frac{1}{2}}.$$

$$|R_\sigma(u)| = \left|\frac{u(x_\mathcal{L}) - u(x_\mathcal{K})}{d_{\mathcal{K}\mathcal{L}}} - \frac{1}{|\sigma|}\int_\sigma \nabla u \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx\right|, \quad \forall \sigma \in \mathcal{E}_{int}.$$

Theorem (Error estimate - Version 1)

Assume $u \in \mathcal{C}^2(\overline{\Omega})$, there exists $C > 0$ depending only on Ω s.t.

$$(\|\mathbb{P}^{\mathcal{T}} u - u^{\mathcal{T}}\|_{L^2} =) \qquad \|e^{\mathcal{T}}\|_{L^2} \leq \mathrm{diam}(\Omega)\|e^{\mathcal{T}}\|_{1,\mathcal{T}} \leq C\mathrm{size}(\mathcal{T})\|D^2 u\|_{L^\infty},$$

$$\|u - u^{\mathcal{T}}\|_{L^2} \leq C\mathrm{size}(\mathcal{T})\|D^2 u\|_{L^\infty}.$$

Main tool : Consistency error terms estimate

For $u \in \mathcal{C}^2(\overline{\Omega})$, $\quad |R_\sigma(u)| \leq C\|D^2 u\|_\infty \mathrm{size}(\mathcal{T})$.

RECALL

$$\|e^{\mathcal{T}}\|_{1,\mathcal{T}} \leq \left(\sum_{\sigma\in\mathcal{E}} d_\sigma|\sigma||R_\sigma(u)|^2\right)^{\frac{1}{2}}.$$

$$|R_\sigma(u)| = \left|\frac{u(x_{\mathcal{L}}) - u(x_{\mathcal{K}})}{d_{\mathcal{KL}}} - \frac{1}{|\sigma|}\int_\sigma \nabla u\cdot\boldsymbol{\nu}_{\mathcal{KL}}\,dx\right|, \quad \forall\sigma\in\mathcal{E}_{int}.$$

THEOREM (ERROR ESTIMATE - Version 2)

Assume $u \in H^2(\Omega)$, there exists $C>0$ depending only on Ω and $\mathrm{reg}(\mathcal{T})$ s.t.

$$(\|\mathbb{P}^{\mathcal{T}}u - u^{\mathcal{T}}\|_{L^2} =) \qquad \|e^{\mathcal{T}}\|_{L^2} \leq \mathrm{diam}(\Omega)\|e^{\mathcal{T}}\|_{1,\mathcal{T}} \leq C\mathrm{size}(\mathcal{T})\|D^2u\|_{L^2},$$

$$\|u - u^{\mathcal{T}}\|_{L^2} \leq C\mathrm{size}(\mathcal{T})\|D^2u\|_{L^2}.$$

MAIN TOOL : CONSISTENCY ERROR TERMS ESTIMATE

For $u\in H^2(\Omega)$, $|R_\sigma(u)| \leq C\mathrm{size}(\mathcal{T})\left(\frac{1}{|\mathcal{D}|}\int_{\mathcal{D}}|D^2u|^2\,dx\right)^{\frac{1}{2}}$, ▸ Proof

where $C>0$ only depends on $\mathrm{reg}(\mathcal{T}) \stackrel{\text{def}}{=} \sup_{\sigma\in\mathcal{E}}\left(|\sigma|/d_{\mathcal{K}\sigma} + |\sigma|/d_{\mathcal{L}\sigma}\right)$.

- In practice, we observe a super-convergence phenomenon

$$\|e^{\mathcal{T}}\|_{L^2(\Omega)} \sim C\text{size}(\mathcal{T})^2,$$

the same as for $\mathbb{P}^1$ finite element approximation (Aubin–Nitschze trick).

$\rightsquigarrow$ Still an open problem up to now.

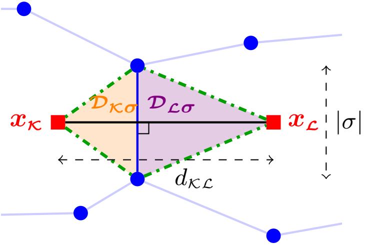
- Discrete functional analysis
  - Discrete Poincaré inequalities **(Eymard-Gallouët-Herbin, '00)** **(Omnes-Le, '13)**
  - Discrete Gagliardo-Nirenberg-Sobolev embeddings **(Bessemoulin–Chatard - Chainais–Hillairet - Filbet, '12)**
  - Discrete Besov estimates **(Andreianov -B. - Hubert,'07)**
  - Discrete Aubin–Lions–Simon lemma **(Gallouët-Latché, '12)**

We need to build the linear system $Au^{\mathcal{T}} = b$ to be solved.

GENERAL PHILOSOPHY : LOOP OVER EDGES

- If $\sigma = \mathcal{K}|\mathcal{L}$ is an interior edge, we define *the transmissivity* $\tau_\sigma \stackrel{\text{def}}{=} \dfrac{|\sigma|}{d_{\mathcal{K}\mathcal{L}}}$, and we assemble the contributions of the flux

$$F_{\mathcal{K}\mathcal{L}}(u^{\mathcal{T}}) = -|\sigma| \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}}} = \tau_\sigma (u_{\mathcal{K}} - u_{\mathcal{L}}).$$



$$\begin{cases} A(\mathcal{K},\mathcal{K}) \leftarrow A(\mathcal{K},\mathcal{K}) + \tau_\sigma, \\ A(\mathcal{K},\mathcal{L}) \leftarrow A(\mathcal{K},\mathcal{L}) - \tau_\sigma, \\ A(\mathcal{L},\mathcal{L}) \leftarrow A(\mathcal{L},\mathcal{L}) + \tau_\sigma, \\ A(\mathcal{L},\mathcal{K}) \leftarrow A(\mathcal{L},\mathcal{K}) - \tau_\sigma. \end{cases}$$

REQUIRED DATA STRUCTURE :

- Connectivity informations for each edge
- Positions of centers and vertices of the mesh.

We need to build the linear system $Au^{\mathcal{T}} = b$ to be solved.

General philosophy : loop over edges

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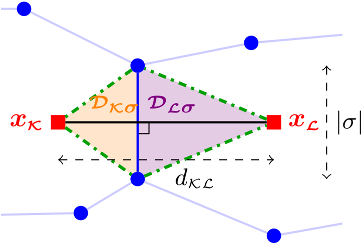


$$\begin{cases} b(\mathcal{K}) \hookleftarrow b(\mathcal{K}) + \displaystyle\int_{\mathcal{D}_{\mathcal{K}\sigma}} f(x)\,dx, \\ b(\mathcal{L}) \hookleftarrow b(\mathcal{L}) + \displaystyle\int_{\mathcal{D}_{\mathcal{L}\sigma}} f(x)\,dx. \end{cases}$$

or any quadrature approximation, e.g.

$$\begin{cases} b(\mathcal{K}) \hookleftarrow b(\mathcal{K}) + |\mathcal{D}_{\mathcal{K}\sigma}| f(x_{\mathcal{K}}), \\ b(\mathcal{L}) \hookleftarrow b(\mathcal{L}) + |\mathcal{D}_{\mathcal{L}\sigma}| f(x_{\mathcal{L}}). \end{cases}$$

Required data structure :

- Connectivity informations for each edge
- Positions of centers and vertices of the mesh.

The problem under study

$$\begin{cases} -\mathrm{div}\,(k(x)\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

with $k \in L^\infty(\Omega, \mathbb{R})$ and $\inf_\Omega k > 0$.

TPFA scheme

General structure unchanged

$$\forall \mathcal{K} \in \mathcal{T}, \quad |\mathcal{K}| f_\mathcal{K} = \sum_{\sigma \in \mathcal{E}_\mathcal{K}} F_{\mathcal{K},\sigma}(u^\mathcal{T}),$$

but we need to adapt the numerical flux definitions

$$F_{\mathcal{K},\sigma}(u^\mathcal{T}) = |\sigma| k_\sigma \frac{u_\mathcal{L} - u_\mathcal{K}}{d_{\mathcal{K}\mathcal{L}}}.$$

Question

How to choose the coefficient k_σ ?

SAME STRATEGY AS IN 1D

- The solution u is "continuous" (in the trace sense on edges).
- The gradient of u **is not** continuous.
- However, the total flux $k(x)\nabla u(x) \cdot \boldsymbol{\nu}$ is (weakly) continuous across edges.

- We introduce an artificial unknown on each edge u_σ.
- We define the fluxes across σ coming from $\mathcal{K}$ and from $\mathcal{L}$

$$F_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = |\sigma| k_{\mathcal{K}} \frac{u_\sigma - u_{\mathcal{K}}}{d_{\mathcal{K}\sigma}}, \quad F_{\mathcal{L},\sigma}(u^{\mathcal{T}}) = |\sigma| k_{\mathcal{L}} \frac{u_\sigma - u_{\mathcal{L}}}{d_{\mathcal{L}\sigma}}.$$

- We impose local conservativity (= total flux continuity)

$$F_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = -F_{\mathcal{L},\sigma}(u^{\mathcal{T}}).$$

- We deduce the value of u_σ and then the formula for the numerical flux

$$\Longrightarrow u_\sigma = \frac{\frac{k_{\mathcal{K}}}{d_{\mathcal{K}\sigma}} u_{\mathcal{K}} + \frac{k_{\mathcal{L}}}{d_{\mathcal{L}\sigma}} u_{\mathcal{L}}}{\frac{k_{\mathcal{K}}}{d_{\mathcal{K}\sigma}} + \frac{k_{\mathcal{L}}}{d_{\mathcal{L}\sigma}}},$$

$$\Longrightarrow F_{\mathcal{K}\mathcal{L}}(u^{\mathcal{T}}) = |\sigma| \left(\frac{d_{\mathcal{K}\mathcal{L}}}{\frac{d_{\mathcal{K}\sigma}}{k_{\mathcal{K}}} + \frac{d_{\mathcal{L}\sigma}}{k_{\mathcal{L}}}} \right) \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}}}$$

Consistency estimate

(Jaffré - Roberts, '05) (Angot - B. - Hubert, '09)

Framework

- A 2D porous matrix (constant, isotropic permeability $k = 1$).
- Small thickness ($= b$) fractures/barriers inside the domain for which permeability is very different from the one in the ambient medium.

The model

- We "replace" the fractures/barriers by hypersurfaces, neglecting their thickness b.
- We account for the flow inside fractures through an asymptotic model
  - **Assumption 1 :** The flow is mostly transverse to the fracture/barrier direction.
  - **Assumption 2 :** Pressure is essentially linear in the transverse direction.
- We arrive to the following transmission conditions

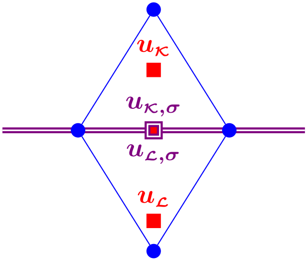
$$\boxed{\begin{aligned} [\![\nabla u \cdot \boldsymbol{n}]\!] &= 0, \\ \nabla u \cdot \boldsymbol{n} &= -\frac{k}{b}[\![u]\!]. \end{aligned}}$$

$$\begin{cases} [\![\nabla u \cdot \boldsymbol{n}]\!] = 0, \\ \nabla u \cdot \boldsymbol{n} = -k \dfrac{[\![u]\!]}{b}. \end{cases} \tag{S}$$

Recall the flux definitions

$$F_{\mathcal{K},\sigma} \stackrel{\text{def}}{=} |\sigma| \frac{u_{\mathcal{K},\sigma} - u_{\mathcal{K}}}{d_{\mathcal{K}\sigma}},$$

$$F_{\mathcal{L},\sigma} \stackrel{\text{def}}{=} |\sigma| \frac{u_{\mathcal{L},\sigma} - u_{\mathcal{L}}}{d_{\mathcal{L}\sigma}}.$$



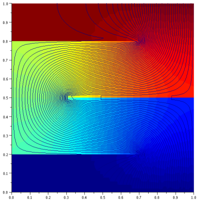
Conditions (S) lead to $F_{\mathcal{K},\mathcal{L}} \stackrel{\text{def}}{=} F_{\mathcal{K},\sigma} = -F_{\mathcal{L},\sigma} = -|\sigma| k \dfrac{u_{\mathcal{L},\sigma} - u_{\mathcal{K},\sigma}}{b}$.
We can then eliminate $u_{\mathcal{K},\sigma}$ and $u_{\mathcal{L},\sigma}$, and finally obtain

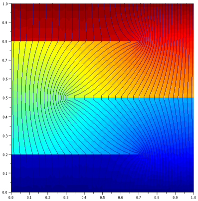
$$\boxed{F_{\mathcal{K},\mathcal{L}} = -|\sigma| \frac{\beta d_{\mathcal{K}\mathcal{L}}}{1 + \beta d_{\mathcal{K}\mathcal{L}}} \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}}}}$$

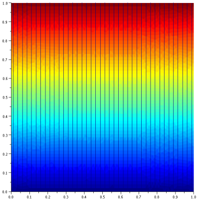
with $\beta = \frac{k}{b}$.

NUMERICAL RESULTS :

Dirichlet BC (given pressure) on top and bottom sides.
Neumann BC (impermeable walls) elsewhere.





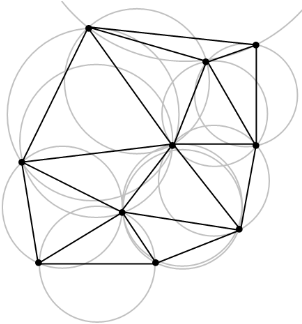


$\beta = 0$	$\beta = 1$	$\beta = +\infty$
Impermeable barriers	Intermediate properties	No fracture/barrier limit

- Cartesian meshes : Control volumes are rectangular parallelepipeds thus choosing $x_{\mathcal{K}}$ as the mass center is OK

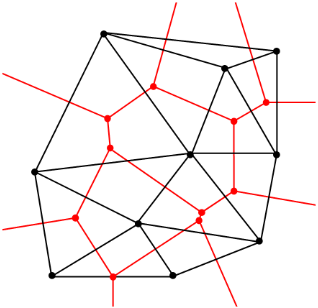
- Cartesian meshes :
- Conforming triangular meshes :
  We take $x_\mathcal{K}$ =circumcenter ; **BUT :**
  - It is not guaranteed that $x_\mathcal{K} \in \mathcal{K}$ (even $x_\mathcal{K} \in \Omega$ is not sure).
  - We can have $x_\mathcal{K} = x_\mathcal{L}$ for $\mathcal{K} \neq \mathcal{L} \Rightarrow d_{\mathcal{K}\mathcal{L}} = 0$!
  - However, the scheme still works if

$$(x_\mathcal{L} - x_\mathcal{K}) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} > 0 \quad \Leftrightarrow \text{Delaunay condition}$$



  - For almost any point distribution in Ω, there exists a unique corresponding Delaunay triangulation.

- Dual construction :
  Voronoï diagram of a set of point.



- There exists efficient algorithms for Delaunay triangulation and Voronoi diagrams.

- For a non conforming triangle mesh : orthogonality condition is impossible to fulfill.
- For a non Cartesian quadrangle mesh : orthogonality condition is impossible to fulfill.
- The homogeneous anisotropic case :

$$-\mathrm{div}(A\nabla u) = f,$$

the admissibility condition becomes *A-orthogonality*

$$x_{\mathcal{L}} - x_{\mathcal{K}} \,/\!\!/\, A\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} \iff A^{-1}(x_{\mathcal{L}} - x_{\mathcal{K}}) \perp \sigma.$$

$\leadsto$ thus the mesh needs to be adapted to the PDE under study.

- The heterogeneous anisotropic case :

$$-\mathrm{div}(A(x)\nabla u) = f,$$

the orthogonality condition will depend on x ...

- Nonlinear problems :

$$-\mathrm{div}(\varphi(x, \nabla u)) = f,$$

it is impossible to approximate fluxes by using only two points since a complete gradient approximation is necessary.

The discrete gradient given by TPFA is not useful

- It is only an approximation of the gradient in the normal direction at each edge.
- Gradient convergence is always weak.

Summary

- We need more than 2 unknowns to build suitable flux approximations.
- Approximation of the gradient of the solution in all directions is necessary.

- Cells-centered schemes : We use unknowns in the neighboring control volumes.
- Primal/dual schemes : We use new unknowns on vertices (dual mesh).
- Mimetic/hybrid/mixed schemes : We use new unknowns on edges/faces.

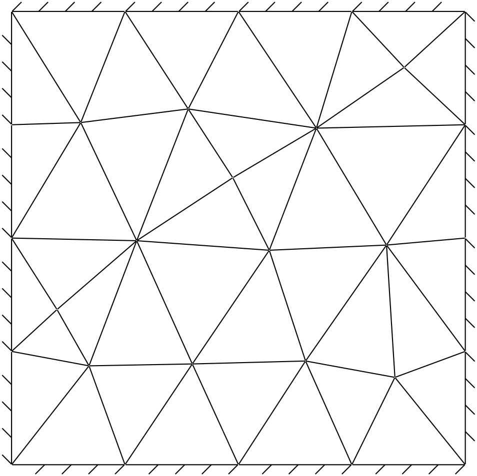
(Hermeline '00) (Domelevo-Omnes '05) (Andreianov-Boyer-Hubert '07)

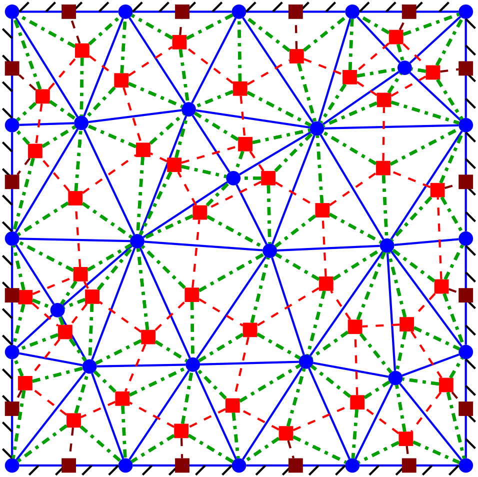
- Scalar elliptic problem

$$-\mathrm{div}(A(x)\nabla u) = f, \text{ in } \Omega,$$

with homogeneous Dirichlet boundary conditions and $x \mapsto A(x) \in M_2(\mathbb{R})$ be a bounded, uniformly coercive matrix-valued function.

- General meshes
  - Possibly non conforming meshes
  - Without the orthogonality condition
- Basic ideas
  - To consider unknowns at the center of each control volume but also on **vertices**.
  - To add new discrete balance equations associated with each vertex.
- It is more expensive than TPFA # unknowns ($\approx \times 2$) but much more robust and efficient.





■ Primal unknown $\boldsymbol{u}_{\mathcal{K}}$

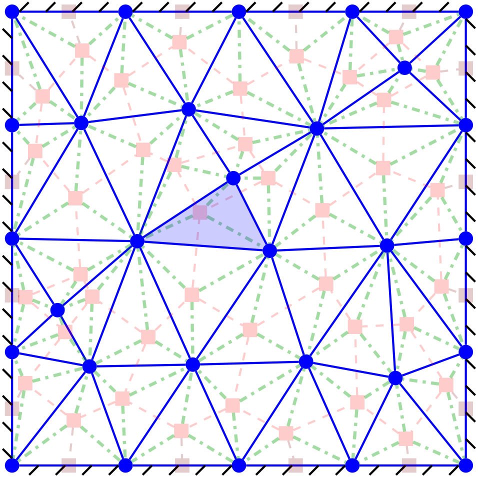
Primal control vol. $\mathcal{K} \in \mathfrak{M}$

● Dual unknown $\boldsymbol{u}_{\mathcal{K}^*}$

Dual control vol. $\mathcal{K}^* \in \mathfrak{M}^*$

Diamond cells $\mathcal{D} \in \mathfrak{D}$

APPROXIMATE SOLUTION : $u^{\mathcal{T}} = \Big((u_{\mathcal{K}})_{\mathcal{K}}, (u_{\mathcal{K}^*})_{\mathcal{K}^*} \Big) \in \mathbb{R}^{\mathcal{T}} = \mathbb{R}^{\mathfrak{M}} \times \mathbb{R}^{\mathfrak{M}^*}$



Primal unknown $u_{\mathcal{K}}$

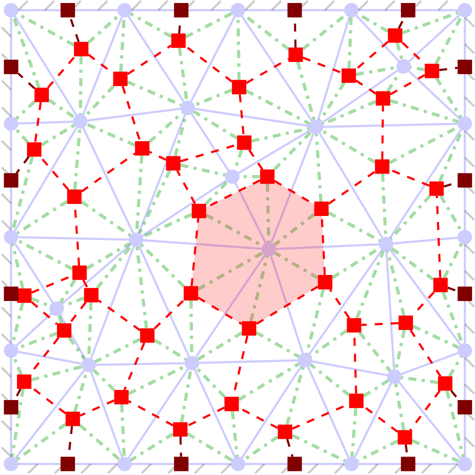
Primal control vol. $\mathcal{K} \in \mathfrak{M}$

Dual unknown $u_{\mathcal{K}^*}$

Dual control vol. $\mathcal{K}^* \in \mathfrak{M}^*$

Diamond cells $\mathcal{D} \in \mathfrak{D}$

APPROXIMATE SOLUTION : $u^{\mathcal{T}} = \Big((u_{\mathcal{K}})_{\mathcal{K}}, (u_{\mathcal{K}^*})_{\mathcal{K}^*} \Big) \in \mathbb{R}^{\mathcal{T}} = \mathbb{R}^{\mathfrak{M}} \times \mathbb{R}^{\mathfrak{M}^*}$



Primal unknown $u_{\mathcal{K}}$

Primal control vol. $\mathcal{K} \in \mathfrak{M}$

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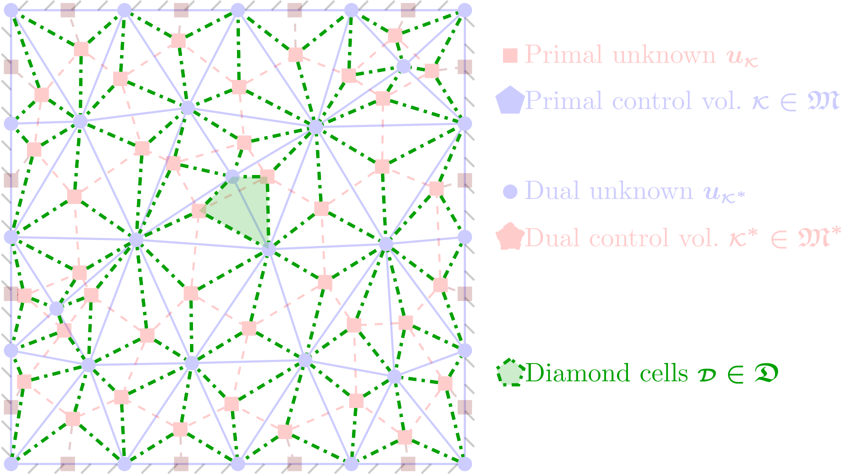
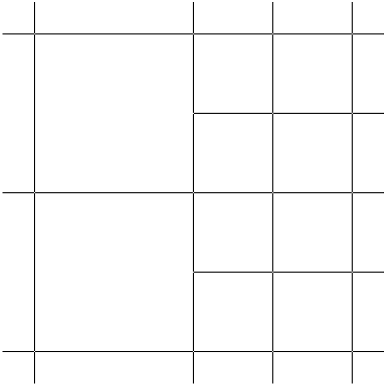
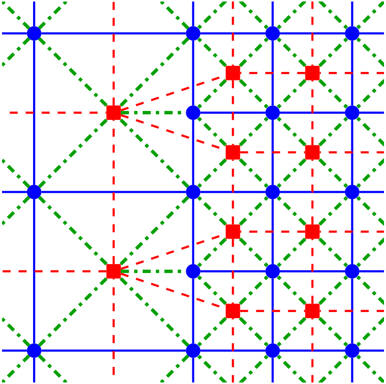


APPROXIMATE SOLUTION : $u^{\mathcal{T}} = \Big((u_{\mathcal{K}})_{\mathcal{K}}, (u_{\mathcal{K}^*})_{\mathcal{K}^*} \Big) \in \mathbb{R}^{\mathcal{T}} = \mathbb{R}^{\mathfrak{M}} \times \mathbb{R}^{\mathfrak{M}^*}$

The DDFV meshes and unknowns





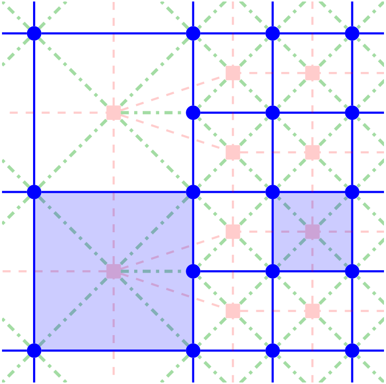
■ Primal unknown $u_{\mathcal{K}}$

Primal control vol. $\mathcal{K} \in \mathfrak{M}$

● Dual unknown $u_{\mathcal{K}^*}$

Dual control vol. $\mathcal{K}^* \in \mathfrak{M}^*$

Diamond cells $\mathcal{D} \in \mathfrak{D}$



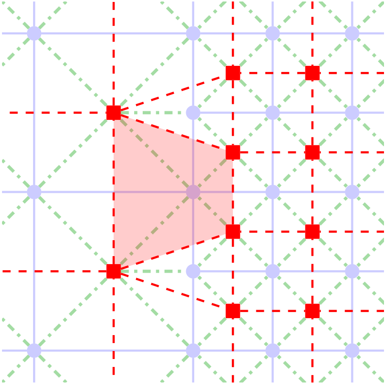
■ Primal unknown $u_{\mathcal{K}}$

⬠ Primal control vol. $\mathcal{K} \in \mathfrak{M}$

● Dual unknown $u_{\mathcal{K}^*}$

⬟ Dual control vol. $\mathcal{K}^* \in \mathfrak{M}^*$

⬟ Diamond cells $\mathcal{D} \in \mathfrak{D}$



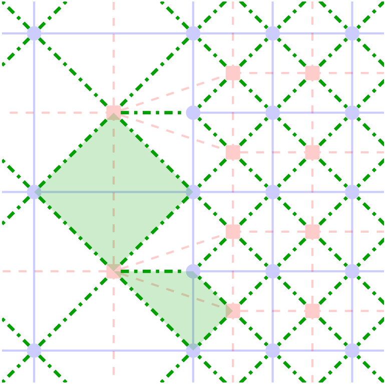
■ Primal unknown $u_{\mathcal{K}}$

Primal control vol. $\mathcal{K} \in \mathfrak{M}$

● Dual unknown $u_{\mathcal{K}^*}$

Dual control vol. $\mathcal{K}^* \in \mathfrak{M}^*$

Diamond cells $\mathcal{D} \in \mathfrak{D}$



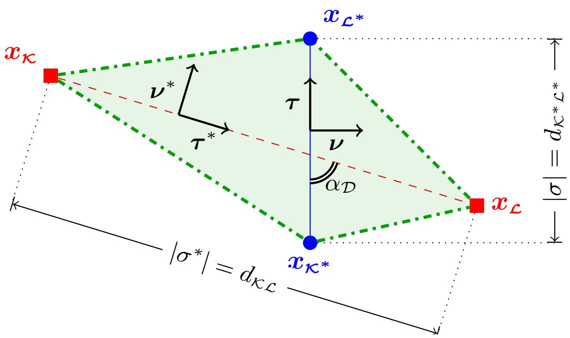
Primal unknown u_κ

Primal control vol. $\kappa \in \mathfrak{M}$

Dual unknown u_{κ^*}

Dual control vol. $\kappa^* \in \mathfrak{M}^*$

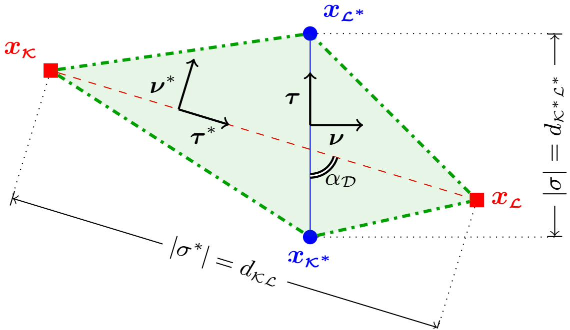
Diamond cells $\mathcal{D} \in \mathfrak{D}$



MESH REGULARITY MEASUREMENT

$$\sin\alpha_{\mathcal{T}} \stackrel{\text{def}}{=} \min_{\mathcal{D}\in\mathfrak{D}} |\sin\alpha_{\mathcal{D}}|,$$

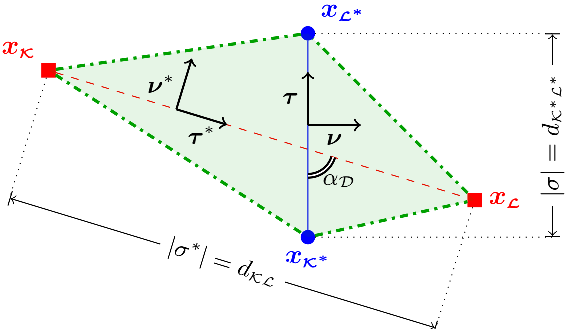
$$\operatorname{reg}(\mathcal{T}) \stackrel{\text{def}}{=} \max\left(\frac{1}{\alpha_{\mathcal{T}}}, \max_{\substack{\mathcal{K}\in\mathfrak{M}\\ \mathcal{D}\in\mathfrak{D}_{\mathcal{K}}}} \frac{\operatorname{diam}(\mathcal{K})}{\operatorname{diam}(\mathcal{D})}, \max_{\substack{\mathcal{K}^*\in\mathfrak{M}^*\\ \mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}}} \frac{\operatorname{diam}(\mathcal{K}^*)}{\operatorname{diam}(\mathcal{D})}, \dots\right).$$



Discrete Gradient

$$\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \stackrel{\text{def}}{=} \frac{1}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{|\sigma^*|} \boldsymbol{\nu} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{|\sigma|} \boldsymbol{\nu}^* \right).$$

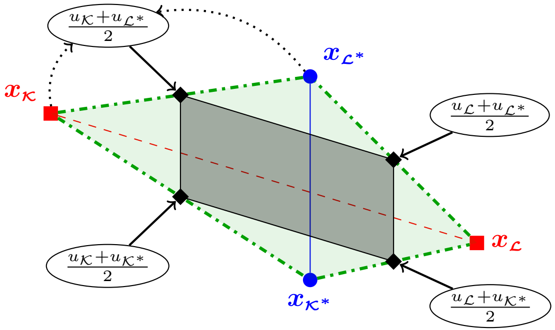
Comes from
$$\begin{cases} \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_{\mathcal{L}} - x_{\mathcal{K}}) = u_{\mathcal{L}} - u_{\mathcal{K}}, \\ \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_{\mathcal{L}^*} - x_{\mathcal{K}^*}) = u_{\mathcal{L}^*} - u_{\mathcal{K}^*}. \end{cases}$$



DISCRETE GRADIENT

$$\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \stackrel{\text{def}}{=} \frac{1}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{|\sigma^*|} \boldsymbol{\nu} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{|\sigma|} \boldsymbol{\nu}^* \right).$$

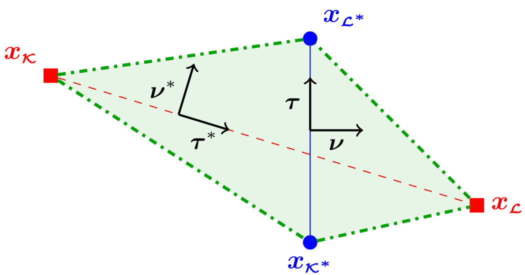
EQUIVALENT DEFINITION $\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} = \dfrac{1}{2|\mathcal{D}|} \Big(|\sigma|(u_{\mathcal{L}} - u_{\mathcal{K}}) \boldsymbol{\nu} + |\sigma^*|(u_{\mathcal{L}^*} - u_{\mathcal{K}^*}) \boldsymbol{\nu}^* \Big),$



DISCRETE GRADIENT

$$\nabla_\mathcal{D}^\mathcal{T} u^\mathcal{T} \stackrel{\text{def}}{=} \frac{1}{\sin \alpha_\mathcal{D}} \left(\frac{u_\mathcal{L} - u_\mathcal{K}}{|\sigma^*|} \boldsymbol{\nu} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{|\sigma|} \boldsymbol{\nu}^* \right).$$

STILL ANOTHER DEFINITION $\nabla_\mathcal{D}^\mathcal{T} u^\mathcal{T} = \nabla \left(\Pi_\mathcal{D} u^\mathcal{T} \right)$, with $\Pi_\mathcal{D} u^\mathcal{T}$ affine in 



Discrete Gradient

$$\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \stackrel{\text{def}}{=} \frac{1}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{|\sigma^*|} \boldsymbol{\nu} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{|\sigma|} \boldsymbol{\nu}^* \right).$$

DDFV fluxes

$$\text{Across the primal edge } \sigma : F_{\mathcal{K}\mathcal{L}}(u^{\mathcal{T}}) = -|\sigma| \big(A_{\mathcal{D}} \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}, \boldsymbol{\nu} \big),$$

$$\text{Across the dual edge } \sigma^* : F_{\mathcal{K}^*\mathcal{L}^*}(u^{\mathcal{T}}) = -|\sigma^*| \big(A_{\mathcal{D}} \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}, \boldsymbol{\nu}^* \big).$$

Finite Volume Formulation : Find $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}} = \mathbb{R}^{\mathfrak{M}} \times \mathbb{R}^{\mathfrak{M}^*}$ such that

$$\begin{cases} -\displaystyle\sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} |\sigma| \left(A_{\mathcal{D}} \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}, \boldsymbol{\nu}_{\mathcal{K}} \right) = |\mathcal{K}| f_{\mathcal{K}}, \quad \forall \mathcal{K} \in \mathfrak{M}, \\ -\displaystyle\sum_{\sigma^* \in \mathcal{E}_{\mathcal{K}^*}} |\sigma^*| \left(A_{\mathcal{D}} \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}, \boldsymbol{\nu}_{\mathcal{K}^*} \right) = |\mathcal{K}^*| f_{\mathcal{K}^*}, \forall \mathcal{K}^* \in \mathfrak{M}^*, \end{cases} \tag{DDFV}$$

with $A_{\mathcal{D}} = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} A(x)\, dx$.

Finite Volume Formulation : Find $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}} = \mathbb{R}^{\mathfrak{M}} \times \mathbb{R}^{\mathfrak{M}^*}$ such that

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with $A_{\mathcal{D}} = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} A(x)\, dx$.

Discrete divergence operator

Given a discrete vector field $\xi^{\mathfrak{D}} = (\xi^{\mathcal{D}})_{\mathcal{D} \in \mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}$, we set

$$\mathbf{div}^{\mathcal{K}} \xi^{\mathfrak{D}} \stackrel{\text{def}}{=} \frac{1}{|\mathcal{K}|} \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} |\sigma| \left(\xi^{\mathcal{D}}, \boldsymbol{\nu}_{\mathcal{K}} \right), \quad \forall \mathcal{K} \in \mathfrak{M},$$

$$\mathbf{div}^{\mathcal{K}^*} \xi^{\mathfrak{D}} \stackrel{\text{def}}{=} \frac{1}{|\mathcal{K}^*|} \sum_{\sigma^* \in \mathcal{E}_{\mathcal{K}^*}} |\sigma^*| \left(\xi^{\mathcal{D}}, \boldsymbol{\nu}_{\mathcal{K}^*} \right), \quad \forall \mathcal{K}^* \in \mathfrak{M}^*,$$

which defines an operator

$$\mathbf{div}^{\mathcal{T}} : \xi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}} \mapsto \left((\mathbf{div}^{\mathcal{K}} \xi^{\mathfrak{D}})_{\mathcal{K} \in \mathfrak{M}}, (\mathbf{div}^{\mathcal{K}^*} \xi^{\mathfrak{D}})_{\mathcal{K}^* \in \mathfrak{M}^*} \right) \in \mathbb{R}^{\mathcal{T}}.$$

FINITE VOLUME FORMULATION : Find $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}} = \mathbb{R}^{\mathfrak{M}} \times \mathbb{R}^{\mathfrak{M}^*}$ such that

$$\begin{cases} -\displaystyle\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} |\sigma| \left(A_{\mathcal{D}} \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}, \boldsymbol{\nu}_{\mathcal{K}} \right) = |\mathcal{K}| f_{\mathcal{K}}, \quad \forall \mathcal{K} \in \mathfrak{M}, \\ -\displaystyle\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} |\sigma^*| \left(A_{\mathcal{D}} \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}, \boldsymbol{\nu}_{\mathcal{K}^*} \right) = |\mathcal{K}^*| f_{\mathcal{K}^*}, \forall \mathcal{K}^* \in \mathfrak{M}^*, \end{cases} \tag{DDFV}$$

with $A_{\mathcal{D}} = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} A(x)\, dx$.

$$\boxed{\text{(DDFV)} \Longleftrightarrow \text{Find } u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}} \text{ such that } -\mathbf{div}^{\boldsymbol{\mathcal{T}}}(A^{\mathfrak{D}} \nabla^{\mathfrak{D}} u^{\mathcal{T}}) = f^{\mathcal{T}}.}$$

FINITE VOLUME FORMULATION : Find $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}} = \mathbb{R}^{\mathfrak{M}} \times \mathbb{R}^{\mathfrak{M}^*}$ such that

$$\begin{cases} -\displaystyle\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} |\sigma| \left(A_{\mathcal{D}}\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}}, \boldsymbol{\nu}_{\mathcal{K}}\right) = |\mathcal{K}| f_{\mathcal{K}}, \quad \forall \mathcal{K} \in \mathfrak{M}, \\ -\displaystyle\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} |\sigma^*| \left(A_{\mathcal{D}}\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}}, \boldsymbol{\nu}_{\mathcal{K}^*}\right) = |\mathcal{K}^*| f_{\mathcal{K}^*}, \ \forall \mathcal{K}^* \in \mathfrak{M}^*, \end{cases} \qquad \text{(DDFV)}$$

with $A_{\mathcal{D}} = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} A(x)\,dx$.

$$\boxed{\text{(DDFV)} \Longleftrightarrow \text{Find } u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}} \text{ such that } -\mathbf{div}^{\boldsymbol{\mathcal{T}}}(A^{\mathfrak{D}}\nabla^{\mathfrak{D}} u^{\mathcal{T}}) = f^{\mathcal{T}}.}$$

PROPOSITION (DISCRETE DUALITY FORMULA / STOKES FORMULA)

For any $\xi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}$ $v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, we have

$$\sum_{\mathcal{K}\in\mathfrak{M}} |\mathcal{K}| \mathbf{div}^{\boldsymbol{\mathcal{K}}}(\xi^{\mathfrak{D}}) v_{\mathcal{K}} + \sum_{\mathcal{K}^*\in\mathfrak{M}^*} |\mathcal{K}^*| \mathbf{div}^{\boldsymbol{\mathcal{K}^*}}(\xi^{\mathfrak{D}}) v_{\mathcal{K}^*} = -2\sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}| \left(\xi^{\mathcal{D}}, \nabla^{\mathcal{T}}_{\mathcal{D}} v^{\mathcal{T}}\right).$$

EQUIVALENT FORMULATION OF DDFV

Find $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ such that, for any test function $v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, we have

$$2\sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}| \left(A_{\mathcal{D}}\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}}, \nabla^{\mathcal{T}}_{\mathcal{D}} v^{\mathcal{T}}\right) = \sum_{\mathcal{K}\in\mathfrak{M}} |\mathcal{K}| f_{\mathcal{K}} v_{\mathcal{K}} + \sum_{\mathcal{K}^*\in\mathfrak{M}^*} |\mathcal{K}^*| f_{\mathcal{K}^*} v_{\mathcal{K}^*}.$$

4. THE DDFV METHOD
   - Derivation of the scheme
   - Analysis of the DDFV scheme
   - Implementation
   - The m-DDFV scheme

5. A REVIEW OF SOME OTHER MODERN METHODS
   - General presentation
   - MPFA schemes
   - Diamond schemes
   - Nonlinear monotone FV schemes
   - Mimetic schemes
   - Mixed finite volume methods
   - SUCCES / SUSHI schemes

- Use the discrete integration by parts formula with $v^{\mathcal{T}} = u^{\mathcal{T}}$

$$2 \sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}| \left(A_{\mathcal{D}} \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}, \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}\right) = \sum_{\mathcal{K}\in\mathfrak{M}} |\mathcal{K}| f_{\mathcal{K}} u_{\mathcal{K}} + \sum_{\mathcal{K}^*\in\mathfrak{M}^*} |\mathcal{K}^*| f_{\mathcal{K}^*} u_{\mathcal{K}^*}.$$

- It follows

$$\alpha \|u^{\mathcal{T}}\|_{1,\mathcal{T}}^2 \leq \|f\|_{L^2} (\|u^{\mathfrak{M}}\|_{L^2} + \|u^{\mathfrak{M}^*}\|_{L^2}).$$

THEOREM (DISCRETE POINCARÉ INEQUALITY Proof**)**

There exists a $C > 0$ depending only on Ω and $\mathrm{reg}(\mathcal{T})$ such that

$$\|u^{\mathfrak{M}}\|_{L^2} + \|u^{\mathfrak{M}^*}\|_{L^2} \leq C \|u^{\mathcal{T}}\|_{1,\mathcal{T}}, \quad \forall u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

CONCLUSION : The approximate solution satisfies $\|u^{\mathcal{T}}\|_{1,\mathcal{T}} \leq C\|f\|_{L^2}$.

Theorem

Let $(\mathcal{T}_n)_n$ be a family of DDFV meshes, such that $\mathrm{size}(\mathcal{T}_n) \xrightarrow[n\to\infty]{} 0$ *and* $(\mathrm{reg}(\mathcal{T}_n))_n$ *is bounded.*
Then, the sequence of approximate solutions $u^{\mathcal{T}_n}$ converges towards the exact solution in the following sense

$$u^{\mathfrak{M}_n} \xrightarrow[n\to\infty]{} u \text{ in } L^2(\Omega),$$

$$u^{\mathfrak{M}_n^*} \xrightarrow[n\to\infty]{} u \text{ in } L^2(\Omega),$$

$$\nabla^{\mathcal{T}_n} u^{\mathcal{T}_n} \xrightarrow[n\to\infty]{} \nabla u \text{ in } (L^2(\Omega))^2.$$

Remark : We have **strong** convergence of the gradients.

Proof

ASSUME THAT A IS SMOOTH WITH RESPECT TO x

- Laplace equation
  - First order convergence for $u^{\mathcal{T}}$ and $\nabla^{\mathcal{T}} u^{\mathcal{T}}$

    (Domelevo - Omnès, 05)
  - Some super-convergence results of $u^{\mathcal{T}}$ in L^2

    (Omnes, 10)
- General case (even for nonlinear Leray-Lions operator)

  (Andreianov - B. - Hubert, '07)

THEOREM

Assume that $u \in H^2(\Omega)$ *and* $x \mapsto A(x)$ *is Lipschitz continuous, then there exists* $C(\mathrm{reg}(\mathcal{T})) > 0$ *such that*

$$\|u - u^{\mathfrak{M}}\|_{L^2} + \|u - u^{\mathfrak{M}^*}\|_{L^2} + \|\nabla u - \nabla^{\mathcal{T}} u^{\mathcal{T}}\|_{L^2} \leq C \,\mathrm{size}(\mathcal{T}).$$

STOKES PROBLEM

The DDFV method applied to the Stokes problem is (almost) **inf-sup stable** and first-order convergent (in L^2 for the pressure, in H^1 for the velocity).

(Delcourte, '07) (Krell,'10)
(Krell-Manzini, '12) (B.-Krell-Nabet, '13)

- The matrix is built **through a loop over primal edges** (that is diamond cells). For each such edge/diamond, we compute 4×4 terms.
- Stencil :
  - does not depend on the permeability tensor.
  - The row corresponding to the unknown $u_{\mathcal{K}}$ has at most $2N + 1$ non zero entries, where N is the number of edges of $\mathcal{K}$.
- The matrix is symmetric positive definite.
- In the case of an orthogonal admissible mesh,

$$\text{DDFV} \Longleftrightarrow \text{TPFA on the primal mesh} + \text{TPFA on the dual mesh.}$$

- In the nonlinear case $-\mathrm{div}(\varphi(x, \nabla u)) = f$, we can adapt the decomposition-coordination method of **Glowinski** to obtain a suitable nonlinear solver that can be proved to be convergent.

(B.-Hubert '08)

(B. - Hubert, '08)

Goals

- To take into account possible permeability discontinuities in the problem without loss of accuracy.
- We allow (full tensor) permeability jumps across
  - Primal edges.
  - Dual edges.
  - Both primal and dual edges.
- Same stencil as for the standard DDFV method.

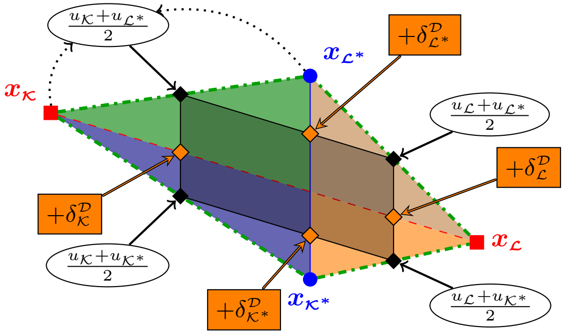
(B. - Hubert, '08)

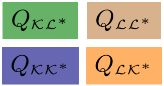
Goals

- To take into account possible permeability discontinuities in the problem without loss of accuracy.
- We allow (full tensor) permeability jumps across
  - Primal edges.
  - Dual edges.
  - Both primal and dual edges.
- Same stencil as for the standard DDFV method.

General principle

- We want to mimick the harmonic mean-value formula that we obtained for TPFA.
  - We need to introduce artificial edges unknowns.
  - We impose local conservativity of some well-chosen numerical fluxes.
  - We eliminate those additional unknowns so that we finally get suitable numerical fluxes formulas
- The coupling between primal and dual unknowns and equations needs a particular care.

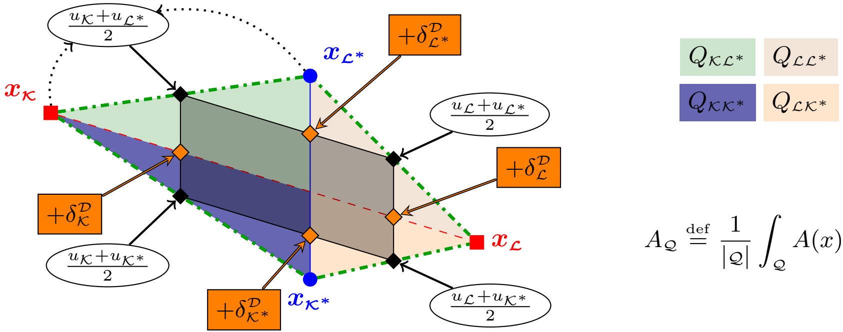




$$A_{\mathcal{Q}} \stackrel{\text{def}}{=} \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} A(x)\,dx$$

STRATEGY

- We add a value $\delta^{\mathcal{D}}_{\bullet}$ to the value of $\Pi_{\mathcal{D}} u^{\mathcal{T}}$ at the points ◆.
- With these new values at hand, we build affine functions on each quarter diamond.
- The gradients of these new functions are used as new discrete gradients in DDFV.
- We eventually eliminate the values $\delta^{\mathcal{D}} = {}^t(\delta^{\mathcal{D}}_{\mathcal{K}}, \delta^{\mathcal{D}}_{\mathcal{L}}, \delta^{\mathcal{D}}_{\mathcal{K}^*}, \delta^{\mathcal{D}}_{\mathcal{L}^*}) \in \mathbb{R}^4$ by imposing suitable conservativity conditions.



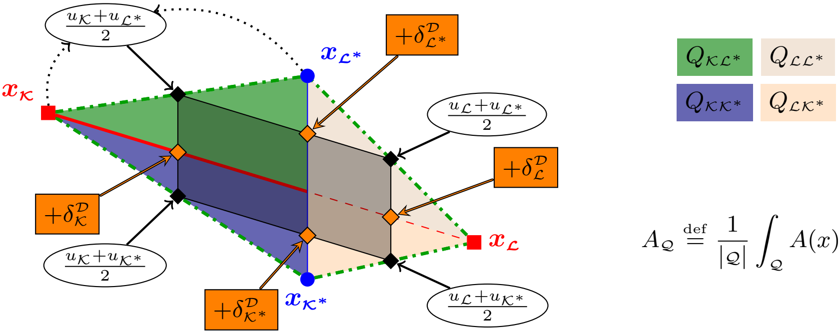
$$A_{\mathcal{Q}} \stackrel{\text{def}}{=} \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} A(x)\,dx$$

New gradients on each quarter diamond

$$\nabla^{\mathcal{N}}_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} u^{\mathcal{T}} \stackrel{\text{def}}{=} \nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} \delta^{\mathcal{D}},$$

with

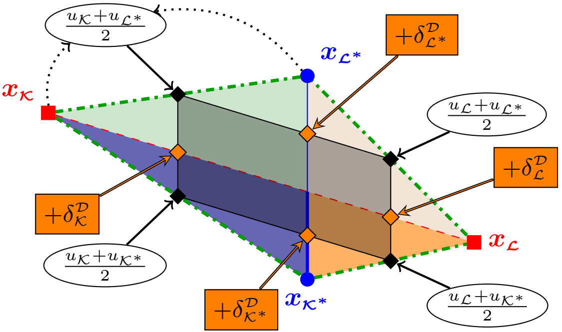
$$B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} \stackrel{\text{def}}{=} \frac{1}{|\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}|}\left(|\sigma_{\mathcal{K}}|\boldsymbol{\nu}^*, 0, |\sigma_{\mathcal{K}^*}|\boldsymbol{\nu}, 0\right).$$

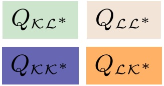


$$A_\mathcal{Q} \stackrel{\text{def}}{=} \frac{1}{|\mathcal{Q}|}\int_\mathcal{Q} A(x)\,dx$$

Write local conservativity between quarter diamonds

$$\left(A_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\nabla_\mathcal{D}^\mathcal{T} u^\mathcal{T} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\delta^\mathcal{D}), \boldsymbol{\nu}^*\right) = \left(A_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\nabla_\mathcal{D}^\mathcal{T} u^\mathcal{T} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\delta^\mathcal{D}), \boldsymbol{\nu}^*\right)$$



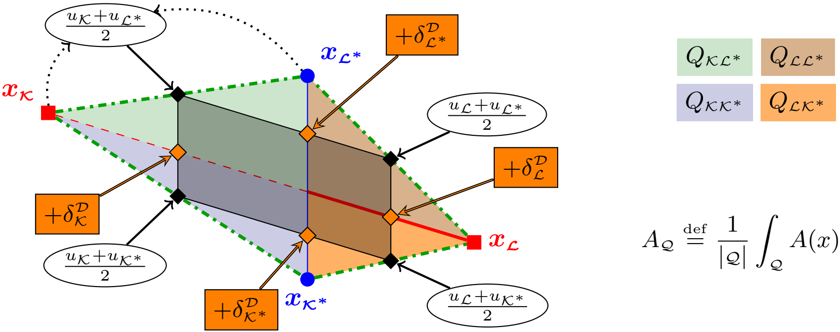


$$A_{\mathcal{Q}} \stackrel{\text{def}}{=} \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} A(x)\,dx$$

WRITE LOCAL CONSERVATIVITY BETWEEN QUARTER DIAMONDS

$$\left(A_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}^*\right) = \left(A_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}^*\right)$$

$$\left(A_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}\right) = \left(A_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}\right)$$



$$A_{\mathcal{Q}} \stackrel{\text{def}}{=} \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} A(x)\,dx$$

Write local conservativity between quarter diamonds

$$\left(A_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}}u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}^*\right) = \left(A_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\nabla^{\mathcal{T}}_{\mathcal{D}}u^{\mathcal{T}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\delta^{\mathcal{D}}), \boldsymbol{\nu}^*\right)$$

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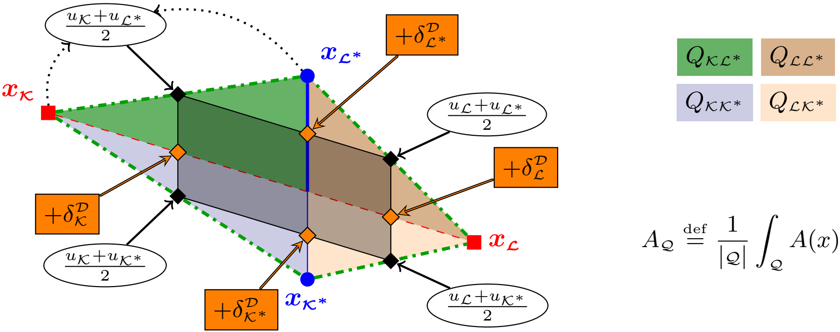

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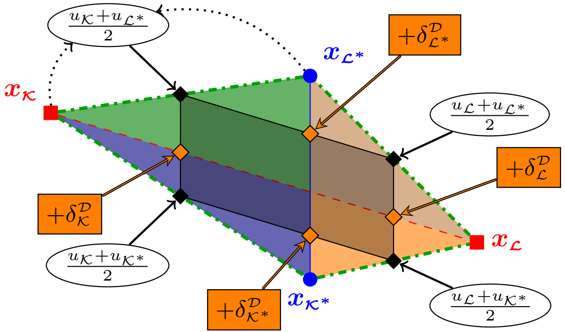

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$$\iff \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}|\, {}^t B_{\mathcal{Q}}.A_{\mathcal{Q}}(\nabla^{\mathcal{T}}_{\mathcal{D}}u^{\mathcal{T}} + B_{\mathcal{Q}}\delta^{\mathcal{D}}) = 0.$$

Proposition

For any $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, *and any diamond* $\mathcal{D}$, *there exists a* **unique** $\delta^{\mathcal{D}} \in \mathbb{R}^4$ *satisfying the local flux conservativity property*

$$\sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}| {}^t B_{\mathcal{Q}} . A_{\mathcal{Q}} (\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}) = 0,$$

and the map $\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} \mapsto \delta^{\mathcal{D}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}})$ *is linear.*

Remark : This strategy applies to the non-linear case where the permeability-map $\xi \mapsto A_{\mathcal{Q}}.\xi$ is now a monotone map

$$\xi \in \mathbb{R}^2 \mapsto \varphi_{\mathcal{Q}}(\xi) \in \mathbb{R}^2.$$

The m-DDFV scheme

We replace in the DDFV scheme the approximate permeability $A_{\mathcal{D}}$ with the following new map

$$A_{\mathcal{D}}^{\mathcal{N}}.\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \stackrel{\text{def}}{=} \frac{1}{|\mathcal{D}|} \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}| A_{\mathcal{Q}} (\underbrace{\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}})}_{=\nabla_{\mathcal{Q}}^{\mathcal{N}} u^{\mathcal{T}}}),$$

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Discrete duality formulation on diamond cells

$$2\sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}| \left(A_{\mathcal{D}}^{\mathcal{N}}.\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}, \nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}}\right) = \int_\Omega f v^{\mathfrak{M}} dx + \int_\Omega f v^{\mathfrak{M}^*} dx, \quad \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

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Discrete duality formulation on quarter diamonds

$$2\sum_{\mathcal{Q}\in\mathfrak{Q}} |\mathcal{Q}| \left(A_{\mathcal{Q}}.\nabla_{\mathcal{Q}}^{\mathcal{N}} u^{\mathcal{T}}, \nabla_{\mathcal{Q}}^{\mathcal{N}} v^{\mathcal{T}}\right) = \int_\Omega f v^{\mathfrak{M}} dx + \int_\Omega f v^{\mathfrak{M}^*} dx, \quad \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

Remarks :

- Same stencil as DDFV.
- All the maps $\xi \mapsto \delta^{\mathcal{D}}(\xi)$ can be pre-computed offline, in a parallel fashion $\Rightarrow$ almost no additional computational cost.

Assume that $x \mapsto A(x)$ is **constant on each primal control volume**, we recover the schemes already introduced in **Hermeline (03)**.
Explicit formulas for $A_{\mathcal{D}}^{\mathcal{N}}$ are available

$$(A_{\mathcal{D}}^{\mathcal{N}}\boldsymbol{\nu},\boldsymbol{\nu}) = \frac{(|\sigma_{\mathcal{K}}|+|\sigma_{\mathcal{L}}|)(A_{\mathcal{K}}\boldsymbol{\nu},\boldsymbol{\nu})(A_{\mathcal{L}}\boldsymbol{\nu},\boldsymbol{\nu})}{|\sigma_{\mathcal{L}}|(A_{\mathcal{K}}\boldsymbol{\nu},\boldsymbol{\nu})+|\sigma_{\mathcal{K}}|(A_{\mathcal{L}}\boldsymbol{\nu},\boldsymbol{\nu})},$$

$$(A_{\mathcal{D}}^{\mathcal{N}}\boldsymbol{\nu}^*,\boldsymbol{\nu}^*) = \frac{|\sigma_{\mathcal{L}}|(A_{\mathcal{L}}\boldsymbol{\nu}^*,\boldsymbol{\nu}^*)+|\sigma_{\mathcal{K}}|(A_{\mathcal{K}}\boldsymbol{\nu}^*,\boldsymbol{\nu}^*)}{|\sigma_{\mathcal{K}}|+|\sigma_{\mathcal{L}}|} - \frac{|\sigma_{\mathcal{K}}||\sigma_{\mathcal{L}}|}{|\sigma_{\mathcal{K}}|+|\sigma_{\mathcal{L}}|}\frac{((A_{\mathcal{K}}\boldsymbol{\nu},\boldsymbol{\nu}^*)-(A_{\mathcal{L}}\boldsymbol{\nu},\boldsymbol{\nu}^*))^2}{|\sigma_{\mathcal{L}}|(A_{\mathcal{K}}\boldsymbol{\nu},\boldsymbol{\nu})+|\sigma_{\mathcal{K}}|(A_{\mathcal{L}}\boldsymbol{\nu},\boldsymbol{\nu})},$$

$$(A_{\mathcal{D}}^{\mathcal{N}}\boldsymbol{\nu},\boldsymbol{\nu}^*) = \frac{|\sigma_{\mathcal{L}}|(A_{\mathcal{L}}\boldsymbol{\nu},\boldsymbol{\nu}^*)(A_{\mathcal{K}}\boldsymbol{\nu},\boldsymbol{\nu})+|\sigma_{\mathcal{K}}|(A_{\mathcal{K}}\boldsymbol{\nu},\boldsymbol{\nu}^*)(A_{\mathcal{L}}\boldsymbol{\nu},\boldsymbol{\nu})}{|\sigma_{\mathcal{L}}|(A_{\mathcal{K}}\boldsymbol{\nu},\boldsymbol{\nu})+|\sigma_{\mathcal{K}}|(A_{\mathcal{L}}\boldsymbol{\nu},\boldsymbol{\nu})}.$$

Theorem

The m-DDFV scheme has a **unique** *solution $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ which depends continuously on the data.*

Theorem

Assume that $x \mapsto A(x)$ is smooth on each quarter diamond, and that u belongs to H^2 on each quarter diamond $\mathcal{Q}$, then we have

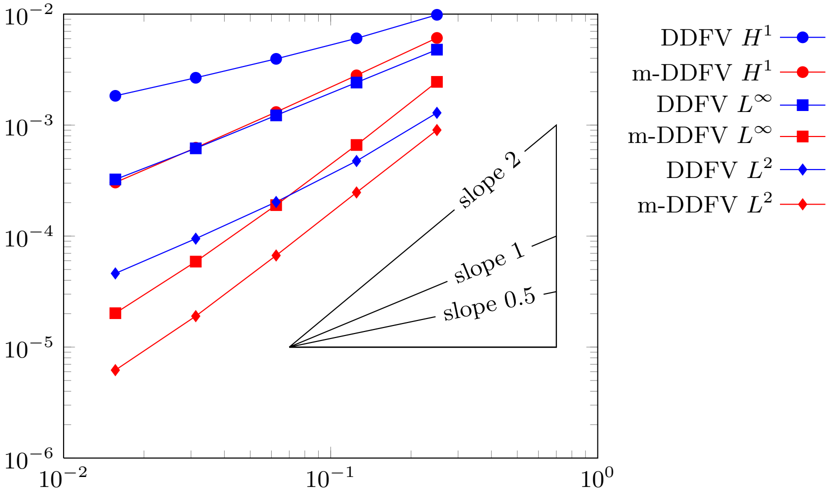
$$\|u - u^{\mathfrak{M}}\|_{L^2} + \|u - u^{\mathfrak{M}^*}\|_{L^2} + \|\nabla u - \nabla^{\mathcal{N}} u^{\mathcal{T}}\|_{L^2} \leq C\,h.$$

$\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$ with $\Omega_1 =]0, 0.5[\times]0, 1[$ and $\Omega_2 =]0.5, 1[\times]0, 1[$

A linear example :

$$-\mathrm{div}(A(x)\nabla u) = f, \quad \text{with } A(x) = \mathrm{Id} \text{ in } \Omega_1, \; A(x) = \begin{pmatrix} 15 & 20 \\ 20 & 40 \end{pmatrix} \text{ in } \Omega_2.$$

- DDFV : order $\frac{1}{2}$ in the H^1 norm
- m-DDFV : order 1 in the H^1 norm


10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-2}
10^{-1}
10^{0}
slope 2
slope 1
slope 0.5
DDFV H^1
m-DDFV H^1
DDFV L^∞
m-DDFV L^∞
DDFV L^2
m-DDFV L^2

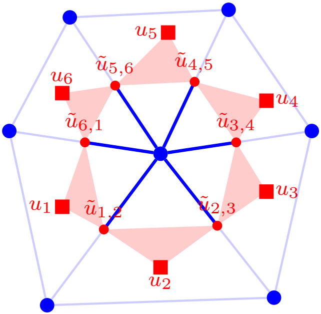
Linear problems $-\mathrm{div}(A(x)\nabla u) = f$

- Fully cell-centered schemes
  - MPFA (Aavatsmark et al. '98 → '08) (Edwards et al. '06,'08)
  - Diamond schemes (Coudière-Vila-Villedieu '99, '00) (Manzini et al ... '04 → '07)
  - SUSHI (barycentric version) = SUCCES (Eymard-Gallouët-Herbin '08)
- Nonlinear monotone finite volume
  - Nonlinear diamond schemes (Bertolazzo-Manzini '07)
  - NMFV (Le Potier '05) (Lipnikov et al '07)
- Schemes on primal/dual meshes
  - DDFV (Hermeline '00) (Domelevo-Omnes '05) (Pierre '06) (Andreianov-B.-Hubert '07)
  - m-DDFV (Hermeline '03) (B.-Hubert '08)
- Hybrid and mixed schemes
  - Mimetic schemes (Brezzi, Lipnikov et al '05 → '08) (Manzini et al '07-'08)
  - Mixed finite volumes (Droniou-Eymard '06)
  - SUSHI (hybrid version) (Eymard-Gallouet-Herbin '08)

Recent review paper : (Droniou, '13)

O SCHEME

(Aavatsmark et al. '98 → '08)

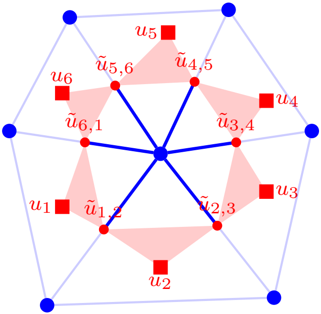


- Intermediate unknowns : $\tilde{u}_{ij}$
- We compute a gradient on each red triangle T_i

$$\nabla_i u^{\mathcal{T}} = \frac{\tilde{u}_{i,i+1} - u_i}{2|T_i|}(\tilde{x}_{i-1,i} - x_i)^{\perp} + \frac{\tilde{u}_{i-1,i} - u_i}{2|T_i|}(\tilde{x}_{i,i+1} - x_i)^{\perp}.$$

O SCHEME

(Aavatsmark et al. '98 → '08)



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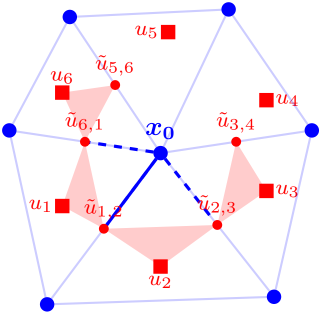
- We write flux continuity at mid-edges

$$\boxed{F_{i,i+1} \stackrel{\text{def}}{=} (A_i \nabla_i u^{\mathcal{T}}) \cdot \boldsymbol{\nu}_{i,i+1} = (A_{i+1} \nabla_{i+1} u^{\mathcal{T}}) \cdot \boldsymbol{\nu}_{i,i+1}, \quad \forall i.}$$

- Given the $(u_i)_i$, we deduce the $(\tilde{u}_{i,i+1})_i$ then the **semi-fluxes** $(F_{i,i+1})_i$.

U SCHEME : **Let us compute F_{12}** **(Aavatsmark et al. '98 → '08)**

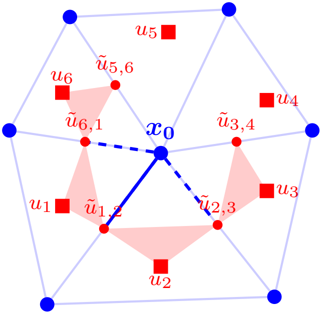


- Intermediate unknowns : $\tilde{u}_{ij}$
- We compute a gradient on each red triangle T_i

$$\nabla_i u^{\mathcal{T}} = \frac{\tilde{u}_{i,i+1} - u_i}{2|T_i|}(\tilde{x}_{i-1,i} - x_i)^{\perp} + \frac{\tilde{u}_{i-1,i} - u_i}{2|T_i|}(\tilde{x}_{i,i+1} - x_i)^{\perp}.$$

⇝ This gives birth to an affine function on each control volume.

U scheme : **Let us compute F_{12}** **(Aavatsmark et al. '98 → '08)**



- Intermediate unknowns : $\tilde{u}_{ij}$
- We compute a gradient on each red triangle T_i

$$\nabla_i u^{\mathcal{T}} = \frac{\tilde{u}_{i,i+1} - u_i}{2|T_i|}(\tilde{x}_{i-1,i} - x_i)^\perp + \frac{\tilde{u}_{i-1,i} - u_i}{2|T_i|}(\tilde{x}_{i,i+1} - x_i)^\perp.$$

⇝ This gives birth to an affine function on each control volume.

- We write fluxes continuity for F_{12}, F_{23} and F_{61}

$$\boxed{F_{i,i+1} \stackrel{\text{def}}{=} (A_i \nabla_i u^{\mathcal{T}}) \cdot \boldsymbol{\nu}_{i,i+1} = (A_{i+1} \nabla_{i+1} u^{\mathcal{T}}) \cdot \boldsymbol{\nu}_{i,i+1}, \quad i \in \{1,2,6\}.}$$

- We need two additional equations. We write :

$$U_2(x_0) = U_3(x_0), \quad \text{and} \quad U_1(x_0) = U_6(x_0).$$

- Given the $(u_i)_i$, we compute the $(\tilde{u}_{i,i+1})_i$ then the semi-fluxes F_{12}.
- We do the same for the other fluxes.

Properties

- In general
  - the final linear system is not symmetric.
  - No coercivity/stability for high anisotropies/heterogeneies/mesh distorsion.
- There exists a stabilized/symmetric version on quadrangles

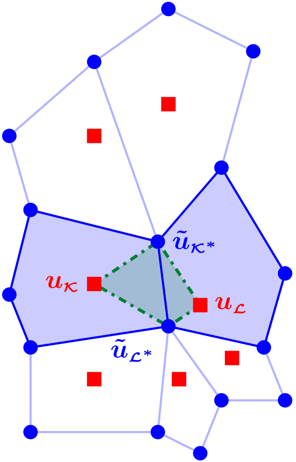
  (Le Potier, '05).
- Stencil :
  - The O scheme has a much too large stencil.
  - For the U scheme on conforming triangles : one flux depends on 6 unknowns.
  - In general, the equation on a control volume κ depends on κ, its neighbors and the neighbors of its neighbors.
  - Other variants : G scheme, L scheme, ...
- No complete gradient reconstruction.
- No discrete maximum principle for basic methods. Some improvements possible to achieve this goal.
- Convergence in the general case provided that a geometric condition for coercivity holds true

  (Agelas-Masson, '08), (Agelas-DiPietro-Droniou, '10)

  (Klausen-Stephansen, '12) (Stephansen, '12)

(Coudière-Vila-Villedieu, '99, '00)

- Intermediate unknowns at vertices $\tilde{u}_{\mathcal{K}^*}$, $\tilde{u}_{\mathcal{L}^*}$.



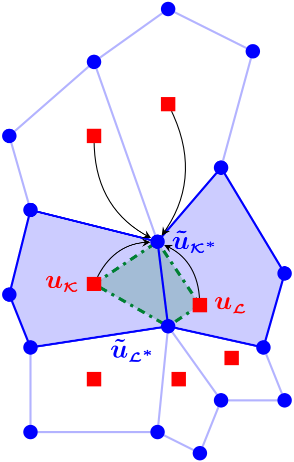
- Discrete gradient on the diamond cell $\mathcal{D}$:

$$\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} = \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}} \sin \alpha_{\mathcal{D}}} \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} + \frac{\tilde{u}_{\mathcal{L}^*} - \tilde{u}_{\mathcal{K}^*}}{d_{\mathcal{K}^*\mathcal{L}^*} \sin \alpha_{\mathcal{D}}} \boldsymbol{\nu}_{\mathcal{K}^*\mathcal{L}^*}.$$

$$\Leftrightarrow \begin{cases} \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_{\mathcal{L}} - x_{\mathcal{K}}) = u_{\mathcal{L}} - u_{\mathcal{K}}, \\ \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_{\mathcal{L}^*} - x_{\mathcal{K}^*}) = \tilde{u}_{\mathcal{L}^*} - \tilde{u}_{\mathcal{K}^*}. \end{cases}$$

(Coudière-Vila-Villedieu, '99, '00)

- Intermediate unknowns at vertices $\tilde{u}_{\mathcal{K}^*}$, $\tilde{u}_{\mathcal{L}^*}$.



- Discrete gradient on the diamond cell $\mathcal{D}$:

$$\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} = \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}} \sin \alpha_{\mathcal{D}}} \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} + \frac{\tilde{u}_{\mathcal{L}^*} - \tilde{u}_{\mathcal{K}^*}}{d_{\mathcal{K}^*\mathcal{L}^*} \sin \alpha_{\mathcal{D}}} \boldsymbol{\nu}_{\mathcal{K}^*\mathcal{L}^*}.$$

$$\Leftrightarrow \left\{ \begin{array}{l} \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_{\mathcal{L}} - x_{\mathcal{K}}) = u_{\mathcal{L}} - u_{\mathcal{K}}, \\ \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_{\mathcal{L}^*} - x_{\mathcal{K}^*}) = \tilde{u}_{\mathcal{L}^*} - \tilde{u}_{\mathcal{K}^*}. \end{array} \right.$$

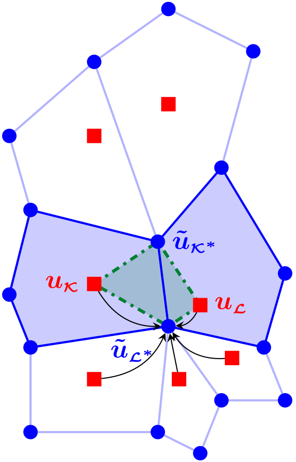
- Here, $\tilde{u}_{\mathcal{K}^*}$ and $\tilde{u}_{\mathcal{L}^*}$ are directly expressed with

$$\tilde{u}_{\mathcal{K}^*} = \sum_{\mathcal{M}} \underbrace{\gamma_{\mathcal{M},\mathcal{K}^*}}_{\geq 0} u_{\mathcal{M}}, \quad \tilde{u}_{\mathcal{L}^*} = \sum_{\mathcal{M}} \underbrace{\gamma_{\mathcal{M},\mathcal{L}^*}}_{\geq 0} u_{\mathcal{M}},$$

with $\displaystyle\sum_{\mathcal{M}} \gamma_{\mathcal{M},\mathcal{K}^*} = 1, \sum_{\mathcal{M}} \gamma_{\mathcal{M},\mathcal{K}^*} x_{\mathcal{M}} = x_{\mathcal{K}^*}.$

(Coudière-Vila-Villedieu, '99, '00)

- Intermediate unknowns at vertices $\tilde{u}_{\mathcal{K}^*}$, $\tilde{u}_{\mathcal{L}^*}$.



- Discrete gradient on the diamond cell $\mathcal{D}$:

$$\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} = \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}} \sin\alpha_{\mathcal{D}}} \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} + \frac{\tilde{u}_{\mathcal{L}^*} - \tilde{u}_{\mathcal{K}^*}}{d_{\mathcal{K}^*\mathcal{L}^*} \sin\alpha_{\mathcal{D}}} \boldsymbol{\nu}_{\mathcal{K}^*\mathcal{L}^*}.$$

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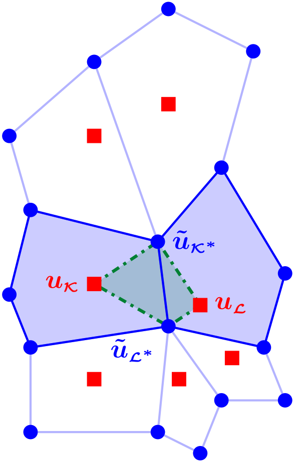
- Here, $\tilde{u}_{\mathcal{K}^*}$ and $\tilde{u}_{\mathcal{L}^*}$ are directly expressed with

$$\tilde{u}_{\mathcal{K}^*} = \sum_{\mathcal{M}} \underbrace{\gamma_{\mathcal{M},\mathcal{K}^*}}_{\geq 0} u_{\mathcal{M}}, \quad \tilde{u}_{\mathcal{L}^*} = \sum_{\mathcal{M}} \underbrace{\gamma_{\mathcal{M},\mathcal{L}^*}}_{\geq 0} u_{\mathcal{M}},$$

with $\displaystyle\sum_{\mathcal{M}} \gamma_{\mathcal{M},\mathcal{K}^*} = 1$, $\displaystyle\sum_{\mathcal{M}} \gamma_{\mathcal{M},\mathcal{K}^*} x_{\mathcal{M}} = x_{\mathcal{K}^*}$.

(Coudière-Vila-Villedieu, '99, '00)

- Intermediate unknowns at vertices $\tilde{u}_{\mathcal{K}^*}$, $\tilde{u}_{\mathcal{L}^*}$.



- Discrete gradient on the diamond cell $\mathcal{D}$:

$$\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} = \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K}\mathcal{L}} \sin \alpha_{\mathcal{D}}} \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} + \frac{\tilde{u}_{\mathcal{L}^*} - \tilde{u}_{\mathcal{K}^*}}{d_{\mathcal{K}^*\mathcal{L}^*} \sin \alpha_{\mathcal{D}}} \boldsymbol{\nu}_{\mathcal{K}^*\mathcal{L}^*}.$$

$$\Leftrightarrow \left\{ \begin{array}{l} \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_{\mathcal{L}} - x_{\mathcal{K}}) = u_{\mathcal{L}} - u_{\mathcal{K}}, \\ \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_{\mathcal{L}^*} - x_{\mathcal{K}^*}) = \tilde{u}_{\mathcal{L}^*} - \tilde{u}_{\mathcal{K}^*}. \end{array} \right.$$

- Here, $\tilde{u}_{\mathcal{K}^*}$ and $\tilde{u}_{\mathcal{L}^*}$ are directly expressed with

$$\tilde{u}_{\mathcal{K}^*} = \sum_{\mathcal{M}} \underbrace{\gamma_{\mathcal{M},\mathcal{K}^*}}_{\geq 0} u_{\mathcal{M}}, \quad \tilde{u}_{\mathcal{L}^*} = \sum_{\mathcal{M}} \underbrace{\gamma_{\mathcal{M},\mathcal{L}^*}}_{\geq 0} u_{\mathcal{M}},$$

with $\sum_{\mathcal{M}} \gamma_{\mathcal{M},\mathcal{K}^*} = 1$, $\sum_{\mathcal{M}} \gamma_{\mathcal{M},\mathcal{K}^*} x_{\mathcal{M}} = x_{\mathcal{K}^*}$.

- The numerical flux then reads (formally the same as for DDFV)

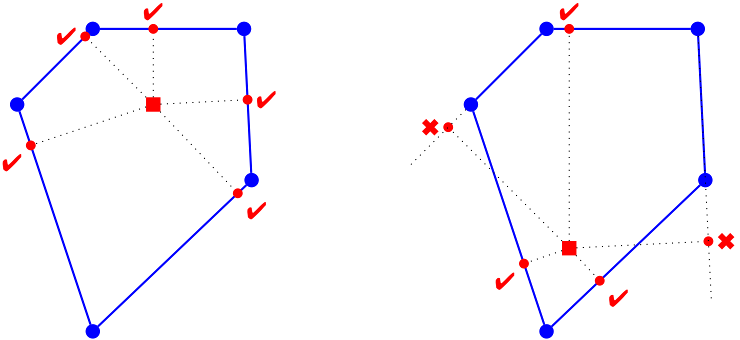
$$\boxed{F_{\mathcal{K}\mathcal{L}} = -|\sigma| \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}.}$$

PROPERTIES

- The weights $\gamma_{\mathcal{M},\mathcal{K}^*}$ are computed through a least-square procedure.
- Finite volume consistance is proved.
- Coercivity/stability is not ensured excepted for meshes not too far from Cartesian rectangle meshes.
- In the case where coercivity holds, we deduce the standard first order error estimates for u and ∇u.
- The scheme can be written on general meshes but is not supported by convergence analysis.
- In general, the linear system to be solved **is not symmetric**.

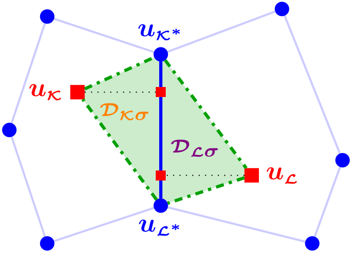
(Manzini et al ... '04 → '07)

- Assume that orthogonal projection of centers belong to the edges.



- With this assumption, the authors provide an algorithm to compute weights $\gamma_{\mathcal{M},\mathcal{K}^*}$, such that

$$\gamma_{\mathcal{M},\mathcal{K}^*} \geq C_0 > 0, \quad \forall \mathcal{M} \text{ containing } x_{\mathcal{K}^*}.$$

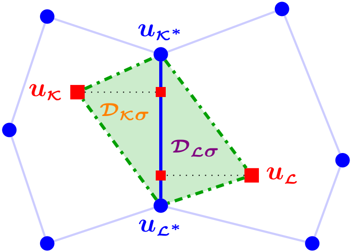


(Manzini et al ... '04 → '07)

Define a discrete gradient on both half-diamond cells $\mathcal{D}_{\mathcal{K}\sigma}$ and $\mathcal{D}_{\mathcal{L}\sigma}$ from the three values at our disposal:

$$\nabla_{\mathcal{D}_{\mathcal{K}\sigma}} u^{\mathcal{T}}, \quad \nabla_{\mathcal{D}_{\mathcal{L}\sigma}} u^{\mathcal{T}}.$$

(Manzini et al ... '04 → '07)



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$$\nabla_{\mathcal{D}_{\mathcal{K}\sigma}} u^{\mathcal{T}}, \quad \nabla_{\mathcal{D}_{\mathcal{L}\sigma}} u^{\mathcal{T}}.$$

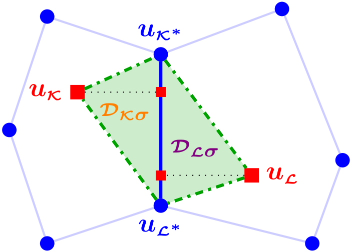
- We compute the corresponding fluxes

$$-|\sigma| \nabla_{\mathcal{D}_{\mathcal{K}\sigma}} u^{\mathcal{T}} \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} = \underbrace{\alpha_{\mathcal{K}}^{\mathcal{K}}}_{>0} (u_{\mathcal{K}} - u_{\mathcal{L}}) + \sum_{\mathcal{M} \neq \mathcal{L}} \underbrace{\alpha_{\mathcal{M}}^{\mathcal{K}}}_{\geq 0} (u_{\mathcal{K}} - u_{\mathcal{M}}),$$

$$-|\sigma| \nabla_{\mathcal{D}_{\mathcal{L}\sigma}} u^{\mathcal{T}} \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} = \underbrace{\alpha_{\mathcal{L}}^{\mathcal{L}}}_{>0} (u_{\mathcal{K}} - u_{\mathcal{L}}) + \sum_{\mathcal{M} \neq \mathcal{K}} \underbrace{\alpha_{\mathcal{M}}^{\mathcal{L}}}_{\geq 0} (u_{\mathcal{M}} - u_{\mathcal{L}}).$$

We set $\alpha = \min(\alpha_{\mathcal{K}}^{\mathcal{K}}, \alpha_{\mathcal{L}}^{\mathcal{L}}) > 0$.

(Manzini et al ... '04 → '07)



Define a discrete gradient on both half-diamond cells $\mathcal{D}_{\mathcal{K}\sigma}$ and $\mathcal{D}_{\mathcal{L}\sigma}$ from the three values at our disposal:

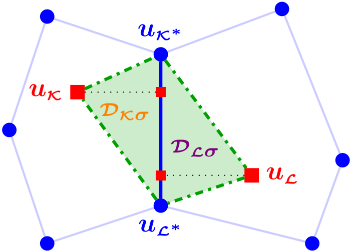
$$\nabla_{\mathcal{D}_{\mathcal{K}\sigma}} u^{\mathcal{T}}, \quad \nabla_{\mathcal{D}_{\mathcal{L}\sigma}} u^{\mathcal{T}}.$$

- We compute the corresponding fluxes

$$-|\sigma|\nabla_{\mathcal{D}_{\mathcal{K}\sigma}} u^{\mathcal{T}} \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} = \alpha(u_{\mathcal{K}} - u_{\mathcal{L}}) + \underbrace{\sum_{\mathcal{M}} \underbrace{(\alpha^{\mathcal{K}}_{\mathcal{M}} - \alpha\delta_{\mathcal{M}\mathcal{L}})}_{\geq 0}(u_{\mathcal{K}} - u_{\mathcal{M}})}_{\stackrel{\text{def}}{=} g_{\mathcal{K}}(u)},$$

$$-|\sigma|\nabla_{\mathcal{D}_{\mathcal{L}\sigma}} u^{\mathcal{T}} \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} = \alpha(u_{\mathcal{K}} - u_{\mathcal{L}}) + \underbrace{\sum_{\mathcal{M}} \underbrace{(\alpha^{\mathcal{K}}_{\mathcal{M}} - \alpha\delta_{\mathcal{M}\mathcal{K}})}_{\geq 0}(u_{\mathcal{M}} - u_{\mathcal{L}})}_{\stackrel{\text{def}}{=} g_{\mathcal{L}}(u)}.$$

(Manzini et al ... '04 → '07)



Define a discrete gradient on both half-diamond cells $\mathcal{D}_{\mathcal{K}\sigma}$ and $\mathcal{D}_{\mathcal{L}\sigma}$ from the three values at our disposal:

$$\nabla_{\mathcal{D}_{\mathcal{K}\sigma}} u^{\mathcal{T}}, \quad \nabla_{\mathcal{D}_{\mathcal{L}\sigma}} u^{\mathcal{T}}.$$

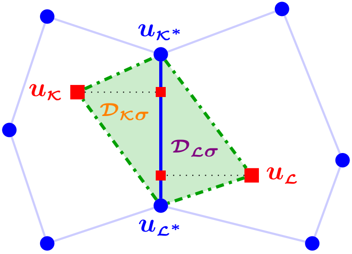
$$-|\sigma|\nabla_{\mathcal{D}_{\mathcal{K}\sigma}} u^{\mathcal{T}} \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} = \alpha(u_{\mathcal{K}} - u_{\mathcal{L}}) + g_{\mathcal{K}}(u^{\mathcal{T}}),$$
$$-|\sigma|\nabla_{\mathcal{D}_{\mathcal{L}\sigma}} u^{\mathcal{T}} \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} = \alpha(u_{\mathcal{K}} - u_{\mathcal{L}}) + g_{\mathcal{L}}(u^{\mathcal{T}}).$$

- We set $\omega_{\mathcal{D}}(u^{\mathcal{T}}) = \dfrac{|g_{\mathcal{L}}(u^{\mathcal{T}})|}{|g_{\mathcal{K}}(u^{\mathcal{T}})| + |g_{\mathcal{L}}(u^{\mathcal{T}})|}$ and we consider now the following gradient on the diamond cell

$$\nabla_{\mathcal{D}} u^{\mathcal{T}} = \omega_{\mathcal{D}}(u^{\mathcal{T}})\nabla_{\mathcal{D}_{\mathcal{K}\sigma}} u^{\mathcal{T}} + (1 - \omega_{\mathcal{D}}(u^{\mathcal{T}}))\nabla_{\mathcal{D}_{\mathcal{L}\sigma}} u^{\mathcal{T}}.$$

$$\Rightarrow F_{\mathcal{K}\mathcal{L}}(u^{\mathcal{T}}) \stackrel{\text{def}}{=} -|\sigma|\nabla_{\mathcal{D}} u^{\mathcal{T}} \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} = \alpha(u_{\mathcal{K}} - u_{\mathcal{L}}) + \underbrace{\frac{g_{\mathcal{K}}(u^{\mathcal{T}})|g_{\mathcal{L}}(u^{\mathcal{T}})| + g_{\mathcal{L}}(u^{\mathcal{T}})|g_{\mathcal{K}}(u^{\mathcal{T}})|}{|g_{\mathcal{K}}(u^{\mathcal{T}})| + |g_{\mathcal{L}}(u^{\mathcal{T}})|}}_{=\mathbf{T}}.$$

(Manzini et al ... '04 → '07)



Define a discrete gradient on both half-diamond cells $\mathcal{D}_{\mathcal{K}\sigma}$ and $\mathcal{D}_{\mathcal{L}\sigma}$ from the three values at our disposal:

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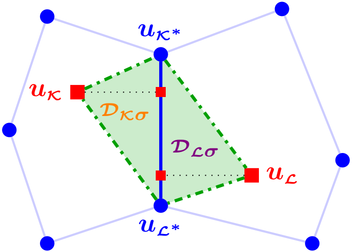
- If $g_{\mathcal{K}}(u^{\mathcal{T}}) g_{\mathcal{L}}(u^{\mathcal{T}}) < 0$:

$$T = 0.$$

- If $g_{\mathcal{K}}(u^{\mathcal{T}}) g_{\mathcal{L}}(u^{\mathcal{T}}) > 0$:

$$T = \frac{2 g_{\mathcal{K}}(u^{\mathcal{T}}) g_{\mathcal{L}}(u^{\mathcal{T}})}{g_{\mathcal{K}}(u^{\mathcal{T}}) + g_{\mathcal{L}}(u^{\mathcal{T}})} \Rightarrow \begin{cases} T \text{ is a } \textbf{nonnegative} \text{ multiple of } g_{\mathcal{K}}(u^{\mathcal{T}}) \\ T \text{ is a } \textbf{nonnegative} \text{ multiple of } g_{\mathcal{L}}(u^{\mathcal{T}}) \end{cases}$$



(Manzini et al ... '04 → '07)

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$$F_{\mathcal{K},\mathcal{L}} = \alpha(u_{\mathcal{K}} - u_{\mathcal{L}}) + \underbrace{C_{\mathcal{K}}}_{\geq 0} \sum_{\mathcal{M}} (\alpha^{\mathcal{K}}_{\mathcal{M}} - \alpha\delta_{\mathcal{M}\mathcal{L}})(u_{\mathcal{K}} - u_{\mathcal{M}}),$$

$$F_{\mathcal{K},\mathcal{L}} = \alpha(u_{\mathcal{K}} - u_{\mathcal{L}}) + \underbrace{C_{\mathcal{L}}}_{\geq 0} \sum_{\mathcal{M}} (\alpha^{\mathcal{K}}_{\mathcal{M}} - \alpha\delta_{\mathcal{M}\mathcal{K}})(u_{\mathcal{M}} - u_{\mathcal{L}}).$$

(Manzini et al ... '04 → '07)

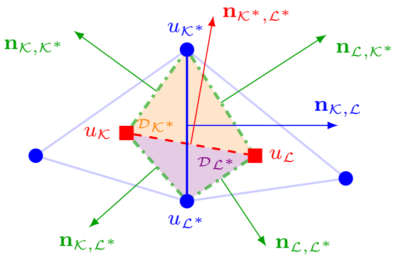
PROPERTIES

- Consistency OK.
- There exists at least one solution of the scheme.
- Quasi-uniqueness : all solutions belong to a ball of radius $O(\text{size}(\mathcal{T})^2)$.
- Solving the scheme can be done by using an iterative solver which necessitates the solution of a (unique) definite positive system.

- The converged solution satisfies the discrete maximum principle.
- Each solver iterate does not satisfy the discrete maximum principle.

- No convergence analysis available.
- In practice, we observe standard second order for u in the L^2 norm.

TO SIMPLIFY A LITTLE : $A(x) = A$ **(Le Potier '05-..) (Lipnikov et al '07-...)**



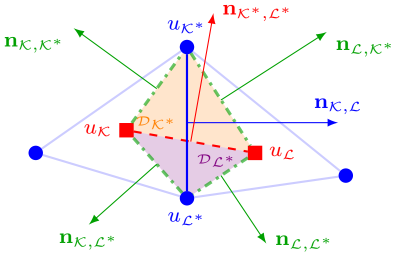
- Triangle mesh.
- The centres $x_{\mathcal{K}}$ and $x_{\mathcal{L}}$ shall be determined in the sequel.
- The vertex values $u_{\mathcal{K}^*}$ and $u_{\mathcal{L}^*}$ are given by

$$u_{\mathcal{K}^*} = \sum_{\mathcal{M}} \underbrace{\gamma_{\mathcal{M}\mathcal{K}^*}}_{0\leq\cdot\leq 1} u_{\mathcal{M}}.$$

- Basic geometry properties :

$$\mathbf{n}_{\mathcal{K},\mathcal{K}^*} + \mathbf{n}_{\mathcal{L},\mathcal{K}^*} + \mathbf{n}_{\mathcal{K}^*,\mathcal{L}^*} = 0,$$
$$\mathbf{n}_{\mathcal{K},\mathcal{L}^*} + \mathbf{n}_{\mathcal{L},\mathcal{L}^*} - \mathbf{n}_{\mathcal{K}^*,\mathcal{L}^*} = 0,$$
$$\mathbf{n}_{\mathcal{K},\mathcal{K}^*} + \mathbf{n}_{\mathcal{K},\mathcal{L}^*} + \mathbf{n}_{\mathcal{K},\mathcal{L}} = 0,$$
$$\mathbf{n}_{\mathcal{L},\mathcal{K}^*} + \mathbf{n}_{\mathcal{L},\mathcal{L}^*} - \mathbf{n}_{\mathcal{K},\mathcal{L}} = 0,$$

TO SIMPLIFY A LITTLE : $A(x) = A$ **(Le Potier '05-..) (Lipnikov et al '07-...)**



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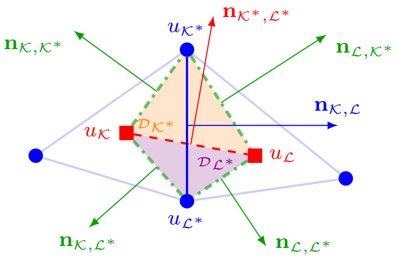
$$u_{\mathcal{K}^*} = \sum_{\mathcal{M}} \underbrace{\gamma_{\mathcal{M}\mathcal{K}^*}}_{0\leq\cdot\leq 1} u_{\mathcal{M}}.$$

- We define one discrete gradient for each half-diamond

$$\nabla_{\mathcal{D}_{\mathcal{K}^*}} u^{\mathcal{T}} = C_{\mathcal{K}^*}\Big(- u_{\mathcal{L}}\mathbf{n}_{\mathcal{K},\mathcal{K}^*} - u_{\mathcal{K}}\mathbf{n}_{\mathcal{L},\mathcal{K}^*} + u_{\mathcal{K}^*}(\mathbf{n}_{\mathcal{K},\mathcal{K}^*} + \mathbf{n}_{\mathcal{L},\mathcal{K}^*})\Big),$$
$$\nabla_{\mathcal{D}_{\mathcal{L}^*}} u^{\mathcal{T}} = C_{\mathcal{L}^*}\Big(- u_{\mathcal{L}}\mathbf{n}_{\mathcal{K},\mathcal{L}^*} - u_{\mathcal{K}}\mathbf{n}_{\mathcal{L},\mathcal{L}^*} + u_{\mathcal{L}^*}(\mathbf{n}_{\mathcal{K},\mathcal{L}^*} + \mathbf{n}_{\mathcal{L},\mathcal{L}^*})\Big).$$

TO SIMPLIFY A LITTLE : $A(x) = A$ **(Le Potier '05-..) (Lipnikov et al '07-...)**



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$$\nabla_{\mathcal{D}_{\mathcal{L}^*}} u^{\mathcal{T}} = C_{\mathcal{L}^*}\Big(- u_{\mathcal{L}}\mathbf{n}_{\mathcal{K},\mathcal{L}^*} - u_{\mathcal{K}}\mathbf{n}_{\mathcal{L},\mathcal{L}^*} + u_{\mathcal{L}^*}(\mathbf{n}_{\mathcal{K},\mathcal{L}^*} + \mathbf{n}_{\mathcal{L},\mathcal{L}^*})\Big).$$

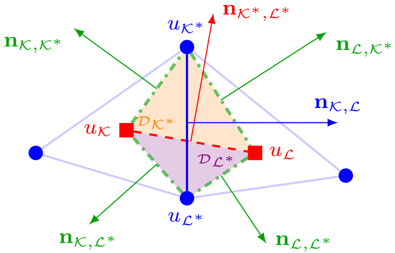
- We look for a (nonlinear) combination

$$F_{\mathcal{K},\mathcal{L}} \stackrel{\text{def}}{=} -\mu|\sigma|(A\nabla_{\mathcal{D}_{\mathcal{K}^*}} u^{\mathcal{T}}) \cdot \mathbf{n}_{\mathcal{K},\mathcal{L}} - (1-\mu)|\sigma|(A\nabla_{\mathcal{D}_{\mathcal{L}^*}} u^{\mathcal{T}}) \cdot \mathbf{n}_{\mathcal{K},\mathcal{L}}$$

so that $F_{\mathcal{K}\mathcal{L}}$ **looks like** a Two-Point flux depending only on $u_{\mathcal{K}}$ and $u_{\mathcal{L}}$

$$\mu C_{\mathcal{K}^*} u_{\mathcal{K}^*} A(\mathbf{n}_{\mathcal{K},\mathcal{K}^*} + \mathbf{n}_{\mathcal{L},\mathcal{K}^*}) \cdot \mathbf{n}_{\mathcal{K},\mathcal{L}} + (1-\mu) C_{\mathcal{L}^*} u_{\mathcal{L}^*} A(\mathbf{n}_{\mathcal{K},\mathcal{L}^*} + \mathbf{n}_{\mathcal{L},\mathcal{L}^*}) \cdot \mathbf{n}_{\mathcal{K},\mathcal{L}} = 0.$$

TO SIMPLIFY A LITTLE : $A(x) = A$ **(Le Potier '05-..) (Lipnikov et al '07-...)**



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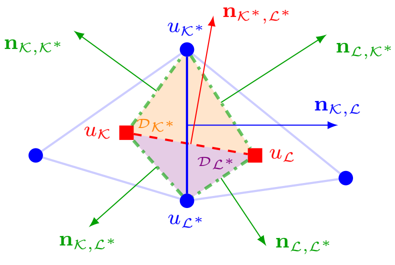
$$\mu C_{\mathcal{K}^*} u_{\mathcal{K}^*} - (1-\mu) C_{\mathcal{L}^*} u_{\mathcal{L}^*} = 0.$$

TO SIMPLIFY A LITTLE : $A(x) = A$ **(Le Potier '05-..) (Lipnikov et al '07-...)**



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so that $F_{\mathcal{K}\mathcal{L}}$ **looks like** a Two-Point flux depending only on $u_{\mathcal{K}}$ and $u_{\mathcal{L}}$

$$\mu(u^{\mathcal{T}}) = \frac{C_{\mathcal{L}^*}u_{\mathcal{L}^*}}{C_{\mathcal{K}^*}u_{\mathcal{K}^*} + C_{\mathcal{L}^*}u_{\mathcal{L}^*}}.$$

SUMMARY : **(Le Potier '05) (Lipnikov et al '07)**

- The numerical flux is written as a nonlinear two point flux

$$F_{\mathcal{K},\mathcal{L}} = \tau_{\mathcal{K},\sigma}(u^{\mathcal{T}})|\sigma|\, u_{\mathcal{K}} - \tau_{\mathcal{L},\sigma}(u^{\mathcal{T}})|\sigma|\, u_{\mathcal{L}},$$

with

$$\tau_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = \mu(u^{\mathcal{T}})C_{\mathcal{K}^*}(A\mathbf{n}_{\mathcal{L},\mathcal{K}^*})\cdot \mathbf{n}_{\mathcal{K},\mathcal{L}} + (1-\mu(u^{\mathcal{T}}))C_{\mathcal{L}^*}(A\mathbf{n}_{\mathcal{L},\mathcal{L}^*})\cdot \mathbf{n}_{\mathcal{K},\mathcal{L}},$$

$$\tau_{\mathcal{L},\sigma}(u^{\mathcal{T}}) = -\mu(u^{\mathcal{T}})C_{\mathcal{K}^*}A\mathbf{n}_{\mathcal{K},\mathcal{K}^*}\cdot \mathbf{n}_{\mathcal{K},\mathcal{L}} - (1-\mu(u^{\mathcal{T}}))C_{\mathcal{L}^*}A\mathbf{n}_{\mathcal{L},\mathcal{L}^*}\cdot \mathbf{n}_{\mathcal{K},\mathcal{L}},$$

SUMMARY : **(Le Potier '05) (Lipnikov et al '07)**

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with

$$\tau_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = C_{\mathcal{K}^*}C_{\mathcal{L}^*}\frac{u_{\mathcal{L}^*}\left(A\mathbf{n}_{\mathcal{L},\mathcal{K}^*}\cdot \mathbf{n}_{\mathcal{K},\mathcal{L}}\right) + u_{\mathcal{K}^*}\left(A\mathbf{n}_{\mathcal{L},\mathcal{L}^*}\cdot \mathbf{n}_{\mathcal{K},\mathcal{L}}\right)}{u_{\mathcal{K}^*}C_{\mathcal{K}^*} + u_{\mathcal{L}^*}C_{\mathcal{L}^*}}.$$

$$\tau_{\mathcal{L},\sigma}(u^{\mathcal{T}}) = -C_{\mathcal{K}^*}C_{\mathcal{L}^*}\frac{u_{\mathcal{L}^*}\left(A\mathbf{n}_{\mathcal{K},\mathcal{K}^*}\cdot \mathbf{n}_{\mathcal{K},\mathcal{L}}\right) + u_{\mathcal{K}^*}\left(A\mathbf{n}_{\mathcal{K},\mathcal{L}^*}\cdot \mathbf{n}_{\mathcal{K},\mathcal{L}}\right)}{u_{\mathcal{K}^*}C_{\mathcal{K}^*} + u_{\mathcal{L}^*}C_{\mathcal{L}^*}}.$$

Summary : **(Le Potier '05) (Lipnikov et al '07)**

- The numerical flux is written as a nonlinear two point flux

$$F_{\mathcal{K},\mathcal{L}} = \tau_{\mathcal{K},\sigma}(u^{\mathcal{T}})|\sigma|\, u_{\mathcal{K}} - \tau_{\mathcal{L},\sigma}(u^{\mathcal{T}})|\sigma|\, u_{\mathcal{L}},$$

with

$$\tau_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = C_{\mathcal{K}^*}C_{\mathcal{L}^*}\frac{u_{\mathcal{L}^*}\left(A\mathbf{n}_{\mathcal{L},\mathcal{K}^*}\cdot\mathbf{n}_{\mathcal{K},\mathcal{L}}\right) + u_{\mathcal{K}^*}\left(A\mathbf{n}_{\mathcal{L},\mathcal{L}^*}\cdot\mathbf{n}_{\mathcal{K},\mathcal{L}}\right)}{u_{\mathcal{K}^*}C_{\mathcal{K}^*} + u_{\mathcal{L}^*}C_{\mathcal{L}^*}}.$$

$$\tau_{\mathcal{L},\sigma}(u^{\mathcal{T}}) = -C_{\mathcal{K}^*}C_{\mathcal{L}^*}\frac{u_{\mathcal{L}^*}\left(A\mathbf{n}_{\mathcal{K},\mathcal{K}^*}\cdot\mathbf{n}_{\mathcal{K},\mathcal{L}}\right) + u_{\mathcal{K}^*}\left(A\mathbf{n}_{\mathcal{K},\mathcal{L}^*}\cdot\mathbf{n}_{\mathcal{K},\mathcal{L}}\right)}{u_{\mathcal{K}^*}C_{\mathcal{K}^*} + u_{\mathcal{L}^*}C_{\mathcal{L}^*}}.$$

- We need now to show that, with suitable assumptions, we have

$$u^{\mathcal{T}} \geq 0 \Longrightarrow \tau_{\mathcal{K},\sigma}(u^{\mathcal{T}}) \geq 0 \ \text{ and } \ \tau_{\mathcal{L},\sigma}(u^{\mathcal{T}}) \geq 0.$$

SUMMARY : **(Le Potier '05) (Lipnikov et al '07)**

- The numerical flux is written as a nonlinear two point flux

$$F_{\mathcal{K},\mathcal{L}} = \tau_{\mathcal{K},\sigma}(u^{\mathcal{T}})|\sigma|\, u_{\mathcal{K}} - \tau_{\mathcal{L},\sigma}(u^{\mathcal{T}})|\sigma|\, u_{\mathcal{L}},$$

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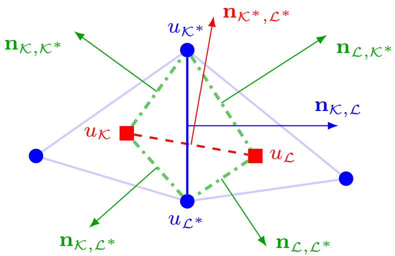
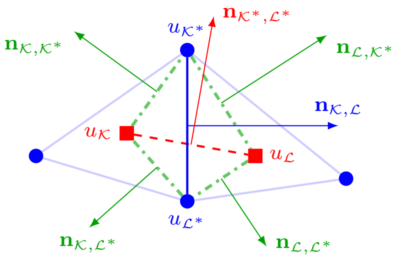
$$A\mathbf{n}_{\mathcal{L},\mathcal{K}^*}\cdot\mathbf{n}_{\mathcal{K},\mathcal{L}} \geq 0, \quad A\mathbf{n}_{\mathcal{L},\mathcal{L}^*}\cdot\mathbf{n}_{\mathcal{K},\mathcal{L}} \geq 0,$$

$$A\mathbf{n}_{\mathcal{K},\mathcal{K}^*}\cdot\mathbf{n}_{\mathcal{K},\mathcal{L}} \leq 0, \quad A\mathbf{n}_{\mathcal{K},\mathcal{L}^*}\cdot\mathbf{n}_{\mathcal{K},\mathcal{L}} \leq 0.$$

To this end, we show that a suitable choice of the centers $x_{\mathcal{K}}$ exists, depending only on A.



We require $\begin{cases} A\mathbf{n}_{\mathcal{L},\mathcal{K}^*} \cdot \mathbf{n}_{\mathcal{K},\mathcal{L}} \geq 0, & A\mathbf{n}_{\mathcal{L},\mathcal{L}^*} \cdot \mathbf{n}_{\mathcal{K},\mathcal{L}} \geq 0, \\ A\mathbf{n}_{\mathcal{K},\mathcal{K}^*} \cdot \mathbf{n}_{\mathcal{K},\mathcal{L}} \leq 0, & A\mathbf{n}_{\mathcal{K},\mathcal{L}^*} \cdot \mathbf{n}_{\mathcal{K},\mathcal{L}} \leq 0. \end{cases}$



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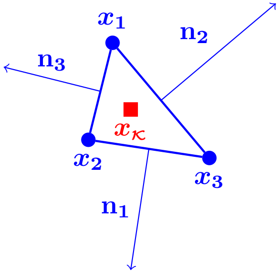


PROPOSITION

The above inequalities hold if we set

$$x_{\mathcal{K}}\stackrel{\text{def}}{=}\frac{\|\mathbf{n}_1\|_A\,x_1+\|\mathbf{n}_2\|_A\,x_2+\|\mathbf{n}_3\|_A\,x_3}{\|\mathbf{n}_1\|_A+\|\mathbf{n}_2\|_A+\|\mathbf{n}_3\|_A},$$

with

$$\|\xi\|_A=\sqrt{A\xi\cdot\xi},\quad\forall\xi\in\mathbb{R}^2.$$

Properties

- Existence of a solution of the nonlinear problem is not known.
- No convergence result because of a lack of coercitivty.

- In practice, there exists a nonlinear iterative solver that preserves positivity of the approximations all along iterations.
- The linearized system to be solved changes at each iteration and is not symmetric.

- With many efforts, the principle of the scheme can be generalised to more general polygonal meshes in the case where
  - $A(x)$ is isotropic.
  - The mesh is regular and the control volumes are star-shaped.
- Extension to 3D for tetrahedral meshes.

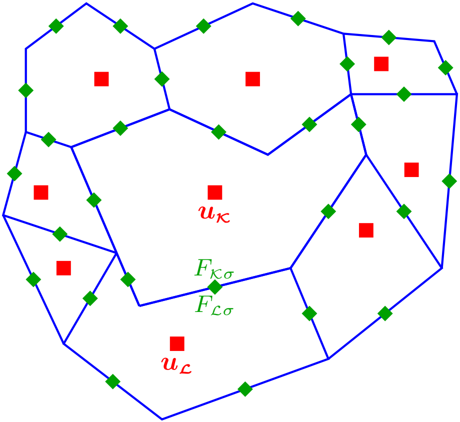
Slightly different approaches

- Nonlinear corrections of general linear schemes

(Burman-Ern,'04) (Le Potier, '10)

(Droniou-Le Potier, '11), (Cancès-Cathala-Le Potier, 13)

(Lipnikov et al '05 → '08) (Manzini '08)



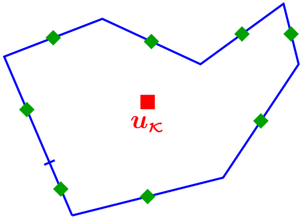
- A scalar unknown $u_\mathcal{K}$ for each control volume $\kappa \in \mathcal{T}$.
- Two **scalar** fluxes unknowns $F_{\mathcal{K},\sigma}$ and $F_{\mathcal{L},\sigma}$ for each edge $\sigma \in \mathcal{E}$.
- They are related through conservativity relations

$$F_{\mathcal{K},\sigma} + F_{\mathcal{L},\sigma} = 0.$$

- Let $\mathbb{R}^\mathcal{T}$ (resp. $\mathbb{R}^\mathcal{E}$) be the set of cell-centered (resp. edge-centered) unknowns.

Basic idea : Try to mimick properties of the continuous problem through the Green formula

$$\int_{\mathcal{K}} A_{\mathcal{K}}^{-1}(A_{\mathcal{K}}\nabla u)\cdot\xi\,dx + \int_{\mathcal{K}} u(\mathrm{div}\xi)\,dx = \int_{\partial\mathcal{K}} u(\xi\cdot\boldsymbol{\nu})\,ds.$$



- For any $F\in\mathbb{R}^{\mathcal{E}}$, we define a discrete divergence operator

$$\mathrm{div}^{\mathcal{K}} F = \frac{1}{|\mathcal{K}|}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \underbrace{|\sigma| F_{\mathcal{K},\sigma}}_{\text{change of notation}}$$

- We suppose given a “scalar product” $(\cdot,\cdot)_{A^{-1},\mathcal{K}}$ on the set of edge unknowns ◆ supposed to approximate $\int_{\mathcal{K}} A_{\mathcal{K}}^{-1}F\cdot G\,dx$.

Mimetic schemes for $-\mathrm{div}(A(x)\nabla\cdot)$

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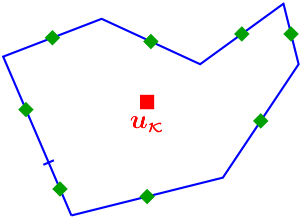

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Assumptions

Coercivity : $\underline{C}|\mathcal{K}|\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}|F_{\mathcal{K},\sigma}|^2 \leq (F,F)_{A^{-1},\mathcal{K}} \leq \overline{C}|\mathcal{K}|\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}|F_{\mathcal{K},\sigma}|^2, \quad \forall\mathcal{K},$

Consistency : $\left((A_{\mathcal{K}}\nabla\varphi),G\right)_{A^{-1},\mathcal{K}} + \int_{\mathcal{K}}\varphi\,\mathrm{div}^{\mathcal{K}}G\,dx$

$$= \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} G_{\mathcal{K},\sigma}\left(\int_\sigma\varphi\right), \forall\mathcal{K}, \forall G\in\mathbb{R}^{\mathcal{E}}, \forall\varphi \text{ affine}.$$

GLOBAL SCALAR PRODUCTS

$$(F,G)_{A^{-1}} = \sum_{\mathcal{K}} (F,G)_{A^{-1},\mathcal{K}},$$

$$(u,v) = \sum_{\mathcal{K}} |\mathcal{K}| u_{\mathcal{K}} v_{\mathcal{K}},$$

APPROXIMATE FLUX OPERATOR : $\Phi : u \in \mathbb{R}^{\mathcal{T}} \mapsto \Phi u \in \mathbb{R}^{\mathcal{E}} \approx (A\nabla u)\cdot \boldsymbol{\nu}$ defined by duality

$$(G, \Phi u)_{A^{-1}} = -(u, \mathrm{div}^{\mathcal{K}} G), \quad \forall u \in \mathbb{R}^{\mathcal{T}}_0, \forall G \in \mathbb{R}^{\mathcal{E}}.$$

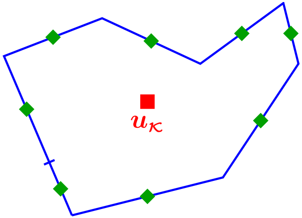
MFD SCHEME

Find $u \in \mathbb{R}^{\mathcal{T}}$ such that

$$-\mathrm{div}^{\mathcal{K}}(\Phi u) = f_{\mathcal{K}}, \quad \forall \mathcal{K}.$$

SUMMARY

The main point remains to find suitable scalar products $(\cdot,\cdot)_{A^{-1},\mathcal{K}}$ satisfying consistency and coercivity properties.



We look for

$$(F,G)_{A^{-1},\mathcal{K}} \stackrel{\text{def}}{=} {}^t\Big(F_{\mathcal{K},\sigma}\Big)_\sigma M_\mathcal{K} \Big(G_{\mathcal{K},\sigma}\Big)_\sigma,$$

with $M_\mathcal{K}$ is a $m\times m$ positive definite matrix.

Definitions

$$R_\mathcal{K} = \begin{pmatrix} |\sigma_1|^t(x_{\sigma_1}-x_\mathcal{K}) \\ \vdots \\ |\sigma_m|^t(x_{\sigma_m}-x_\mathcal{K}) \end{pmatrix}, \quad N_\mathcal{K} = \begin{pmatrix} {}^t\boldsymbol{\nu}_{\sigma_1} \\ \vdots \\ {}^t\boldsymbol{\nu}_{\sigma_m} \end{pmatrix} A_\mathcal{K}, \quad \text{of size} \;\; m\times 2,$$

Proposition

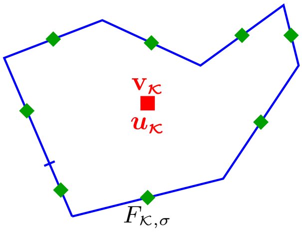
Conistency condition is equivalent to

$$M_\mathcal{K} N_\mathcal{K} = R_\mathcal{K} \iff M_\mathcal{K} = \frac{1}{|\mathcal{K}|} R_\mathcal{K} A_\mathcal{K}^{-1\,t} R_\mathcal{K} + C_\mathcal{K} U_\mathcal{K} {}^t C_\mathcal{K},$$

where $C_\mathcal{K}$ is a $m\times(m-2)$ matrix such that ${}^tC_\mathcal{K}\, N_\mathcal{K} = 0$ and $U_\mathcal{K}$ is any $(m-2)\times(m-2)$ positive definite matrix.

Properties

- Control volumes needs to be star-shaped with respect to their mass center.
- Total number of unknowns is the sum of the number of control volumes and the number of edges.
- The linear system to be solved is of saddle-point kind.
- Those schemes can be seen as a generalisations of mixed finite elements with suitable quadrature formulas.
- With reasonnable regularity assumptions on mesh families and on $x \mapsto A(x)$, one can show second order convergence in the L^2 norm and first order convergence in the H^1 norm.
- When $x \mapsto A(x)$ is discontinuous : no complete analysis up to now.



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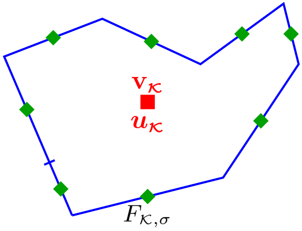

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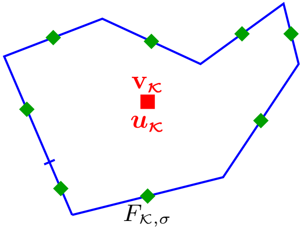

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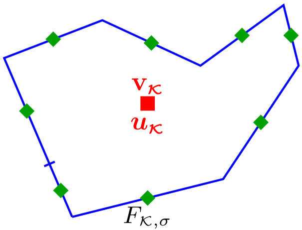
- A simple formula

$$|\mathcal{K}|\xi = \int_\mathcal{K} \underbrace{\mathrm{div}\Big((x - x_\mathcal{K}) \otimes \xi\Big)}_{=\xi} \, dx = \int_{\partial\mathcal{K}} (\xi \cdot \boldsymbol{\nu})(x - x_\mathcal{K}) \, dx, \quad \forall \xi \in \mathbb{R}^2.$$



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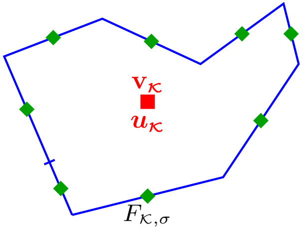
Idea : Apply this to $\xi = A\nabla u$ ...



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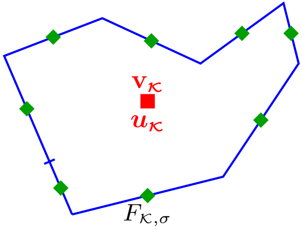

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- A simple formula

$$|\mathcal{K}|A_\mathcal{K}\mathbf{v}_\mathcal{K} = -\sum_{\sigma\in\mathcal{E}_\mathcal{K}} F_{\mathcal{K},\sigma}(x_\sigma - x_\mathcal{K}).$$

- Flux balance equation $\displaystyle\sum_{\sigma\in\mathcal{E}_\mathcal{K}} F_{\mathcal{K},\sigma} = |\mathcal{K}|f_\mathcal{K}.$

PROPERTIES

- We have 3 scalar unknowns by control volume and two (in fact one ...) by edge.
- For a conforming mesh made of triangles, no need of penalisation term. In the other cases, this term is need to ensure well-posedness of the scheme.
- No particular difficulties to deal with non-linear equations $-\mathrm{div}(\varphi(x, \nabla u)) = f$.
- Convergence result for the solution $u^{\mathcal{T}} = (u_{\mathcal{K}})_{\mathcal{K}}$ and its gradient $\mathbf{v}^{\mathcal{T}} = (\mathbf{v}_{\mathcal{K}})_{\mathcal{K}}$, for any mesh and any data.
  - Poincare inequality.
  - *A priori* estimate.
  - Compactness.
  - Convergence.
- For smooth solutions
  - On general meshes : Theoretical error estimates in $O(\sqrt{\mathrm{size}(\mathcal{T})})$.
  - On conforming triangle meshes : Error estimates in $O(\mathrm{size}(\mathcal{T}))$.

Some remarks on this approach

- The number of unknowns is very large.
- The linear system to be solved has no simple structure (in particular it is not positive definite).

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Hybridisation

Elimination of cell-centered unknowns $(u_\mathcal{K})_\mathcal{K}$ and $(v_\mathcal{K})_\mathcal{K}$ in order to transform the system into a smaller and definite positive system.

- We define a new scalar unknown for each edge by

$$u_\sigma \stackrel{\text{def}}{=} u_\mathcal{K} + \mathbf{v}_\mathcal{K}\cdot(x_\mathcal{K}-x_\sigma) - \nu_\mathcal{K}|\mathcal{K}|F_{\mathcal{K},\sigma} = u_\mathcal{L} + \mathbf{v}_\mathcal{L}\cdot(x_\mathcal{L}-x_\sigma) - \nu_\mathcal{L}|\mathcal{L}|F_{\mathcal{L},\sigma}.$$

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$$\mathbf{v}_\mathcal{K} = -\frac{1}{|\mathcal{K}|}\sum_{\sigma\in\mathcal{E}_\mathcal{K}} F_{\mathcal{K},\sigma}A_\mathcal{K}^{-1}(x_\sigma - x_\mathcal{K}).$$

- Thus
$$(u_\sigma - u_\mathcal{K})_{\sigma\in\mathcal{E}_\mathcal{K}} = B_\mathcal{K}(F_{\mathcal{K},\sigma})_{\sigma\in\mathcal{E}_\mathcal{K}},$$
where $B_\mathcal{K}$ is positive definite and depends only on the geometry of $\mathcal{K}$ and $\nu_\mathcal{K}$.

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- Thus $$(F_{\mathcal{K},\sigma})_{\sigma \in \mathcal{E}_\mathcal{K}} = B_\mathcal{K}^{-1} (u_\sigma - u_\mathcal{K})_{\sigma \in \mathcal{E}_\mathcal{K}}.$$
- Elimination of $u_\mathcal{K}$

$$\sum_{\sigma \in \mathcal{E}_\mathcal{K}} F_{\mathcal{K},\sigma} = |\mathcal{K}| f_\mathcal{K}$$

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$$u_\sigma \stackrel{\text{def}}{=} u_\mathcal{K} + \mathbf{v}_\mathcal{K} \cdot (x_\mathcal{K} - x_\sigma) - \nu_\mathcal{K} |\mathcal{K}| F_{\mathcal{K},\sigma} = u_\mathcal{L} + \mathbf{v}_\mathcal{L} \cdot (x_\mathcal{L} - x_\sigma) - \nu_\mathcal{L} |\mathcal{L}| F_{\mathcal{L},\sigma}.$$

- We use

$$\mathbf{v}_\mathcal{K} = -\frac{1}{|\mathcal{K}|} \sum_{\sigma \in \mathcal{E}_\mathcal{K}} F_{\mathcal{K},\sigma} A_\mathcal{K}^{-1} (x_\sigma - x_\mathcal{K}).$$

- Thus $$(F_{\mathcal{K},\sigma})_{\sigma \in \mathcal{E}_\mathcal{K}} = B_\mathcal{K}^{-1} (u_\sigma - u_\mathcal{K})_{\sigma \in \mathcal{E}_\mathcal{K}}.$$
- Elimination of $u_\mathcal{K}$

$$\sum_{\sigma \in \mathcal{E}_\mathcal{K}} c_{\mathcal{K},\sigma} (u_\sigma - u_\mathcal{K}) = |\mathcal{K}| f_\mathcal{K}$$

Some remarks on this approach

- The number of unknowns is very large.
- The linear system to be solved has no simple structure (in particular it is not positive definite).

Hybridisation

Elimination of cell-centered unknowns $(u_\mathcal{K})_\mathcal{K}$ and $(v_\mathcal{K})_\mathcal{K}$ in order to transform the system into a smaller and definite positive system.

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- Elimination of $u_\mathcal{K}$ $$\Rightarrow u_\mathcal{K} = b_\mathcal{K} + \sum_{\sigma \in \mathcal{E}_\mathcal{K}} \tilde{c}_{\mathcal{K},\sigma} u_\sigma.$$

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- Thus $\qquad (F_{\mathcal{K},\sigma})_{\sigma\in\mathcal{E}_\mathcal{K}} = B_\mathcal{K}^{-1}(u_\sigma - u_\mathcal{K})_{\sigma\in\mathcal{E}_\mathcal{K}}.$
- Elimination of $u_\mathcal{K}$ $\quad \Rightarrow u_\mathcal{K} = b_\mathcal{K} + \sum_{\sigma\in\mathcal{E}_\mathcal{K}} \tilde{c}_{\mathcal{K},\sigma} u_\sigma.$

$$\Rightarrow (F_{\mathcal{K},\sigma})_{\sigma\in\mathcal{E}_\mathcal{K}} = C_\mathcal{K}(u_\sigma - u_{\sigma'})_{\sigma,\sigma'\in\mathcal{E}_\mathcal{K}} + (G_{\mathcal{K},\sigma})_{\sigma\in\mathcal{E}_\mathcal{K}}.$$

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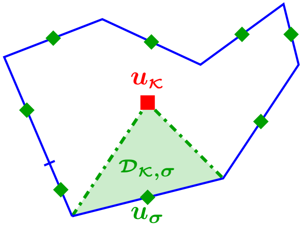
- $F_{\mathcal{K},\sigma} + F_{\mathcal{L},\sigma} = 0 \Rightarrow$ One equation for each edge satisfied by $(u_\sigma)_\sigma$.

(Eymard-Gallouët-Herbin '08-..)



- Unknowns : cell-centered $u_{\mathcal{K}}$ and edge-centered u_σ.
- The set of edges is separated into two parts

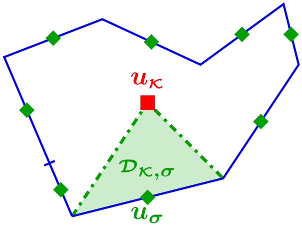
$$\mathcal{E} = \mathcal{E}_B \cup \mathcal{E}_H.$$

- The unknowns u_σ corresponding to $\sigma \in \mathcal{E}_B$ are eliminated by a barycentric formula $u_\sigma = \sum_{\mathcal{K}} \gamma_{\mathcal{K}}^\sigma u_{\mathcal{K}}$.
- The unknowns u_σ corresponding to $\sigma \in \mathcal{E}_H$ are **free**.
- A "geometric" formula $\Longrightarrow$ definition of a discrete gradient

$$|\mathcal{K}|\xi = \int_{\mathcal{K}} \nabla\Big(\xi \cdot (x - x_{\mathcal{K}})\Big)\, dx = \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} \int_\sigma \Big(\xi \cdot (x - x_{\mathcal{K}})\Big)\boldsymbol{\nu}_{\mathcal{K},\sigma}\, dx$$

$$= \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} |\sigma| \Big(\xi \cdot (x_\sigma - x_{\mathcal{K}})\Big)\boldsymbol{\nu}_{\mathcal{K},\sigma}.$$

(Eymard-Gallouët-Herbin '08-..)



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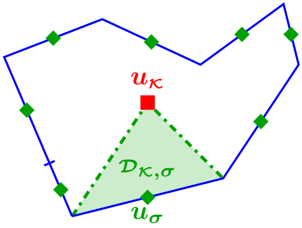
$$|\kappa|\nabla u \approx \sum_{\sigma \in \mathcal{E}_\mathcal{K}} |\sigma|(u(x_\sigma) - u(x_\mathcal{K}))\boldsymbol{\nu}_{\mathcal{K},\sigma}.$$

(Eymard-Gallouët-Herbin '08-..)



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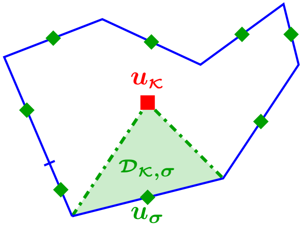
$$\nabla_\mathcal{K} u^\mathcal{T} \stackrel{\text{def}}{=} \frac{1}{|\mathcal{K}|} \sum_{\sigma \in \mathcal{E}_\mathcal{K}} |\sigma|(u_\sigma - u_\mathcal{K})\boldsymbol{\nu}_{\mathcal{K},\sigma}.$$

(Eymard-Gallouët-Herbin '08-..)



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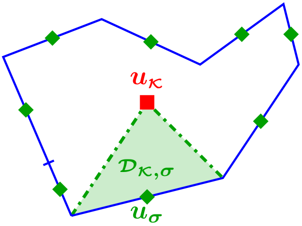
$$\nabla_{\mathcal{K}} u^{\mathcal{T}} \stackrel{\text{def}}{=} \frac{1}{|\mathcal{K}|} \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} |\sigma| (u_\sigma - u_{\mathcal{K}}) \boldsymbol{\nu}_{\mathcal{K},\sigma}.$$
- Consistency error $R_{\mathcal{K},\sigma}(u^{\mathcal{T}}) = \dfrac{\alpha}{d_{\mathcal{K},\sigma}} \Big(u_\sigma - u_{\mathcal{K}} - \nabla_{\mathcal{K}} u^{\mathcal{T}} \cdot (x_\sigma - x_{\mathcal{K}}) \Big)$.

(Eymard-Gallouët-Herbin '08-..)



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- On each triangle (=half-diamond) $\mathcal{D}_{\mathcal{K},\sigma}$ we define a **stabilised** discrete gradient
$$\nabla_{\mathcal{K},\sigma} u^\mathcal{T} = \nabla_\mathcal{K} u^\mathcal{T} + R_{\mathcal{K},\sigma}(u^\mathcal{T})\boldsymbol{\nu}_{\mathcal{K},\sigma}.$$

- The scheme is then written under *variationnal* form

$$\int_\Omega (A(x)\nabla^\tau u^\mathcal{T}) \cdot \nabla^\tau v^\mathcal{T}\, dx = \sum_\mathcal{K} |\mathcal{K}| v_\mathcal{K} f_\mathcal{K}, \quad \forall v^\mathcal{T} = ((v_\mathcal{K})_\mathcal{K}, (v_\sigma)_\sigma).$$

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$$\int_\Omega (A(x)\nabla^\tau u^\mathcal{T}) \cdot \nabla^\tau v^\mathcal{T}\, dx = \sum_\mathcal{K} |\kappa| v_\mathcal{K} f_\mathcal{K}, \quad \forall v^\mathcal{T} = ((v_\mathcal{K})_\mathcal{K}, (v_\sigma)_\sigma).$$

- However, we can write it under a more standard FV form

$$\sum_\mathcal{K} \sum_{\sigma\in\mathcal{E}_\mathcal{K}} F_{\mathcal{K},\sigma}(u^\mathcal{T})(v_\mathcal{K} - v_\sigma) = \sum_\mathcal{K} |\kappa| v_\mathcal{K} f_\mathcal{K}, \quad \forall v^\mathcal{T} = ((v_\mathcal{K})_\mathcal{K}, (v_\sigma)_\sigma),$$

with $F_{\mathcal{K},\sigma}(u^\mathcal{T}) = \sum_{\sigma'\in\mathcal{E}_\mathcal{K}} \alpha_\mathcal{K}^{\sigma,\sigma'}(u_\mathcal{K} - u_{\sigma'})$, and $\alpha_\mathcal{K}^{\sigma,\sigma'}$ depends on the data.

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- For edges $\sigma \in \mathcal{E}_H$, the unknown v_σ is a degree of freedom, thus

$$F_{\mathcal{K},\sigma}(u^\mathcal{T}) + F_{\mathcal{L},\sigma}(u^\mathcal{T}) = 0.$$

- For edges $\sigma \in \mathcal{E}_B$, **this local consistency property does not hold anymore**.

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- However, we can write it under a more standard FV form

$$\sum_\mathcal{K}\sum_{\sigma\in\mathcal{E}_\mathcal{K}} F_{\mathcal{K},\sigma}(u^\mathcal{T})(v_\mathcal{K}-v_\sigma) = \sum_\mathcal{K} |\kappa| v_\mathcal{K} f_\mathcal{K}, \quad \forall v^\mathcal{T} = ((v_\mathcal{K})_\mathcal{K}, (v_\sigma)_\sigma),$$

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- For edges $\sigma\in\mathcal{E}_H$, the unknown v_σ is a degree of freedom, thus

$$F_{\mathcal{K},\sigma}(u^\mathcal{T}) + F_{\mathcal{L},\sigma}(u^\mathcal{T}) = 0.$$

- For edges $\sigma\in\mathcal{E}_B$, **this local consistency property does not hold anymore**.

POSSIBLE STRATEGY TO CHOOSE BARYCENTRIC/HYBRID EDGES

- We decide that $\sigma\in\mathcal{E}_B$, if the permeability tensor A is smooth near σ.
- We decide that $\sigma\in\mathcal{E}_H$, if A is discontinuous across σ in order to ensure a good accuracy and local conservativity.

Properties

(Eymard-Gallouët-Herbin '08)

- Barycentric/hybrid scheme adapted to the properties of the permeability tensor which is intermediate between standard FV and mixed FE methods.
- Local conservativity is only ensured on **hybrid** edges.
  - Fully barycentric scheme :

    Few unknowns / Large stencil / No local conservativity
  - Fully hybrid scheme :

    Many unknowns / Large stencil / Local conservativity
- In the barycentric case, the notion of *local flux* across edges is not really clear.
- The linear system to be solved is symmetric.
- Existence and uniqueness of the solution holds true without any assumption.
- Convergence theorem in the general case.
- Error estimate in $O(\text{size}(\mathcal{T}))$ for u and ∇u in the case of a smooth isotropic permeability.

(Droniou–Eymard–Gallouët–Herbin '09)

Theorem (Simplified statement)

The three approaches

- *Mimetic*
- *Mixed FV*
- *SUCCES*

are **algebraically** *equivalent (for a suitable choice of the numerical parameters).*

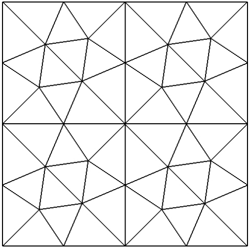
(Herbin-Hubert, '08)

`http://www.latp.univ-mrs.fr/fvca5`
Proceedings edited by Wiley
Ed. : Robert Eymard and Jean-Marc Hérard

- 19 contributions.
- 9 test cases.
- 10 different mesh families.
- Some properties/quantities to be compared :
  - Number of unknowns / of non-zero entries of the matrix.
  - Local conservativity or not.
  - L^∞/L^2 error for u and ∇u.
  - Approximation error for fluxes at interfaces.
  - Monotony / Discrete maximum principle.
  - Total energy balance.

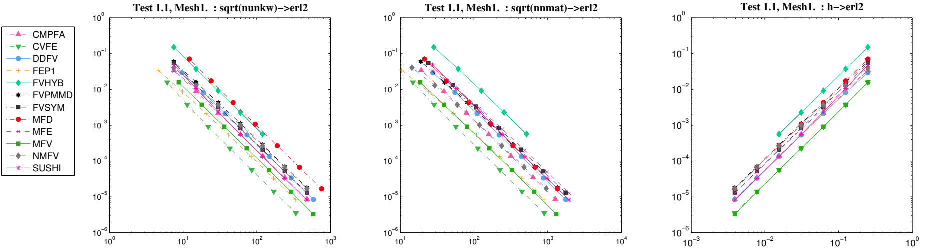
$$-\mathrm{div}(A\nabla u) = f \text{ in } \Omega$$

with $A = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$ and $u(x,y) = 16x(1-x)y(1-y)$.

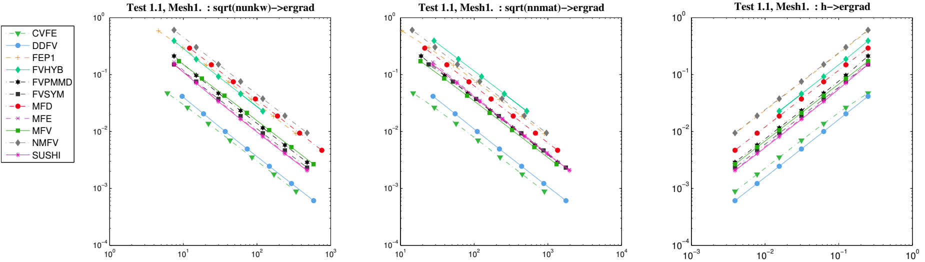


Mesh1

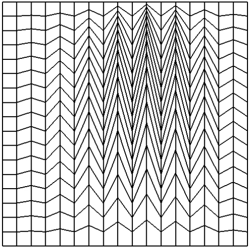
L^2 error on u (second order)



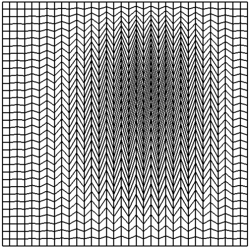
on ∇u (first order)



Test #1 : Moderate anisotropy



Mesh4_1



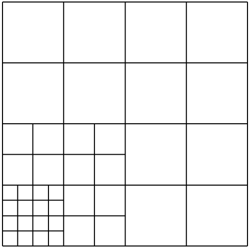
Mesh4_2

Minimum and maximum values of the approximate solution

	mesh 4_1		mesh 4_2	
	umin	umax	umin	umax
CMPFA	9.95E-03	1.00E+00	2.73E-03	9.99E-01
CVFE	0.00E+00	8.43E-01	0.00E+00	9.14E-01
DDFV	1.33E-02	9.96E-01	3.63E-03	9.99E-01
FEQ1	0.00E+00	8.61E-01	0.00E+00	9.37E-01
FVHYB	2.14E-03	9.84E-01	7.16E-04	9.93E-01
FVSYM	7.34E-03	9.59E-01	2.33E-03	9.89E-01
MFD	6.64E-03	9.71E-01	1.50E-03	9.93E-01
MFV	1.08E-02	9.42E-01	3.34E-03	9.82E-01
NMFV	1.30E-02	**1.11E+00**	3.61E-03	**1.04E+00**
SUSHI	7.64E-03	8.88E-01	2.33E-03	9.61E-01

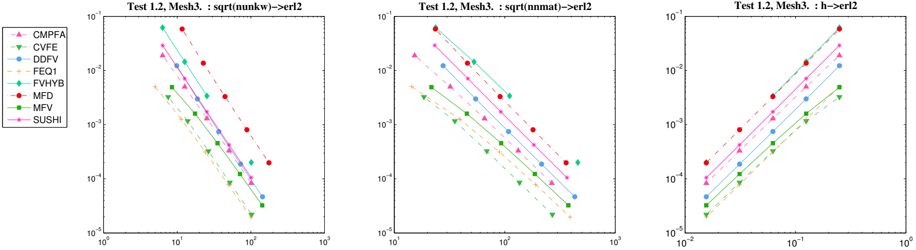
$$-\mathrm{div}(A\nabla u) = f \text{ in } \Omega$$

with $A = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$ and $u(x,y) = \sin\left((1-x)(1-y)\right) + (1-x)^3(1-y)^2.$

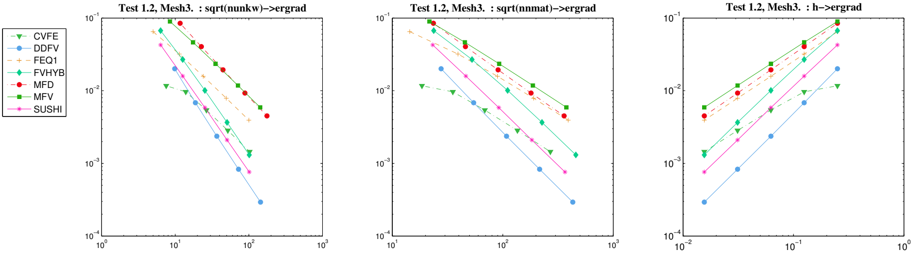


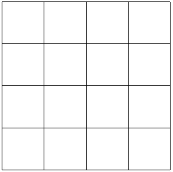
Mesh3

L^2 error on the solution u (second order)



on the gradient (half and first order)





$$-\mathrm{div}(A\nabla u) = 0 \text{ in } \Omega$$

with $A = R_\theta \begin{pmatrix} 1 & 0 \\ 0 & 10^{-3} \end{pmatrix} R_\theta^{-1}$, $\theta = 40^\circ$

The boundary data $\bar{u}$ is continuous and piecewise affine on $\partial\Omega$:

$$\bar{u}(x,y) = \begin{cases} 1 & \text{on } (0,.2)\times\{0.\}\cup\{0.\}\times(0,.2) \\ 0 & \text{on } (.8,1.)\times\{1.\}\cup\{1.\}\times(.8,1.) \\ \frac{1}{2} & \text{on } ((.3,1.)\times\{0\}\cup\{0\}\times(.3,1.) \\ \frac{1}{2} & \text{on } (0.,.7)\times\{1.\}\cup\{1.\}\times(0.,0.7) \end{cases}$$

Minimum and maximum values of the approximate solution

	umin_i	umax_i	i
CMPFA	6.90E-02	9.31E-01	1
	9.83E-04	9.99E-01	7
CVFE	0.00E+00	1.00E+00	1
	0.00E+00	1.00E+00	7
DDFV	**-4.72E-03**	1.00E+00	1
	-5.31E-04	1.00E+00	7
FEQ1	0.00E+00	1.00E+00	1
	0.00E+00	1.00E+00	7
FVHYB	**-1.75E-01**	**1.17E+00**	1
	-1.00E-03	1.00E+00	6
FVSYM	6.85E-02	9.32E-01	1
	4.92E-04	9.99E-01	8
MFD	7.56E-02	9.24E-01	1
	8.01E-04	9.99E-01	8
MFE	3.12E-02	9.69E-01	1
	5.08E-04	9.99E-01	8
MFV	1.22E-02	8.78E-01	1
	7.92E-04	9.99E-01	7
NMFV	1.11e-01	8.88e-01	1
	1.28E-03	9.99E-01	7
SUSHI	6.03E-02	9.40E-01	1
	8.52E-04	9.99E-01	7

THE ENERGIES

	ener1	eren	i
CMPFA	N/A N/A	N/A N/A	
CVFE	2.24E-01 2.42E-01	8.42E-02 3.33E-03	1 7
DDFV	2.14E-01 2.42E-01	9.60E-02 7.11E-06	1 7
FEQ1	2.21E-01 2.44E-01	3.67E-01 3.17E-02	1 7
FVHYB	2.13E-01 2.42E-01	2.55E-01 8.19E-03	1 6
FVSYM	2.20E-01 2.42E-01	0.00E+00 0.00E+00	1 8
MFD	**1.91E-01** 2.42E-01	1.87E-14 3.70E-14	1 8
MFE	**1.25E-01** 2.41E-01	2.46E-02 2.91E-03	1 8
MFV	**4.85E-01** 2.42E-01	8.23E-07 9.74E-06	1 7
NMFV	2.33e-01 2.45E-01	1.45e-01 1.94E-02	1 7
SUSHI	2.25E-01 2.43E-01	3.01E-01 1.28E-02	1 7

Volume energy

$$\text{ener1} \approx \int_\Omega A\nabla u \cdot \nabla u \, dx,$$

Boundary energy

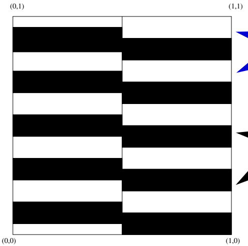
$$\text{ener2} \approx \int_{\partial\Omega} A\nabla u \cdot \boldsymbol{\nu} \, dx.$$

For the continuous solution we have

$$\text{eren1} = \text{eren2}.$$

We compute, at the discrete level, the error

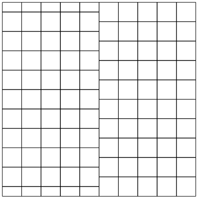
$$\text{eren} = \text{ener1} - \text{ener2}.$$



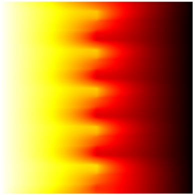
$$-\mathrm{div}(A\nabla u) = 0 \text{ in } \Omega$$
$$u = \bar{u} \text{ on } \partial\Omega$$

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \text{ with } \begin{cases} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^2 \\ 10 \end{pmatrix} \text{ in } \Omega_1, \\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^{-2} \\ 10^{-3} \end{pmatrix} \text{ in } \Omega_2 \end{cases}$$

and $\bar{u}(x,y) = 1 - x$.



mesh5



Maximum principle

- Satisfied by all the methods presented here.

Values of the energies

	ener1 mesh5	eren mesh5	ener1 mesh5_ref	eren mesh5_ref
CVFE	45.9	1.04E-02	**43.3**	6.25E-04
DDFV	**42.1**	3.65E-02	**43.2**	1.27E-03
FVHYB	**41.4**	6.12E-02	/	/
MFD-BLS	33.9	7.93E-14	**43.2**	2.84E-12
MFD	31.4	1.16E-12	**43.2**	4.71E-14
MFV	49.9	4.21E-05	**43.2**	1.88E-05
NMFV	/	/	**43.2**	5.92E-04
SUSHI	39.1	6.67E-02	**43.1**	8.88E-04

BOUNDARY FLUXES APPROXIMATION

$$\text{Flux across } \{x=0\} : \int_{\partial\Omega\cap\{x=0\}} A\nabla u \cdot \boldsymbol{\nu},$$

	flux0 mesh5	flux0 mesh5_ref	flux1 mesh5	flux1 mesh5_ref	fluy0 mesh5	fluy0 mesh5_ref	fluy1 mesh5
CMPFA	-45.2	**-42.1**	46.1	**44.4**	-0.95	**-2.33**	4.84E-04
CVFE	-46.6	**-42.2**	48.5	**44.5**	0.87	-2.25	8.02E-04
DDFV	-40.0	**-42.1**	41.8	**44.4**	-1.81	**-2.33**	9.08E-04
FEQ1	/	**-42.2**	/	**44.5**	/	-2.16	/
FVHYB	-44.3	/	46.3	/	0.49	/	1.55E-04
MFD	-29.7	**-42.1**	34.1	**44.4**	-4.37	**-2.33**	1.01E-03
MFV	-44.0	**-42.1**	50.3	**44.4**	-8.03	**-2.33**	1.72E+00
NMFV	**-43.2**	**-42.1**	**44.5**	**44.4**	-1.23	**-2.33**	2.32E-04
SUSHI	**-40.9**	**-42.1**	**43.1**	**44.4**	**-2.21**	**-2.33**	6.94E-04

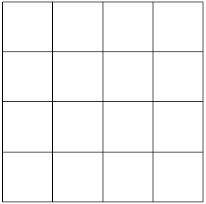
Test #5 : Heterogeneous rotating anisotropy

$$-\mathrm{div}(A\nabla u) = f \text{ in } \Omega$$

with

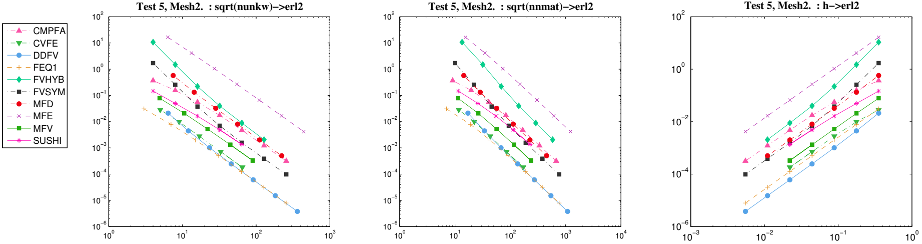
$$A = \frac{1}{(x^2+y^2)}\begin{pmatrix} 10^{-3}x^2 + y^2 & (10^{-3}-1)xy \\ (10^{-3}-1)xy & x^2 + 10^{-3}y^2 \end{pmatrix}$$

and $u(x,y) = \sin \pi x \sin \pi y$.

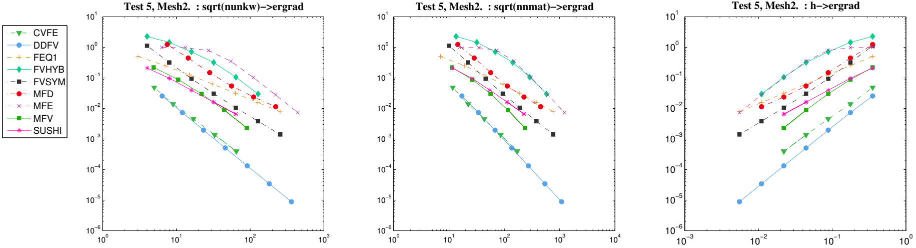


`mesh2`

L^2 error on u (second order)



and on ∇u (first and second order)



SOME SCHEMES DO NOT SATISFY THE DISCRETE MAXIMUM PRINCIPLE

	umin	umax
CMPFA	**-1.06E-01**	**1.09E+00**
FEQ1	0.00E+00	**1.05E+00**
FVHYB	**-1.92E+01**	**5.38E+00**
FVSYM	**-8.67E-01**	**2.57E+00**
MFE	**-1.62E+00**	**1.90E+01**

Is there any conclusion ?

How to choose a scheme ?

- Do I really need to use a general mesh (non conforming, distorded, ...) ?
- Do I need to ensure monotony/nonnegativity ?
- Do I need an accurate approximation of the gradient of u, of the fluxes ?
- Do I need an accurate approximation of the energies ?

- Is there any other equation that is coupled with the elliptic system under study ?
- In this case, what are the constraints related to this coupling ? to an existing code ?

- Do I really need theorems ?

THE END

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The International Symposium of Finite Volumes for Complex Applications

FVCA VII, Berlin, Germany,
June 16-20, 2014

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Let $u^{\mathcal{T}} = (u_i)_i$ satisfying $Au^{\mathcal{T}} = (h_i f_i)_i$.

- We assume $(f_i)_i \geq 0$, we want to show that $(u_i)_i \geq 0$.

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- We assume $(f_i)_i \geq 0$, we want to show that $(u_i)_i \geq 0$.
- Assume that $u^{\mathcal{T}} \ngeq 0$, then we set

$$i_0 = \min\{1 \leq i \leq N, \quad \text{s.t.} \quad u_i = \min u^{\mathcal{T}}\}.$$

By assumption, we have $u_{i_0} < 0 = u_0$ so that

$$u_{i_0} < u_{i_0-1}, \quad \text{and} \quad u_{i_0} \leq u_{i_0+1}.$$

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- Assume that $u^{\mathcal{T}} \not\geq 0$, then we set

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By assumption, we have $u_{i_0} < 0 = u_0$ so that

$$u_{i_0} < u_{i_0-1}, \quad \text{and} \quad u_{i_0} \leq u_{i_0+1}.$$

- We look at the equation $\#i_0$ of the system

$$\underbrace{-k_{i_0+1/2}\frac{u_{i_0+1} - u_{i_0}}{h_{i_0+1/2}}}_{\leq 0} + \underbrace{k_{i_0-1/2}\frac{u_{i_0} - u_{i_0-1}}{h_{i_0-1/2}}}_{\boxed{<0}} = \underbrace{h_{i_0} f_{i_0}}_{\geq 0}.$$

$\Rightarrow$ Contradiction.

Back

Let $u^\mathcal{T}$ satisfying $Au^\mathcal{T} = (h_i f_i)_i$.

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- This vector satisfies $\|v^\mathcal{T}\|_\infty \leq C$ and $v_i \approx v(x_i)$ where v is the solution of

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$$A\big(\|f\|_\infty v^{\mathcal{T}} - u^{\mathcal{T}}\big) = \big(h_i(\|f\|_\infty - f_i)\big)_i \geq 0.$$

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- By linearity, we have

$$A\big(\|f\|_\infty v^{\mathcal{T}} - u^{\mathcal{T}}\big) = \big(h_i(\|f\|_\infty - f_i)\big)_i \geq 0.$$

- By using the discrete maximum principle, we get

$$u^{\mathcal{T}} \leq \|f\|_\infty v^{\mathcal{T}},$$

and thus

$$u^{\mathcal{T}} \leq C\|f\|_\infty.$$

- The same reasonning applied to $-u^{\mathcal{T}}$ let us conclude.

Back

- For simplicity, we assume that Ω is convex.
- Let $\xi \in \mathbb{R}^2$ be any unitary vector. For any $x \in \Omega$ let $y(x)$ be the *projection* of x onto $\partial\Omega$ in the direction ξ.
- For any $\sigma \in \mathcal{E}$, we set

$$\chi_\sigma(x, y) = \begin{cases} 1 & \text{if } [x, y] \cap \sigma \neq \emptyset \\ 0 & \text{if } [x, y] \cap \sigma = \emptyset. \end{cases}$$

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- Let us define the set

$$\tilde{\Omega} = \Big\{ x \in \Omega, \quad [x, y(x)] \text{ does not contain any vertex of the mesh} \\ \text{and any edge of the mesh} \Big\}.$$

Observe that $\tilde{\Omega}^c$ has a zero Lebesgue measure in Ω.

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  Observe that $\tilde{\Omega}^c$ has a zero Lebesgue measure in Ω.
- Let us take $\kappa \in \mathcal{T}$ and $x \in \kappa \cap \tilde{\Omega}$.
  By following the segment $[x, y(x)]$ from x to $y(x)$, we encounter a finite number of control volumes denoted by $(\kappa_i)_{1 \leq i \leq m}$ with $\kappa_1 = \kappa$ and κ_m is a boundary control volume.

- We write a telescoping sum

$$u_{\mathcal{K}} = u_{\mathcal{K}_1} = \sum_{i=1}^{m-1} (u_{\mathcal{K}_i} - u_{\mathcal{K}_{i+1}}) + u_{\mathcal{K}_m},$$

$$|u_{\mathcal{K}}| \leq \sum_{i=1}^{m-1} |u_{\mathcal{K}_i} - u_{\mathcal{K}_{i+1}}| + |0 - u_{\mathcal{K}_m}|,$$

- Thus, we get

$$\boxed{|u_{\mathcal{K}}| \leq \sum_{\sigma\in\mathcal{E}} d_\sigma |D_\sigma(u^{\mathcal{T}})| \chi_\sigma(x, y(x)).}$$

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- Thus, we get

$$|u_\mathcal{K}| \leq \sum_{\sigma\in\mathcal{E}} d_\sigma \sqrt{c_\sigma} \frac{1}{\sqrt{c_\sigma}} |D_\sigma(u^\mathcal{T})| \chi_\sigma(x, y(x)).$$

with $c_\sigma = |\boldsymbol{\nu}_\sigma \cdot \xi|$ (which is non zero, since $x \in \tilde{\Omega}$).

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- Thus, we get

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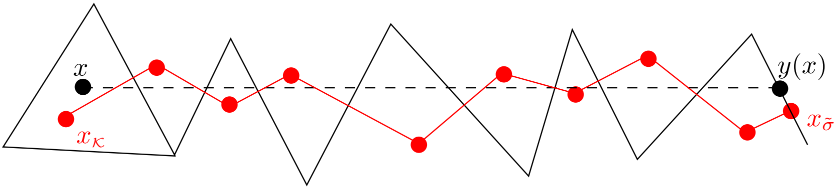
- We use Cauchy-Schwarz inequality

$$|u_{\mathcal{K}}|^2 \leq \left(\sum_{\sigma\in\mathcal{E}} d_\sigma c_\sigma \chi_\sigma(x, y(x))\right) \left(\sum_{\sigma\in\mathcal{E}} \frac{d_\sigma}{c_\sigma} |D_\sigma(u^{\mathcal{T}})|^2 \chi_\sigma(x, y(x))\right).$$

$$|u_\mathcal{K}|^2 \leq \left(\sum_{\sigma\in\mathcal{E}} d_\sigma c_\sigma \chi_\sigma(x,y(x))\right)\left(\sum_{\sigma\in\mathcal{E}} \frac{d_\sigma}{c_\sigma}|D_\sigma(u^\mathcal{T})|^2 \chi_\sigma(x,y(x))\right).$$

FIRST TERM ESTIMATE Let $\tilde{\sigma} \in \mathcal{E}_{ext}$ be such that $y(x) \in \tilde{\sigma}$.



$$\sum_{i=1}^{m-1}(x_{\mathcal{K}_i} - x_{\mathcal{K}_{i+1}}) + x_{\mathcal{K}_m} - x_{\tilde{\sigma}} = x_\mathcal{K} - x_{\tilde{\sigma}},$$

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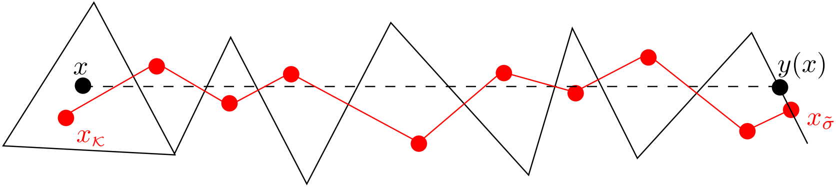

$$\sum_{i=1}^{m-1}(x_{\mathcal{K}_i} - x_{\mathcal{K}_{i+1}})\cdot\xi + (x_{\mathcal{K}_m} - x_{\tilde{\sigma}})\cdot\xi = (x_{\mathcal{K}} - x_{\tilde{\sigma}})\cdot\xi,$$

$$|u_{\mathcal{K}}|^2 \leq \left(\sum_{\sigma\in\mathcal{E}} d_\sigma c_\sigma \chi_\sigma(x,y(x))\right)\left(\sum_{\sigma\in\mathcal{E}} \frac{d_\sigma}{c_\sigma}|D_\sigma(u^{\mathcal{T}})|^2\chi_\sigma(x,y(x))\right).$$

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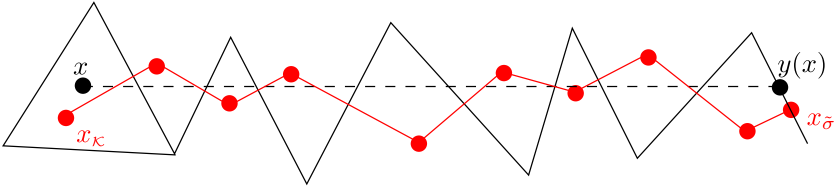
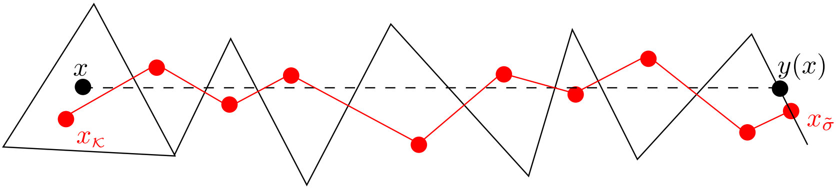


$$-\sum_{i=1}^{m-1} \underbrace{d_{\mathcal{K}_i\mathcal{K}_{i+1}}}_{d_\sigma} \underbrace{\boldsymbol{\nu}_{\mathcal{K}_i\mathcal{K}_{i+1}}\cdot\xi}_{=c_\sigma} - d_{\mathcal{K}_m\tilde{\sigma}}\boldsymbol{\nu}_{\mathcal{K}_m\tilde{\sigma}}\cdot\xi = (x_{\mathcal{K}} - x_{\tilde{\sigma}})\cdot\xi,$$

$$|u_\mathcal{K}|^2 \leq \left(\sum_{\sigma\in\mathcal{E}} d_\sigma c_\sigma \chi_\sigma(x,y(x))\right)\left(\sum_{\sigma\in\mathcal{E}} \frac{d_\sigma}{c_\sigma}|D_\sigma(u^\mathcal{T})|^2 \chi_\sigma(x,y(x))\right).$$

FIRST TERM ESTIMATE Let $\tilde{\sigma} \in \mathcal{E}_{ext}$ be such that $y(x) \in \tilde{\sigma}$.



$$\sum_{\sigma\in\mathcal{E}} d_\sigma c_\sigma \chi_\sigma(x,y(x)) = \left|(x_\mathcal{K} - x_{\tilde{\sigma}})\cdot\xi\right| \leq \operatorname{diam}(\Omega).$$

$$|u_\mathcal{K}|^2 \le \mathrm{diam}(\Omega)\left(\sum_{\sigma\in\mathcal{E}} \frac{d_\sigma}{c_\sigma}|D_\sigma(u^\mathcal{T})|^2\chi_\sigma(x,y(x))\right), \quad \forall \kappa \in \mathcal{T}, \forall x \in \kappa \cap \tilde{\Omega}.$$

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We integrate this formula on $\mathcal{K} \cap \tilde{\Omega}$ with respect to x, then we sum over $\mathcal{K} \in \mathcal{T}$

$$\sum_{\mathcal{K} \in \mathcal{T}} |\mathcal{K}| |u_{\mathcal{K}}|^2 \leq \mathrm{diam}(\Omega) \sum_{\sigma \in \mathcal{E}} \frac{d_\sigma}{c_\sigma} |D_\sigma(u^{\mathcal{T}})|^2 \left(\int_\Omega \chi_\sigma(x, y(x))\, dx \right),$$

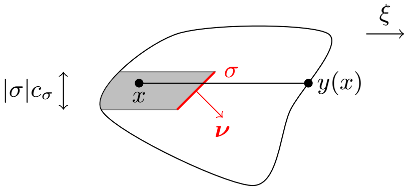
$$|u_\kappa|^2 \leq \mathrm{diam}(\Omega) \left(\sum_{\sigma\in\mathcal{E}} \frac{d_\sigma}{c_\sigma} |D_\sigma(u^\mathcal{T})|^2 \chi_\sigma(x, y(x)) \right), \quad \forall \kappa \in \mathcal{T}, \forall x \in \kappa \cap \tilde{\Omega}.$$

We integrate this formula on $\kappa \cap \tilde{\Omega}$ with respect to x, then we sum over $\kappa \in \mathcal{T}$

$$\sum_{\kappa\in\mathcal{T}} |\kappa||u_\kappa|^2 \leq \mathrm{diam}(\Omega) \sum_{\sigma\in\mathcal{E}} \frac{d_\sigma}{c_\sigma} |D_\sigma(u^\mathcal{T})|^2 \left(\int_\Omega \chi_\sigma(x, y(x))\, dx \right),$$

Estimate of each integral

$$\int_\Omega \chi_\sigma(x, y(x))\, dx \leq \mathrm{diam}(\Omega) |\sigma| c_\sigma.$$



$$|u_\mathcal{K}|^2 \leq \operatorname{diam}(\Omega)\left(\sum_{\sigma\in\mathcal{E}} \frac{d_\sigma}{c_\sigma}|D_\sigma(u^\mathcal{T})|^2\chi_\sigma(x,y(x))\right), \quad \forall \mathcal{K}\in\mathcal{T}, \forall x\in \mathcal{K}\cap\tilde{\Omega}.$$

We integrate this formula on $\mathcal{K}\cap\tilde{\Omega}$ with respect to x, then we sum over $\mathcal{K}\in\mathcal{T}$

$$\sum_{\mathcal{K}\in\mathcal{T}} |\mathcal{K}||u_\mathcal{K}|^2 \leq \operatorname{diam}(\Omega)\sum_{\sigma\in\mathcal{E}} \frac{d_\sigma}{c_\sigma}|D_\sigma(u^\mathcal{T})|^2\left(\int_\Omega \chi_\sigma(x,y(x))\,dx\right),$$

Estimate of each integral

$$\int_\Omega \chi_\sigma(x,y(x))\,dx \leq \operatorname{diam}(\Omega)|\sigma|c_\sigma.$$

Conclusion

$$\|u^\mathcal{T}\|_{L^2}^2 = \sum_{\mathcal{K}\in\mathcal{T}} |\mathcal{K}||u_\mathcal{K}|^2 \leq \operatorname{diam}(\Omega)^2\sum_{\sigma\in\mathcal{E}} |\sigma|d_\sigma|D_\sigma(u^\mathcal{T})|^2 = \operatorname{diam}(\Omega)^2\|u^\mathcal{T}\|_{1,\mathcal{T}}^2.$$

Back

- Let $u^n \in L^2(\mathbb{R}^2)$ be the extension by 0 of $u^{\mathcal{T}_n} \in L^2(\Omega)$.
- In order to use the Kolmogorov theorem we need a translation estimate

$$\|u^n(\cdot + \eta) - u^n\|_{L^2} \xrightarrow[\eta \to 0]{} 0, \quad \text{unif. with respect to } n.$$

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$$\|u^n(\cdot + \eta) - u^n\|_{L^2} \xrightarrow[\eta \to 0]{} 0, \quad \text{unif. with respect to } n.$$

- Let $\eta \in \mathbb{R}^2 \setminus \{0\}$. A standard computation (telescoping sum) leads to

$$|u^n(x+\eta) - u^n(x)| \leq \sum_{\sigma \in \mathcal{E}} d_\sigma |D_\sigma(u^{\mathcal{T}_n})| \chi_\sigma(x, x+\eta),$$

then by the Cauchy-Schwarz inequality

$$|u^n(x+\eta) - u^n(x)|^2 \leq \left(\sum_{\sigma \in \mathcal{E}} \chi_\sigma(x, x+\eta) d_\sigma c_\sigma \right) \times \left(\sum_{\sigma \in \mathcal{E}} \chi_\sigma(x, x+\eta) \frac{d_\sigma}{c_\sigma} |D_\sigma(u^{\mathcal{T},n})|^2 \right),$$

with $c_\sigma = |\boldsymbol{\nu}_\sigma \cdot \frac{\eta}{|\eta|}|$.

$$|u^n(x+\eta)-u^n(x)|^2 \leq \left(\sum_{\sigma\in\mathcal{E}} \chi_\sigma(x,x+\eta) d_\sigma c_\sigma\right) \times \left(\sum_{\sigma\in\mathcal{E}} \chi_\sigma(x,x+\eta)\frac{d_\sigma}{c_\sigma}|D_\sigma(u^{\mathcal{T},n})|^2\right),$$

with $c_\sigma = |\boldsymbol{\nu}\cdot\frac{\eta}{|\eta|}|$.

Estimate of the first term

- Without loss of generalities we assume that $[x,x+\eta]\subset\Omega$.
- Let $\kappa,\mathcal{L}\in\mathcal{T}$ be such that $x\in\kappa$ and $x+\eta\in\mathcal{L}$. Thus, we have

$$\begin{aligned}\sum_{\sigma\in\mathcal{E}} \chi_\sigma(x,x+\eta) d_\sigma c_\sigma &= \left|(x_\mathcal{L}-x_\mathcal{K})\cdot\frac{\eta}{|\eta|}\right| \leq |x_\mathcal{L}-x_\mathcal{K}| \\ &\leq |x_\mathcal{L}-(x+\eta)|+|(x+\eta)-x|+|x-x_\mathcal{K}| \\ &\leq |\eta| + 2\mathrm{size}(\mathcal{T}_n).\end{aligned}$$

- We integrate with respect to $x\in\mathbb{R}^2$

$$\|u^n(\cdot+\eta)-u^n\|_{L^2}^2 \leq C(|\eta|+\mathrm{size}(\mathcal{T}_n))\sum_{\sigma\in\mathcal{E}}\frac{d_\sigma}{c_\sigma}|D_\sigma(u^{\mathcal{T},n})|^2\left(\int_{\mathbb{R}^2}\chi_\sigma(x,x+\eta)\,dx\right)$$

$$|u^n(x+\eta)-u^n(x)|^2 \leq C(|\eta|+\text{size}(\mathcal{T}_n))\sum_{\sigma\in\mathcal{E}}\left(\chi_\sigma(x,x+\eta)\frac{d_\sigma}{c_\sigma}|D_\sigma(u^{\mathcal{T},n})|^2\right),$$

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- We integrate with respect to $x \in \mathbb{R}^2$

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- Computation of the integral

$$\int_{\mathbb{R}^2} \chi_\sigma(x,x+\eta)\,dx = |\eta||\sigma|c_\sigma.$$

$$|u^n(x+\eta)-u^n(x)|^2 \leq C(|\eta|+\text{size}(\mathcal{T}_n)) \sum_{\sigma\in\mathcal{E}} \left(\chi_\sigma(x,x+\eta)\frac{d_\sigma}{c_\sigma}|D_\sigma(u^{\mathcal{T},n})|^2 \right),$$

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- Computation of the integral

$$\int_{\mathbb{R}^2} \chi_\sigma(x,x+\eta)\,dx = |\eta||\sigma|c_\sigma.$$

Conclusion

$$\|u^n(\cdot+\eta)-u^n\|_{L^2}^2 \leq C|\eta|(|\eta| + \underbrace{\text{size}(\mathcal{T}_n)}_{\leq \text{diam}(\Omega)})\underbrace{\|u^{\mathcal{T}_n}\|_{1,\mathcal{T}_n}^2}_{\text{bounded}}.$$

$$\boxed{\text{Kolmogoroff} \Rightarrow \exists \text{ a subsequence } u^{\varphi(n)} \longrightarrow u \in L^2(\mathbb{R}^2) \text{ with } u = 0 \text{ outside } \Omega.}$$

- By assumption, we have

$$\sup_n \|\nabla^{\mathcal{T}_n} u^{\mathcal{T}_n}\|_{L^2} < +\infty,$$

- There exists $G \in (L^2(\Omega))^2$ such that (up to a subsequence) we have

$$\nabla^{\mathcal{T}_{\varphi(n)}} u^{\mathcal{T}_{\varphi(n)}} \xrightharpoonup[n\to\infty]{} G, \quad \text{in } L^2(\Omega)^2.$$

We want to show that $u \in H_0^1(\Omega)$ and $\nabla u = G$.

- By assumption, we have

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$$\boxed{\text{We want to show that } u \in H^1_0(\Omega) \text{ and } \nabla u = G.}$$

- Let $\Phi \in (\mathcal{C}^\infty(\mathbb{R}^2))^2$ (**we do not assume that** $\Phi = 0$ **on** $\partial\Omega$)

$$I_n \stackrel{\text{def}}{=} \int_\Omega u^{\mathcal{T}_n} (\text{div}\Phi)\, dx \xrightarrow[n\to\infty]{} \int_\Omega u\, (\text{div}\Phi)\, dx.$$

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$$\boxed{\text{We want to show that } u \in H_0^1(\Omega) \text{ and } \nabla u = G.}$$

- Let $\Phi \in (\mathcal{C}^\infty(\mathbb{R}^2))^2$ (**we do not assume that $\Phi = 0$ on $\partial\Omega$**)

$$I_n \stackrel{\text{def}}{=} \int_\Omega u^{\mathcal{T}_n} (\text{div}\Phi)\, dx \xrightarrow[n\to\infty]{} \int_\Omega u\, (\text{div}\Phi)\, dx.$$

- We also have

$$I_n = \sum_{\mathcal{K}\in\mathcal{T}_n} u_{\mathcal{K}}^n \left(\int_{\mathcal{K}} \text{div}\Phi\, dx\right) = \sum_{\mathcal{K}\in\mathcal{T}_n} u_{\mathcal{K}}^n \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \left(\int_\sigma \Phi\cdot\boldsymbol{\nu}_{\mathcal{K}\sigma}\, dx\right).$$

- We sum over the edges in the mesh

$$I_n = \sum_{\sigma \in \mathcal{E}_{int}} (u_{\mathcal{K}}^n - u_{\mathcal{L}}^n) \left(\int_\sigma \Phi \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} \right) + \sum_{\sigma \in \mathcal{E}_{ext}} (u_{\mathcal{K}}^n - 0) \left(\int_\sigma \Phi \cdot \boldsymbol{\nu}_{\mathcal{K}\sigma} \right).$$

- We sum over the edges in the mesh

$$I_n = \sum_{\sigma \in \mathcal{E}_{int}} (u_{\mathcal{K}}^n - u_{\mathcal{L}}^n) \boldsymbol{\nu}_{\mathcal{KL}} \cdot \left(\int_\sigma \Phi \right) + \sum_{\sigma \in \mathcal{E}_{ext}} (u_{\mathcal{K}}^n - 0) \boldsymbol{\nu}_{\mathcal{K}\sigma} \cdot \left(\int_\sigma \Phi \right).$$

- We sum over the edges in the mesh

$$I_n = \sum_{\sigma\in\mathcal{E}_{int}} \frac{|\sigma| d_{\mathcal{K}\mathcal{L}}}{d} \left(d \frac{u_{\mathcal{K}}^n - u_{\mathcal{L}}^n}{d_{\mathcal{K}\mathcal{L}}} \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} \right) \cdot \left(\frac{1}{|\sigma|} \int_\sigma \Phi \right) + \sum_{\sigma\in\mathcal{E}_{ext}} \frac{|\sigma| d_{\mathcal{K}\sigma}}{d} \left(d \frac{u_{\mathcal{K}}^n - 0}{d_{\mathcal{K}\sigma}} \boldsymbol{\nu}_{\mathcal{K}\sigma} \right) \cdot \left(\frac{1}{|\sigma|} \int_\sigma \Phi \right).$$

- We sum over the edges in the mesh

$$I_n = -\sum_{\sigma\in\mathcal{E}} |\mathcal{D}|\nabla_{\mathcal{D}}^{\mathcal{T}_n} u^{\mathcal{T}_n} \cdot \left(\frac{1}{|\sigma|}\int_\sigma \Phi\right).$$

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$$I_n = -\sum_{\sigma\in\mathcal{E}} |\mathcal{D}| \nabla_{\mathcal{D}}^{\mathcal{T}_n} u^{\mathcal{T}_n} \cdot \left(\frac{1}{|\sigma|} \int_\sigma \Phi \right).$$

- Since Φ is $\mathcal{C}^\infty$

$$\forall \sigma \in \mathcal{E}, \left| \left(\frac{1}{|\sigma|} \int_\sigma \Phi \right) - \left(\frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} \Phi \right) \right| \leq C \|\nabla \Phi\|_\infty \text{size}(\mathcal{T}_n).$$

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- Since $(\nabla^{\mathcal{T}_n} u^{\mathcal{T}_n})_n$ is bounded in L^2, we have

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$$I_n = -\int_\Omega \nabla^{\mathcal{T}_n} u^{\mathcal{T}_n} \cdot \Phi\, dx + O_\Phi(\mathrm{size}(\mathcal{T}_n)).$$

- Conclusion, for any $\Phi \in (\mathcal{C}^\infty(\mathbb{R}^2))^2$, we have

$$I_n \xrightarrow[n\to\infty]{} -\int_\Omega G\cdot\Phi\, dx, \quad \text{and} \quad I_n \xrightarrow[n\to\infty]{} \int_\Omega u\, \mathrm{div}\Phi\, dx.$$

Back

Energy estimate

$$\forall n \geq 0, \quad \|u^{\mathcal{T}_n}\|_{1,\mathcal{T}_n} \leq \mathrm{diam}(\Omega)\|f\|_{L^2}.$$

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Compactness theorem $\Rightarrow$ There exists $u \in H_0^1(\Omega)$ such that

$$u^{\mathcal{T}_{\varphi(n)}} \xrightarrow[n\to\infty]{} u \text{ in } L^2(\Omega),$$

$$\nabla^{\mathcal{T}_{\varphi(n)}} u^{\mathcal{T}_{\varphi(n)}} \xrightharpoonup[n\to\infty]{} \nabla u \text{ in } (L^2(\Omega))^2.$$

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$$\nabla^{\mathcal{T}_{\varphi(n)}} u^{\mathcal{T}_{\varphi(n)}} \rightharpoonup_{n\to\infty} \nabla u \text{ in } (L^2(\Omega))^2.$$

It remains to check that

$$u \text{ solves } -\Delta u = f,$$

that is

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx, \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega).$$

$\rightsquigarrow$ By standard uniqueness arguments, the convergence holds for the whole sequence $(u^{\mathcal{T}_n})_n$.

$$\mathbb{P}^{\mathcal{T}}\varphi \stackrel{\text{def}}{=} (\varphi(x_{\mathcal{K}}))_{\mathcal{K}\in\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}, \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega).$$

Discrete integration by parts with $v^{\mathcal{T}} = \mathbb{P}^{\mathcal{T}}\varphi$

$$\sum_{\sigma\in\mathcal{E}_{int}} d_\sigma|\sigma| D_{\mathcal{K}\mathcal{L}}(u^{\mathcal{T}_n}) \cdot \left(\frac{\varphi(x_{\mathcal{L}}) - \varphi(x_{\mathcal{K}})}{d_{\mathcal{K}\mathcal{L}}} \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\right) = \sum_{\mathcal{K}\in\mathcal{T}_n} |\mathcal{K}| f_{\mathcal{K}} \varphi(x_{\mathcal{K}}).$$

Remark : For n large enough, the boundary terms vanish.

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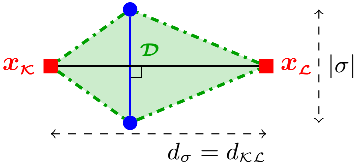
Definition of the discrete gradient

$$\sum_{\sigma\in\mathcal{E}_{int}} \underbrace{\frac{d_\sigma|\sigma|}{d}}_{=|\mathcal{D}|} \nabla_{\mathcal{D}}^{\mathcal{T}_n} u^{\mathcal{T}_n} \cdot \left(\frac{\varphi(x_{\mathcal{L}})-\varphi(x_{\mathcal{K}})}{d_{\mathcal{K}\mathcal{L}}}\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\right) = \sum_{\mathcal{K}\in\mathcal{T}_n} |\mathcal{K}| f_{\mathcal{K}}\varphi(x_{\mathcal{K}}).$$



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Discrete integration by parts with $v^{\mathcal{T}} = \mathbb{P}^{\mathcal{T}}\varphi$

$$\sum_{\sigma\in\mathcal{E}_{int}} d_\sigma |\sigma| D_{\mathcal{K}\mathcal{L}}(u^{\mathcal{T}_n}) \cdot \left(\frac{\varphi(x_{\mathcal{L}}) - \varphi(x_{\mathcal{K}})}{d_{\mathcal{K}\mathcal{L}}}\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\right) = \sum_{\mathcal{K}\in\mathcal{T}_n} |\mathcal{K}| f_{\mathcal{K}}\varphi(x_{\mathcal{K}}).$$

Remark : For n large enough, the boundary terms vanish.

Definition of the discrete gradient

$$\sum_{\sigma\in\mathcal{E}_{int}} \int_{\mathcal{D}} \nabla_{\mathcal{D}}^{\mathcal{T}_n} u^{\mathcal{T}_n} \cdot \left(\frac{\varphi(x_{\mathcal{L}}) - \varphi(x_{\mathcal{K}})}{d_{\mathcal{K}\mathcal{L}}}\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\right) = \sum_{\mathcal{K}\in\mathcal{T}_n} \int_{\mathcal{K}} f(x)\varphi(x_{\mathcal{K}})\, dx.$$

$$\mathbb{P}^{\mathcal{T}}\varphi \stackrel{\text{def}}{=} (\varphi(x_{\mathcal{K}}))_{\mathcal{K}\in\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}, \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega).$$

DISCRETE INTEGRATION BY PARTS WITH $v^{\mathcal{T}} = \mathbb{P}^{\mathcal{T}}\varphi$

$$\sum_{\sigma\in\mathcal{E}_{int}} d_\sigma|\sigma| D_{\mathcal{K}\mathcal{L}}(u^{\mathcal{T}_n}) \cdot \left(\frac{\varphi(x_{\mathcal{L}}) - \varphi(x_{\mathcal{K}})}{d_{\mathcal{K}\mathcal{L}}}\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\right) = \sum_{\mathcal{K}\in\mathcal{T}_n} |\mathcal{K}| f_{\mathcal{K}}\varphi(x_{\mathcal{K}}).$$

Remark : For n large enough, the boundary terms vanish.

DEFINITION OF THE DISCRETE GRADIENT

$$\sum_{\sigma\in\mathcal{E}_{int}} \int_{\mathcal{D}} \nabla_{\mathcal{D}}^{\mathcal{T}_n} u^{\mathcal{T}_n} \cdot \left(\nabla\varphi(x) + \underbrace{\left(\frac{\varphi(x_{\mathcal{L}}) - \varphi(x_{\mathcal{K}})}{d_{\mathcal{K}\mathcal{L}}}\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} - (\nabla\varphi(x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}})\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\right)}_{\stackrel{\text{def}}{=} R_1^n(x)} \right) dx$$

$$= \sum_{\mathcal{K}\in\mathcal{T}} \int_{\mathcal{K}} f(x) \left(\varphi(x) + \underbrace{(\varphi(x_{\mathcal{K}}) - \varphi(x))}_{\stackrel{\text{def}}{=} R_2^n(x)} \right) dx.$$

Summary

$$\int_\Omega \nabla^{\mathcal{T}_n} u^{\mathcal{T}_n} \cdot \nabla\varphi(x)\,dx - \int_\Omega f(x)\,\varphi(x)\,dx$$
$$= -\int_\Omega \nabla^{\mathcal{T}_n} u^{\mathcal{T}_n} \cdot R_1^n(x)\,dx + \int_\Omega f(x) R_2^n(x)\,dx.$$

SUMMARY

$$\int_\Omega \nabla^{\mathcal{T}_n} u^{\mathcal{T}_n} \cdot \nabla\varphi(x)\,dx - \int_\Omega f(x)\,\varphi(x)\,dx$$
$$= -\int_\Omega \nabla^{\mathcal{T}_n} u^{\mathcal{T}_n} \cdot R_1^n(x)\,dx + \int_\Omega f(x) R_2^n(x)\,dx.$$

REMAINDERS ESTIMATES

Recall that φ is smooth

$$|R_1^n(x)| = \left| \frac{\varphi(x_\mathcal{L}) - \varphi(x_\mathcal{K})}{d_{\mathcal{K}\mathcal{L}}} \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} - (\nabla\varphi(x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}})\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} \right| \leq C\|D^2\varphi\|_\infty \mathrm{size}(\mathcal{T}_n).$$

$$|R_2^n(x)| = |\varphi(x_\mathcal{K}) - \varphi(x)| \leq \|D\varphi\|_\infty \mathrm{size}(\mathcal{T}_n).$$

SUMMARY

$$\int_\Omega \nabla^{\mathcal{T}_n} u^{\mathcal{T}_n} \cdot \nabla\varphi(x)\,dx - \int_\Omega f(x)\,\varphi(x)\,dx = -\int_\Omega \nabla^{\mathcal{T}_n} u^{\mathcal{T}_n} \cdot R_1^n(x)\,dx + \int_\Omega f(x) R_2^n(x)\,dx.$$

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$$|R_2^n(x)| = |\varphi(x_{\mathcal{K}}) - \varphi(x)| \leq \|D\varphi\|_\infty \mathrm{size}(\mathcal{T}_n).$$

WE CAN PASS TO THE LIMIT

$$\int_\Omega \nabla u \cdot \nabla\varphi\,dx = \int_\Omega f\varphi\,dx, \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega). \tag{$\star$}$$

Observe that, $(\star)$ still holds for any $\varphi \in H_0^1(\Omega)$.

Strong convergence of the gradient does not hold

Using the discrete integration by parts, we get

$$\sum_{\sigma\in\mathcal{E}} d_\sigma|\sigma||D_\sigma(u^{\mathcal{T}_n})|^2 = \sum_{\mathcal{K}\in\mathcal{T}_n} |\mathcal{K}| f_{\mathcal{K}} u^n_{\mathcal{K}} = \int_\Omega f(x) u^{\mathcal{T}_n}(x)\, dx.$$

Since $\nabla_{\mathcal{D}}^{\mathcal{T}_n} u^{\mathcal{T}_n} = d D_\sigma(u^{\mathcal{T}_n})\boldsymbol{\nu}_\sigma$, we deduce

$$\frac{1}{d}\sum_{\sigma\in\mathcal{E}} \underbrace{\frac{d_\sigma|\sigma|}{d}}_{=|\mathcal{D}|} |\nabla_{\mathcal{D}}^{\mathcal{T}_n} u^{\mathcal{T}_n}|^2 = \int_\Omega f(x) u^{\mathcal{T}_n}(x)\, dx.$$

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Since $\nabla_{\mathcal{D}}^{\mathcal{T}_n} u^{\mathcal{T}_n} = d D_\sigma(u^{\mathcal{T}_n})\boldsymbol{\nu}_\sigma$, we deduce

$$\frac{1}{d}\|\nabla^{\mathcal{T}_n} u^{\mathcal{T}_n}\|_{L^2}^2 = \int_\Omega f(x)u^{\mathcal{T}_n}(x)\,dx.$$

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Since $\nabla_{\mathcal{D}}^{\mathcal{T}_n} u^{\mathcal{T}_n} = d D_\sigma(u^{\mathcal{T}_n})\boldsymbol{\nu}_\sigma$, we deduce

$$\frac{1}{d}\|\nabla^{\mathcal{T}_n} u^{\mathcal{T}_n}\|_{L^2}^2 = \int_\Omega f(x) u^{\mathcal{T}_n}(x)\,dx.$$

We pass to the limit in the right-hand side term to get

$$\lim_{n\to\infty} \|\nabla^{\mathcal{T}_n} u^{\mathcal{T}_n}\|_{L^2}^2 = d\int_\Omega f(x)u(x)\,dx = d\|\nabla u\|_{L^2}^2.$$

$\rightsquigarrow$ For $u \neq 0$ and $d \geq 2$, **we do not have** strong convergence of the gradients.

Back

$$R_\sigma(u) = \frac{u(x_\mathcal{L}) - u(x_\mathcal{K})}{d_{\mathcal{K}\mathcal{L}}} - \frac{1}{|\sigma|}\int_\sigma \nabla u(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\,dx.$$

- Taylor formulas for $u \in \mathcal{C}^2$ and $x \in \sigma$

$$u(x_\mathcal{L}) = u(x) + \nabla u(x)\cdot(x_\mathcal{L} - x) + \int_0^1 (1-t) D^2 u(x + t(x_\mathcal{L} - x)).(x_\mathcal{L} - x)^2\,dt,$$

$$u(x_\mathcal{K}) = u(x) + \nabla u(x)\cdot(x_\mathcal{K} - x) + \int_0^1 (1-t) D^2 u(x + t(x_\mathcal{K} - x)).(x_\mathcal{K} - x)^2\,dt.$$

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$$u(x_\mathcal{K}) = u(x)+\nabla u(x)\cdot(x_\mathcal{K}-x)+\int_0^1 (1-t)D^2u(x+t(x_\mathcal{K}-x)).(x_\mathcal{K}-x)^2\,dt.$$

- By subtraction, and using that $x_\mathcal{L} - x_\mathcal{K} = d_{\mathcal{K}\mathcal{L}}\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}$,

$$u(x_\mathcal{L})-u(x_\mathcal{K}) = d_{\mathcal{K}\mathcal{L}}\nabla u(x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}+\int_0^1 (1-t)D^2u(x+t(x_\mathcal{L}-x)).(x_\mathcal{L}-x)^2\,dt$$
$$- \int_0^1 (1-t)D^2u(x+t(x_\mathcal{K}-x)).(x_\mathcal{K}-x)^2\,dt.$$

$$R_\sigma(u) = \frac{u(x_\mathcal{L}) - u(x_\mathcal{K})}{d_{\mathcal{K}\mathcal{L}}} - \frac{1}{|\sigma|}\int_\sigma \nabla u(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\,dx.$$

- Taylor formulas for $u \in \mathcal{C}^2$ and $x \in \sigma$

$$u(x_\mathcal{L}) = u(x)+\nabla u(x)\cdot(x_\mathcal{L}-x)+\int_0^1 (1-t)D^2u(x+t(x_\mathcal{L}-x)).(x_\mathcal{L}-x)^2\,dt,$$

$$u(x_\mathcal{K}) = u(x)+\nabla u(x)\cdot(x_\mathcal{K}-x)+\int_0^1 (1-t)D^2u(x+t(x_\mathcal{K}-x)).(x_\mathcal{K}-x)^2\,dt.$$

- Conclusion

$$R_\sigma(u) = \underbrace{\frac{1}{d_{\mathcal{K}\mathcal{L}}|\sigma|}\int_\sigma\int_0^1 (1-t)D^2u(x+t(x_\mathcal{L}-x)).(x_\mathcal{L}-x)^2\,dtdx}_{=T_1}$$

$$-\underbrace{\frac{1}{d_{\mathcal{K}\mathcal{L}}|\sigma|}\int_\sigma\int_0^1 (1-t)D^2u(x+t(x_\mathcal{K}-x)).(x_\mathcal{K}-x)^2\,dt\,dx}_{=T_2}.$$

$$T_1 = \frac{1}{d_{\mathcal{KL}}|\sigma|} \int_\sigma \int_0^1 (1-t) D^2 u(x + t(x_{\mathcal{L}} - x)).(x_{\mathcal{L}} - x)^2 \, dt dx.$$

$$T_1 = \frac{1}{d_{\mathcal{K}\mathcal{L}}|\sigma|}\int_\sigma \int_0^1 (1-t)D^2u(x+t(x_\mathcal{L}-x)).(x_\mathcal{L}-x)^2\,dtdx.$$

Jensen inequality

$$|T_1|^2 \leq \frac{1}{d_{\mathcal{K}\mathcal{L}}^2|\sigma|}\int_\sigma \int_0^1 |1-t|^2|D^2u(x+t(x_\mathcal{L}-x))|^2|x_\mathcal{L}-x|^4\,dtdx.$$

$$T_1 = \frac{1}{d_{\mathcal{K}\mathcal{L}}|\sigma|}\int_\sigma \int_0^1 (1-t)D^2u(x+t(x_\mathcal{L}-x)).(x_\mathcal{L}-x)^2\,dtdx.$$

Jensen inequality

$$|T_1|^2 \leq \frac{1}{d_{\mathcal{K}\mathcal{L}}^2|\sigma|}\int_\sigma \int_0^1 |1-t|^2|D^2u(x+t(x_\mathcal{L}-x))|^2|x_\mathcal{L}-x|^4\,dtdx.$$

Change of variables

$$(t,x)\in[0,1]\times\sigma \mapsto y = x+t(x_\mathcal{L}-x)\in\mathcal{D}_\mathcal{L}.$$

The Jacobian determinant is $(1-t)(x_\mathcal{L}-x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} = (1-t)d_{\mathcal{L}\sigma}$.

$$|T_1|^2 \leq \frac{\mathrm{d}_\mathcal{D}^4}{d_{\mathcal{K}\mathcal{L}}^2 d_{\mathcal{L},\sigma}|\sigma|}\int_{\mathcal{D}_\mathcal{L}} |D^2u(y)|^2\,dy \leq C(\mathrm{reg}(\mathcal{T}))\frac{\mathrm{size}(\mathcal{T})^2}{|\mathcal{D}|}\int_\mathcal{D} |D^2u(y)|^2\,dy.$$

Back

$$R_\sigma(u) = \frac{d_{\mathcal{K}\mathcal{L}}}{\frac{d_{\mathcal{K}\sigma}}{k_{\mathcal{K}}} + \frac{d_{\mathcal{L}\sigma}}{k_{\mathcal{L}}}} \frac{u(x_{\mathcal{L}}) - u(x_{\mathcal{K}})}{d_{\mathcal{K}\mathcal{L}}} - \frac{1}{|\sigma|} \int_\sigma k(x) \nabla u(x) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx.$$

"Continuity" of the total flux

$$\int_\sigma k(x)\nabla u(x) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx \stackrel{\text{def}}{=} k_{\mathcal{K}} \int_\sigma \nabla u_{|\mathcal{K}}(x) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx = k_{\mathcal{L}} \int_\sigma \nabla u_{|\mathcal{L}}(x) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx.$$

$$R_\sigma(u) = \frac{d_{\mathcal{K}\mathcal{L}}}{\frac{d_{\mathcal{K}\sigma}}{k_\mathcal{K}} + \frac{d_{\mathcal{L}\sigma}}{k_\mathcal{L}}} \frac{u(x_\mathcal{L}) - u(x_\mathcal{K})}{d_{\mathcal{K}\mathcal{L}}} - \frac{1}{|\sigma|}\int_\sigma k(x)\nabla u(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx.$$

"Continuity" of the total flux

$$\int_\sigma k(x)\nabla u(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx \stackrel{\text{def}}{=} k_\mathcal{K}\int_\sigma \nabla u_{|\mathcal{K}}(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx = k_\mathcal{L}\int_\sigma \nabla u_{|\mathcal{L}}(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx.$$

Taylor formulas around $x \in \sigma$

WARNING : u is not globally smooth but $u_{|\mathcal{K}}$ and $u_{|\mathcal{L}}$ are smooth.

$$u(x_\mathcal{K}) = u(x) + \nabla u_{|\mathcal{K}}(x)\cdot(x_\mathcal{K} - x) + O(\text{size}(\mathcal{T})^2),$$

$$u(x_\mathcal{L}) = u(x) + \nabla u_{|\mathcal{L}}(x)\cdot(x_\mathcal{L} - x) + O(\text{size}(\mathcal{T})^2).$$

$$R_\sigma(u) = \frac{d_{\mathcal{K}\mathcal{L}}}{\frac{d_{\mathcal{K}\sigma}}{k_\mathcal{K}} + \frac{d_{\mathcal{L}\sigma}}{k_\mathcal{L}}} \frac{u(x_\mathcal{L}) - u(x_\mathcal{K})}{d_{\mathcal{K}\mathcal{L}}} - \frac{1}{|\sigma|}\int_\sigma k(x)\nabla u(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\,dx.$$

"Continuity" of the total flux

$$\int_\sigma k(x)\nabla u(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\,dx \stackrel{\text{def}}{=} k_\mathcal{K}\int_\sigma \nabla u_{|\mathcal{K}}(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\,dx = k_\mathcal{L}\int_\sigma \nabla u_{|\mathcal{L}}(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\,dx.$$

Taylor formulas around $x \in \sigma$

$$u(x_\mathcal{K}) = u(x) + \nabla u_{|\mathcal{K}}(x)\cdot(-d_{\mathcal{K}\sigma}\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} + x_\sigma - x) + O(\text{size}(\mathcal{T})^2),$$
$$u(x_\mathcal{L}) = u(x) + \nabla u_{|\mathcal{L}}(x)\cdot(d_{\mathcal{L}\sigma}\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} + x_\sigma - x) + O(\text{size}(\mathcal{T})^2).$$

$$R_\sigma(u) = \frac{d_{\mathcal{K}\mathcal{L}}}{\frac{d_{\mathcal{K}\sigma}}{k_\mathcal{K}} + \frac{d_{\mathcal{L}\sigma}}{k_\mathcal{L}}} \frac{u(x_\mathcal{L}) - u(x_\mathcal{K})}{d_{\mathcal{K}\mathcal{L}}} - \frac{1}{|\sigma|}\int_\sigma k(x)\nabla u(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\,dx.$$

"Continuity" of the total flux

$$\int_\sigma k(x)\nabla u(x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\,dx \overset{\text{def}}{=} k_\mathcal{K}\int_\sigma \nabla u_{|\mathcal{K}}(x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\,dx = k_\mathcal{L}\int_\sigma \nabla u_{|\mathcal{L}}(x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\,dx.$$

Taylor formulas around $x \in \sigma$

$$u(x_\mathcal{L}) - u(x_\mathcal{K}) = d_{\mathcal{L}\sigma}\nabla u_{|\mathcal{L}}(x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} + d_{\mathcal{K}\sigma}\nabla u_{|\mathcal{K}}(x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} + O(\text{size}(\mathcal{T})^2).$$

$$R_\sigma(u) = \frac{d_{\mathcal{K}\mathcal{L}}}{\frac{d_{\mathcal{K}\sigma}}{k_\mathcal{K}} + \frac{d_{\mathcal{L}\sigma}}{k_\mathcal{L}}} \frac{u(x_\mathcal{L}) - u(x_\mathcal{K})}{d_{\mathcal{K}\mathcal{L}}} - \frac{1}{|\sigma|}\int_\sigma k(x)\nabla u(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\,dx.$$

"Continuity" of the total flux

$$\int_\sigma k(x)\nabla u(x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\,dx \overset{\text{def}}{=} k_\mathcal{K}\int_\sigma \nabla u_{|\mathcal{K}}(x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\,dx = k_\mathcal{L}\int_\sigma \nabla u_{|\mathcal{L}}(x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\,dx.$$

Taylor formulas around $x\in\sigma$

$$u(x_\mathcal{L})-u(x_\mathcal{K}) = \frac{d_{\mathcal{L}\sigma}}{k_\mathcal{L}}\left(k_\mathcal{L}\nabla u_{|\mathcal{L}}(x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\right) + \frac{d_{\mathcal{K}\sigma}}{k_\mathcal{K}}\left(k_\mathcal{K}\nabla u_{|\mathcal{K}}(x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\right) + O(\text{size}(\mathcal{T})^2).$$

$$R_\sigma(u) = \frac{d_{\mathcal{K}\mathcal{L}}}{\frac{d_{\mathcal{K}\sigma}}{k_{\mathcal{K}}} + \frac{d_{\mathcal{L}\sigma}}{k_{\mathcal{L}}}} \frac{u(x_{\mathcal{L}}) - u(x_{\mathcal{K}})}{d_{\mathcal{K}\mathcal{L}}} - \frac{1}{|\sigma|} \int_\sigma k(x)\nabla u(x) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx.$$

"Continuity" of the total flux

$$\int_\sigma k(x)\nabla u(x) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx \stackrel{\text{def}}{=} k_{\mathcal{K}} \int_\sigma \nabla u_{|\mathcal{K}}(x) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx = k_{\mathcal{L}} \int_\sigma \nabla u_{|\mathcal{L}}(x) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx.$$

Taylor formulas around $x \in \sigma$

$$u(x_{\mathcal{L}}) - u(x_{\mathcal{K}}) = \frac{d_{\mathcal{L}\sigma}}{k_{\mathcal{L}}} \left(k_{\mathcal{L}} \nabla u_{|\mathcal{L}}(x) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} \right) + \frac{d_{\mathcal{K}\sigma}}{k_{\mathcal{K}}} \left(k_{\mathcal{K}} \nabla u_{|\mathcal{K}}(x) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}} \right) + O(\text{size}(\mathcal{T})^2).$$

We integrate on σ

$$|\sigma|(u(x_{\mathcal{L}}) - u(x_{\mathcal{K}})) = \left(\frac{d_{\mathcal{L}\sigma}}{k_{\mathcal{L}}} + \frac{d_{\mathcal{K}\sigma}}{k_{\mathcal{K}}} \right) \int_\sigma k(x)\nabla u(x) \cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx + O(\text{size}(\mathcal{T})^3).$$

$$R_\sigma(u) = \frac{d_{\mathcal{K}\mathcal{L}}}{\frac{d_{\mathcal{K}\sigma}}{k_\mathcal{K}} + \frac{d_{\mathcal{L}\sigma}}{k_\mathcal{L}}} \frac{u(x_\mathcal{L}) - u(x_\mathcal{K})}{d_{\mathcal{K}\mathcal{L}}} - \frac{1}{|\sigma|}\int_\sigma k(x)\nabla u(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx.$$

"Continuity" of the total flux

$$\int_\sigma k(x)\nabla u(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx \stackrel{\text{def}}{=} k_\mathcal{K}\int_\sigma \nabla u_{|\mathcal{K}}(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx = k_\mathcal{L}\int_\sigma \nabla u_{|\mathcal{L}}(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx.$$

Taylor formulas around $x\in\sigma$

$$u(x_\mathcal{L}) - u(x_\mathcal{K}) = \frac{d_{\mathcal{L}\sigma}}{k_\mathcal{L}}\Big(k_\mathcal{L}\nabla u_{|\mathcal{L}}(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\Big) + \frac{d_{\mathcal{K}\sigma}}{k_\mathcal{K}}\Big(k_\mathcal{K}\nabla u_{|\mathcal{K}}(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\Big) + O(\text{size}(\mathcal{T})^2).$$

We integrate on σ

$$\frac{d_{\mathcal{K}\mathcal{L}}}{\frac{d_{\mathcal{L}\sigma}}{k_\mathcal{L}} + \frac{d_{\mathcal{K}\sigma}}{k_\mathcal{K}}} \frac{u(x_\mathcal{L}) - u(x_\mathcal{K})}{d_{\mathcal{K}\mathcal{L}}} = \frac{1}{|\sigma|}\int_\sigma k(x)\nabla u(x)\cdot \boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}\, dx + O(\text{size}(\mathcal{T})).$$

Back

(Andreianov–Gutnic–Wittbold, '04)

- Regularity of the mesh

$$|\mathcal{D}| = \frac{1}{2}(\sin \alpha_{\mathcal{D}})|\sigma| d_{\mathcal{K}\mathcal{L}} \Rightarrow |\sigma| d_{\mathcal{K}\mathcal{L}} \leq C(\mathrm{reg}(\mathcal{T}))|\mathcal{D}|, \quad \forall \mathcal{D} \in \mathfrak{D}.$$

- We use again the notation $\chi_\sigma(x, y)$, a direction ξ and $y(x)$ the *projection* of $x \in \Omega$ onto $\partial\Omega$ along the direction ξ.
- Telescoping sum for $|u|^2$

$$|u_{\mathcal{K}}|^2 = |u_{\mathcal{K}_1}|^2 = \sum_{i=1}^{m-1} (|u_{\mathcal{K}_i}|^2 - |u_{\mathcal{K}_{i+1}}|^2) + |u_{\mathcal{K}_m}|^2,$$

(Andreianov–Gutnic–Wittbold, '04)

- Regularity of the mesh

$$|\mathcal{D}| = \frac{1}{2}(\sin \alpha_{\mathcal{D}})|\sigma| d_{\mathcal{K}\mathcal{L}} \Rightarrow |\sigma| d_{\mathcal{K}\mathcal{L}} \leq C(\mathrm{reg}(\mathcal{T}))|\mathcal{D}|, \quad \forall \mathcal{D} \in \mathfrak{D}.$$

- We use again the notation $\chi_\sigma(x, y)$, a direction ξ and $y(x)$ the *projection* of $x \in \Omega$ onto $\partial\Omega$ along the direction ξ.
- Telescoping sum for $|u|^2$

$$|u_{\mathcal{K}}|^2 \leq \left(\sum_{i=1}^{m-1} |u_{\mathcal{K}_i} - u_{\mathcal{K}_{i+1}}|(|u_{\mathcal{K}_i}| + |u_{\mathcal{K}_{i+1}}|) \right) + |u_{\mathcal{K}_m}|^2.$$

(Andreianov–Gutnic–Wittbold, '04)

- Regularity of the mesh

$$|\mathcal{D}| = \frac{1}{2}(\sin\alpha_{\mathcal{D}})|\sigma|d_{\mathcal{K}\mathcal{L}} \Rightarrow |\sigma|d_{\mathcal{K}\mathcal{L}} \leq C(\mathrm{reg}(\mathcal{T}))|\mathcal{D}|, \quad \forall \mathcal{D} \in \mathfrak{D}.$$

- We use again the notation $\chi_\sigma(x,y)$, a direction ξ and $y(x)$ the *projection* of $x \in \Omega$ onto $\partial\Omega$ along the direction ξ.
- Telescoping sum for $|u|^2$

$$\sum_{\mathcal{K}\in\mathfrak{M}} |\mathcal{K}||u_{\mathcal{K}}|^2 \leq \sum_{\sigma=\mathcal{K}|\mathcal{L}\in\mathcal{E}} |u_{\mathcal{K}} - u_{\mathcal{L}}|(|u_{\mathcal{K}}| + |u_{\mathcal{L}}|) \underbrace{\left(\int_\Omega \chi_\sigma(x, y(x))\, dx\right)}_{\leq|\sigma|}.$$

(Andreianov–Gutnic–Wittbold, '04)

- Regularity of the mesh
$$|\mathcal{D}| = \frac{1}{2}(\sin\alpha_{\mathcal{D}})|\sigma| d_{\mathcal{K}\mathcal{L}} \Rightarrow |\sigma| d_{\mathcal{K}\mathcal{L}} \leq C(\mathrm{reg}(\mathcal{T}))|\mathcal{D}|, \quad \forall \mathcal{D} \in \mathfrak{D}.$$
- We use again the notation $\chi_\sigma(x,y)$, a direction ξ and $y(x)$ the *projection* of $x \in \Omega$ onto $\partial\Omega$ along the direction ξ.
- Telescoping sum for $|u|^2$
$$\sum_{\mathcal{K}\in\mathfrak{M}} |\mathcal{K}||u_{\mathcal{K}}|^2 \leq \sum_{\sigma\in\mathcal{E}} |\sigma| d_{\mathcal{K}\mathcal{L}} \left| \frac{u_{\mathcal{K}} - u_{\mathcal{L}}}{d_{\mathcal{K}\mathcal{L}}} \right| (|u_{\mathcal{K}}| + |u_{\mathcal{L}}|).$$

(Andreianov–Gutnic–Wittbold, '04)

- Regularity of the mesh

$$|\mathcal{D}| = \frac{1}{2}(\sin\alpha_{\mathcal{D}})|\sigma|d_{\mathcal{KL}} \Rightarrow |\sigma|d_{\mathcal{KL}} \leq C(\mathrm{reg}(\mathcal{T}))|\mathcal{D}|, \quad \forall \mathcal{D} \in \mathfrak{D}.$$

- We use again the notation $\chi_\sigma(x,y)$, a direction ξ and $y(x)$ the *projection* of $x \in \Omega$ onto $\partial\Omega$ along the direction ξ.
- Telescoping sum for $|u|^2$

$$\|u^{\mathfrak{m}}\|_{L^2}^2 \leq \left(\sum_{\sigma\in\mathcal{E}} |\sigma|d_{\mathcal{KL}}\left|\frac{u_{\mathcal{K}}-u_{\mathcal{L}}}{d_{\mathcal{KL}}}\right|^2\right)^{\frac{1}{2}} \left(\sum_{\sigma\in\mathcal{E}} |\sigma|d_{\mathcal{KL}}(|u_{\mathcal{K}}|^2+|u_{\mathcal{L}}|^2)\right)^{\frac{1}{2}}.$$

(Andreianov–Gutnic–Wittbold, '04)

- Regularity of the mesh

$$|\mathcal{D}| = \frac{1}{2}(\sin \alpha_{\mathcal{D}})|\sigma| d_{\mathcal{K}\mathcal{L}} \Rightarrow |\sigma| d_{\mathcal{K}\mathcal{L}} \leq C(\mathrm{reg}(\mathcal{T}))|\mathcal{D}|, \quad \forall \mathcal{D} \in \mathfrak{D}.$$

- We use again the notation $\chi_\sigma(x,y)$, a direction ξ and $y(x)$ the *projection* of $x \in \Omega$ onto $\partial\Omega$ along the direction ξ.
- Telescoping sum for $|u|^2$

$$\|u^{\mathfrak{m}}\|_{L^2}^2 \leq \left(\sum_{\sigma\in\mathcal{E}} |\sigma| d_{\mathcal{K}\mathcal{L}} \left| \frac{u_{\mathcal{K}} - u_{\mathcal{L}}}{d_{\mathcal{K}\mathcal{L}}} \right|^2 \right)^{\frac{1}{2}} \left(\sum_{\sigma\in\mathcal{E}} |\sigma| d_{\mathcal{K}\mathcal{L}} (|u_{\mathcal{K}}|^2 + |u_{\mathcal{L}}|^2) \right)^{\frac{1}{2}}.$$

- To conclude, we need to prove that

$$\sum_{\sigma\in\mathcal{E}} |\sigma| d_{\mathcal{K}\mathcal{L}} (|u_{\mathcal{K}}|^2 + |u_{\mathcal{L}}|^2) \leq C\|u^{\mathfrak{m}}\|_{L^2}^2 + C \sum_{\sigma\in\mathcal{E}} |\sigma| d_{\mathcal{K}\mathcal{L}} \left| \frac{u_{\mathcal{K}} - u_{\mathcal{L}}}{d_{\mathcal{K}\mathcal{L}}} \right|^2.$$

WE WANT TO SHOW

$$\sum_{\sigma\in\mathcal{E}} |\sigma| d_{\mathcal{K}\mathcal{L}} (|u_{\mathcal{K}}|^2 + |u_{\mathcal{L}}|^2) \leq C \|u^{\mathfrak{M}}\|_{L^2}^2 + C \sum_{\sigma\in\mathcal{E}} |\sigma| d_{\mathcal{K}\mathcal{L}} \left| \frac{u_{\mathcal{K}} - u_{\mathcal{L}}}{d_{\mathcal{K}\mathcal{L}}} \right|^2 .$$

WE WANT TO SHOW

$$\sum_{\sigma\in\mathcal{E}} |\sigma| d_{\mathcal{K}\mathcal{L}} (|u_{\mathcal{K}}|^2 + |u_{\mathcal{L}}|^2) \leq C \|u^{\mathfrak{M}}\|_{L^2}^2 + C \sum_{\sigma\in\mathcal{E}} |\sigma| d_{\mathcal{K}\mathcal{L}} \left| \frac{u_{\mathcal{K}} - u_{\mathcal{L}}}{d_{\mathcal{K}\mathcal{L}}} \right|^2 .$$

- We first write

$$\begin{aligned}\sum_{\sigma\in\mathcal{E}} |\sigma| d_{\mathcal{K}\mathcal{L}} (|u_{\mathcal{K}}|^2 + |u_{\mathcal{L}}|^2) &= \sum_{\sigma\in\mathcal{E}} |\sigma| (d_{\mathcal{K}\sigma} + d_{\mathcal{L}\sigma})(|u_{\mathcal{K}}|^2 + |u_{\mathcal{L}}|^2) \\ &= \sum_{\sigma\in\mathcal{E}} |\sigma| \left(d_{\mathcal{K}\sigma}|u_{\mathcal{K}}|^2 + d_{\mathcal{K}\sigma}|u_{\mathcal{L}}|^2 + d_{\mathcal{L}\sigma}|u_{\mathcal{K}}|^2 + d_{\mathcal{L}\sigma}|u_{\mathcal{L}}|^2 \right) \\ &\leq C \sum_{\mathcal{K}\in\mathfrak{M}} |\mathcal{K}| |u_{\mathcal{K}}|^2 + \sum_{\sigma\in\mathcal{E}} |\sigma| (d_{\mathcal{L}\sigma}|u_{\mathcal{K}}|^2 + d_{\mathcal{K}\sigma}|u_{\mathcal{L}}|^2).\end{aligned}$$

We want to show

$$\sum_{\sigma\in\mathcal{E}} |\sigma| d_{\mathcal{K}\mathcal{L}}(|u_\mathcal{K}|^2 + |u_\mathcal{L}|^2) \leq C\|u^{\mathfrak{M}}\|_{L^2}^2 + C\sum_{\sigma\in\mathcal{E}} |\sigma| d_{\mathcal{K}\mathcal{L}} \left|\frac{u_\mathcal{K}-u_\mathcal{L}}{d_{\mathcal{K}\mathcal{L}}}\right|^2.$$

- We first write

$$\begin{aligned}\sum_{\sigma\in\mathcal{E}} |\sigma| d_{\mathcal{K}\mathcal{L}}(|u_\mathcal{K}|^2 + |u_\mathcal{L}|^2) &= \sum_{\sigma\in\mathcal{E}} |\sigma|(d_{\mathcal{K}\sigma}+d_{\mathcal{L}\sigma})(|u_\mathcal{K}|^2 + |u_\mathcal{L}|^2)\\ &= \sum_{\sigma\in\mathcal{E}} |\sigma|(d_{\mathcal{K}\sigma}|u_\mathcal{K}|^2 + d_{\mathcal{K}\sigma}|u_\mathcal{L}|^2 + d_{\mathcal{L}\sigma}|u_\mathcal{K}|^2 + d_{\mathcal{L}\sigma}|u_\mathcal{L}|^2)\\ &\leq C\sum_{\mathcal{K}\in\mathfrak{M}} |\mathcal{K}||u_\mathcal{K}|^2 + \sum_{\sigma\in\mathcal{E}} |\sigma|(d_{\mathcal{L}\sigma}|u_\mathcal{K}|^2 + d_{\mathcal{K}\sigma}|u_\mathcal{L}|^2).\end{aligned}$$

- For the blue terms, we notice now that

$$d_{\mathcal{L}\sigma}|u_\mathcal{K}|^2 \leq \begin{cases} 2^2 d_{\mathcal{L}\sigma}|u_\mathcal{L}|^2, & \text{for } |u_\mathcal{K}|\leq 2|u_\mathcal{L}|,\\ 2^2 d_{\mathcal{L}\sigma}|u_\mathcal{L}-u_\mathcal{K}|^2, & \text{for } |u_\mathcal{K}| > 2|u_\mathcal{L}|.\end{cases}$$

Back

ENERGY ESTIMATES

$$\sup_n \|u^{\mathcal{T}_n}\|_{1,\mathcal{T}_n} = \sup_n \|\nabla^{\mathcal{T}_n} u^{\mathcal{T}_n}\|_{L^2} \leq C(\Omega, f).$$

COMPACTNESS

THEOREM (WEAK-L^2 COMPACTNESS THEOREM)

There exists $u \in H_0^1(\Omega)$ such that (up to a subsequence!)

$$u^{\mathfrak{M}_n} \xrightarrow[n\to\infty]{} u \text{ in } L^2(\Omega),$$

$$u^{\mathfrak{M}_n^*} \xrightarrow[n\to\infty]{} u \text{ in } L^2(\Omega),$$

$$\nabla^{\mathcal{T}_n} u^{\mathcal{T}_n} \rightharpoonup_{n\to\infty} \nabla u \text{ in } (L^2(\Omega))^2.$$

Observe that $u^{\mathfrak{M}_n}$ and $u^{\mathfrak{M}_n^*}$ converge towards the same limit.

PASSING TO THE LIMIT IN THE SCHEME ✔

STRONG CONVERGENCE OF GRADIENTS

We pass to the limit in the formula

$$2\int_\Omega (A(x)\nabla^{\mathcal{T}_n} u^{\mathcal{T}_n}, \nabla^{\mathcal{T}_n} u^{\mathcal{T}_n})\, dx = \int_\Omega f(x) u^{\mathfrak{M}_n}\, dx + \int_\Omega f(x) u^{\mathfrak{M}_n^*}\, dx.$$

Back