Controllability of parabolic PDEs

Methods - Results - Open problems

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Ecole Polytechnique, CMAP, March 2017

GENERAL DYNAMICAL SYSTEM

$$(S) \begin{cases} y' = F(t, y, v), \\ y(0) = y_0, \end{cases}$$

$t \mapsto y(t)$ is the state (possibly infinite dimensional), $t \mapsto v(t)$ is the control.

TYPICAL CONTROLLABILITY QUESTION

For a given initial data y_0, can we find a control v such that the corresponding solution of (S) has a prescribed behavior ?

IN THIS TALK : Let $T > 0$ and y_T a fixed target.

- **Exact controllability** : Can I find a control v such that $y(T) = y_T$?
- **Approximate controllability** : Can I find a control v such that $\|y(T) - y_T\|$ is as small as desired ?

- Only linear problems.

$$(S) \qquad y'(t) + Ay(t) = Bv(t),$$

where $A \in M_n(\mathbb{R})$, $B \in M_{n,m}(\mathbb{R})$, $y(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^m$.

THEOREM (KALMAN CRITERION)

Let $T > 0$. The following propositions are equivalent.

(1) *Problem (S) is exactly controllable at time T.*

(2) *Problem (S) is approximately controllable at time T.*

(3) *The matrices A and B satisfy*

$$\operatorname{rank}(K) = n, \;\; \textit{with} \;\; K = \left(B|AB|\dots|A^{n-1}B\right) \in M_{n,mn}(\mathbb{R}).$$

REMARKS

- Approximate and exact controllability are equivalent.
- The controllability of the system is **independent of** T.
- There exists a generalization of this criterion for time dependent linear ODEs.

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where $A \in M_n(\mathbb{R})$, $B \in M_{n,m}(\mathbb{R})$, $y(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^m$.

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SKETCH OF PROOF

The set of reachable states at time T from an initial data y_0 is affine

$$R_T(y_0) = \left\{y_{v,y_0}(T), \ \ v \in L^2(0,T;\mathbb{R}^m)\right\}.$$

$$(1) \Leftrightarrow R_T(y_0) = \mathbb{R}^n \Leftrightarrow R_T(y_0) \text{ is dense in } \mathbb{R}^n \Leftrightarrow (2).$$

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SKETCH OF PROOF

$(1) \Rightarrow (3)$: Assume that $\text{rank}(K) < n$, there exists $\psi \in \mathbb{R}^n \setminus \{0\}$ such that ${}^t\psi K = 0$.

$$\Longrightarrow \left({}^t\psi P(A)B = 0, \ \ \forall P \in \mathbb{R}[X]\right) \Longrightarrow \left(\forall s \in \mathbb{R}, \ {}^t\psi e^{sA}B = 0\right).$$

Thus, for any control v, we have

$$\frac{d}{dt}({}^t\psi e^{tA}y(t)) = {}^t\psi e^{tA}Bv(t) = 0.$$

It follows that $\psi \perp \left(e^{TA}R_T(y_0) - y_0\right)$.

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where $A \in M_n(\mathbb{R})$, $B \in M_{n,m}(\mathbb{R})$, $y(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^m$.

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SKETCH OF PROOF

$(3) \Rightarrow (1)$: Assume that (S) is not controllable at time T.
Thus, there exists $\psi \neq 0$ such that $\psi \perp \left(R_T(y_0) - e^{-TA}y_0\right)$. By the Duhamel formula,

$$0 = {}^t\psi \int_0^T e^{-(T-t)A} Bv(t)\,dt, \quad \forall v : [0,T] \to \mathbb{R}^m.$$

We take $v(t) = B^* e^{-(T-t)A^*}\psi$ to obtain $0 = \int_0^T \left\| {}^t\psi e^{-(T-t)A} B \right\|_2^2 dt.$

It follows that ${}^t\psi e^{sA} B = 0, \ \ \forall s \in \mathbb{R}$, and then ${}^t\psi K = 0$ which gives $\text{rank}(K) < n$.

$$(S) \qquad y'(t) + Ay(t) = Bv(t),$$

where $A \in M_n(\mathbb{R})$, $B \in M_{n,m}(\mathbb{R})$, $y(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^m$.

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ALTERNATIVE (CONSTRUCTIVE) PROOF IN THE CASE $m = 1$ AND $y_T = 0$

Cascade structure : Use the change of variable $y = Kz$ (K is invertible !)

$$z'(t) + \begin{pmatrix} 0 & \cdots & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & \vdots & a_1 \\ 0 & 1 & \ddots & \vdots & a_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & a_n \end{pmatrix} z(t) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} v(t) \implies \begin{cases} \text{Choose a suitable } z_n \text{ (flat at } T\text{)} \\ \text{then compute } z_{n-1}, \ldots, z_1 \text{ and finally } v \end{cases}$$

Outline

(Jackson-Byrne, 2000)

CANCER THERAPY MODEL

$$\begin{cases} \partial_t d - \text{div}(D_d(x)\nabla d) & + \lambda d & = b(x)v(t,x), & \text{drugs} \\ \partial_t p - \text{div}(D_p(x)\nabla p) & - F_p(p) & = -C_p(d,p), & \text{sensitive cells} \\ \partial_t q - \text{div}(D_q(x)\nabla q) & - F_q(q) & = -C_q(d,q), & \text{unsensitive cells} \end{cases}$$

MAIN FEATURES :

- Reaction-diffusion system
- Possibly different diffusions
- Only one control acting on the drug concentration !
- Importance of coupling / interaction terms. No direct coupling between p and q.
- A reasonable control should be non-negative and bounded.

THIS PROBLEM IS MUCH TOO COMPLEX UP TO NOW

- Linearisation ?
- Same diffusions ?
- 1D case ?
- Relax constraints on the control ?

- Two Hilbert spaces : the state space $(E, \langle .,. \rangle)$ and the control space $(U, [.,.])$.
- $\mathcal{A} : D(\mathcal{A}) \subset E \mapsto E$ is some *elliptic* operator.
- $\mathcal{B} : U \mapsto D(\mathcal{A}^\star)'$ the control operator, $\mathcal{B}^\star$ its adjoint.
- COMPATIBILITY ASSUMPTION : we assume that

$$\left(t \mapsto \mathcal{B}^\star e^{-t\mathcal{A}^\star}\psi\right) \in L^2(0,T;U), \text{ and } \left[\!\left[\mathcal{B}^\star e^{-\cdot\mathcal{A}^\star}\psi\right]\!\right]_{L^2(0,T;U)} \leq C\|\psi\|, \ \forall \psi \in E.$$

Our controlled parabolic problem is

$$(S) \quad \begin{cases} \partial_t y + \mathcal{A}y = \mathcal{B}v & \text{in }]0,T[, \\ y(0) = y_0, \end{cases}$$

Here, $y_0 \in E$ is the initial data, $v \in L^2(]0,T[,U)$ is the control we are looking for.

THEOREM (WELL-POSEDNESS OF (S) IN A DUAL SENSE)

For any $y_0 \in E$ and $v \in L^2(0,T;U)$, there exists a unique $y = y_{v,y_0} \in \mathcal{C}^0([0,T],E)$ such that

$$\langle y(t), \psi \rangle - \left\langle y_0, e^{-t\mathcal{A}^\star}\psi \right\rangle = \int_0^t \left[v(s), \mathcal{B}^\star e^{-(t-s)\mathcal{A}^\star}\psi\right] ds, \ \ \forall t \in [0,T], \forall \psi \in E.$$

Distributed control for scalar equation

$\mathcal{B}^\star = \mathbf{1}_\omega$

$$\begin{cases} \partial_t y - \Delta y = \mathbf{1}_\omega v, & \text{in } \Omega \\ y = 0, & \text{on } \partial\Omega. \end{cases}$$

with $\omega \subset \Omega$ strict subset (the case $\omega = \Omega$ is straightforward).

Dirichlet boundary control for scalar equation

$\mathcal{B}^\star = \mathbf{1}_{\Gamma_0} \partial_n$

$$\begin{cases} \partial_t y - \Delta y = 0, & \text{in } \Omega \\ y = \mathbf{1}_{\Gamma_0} v, & \text{on } \partial\Omega. \end{cases}$$

where $\Gamma_0 \subset \partial\Omega$ is a subset of the boundary.

DISTRIBUTED CONTROL FOR SCALAR EQUATION $\mathcal{B}^\star = \mathbf{1}_\omega$

$$\begin{cases} \partial_t y - \Delta y = \mathbf{1}_\omega v, & \text{in } \Omega \\ y = 0, & \text{on } \partial\Omega. \end{cases}$$

with $\omega \subset \Omega$ strict subset (the case $\omega = \Omega$ is straightforward).

DIRICHLET BOUNDARY CONTROL FOR SCALAR EQUATION $\mathcal{B}^\star = \mathbf{1}_{\Gamma_0}\partial_n$

$$\begin{cases} \partial_t y - \Delta y = 0, & \text{in } \Omega \\ y = \mathbf{1}_{\Gamma_0} v, & \text{on } \partial\Omega. \end{cases}$$

where $\Gamma_0 \subset \partial\Omega$ is a subset of the boundary.

COUPLED PARABOLIC SYSTEM WITH FEW DISTRIBUTED CONTROLS $\mathcal{B}^\star = \mathbf{1}_\omega B^*$

$y(t,x) \in \mathbb{R}^n, A(t,x) \in M_n(\mathbb{R}), B \in M_{n,m}(\mathbb{R})$, with $m < n$

$$\begin{cases} \partial_t y - \Delta y + A(t,x)y = \mathbf{1}_\omega B v, & \text{in } \Omega \\ y = 0, & \text{on } \partial\Omega. \end{cases}$$

COUPLED PARABOLIC SYSTEM WITH FEW BOUNDARY CONTROLS $\mathcal{B}^\star = \mathbf{1}_{\Gamma_0} B^* \partial_n$

$y(t,x) \in \mathbb{R}^n, A(t,x) \in M_n(\mathbb{R}), B \in M_{n,m}(\mathbb{R})$, with $m < n$

$$\begin{cases} \partial_t y - \Delta y + A(t,x)y = 0, & \text{in } \Omega \\ y = \mathbf{1}_{\Gamma_0} B v, & \text{on } \partial\Omega. \end{cases}$$

$$(S)\begin{cases}\partial_t y + \mathcal{A}y = \mathcal{B}v & \text{in }]0,T[,\\ y(0) = y_0.\end{cases}$$

FUNDAMENTAL REMARK FOR PARABOLIC PDES

Due to regularisation effects, not all targets can be reached !

APPROXIMATE (NULL-)CONTROL PROBLEM AT TIME T FROM y_0

For any $\delta > 0$, is there a $v_\delta \in L^2(]0,T[,U)$ such that $\|y_{v_\delta,y_0}(T)\| \leq \delta$?

NULL-CONTROL PROBLEM AT TIME T FROM y_0

Is there a $v \in L^2(]0,T[,U)$ such that $y_{v,y_0}(T) = 0$?

This is equivalent to the control to the trajectories.

(Fattorini-Russell, ’71) (Lebeau-Robbiano, ’95)

(Fursikov-Imanuvilov, ’96) (Alessandrini-Escauriaza, ’08)

(Ammar-Khodja, Benabdallah, González-Burgos, de Teresa, ’11)

$$(S)\begin{cases}\partial_t y + \mathcal{A}y = \mathcal{B}v & \text{in }]0,T[,\\ y(0)=y_0.\end{cases} \qquad (S^*)\begin{cases}-\partial_t q + \mathcal{A}^\star q = 0 & \text{in }]0,T[,\\ q(T)=q_F.\end{cases}$$

THEOREM (APPROXIMATE CONTROLLABILITY AND UNIQUE CONTINUATION)

Let $T>0$ given.

$$\textit{(S) is AC from any initial data} \iff \begin{cases}\textit{Any solution } q \textit{ of } (S^*)\\ \textit{such that } \mathcal{B}^*q(t)=0, \forall t\in[0,T]\\ \textit{satisfies } q\equiv 0.\end{cases}$$

REMARK : This property is, in general, **independent** of T.

SKETCH OF PROOF : Let $P : v \in L^2(0,T;U) \mapsto y_v(T) \in E$

$$\mathrm{Im}(P) \text{ is dense } \iff \ker P^* = \{0\}.$$

$$(S)\begin{cases}\partial_t y + \mathcal{A}y = \mathcal{B}v & \text{in }]0,T[,\\ y(0) = y_0.\end{cases} \qquad (S^*)\begin{cases}-\partial_t q + \mathcal{A}^\star q = 0 & \text{in }]0,T[,\\ q(T) = q_F.\end{cases}$$

THEOREM (NULL CONTROLLABILITY AND OBSERVABILITY)

(S) is NC from any initial data $\iff \begin{cases}\textit{Any solution } q \textit{ of } (S^*)\\ \textit{satisfies } \|q(0)\|^2 \leq C_T^2 \int_0^T \|\mathcal{B}^\star q(t)\|^2\,dt.\end{cases}$

The control v with minimal L^2 norm for a data y_0 satisfies $\|v\|_{L^2(0,T;U)} \leq C_T\|y_0\|$.

REMARK : This property **may depend** of T.

SKETCH OF PROOF : Let $P : v \in L^2(0,T;U) \mapsto y_v(T) \in E$ and $Q = e^{-T\mathcal{A}}$

$$\mathrm{Im}(P) \subset \mathrm{Im}(Q) \iff \|Q^* x\| \leq C\|P^* x\|, \ \forall x.$$

(λ_k, ϕ_k) : eigenelements of $\mathcal{A} = \mathcal{A}^\star = -\partial_x(\gamma(x)\partial_x \bullet)$ with Dirichlet BC.

PROPOSITION

A function $v \in L^2(0,T;U)$ is a null-control for the problem

$$\partial_t y + \mathcal{A}y = \mathcal{B}v, \;\; y(0) = y_0,$$

if and only if

$$-\left\langle y_0, e^{-\lambda_k T}\phi_k \right\rangle = \int_0^T \left[v(t), e^{-\lambda_k (T-t)}\mathcal{B}^\star \phi_k \right] dt, \;\; \forall k \geq 1.$$

This is a **moment problem***.*

STRATEGY : If we are able to build a family of functions $\overline{q_k} \in L^2(0,T;U)$ such that

$$\int_0^T \left[\overline{q_l}(t), e^{-\lambda_k (T-t)}\mathcal{B}^\star \phi_k \right] dt = \delta_{kl}, \;\; \forall k, l \geq 1,$$

then the control problem can be **formally** solved by

$$v(t) = -\sum_{l \geq 1} \left\langle y_0, e^{-\lambda_l T}\phi_l \right\rangle \overline{q_l}(t).$$

(Fattorini-Russell, '71-'74)

THEOREM (BIORTHOGONAL FAMILIES OF EXPONENTIAL FUNCTIONS)

Let $(\sigma_k)_k$ be an increasing sequence of distinct positive numbers.
We assume that, for some $\rho > 0$ and some $\mathcal{N} : (0, +\infty) \to (0, +\infty)$, we have

$$\sum_{k \geq \mathcal{N}(\varepsilon)} \frac{1}{\sigma_k} \leq \varepsilon, \ \forall \varepsilon > 0, \ \text{and } \ \sigma_{k+1} - \sigma_k \geq \rho, \ \ \forall k \geq 1.$$

Then, for any $T > 0$, there exists a sequence of functions $(q_k)_k \subset L^2(0,T)$ such that

$$\int_0^T q_k(s) e^{-(T-s)\sigma_l} \, ds = \delta_{kl}, \ \forall k, l, \ \text{and } \ \|q_k\|_{L^2(0,T)} \leq K_{\varepsilon,T,\mathcal{N},\rho} e^{\varepsilon \sigma_k}, \ \forall \varepsilon > 0, \forall k \geq 1,$$

where $K_{\varepsilon,T,\mathcal{N},\rho}$ only depends on T, ε, ρ, $\mathcal{N}$ but not on the sequence $(\sigma_k)_k$.

REMARKS

- **Good news :** Those conditions are satisfied in 1D for heat-like equations.
- **Bad news :** There are not satisfied in higher dimension.
- The estimates are somehow uniform with respect to the sequence of eigenvalues.
- Extension possible to more general sets of functions $s \mapsto s^j e^{-(T-s)\sigma_l}$.

WHAT WE WANT

$$\int_0^T \left[\overline{q_l}(t), e^{-\lambda_k(T-t)}\mathcal{B}^\star\phi_k\right] dt = \delta_{kl}, \quad \forall k,l \geq 1,$$

$$\text{and set } \; v = -\sum_{l\geq 1} \left\langle y_0, e^{-\lambda_l T}\phi_l \right\rangle \overline{q_l}.$$

CONSTRUCTION

We define

$$\overline{q_k}(t) = q_k(t)\frac{\mathcal{B}^\star\phi_k}{[\![\mathcal{B}^\star\phi_k]\!]^2} \in U,$$

so that $[\![\overline{q_k}]\!]_{L^2(0,T;U)} = \|q_k\|_{L^2(0,T)}\,[\![\mathcal{B}^\star\phi_k]\!]^{-1}$.

- **Distributed control :** $U = L^2(\Omega)$, $\mathcal{B}^\star\phi_k = 1_\omega\phi_k$.
  It can be proved that, for some C_ω we have
  $$[\![\mathcal{B}^\star\phi_k]\!] = \|\phi_k\|_{L^2(\omega)} \geq C_\omega, \quad \forall k \geq 1.$$
- **Boundary control at $x = 1$:** $U = \mathbb{R}$, $\mathcal{B}^\star\phi_k = \gamma(1)\phi_k'(1)$.
  It can be proved that
  $$[\![\mathcal{B}^\star\phi_k]\!] = |\gamma(1)\phi_k'(1)| \geq C\sqrt{\lambda_k}.$$

CONCLUSION : We have $\left[\!\!\left[\left\langle y_0, e^{-\lambda_l T}\phi_l\right\rangle \overline{q_l}\right]\!\!\right]_{L^2(0,T;U)} \leq C_\varepsilon e^{-\lambda_l T}e^{\varepsilon\lambda_l}$, taking $\varepsilon = T/2$ shows that the series that defines v converges !

EXPECTED RESULTS

- Null(?)-controllability results for semi-discrete parabolic equations

$$\begin{cases} \partial_t y^h + \mathcal{A}_h y^h = \mathcal{B}_h v_h \\ y^h(0) = y^{0,h} \in E_h, \end{cases}$$

where $\mathcal{A}_h$ is the finite difference discretisation operator

$$(\mathcal{A}_h y)_i = -\frac{1}{h}\left(\gamma(x_{i+1/2})\frac{y_{i+1}-y_i}{h} - \gamma(x_{i-1/2})\frac{y_i - y_{i-1}}{h}\right),$$

and the discrete control operator is given by

$$\mathcal{B}_h = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\gamma_{N+1/2}}{h_N h_{N+1/2}} \end{pmatrix}, \qquad \text{or} \qquad \mathcal{B}_h = 1_\omega.$$

THEOREM

For any $p > 0$ there exists $C > 0$, $h_0 > 0$ such that for any $h < h_0$, any $y^{0,h}$, there exists a $v_h \in L^2(0, T, U_h)$ such that

$$\|v_h\|_{L^2(0,T;U_h)} \leq C\|y^{0,h}\|_h, \ \ \textit{and} \ \ \|y^h(T)\|_h \leq C\|y^{0,h}\|_h h^p.$$

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TOOLS : BIOTHO. FAMILIES + DISCRETE SPECTRAL PROPERTIES FOR $\mathcal{A}_h$

(Allonsius-B.-Morancey, '16)

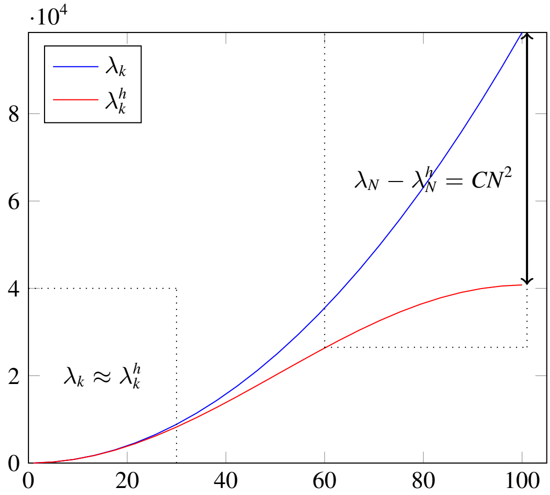
- Uniform growth rate for eigenvalues $\lambda_k^h \geq Ck^2, \ \forall k, \forall h.$
- Uniform spectral gap $\lambda_{k+1}^h - \lambda_k^h \geq \rho, \ \forall h, \ \forall k \leq \frac{C}{h}.$
- Uniform lower bounds for discrete eigenfunctions

$$\|\phi_k^h\|_{L^2(\omega)} \geq C, \ \forall h, \forall k \leq \frac{C}{h} \text{ for distributed control,}$$

$$|\partial_r \phi_k^h| \geq C, \ \forall h, \forall k \leq \frac{C}{h} \text{ for boundary control.}$$

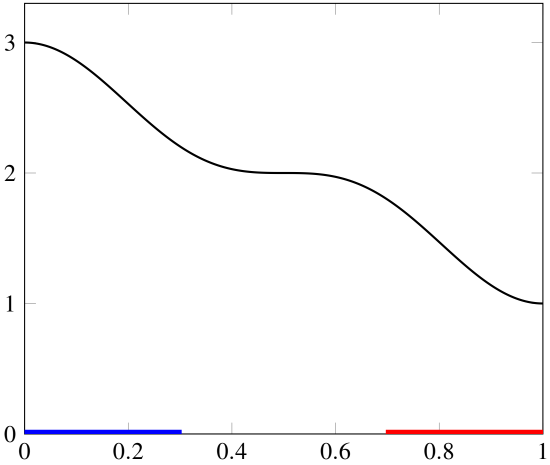
REMARKS

- Those properties are straightforward for the Laplace operator on uniform grids.
- Our results are uniform for a constant portion of the spectrum.

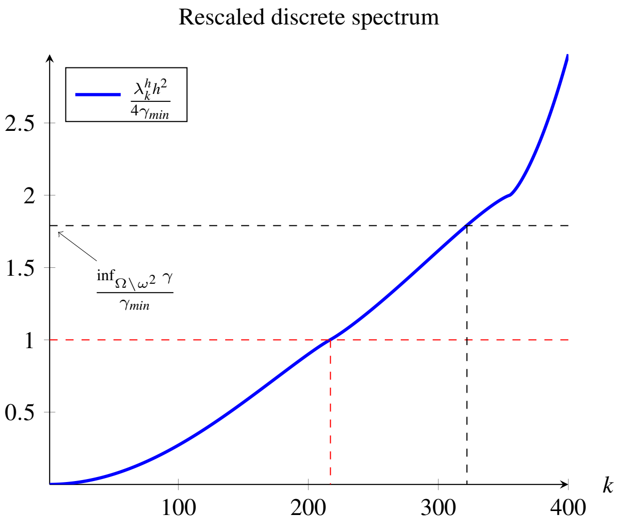


TYPICAL ERROR ESTIMATE (USELESS FOR LARGE k) : $|\lambda_k^h - \lambda_k| \approx Ch^2\lambda_k^2$.
HOWEVER UNIFORM DISCRETE GAP HOLDS : $\inf_{k \leq N} |\lambda_{k+1}^h - \lambda_k^h| \approx C$

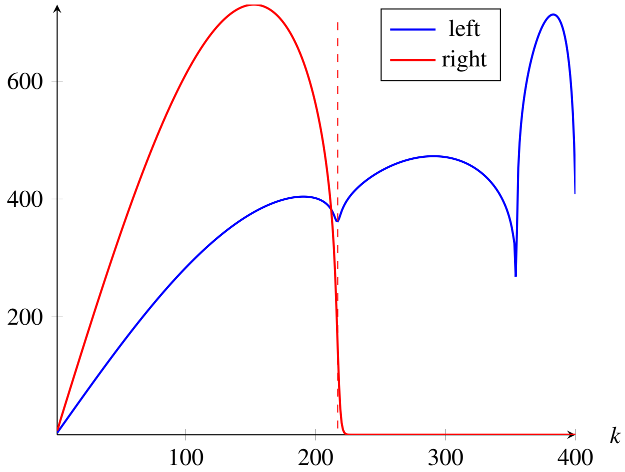
Diffusion coefficient $\gamma(x) = 2 + \cos(\pi x)^3$



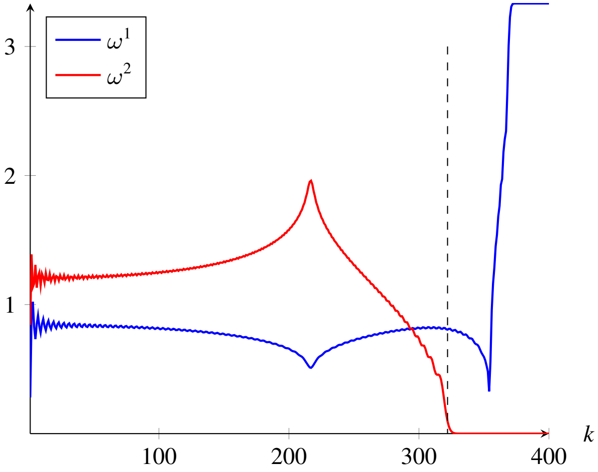
TWO OBSERVATION DOMAINS : $\omega^1 = (0, 0.3)$, $\omega^2 = (0.7, 1)$.

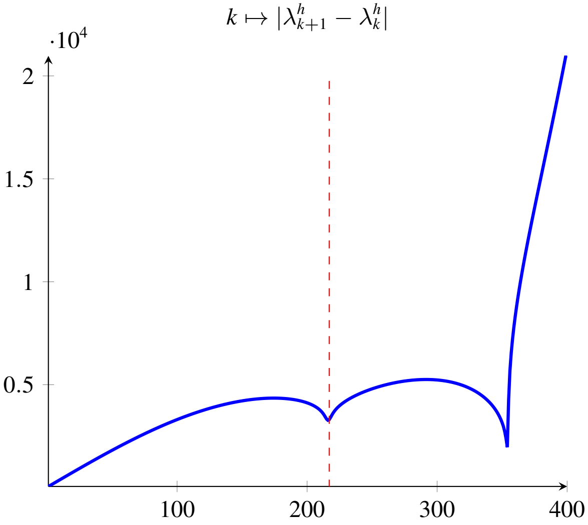

Rescaled discrete spectrum
$\frac{\lambda_k^h h^2}{4\gamma_{min}}$
$\frac{\inf_{\Omega\setminus\omega^2}\gamma}{\gamma_{min}}$
2.5
2
1.5
1
0.5
100
200
300
400
k

Normal derivatives of ϕ_k^h

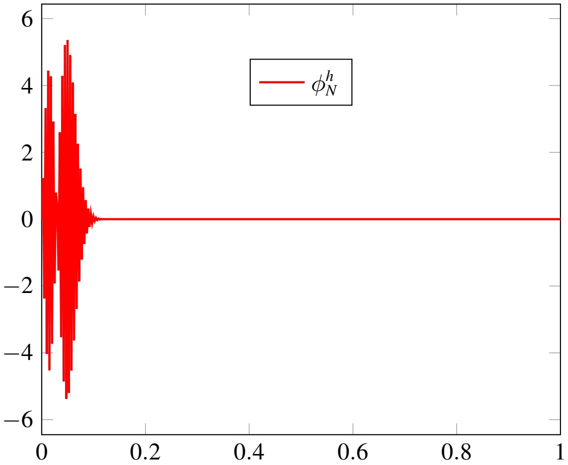


$$k \mapsto \|\phi_k^h\|^2_{L^2(\omega^i)} / |\omega^i|$$



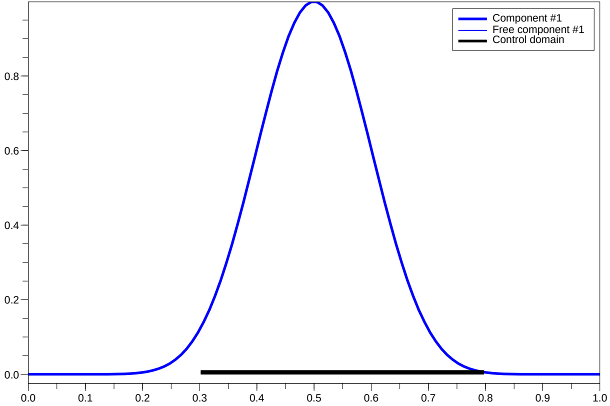

$k \mapsto |\lambda_{k+1}^h - \lambda_k^h|$
$\cdot 10^4$
2
1.5
1
0.5
100
200
300
400

The last discrete eigenfunction



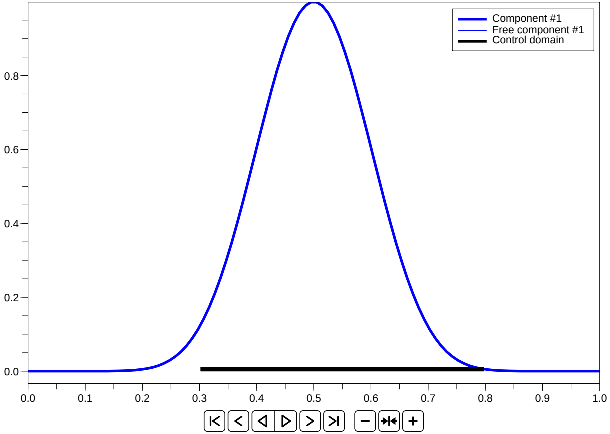
$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

Time t=0

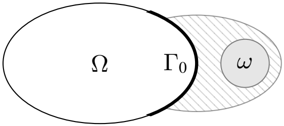


$$\partial_t y - 0.1\partial_x^2 y - 1.5y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

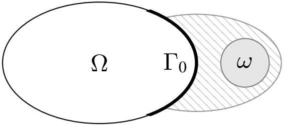
Time t=0



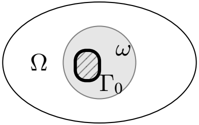
DISTRIBUTED CONTROLLABILITY $\Rightarrow$ BOUNDARY CONTROLLABILITY



DISTRIBUTED CONTROLLABILITY ⇒ BOUNDARY CONTROLLABILITY



BOUNDARY CONTROLLABILITY ⇒ DISTRIBUTED CONTROLLABILITY



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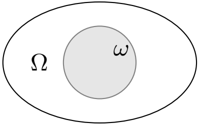

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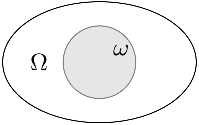

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(1) SPECTRAL INEQUALITY (a.k.a. Lebeau-Robbiano inequality)

$$\|\psi\|_{L^2(\Omega)} \leq Ce^{C\sqrt{\mu}} \|\psi\|_{L^2(\omega)}, \ \ \forall \psi \in E_\mu.$$

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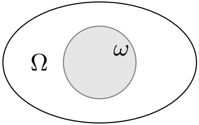

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$$\|e^{-\tau\mathcal{A}} q^F\|^2_{L^2(\Omega)} \leq C \frac{e^{C\sqrt{\mu}}}{\tau} \int_0^\tau \|e^{-s\mathcal{A}} q^F\|^2_{L^2(\omega)}\, ds, \quad \forall q^F \in E_\mu.$$

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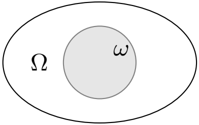

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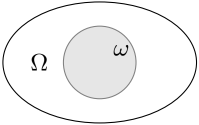

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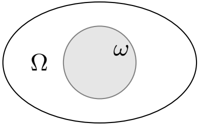

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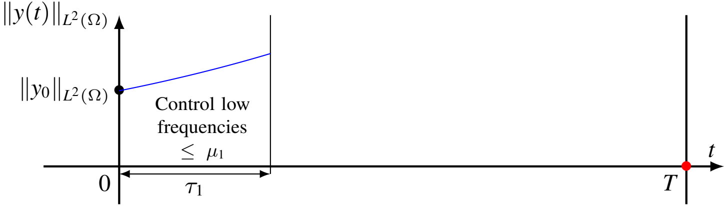
(4) CONSTRUCTION OF THE CONTROL : Time slicing procedure.



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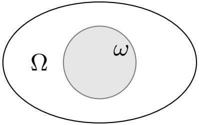

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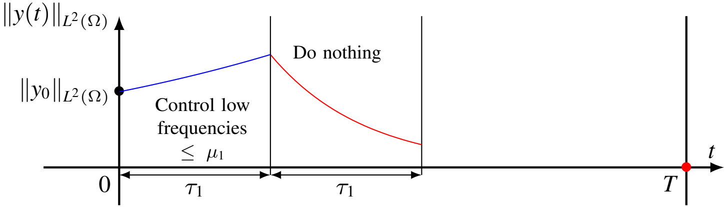

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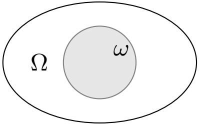

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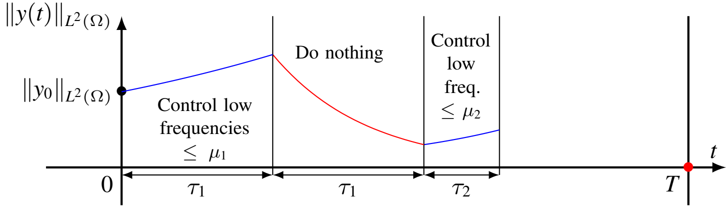

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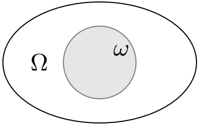

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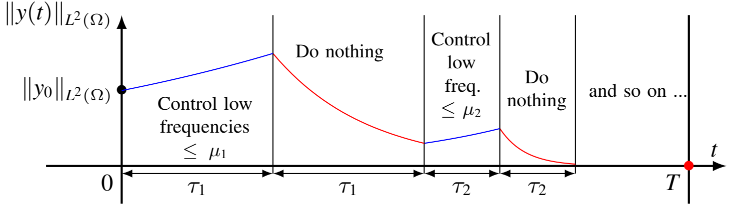

At the end, the control v is shown to satisfy

$$\|v\|_{L^2(]0,T[\times\omega)} \leq C\|y_0\|_{L^2(\Omega)}.$$

CONSEQUENCE

By duality, we obtain the uniform observability inequality for the adjoint system.

THEOREM (LEBEAU-ROBBIANO SPECTRAL INEQUALITY)

There exists $C > 0$ such that

$$\int_\Omega \Big| \sum_{\lambda_k \leq \mu} a_k \phi_k \Big|^2 \leq C e^{C\sqrt{\mu}} \int_\omega \Big| \sum_{\lambda_k \leq \mu} a_k \phi_k \Big|^2, \quad \forall (a_k)_k \in \mathbb{R}^{\mathbb{N}}.$$

AN ELLIPTIC GLOBAL CARLEMAN ESTIMATE IN $]0,T^*[\times\Omega$

For a suitable weight function $(t,x) \mapsto \varphi(t,x)$ (s.t. in particular $\nabla_x \varphi(T^*) = 0$)

$$\begin{aligned} s^3 \|e^{s\varphi} u\|^2_{L^2(\Omega_{T^*})} + s\|e^{s\varphi}\nabla u\|^2_{L^2(\Omega_{T^*})} + s\|e^{s\varphi(0,.)}\partial_t u(0,.)\|^2_{L^2(\Omega)} \\ + s e^{2s\varphi(T^*)}\|\partial_t u(T^*,.)\|^2_{L^2(\Omega)} + s^3 e^{2s\varphi(T^*)}\|u(T^*,.)\|^2_{L^2(\Omega)} \\ \leq C\left(\|e^{s\varphi}(\partial_t^2 + \Delta)u\|^2_{L^2(\Omega_{T^*})} + s e^{2s\varphi(T^*)}\|\nabla_x u(T^*,.)\|^2_{L^2(\Omega)} + s\|e^{s\varphi(0,.)}\partial_t u(0,.)\|^2_{L^2(\omega)} \right), \end{aligned}$$

for any $s \geq s_0$, and all smooth u, with $u(0,.) = 0$, and $u = 0$ on $\partial\Omega$.

STANDARD NOTATION

$$\Omega_T =]0,T[\times\Omega,$$
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AN ELLIPTIC GLOBAL CARLEMAN ESTIMATE IN $]0, T^*[\times\Omega$

For all u such that $u(0,.) = 0$, $u = 0$ on $\partial\Omega$ and $(\partial_t^2 + \Delta)u = 0$, we have

$$s^3 e^{2s\varphi(T^*)} \|u(T^*,.)\|^2_{L^2(\Omega)} \leq C \left(s e^{2s\varphi(T^*)} \|\nabla_x u(T^*,.)\|^2_{L^2(\Omega)} + s \|e^{s\varphi(0,.)} \partial_t u(0,.)\|^2_{L^2(\omega)} \right).$$

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APPLY THIS ESTIMATE TO THE FOLLOWING FUNCTION

$$u(t,x) = \sum_{\lambda_k \leq \mu} a_k \frac{\sinh(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}} \phi_k(x).$$

$$\left.\begin{aligned} \|\nabla_x u(T^*,.)\|^2_{L^2(\Omega)} &= \sum_{\lambda_k \leq \mu} |a_k|^2 |\sinh(\sqrt{\lambda_k} T^*)|^2 \\ &\leq \mu \sum_{\lambda_k \leq \mu} |a_k|^2 \left| \frac{\sinh(\sqrt{\lambda_k} T^*)}{\sqrt{\lambda_k}} \right|^2 \leq \mu \|u(T^*,.)\|^2_{L^2(\Omega)}. \end{aligned}\right\} \Rightarrow \text{Take } s \sim \sqrt{\mu}$$

THEOREM (LEBEAU-ROBBIANO SPECTRAL INEQUALITY)

There exists $C > 0$ such that

$$\int_\Omega \left| \sum_{\lambda_k \leq \mu} a_k \phi_k \right|^2 \leq C e^{C\sqrt{\mu}} \int_\omega \left| \sum_{\lambda_k \leq \mu} a_k \phi_k \right|^2, \quad \forall (a_k)_k \in \mathbb{R}^{\mathbb{N}}.$$

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$$u(t,x) = \sum_{\lambda_k \leq \mu} a_k \frac{\sinh(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}} \phi_k(x).$$

Carleman estimate $\implies \mu e^{2\sqrt{\mu}\varphi(T^*)} |u(T^*,.)|^2_{L^2(\Omega)} \leq C e^{2\sqrt{\mu} \max \varphi(0,.)} |\partial_t u(0,.)|^2_{L^2(\omega)}$.

STRAIGHTFORWARD COMPUTATIONS

$$\|u(T^*,.)\|^2_{L^2(\Omega)} = \sum_{\lambda_k \leq \mu} |a_k|^2 \left| \frac{\sinh(\sqrt{\lambda_k} T^*)}{\sqrt{\lambda_k}} \right|^2 \geq C \sum_{\lambda_k \leq \mu} |a_k|^2 = C \int_\Omega \left| \sum_{\lambda_k \leq \mu} a_k \phi_k \right|^2,$$

$$\|\partial_t u(0,.)\|^2_{L^2(\omega)} = \int_\omega \left| \sum_{\lambda_k \leq \mu} a_k \phi_k \right|^2.$$

A DISCRETE LEBEAU-ROBBIANO INEQUALITY

Let $\mathcal{A}_h$ be a multi-D finite difference discretization of $\mathcal{A}$ and $(\phi_k^h, \lambda_k^h)_{1\leq k\leq N_h}$ be the eigenelements of $\mathcal{A}_h$.
For any $\mu > 0$, we set $E_\mu^h = \mathrm{Span}(\phi_k^h,\ \lambda_k^h \leq \mu)$.

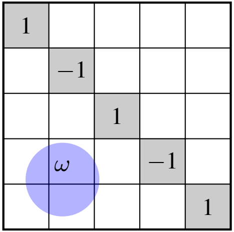
QUESTION

Is it true that

$$\|\psi^h\|_{L^2(\Omega)} \leq Ce^{C\sqrt{\mu}}\|\psi^h\|_{L^2(\omega)}, \quad \forall \psi^h \in E_\mu^h, \qquad (\star)$$

for some C independent of h ?

- Answer 1 : **No** ... for linear algebra reasons.
- Answer 2 : for the 5-point discrete Laplace on a uniform grid **(Kavian-Zuazua)**



There exists a non trivial ϕ_k^h such that $1_\omega \phi_k^h = 0$.

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for some C independent of h ?

(B.-Hubert-Le Rousseau, '09-'11)

THEOREM

Under some standard assumptions, there exist $h_0 > 0$, $C, \tilde{C} > 0$ such that $(\star)$ holds for any $h < h_0$ and any

$$\mu < \tilde{C}/h^2.$$

THEOREM ((SAME AS BEFORE BUT MULTI-D))

For any $p > 0$ there exists $C > 0$, $h_0 > 0$ such that for any $h < h_0$, any $y^{0,h}$, there exists a $v_h \in L^2(0, T, U_h)$ such that

$$\|v_h\|_{L^2(0,T;U_h)} \leq C\|y^{0,h}\|_h, \ \ \text{and}\ \ \|y^h(T)\|_h \leq C\|y^{0,h}\|_h h^p.$$

(Fursikov-Imanuvilov, '96)

Let $\gamma(t) = \frac{1}{\sqrt{t(T-t)}}$. There is a smooth $x \mapsto \beta(x)$ so that, with $\varphi(t,x) = \gamma(t)^2\beta(x)$

THEOREM (PARABOLIC CARLEMAN ESTIMATE)

For any $d \in \mathbb{R}$, there exists $C > 0$ such that for any s large enough, and any smooth function q, such that $q = 0$ on $\partial\Omega$, we have

$$s^d\|e^{s\varphi}\gamma^d q\|^2_{L^2(\Omega_T)} + s^{d-4}\|e^{s\varphi}\gamma^{d-4}\partial_t q\|^2_{L^2(\Omega_T)} + s^{d-4}\|e^{s\varphi}\gamma^{d-4}\Delta q\|^2_{L^2(\Omega_T)}$$
$$\leq C\left(s^d\|e^{s\varphi}\gamma^d q\|^2_{L^2(\omega_T)} + s^{d-3}\|e^{s\varphi}\gamma^{d-3}(-\partial_t q + \Delta q)\|^2_{L^2(\Omega_T)}\right)$$

COROLLARY (OBSERVABILITY)

For any solution of the adjoint problem $-\partial_t q + \Delta q = 0$, $q = 0$ on $\partial\Omega$ we have

$$\|q(0)\|^2_{L^2(\Omega)} \leq C\int_{T/4}^{3T/4}\|q(t)\|^2_{L^2(\Omega)}\,dt \leq C'\int_0^T\int_\omega |q|^2\,dt,$$

by using the parabolic dissipation property and the Carleman estimate.

DISCRETE VERSIONS ...

$$y(t,x) \in \mathbb{R}^n, A(t,x) \in M_n(\mathbb{R}), B \in M_{n,m}(\mathbb{R})$$

DISTRIBUTED CONTROL

$$(S_D) \begin{cases} \partial_t y - \Delta y + A(t,x)y = \mathbf{1}_\omega B v, & \text{in } \Omega \\ y = 0, & \text{on } \partial\Omega. \end{cases}$$

BOUNDARY CONTROL

$$(S_B) \begin{cases} \partial_t y - \Delta y + A(t,x)y = 0, & \text{in } \Omega \\ y = \mathbf{1}_{\Gamma_0} B v, & \text{on } \partial\Omega. \end{cases}$$

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1. In the case $\operatorname{rank}(B) = n$ (in particular $m \geq n$) :
   - Distributed and boundary controllability are equivalent.
   - Controllability proofs works almost the same as in the scalar case (Fursikov-Imanuvilov strategy for instance).
2. In the case $\operatorname{rank}(B) < n$ (important in applications !) :
   - Distributed and boundary controllability **are not** equivalent.
   - Controllability proofs have to be adapted.
   - Many results in 1D. The multi-D case is much more difficult in particular for (S_B).

$$y(t,x) \in \mathbb{R}^n, A(t,x) \in M_n(\mathbb{R}), B \in M_{n,m}(\mathbb{R})$$

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SOME SURPRISING FEATURES I WILL DISCUSS

- The "geometry" of the control domain ω has an influence on the controllability of the system.
- It may exist a minimal time T_0 for the null-controllability
  - For $T > T_0$: the system is null-controllable.
  - For $T < T_0$: the system is not null-controllable.

Outline

(Ammar-Khodja, Benabdallah, Dupaix, González-Burgos, '09) (González-Burgos, de Teresa '10)

$$(S_D)\begin{cases}\partial_t y - \Delta y + Ay = \mathbf{1}_\omega Bv, & \text{in } \Omega\\ y = 0, & \text{on } \partial\Omega.\end{cases}$$

THEOREM

System (S_D) is null-controllable at time T if and only if $\operatorname{rank}(B|AB|\cdots|A^{n-1}B) = n$.

(Ammar-Khodja, Benabdallah, Dupaix, González-Burgos, '09) (González-Burgos, de Teresa '10)

$$(S_D)\begin{cases} \partial_t y - \Delta y + Ay = 1_\omega Bv, & \text{in } \Omega \\ y = 0, & \text{on } \partial\Omega. \end{cases}$$

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SKETCH OF PROOF : in the case $n = 2$, $m = 1$.

- Kalman rank condition $\Rightarrow$ canonical (**cascade**) form $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\text{Adjoint system} \quad \begin{cases} -\partial_t q_1 - \Delta q_1 + q_2 & = 0 \\ -\partial_t q_2 - \Delta q_2 & = 0 \end{cases}$$

- Carleman estimate for q_i, $i = 1, 2$, $d_i \geq 4$

$$s^{d_i}\|e^{s\varphi}\gamma^{d_i}q_i\|^2_{L^2(\Omega_T)} + s^{d_i-4}\|e^{s\varphi}\gamma^{d_i-4}\partial_t q_i\|^2_{L^2(\Omega_T)} + s^{d_i-4}\|e^{s\varphi}\gamma^{d_i-4}\Delta q_i\|^2_{L^2(\Omega_T)}$$
$$\leq C\left(s^{d_i}\|e^{s\varphi}\gamma^{d_i}q_i\|^2_{L^2(\omega_T)} + s^{d_i-3}\|e^{s\varphi}\gamma^{d_i-3}(-\partial_t q_i + \Delta q_i)\|^2_{L^2(\Omega_T)}\right)$$

(Ammar-Khodja, Benabdallah, Dupaix, González-Burgos, '09) (González-Burgos, de Teresa '10)

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$$\varepsilon s^{d_1} \|e^{s\varphi}\gamma^{d_1} q_1\|^2_{L^2(\Omega_T)} + s^{d_2} \|e^{s\varphi}\gamma^{d_2} q_2\|^2_{L^2(\Omega_T)} + s^{d_2-4} \|e^{s\varphi}\gamma^{d_2-4}(|\partial_t q_2| + |\Delta q_2|)\|^2_{L^2(\Omega_T)}$$
$$\leq C\left(\varepsilon s^{d_1}\|e^{s\varphi}\gamma^{d_1} q_1\|^2_{L^2(\omega_T)} + s^{d_2}\|e^{s\varphi}\gamma^{d_2} q_2\|^2_{L^2(\omega_T)} + \varepsilon s^{d_1-3}\|e^{s\varphi}\gamma^{d_1-3} q_2\|^2_{L^2(\Omega_T)}\right)$$

(Ammar-Khodja, Benabdallah, Dupaix, González-Burgos, '09) (González-Burgos, de Teresa '10)

$$(S_D)\begin{cases}\partial_t y - \Delta y + Ay = \mathbf{1}_\omega Bv, & \text{in } \Omega\\ y = 0, & \text{on } \partial\Omega.\end{cases}$$

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$$\text{Adjoint system}\quad\begin{cases}-\partial_t q_1 - \Delta q_1 + q_2 & = 0\\ -\partial_t q_2 - \Delta q_2 & = 0\end{cases}$$

- Carleman estimate for q_i, $i = 1, 2$. **We choose** $d_1 = 7$ **and** $d_2 = 4$

$$s^7\|e^{s\varphi}\gamma^7 q_1\|^2_{L^2(\Omega_T)} + s^4\|e^{s\varphi}\gamma^4 q_2\|^2_{L^2(\Omega_T)} + \|e^{s\varphi}(|\partial_t q_2| + |\Delta q_2|)\|^2_{L^2(\Omega_T)}$$
$$\leq C\left(s^7\|e^{s\varphi}\gamma^7 q_1\|^2_{L^2(\omega_T)} + s^4\|e^{s\varphi}\gamma^4 q_2\|^2_{L^2(\omega_T)}\right)$$

Eliminate the last term : $\displaystyle\int_{\omega_T} q_2^2 = \int_{\omega_T} q_2(\partial_t q_1 + \Delta q_1) \sim \int_{\omega_T} q_1(-\partial_t q_2 + \Delta q_2)$

(Fernández-Cara, González-Burgos, de Teresa, '10) (Ammar-Khodja, Benabdallah, González-Burgos, de Teresa, '11)

$$(S_B) \begin{cases} \partial_t y - \partial_x^2 y + Ay = 0, \quad \text{in }]0,\pi[\\ y(t,0) = Bv, \quad y(t,\pi) = 0. \end{cases}$$

MAIN ISSUE : Carleman-like methods are useless !
MOMENTS METHOD $\Longrightarrow$ restriction to the 1D case.

(Fernández-Cara, González-Burgos, de Teresa, '10) (Ammar-Khodja, Benabdallah, González-Burgos, de Teresa, '11)

$$(S_B)\begin{cases} \partial_t y - \partial_x^2 y + Ay = 0, \text{ in }]0,\pi[\\ y(t,0) = Bv, \quad y(t,\pi) = 0. \end{cases}$$

THEOREM

System (S_B) is null-controllable at time T if and only if

$$\mathrm{rank}(B_k|A_kB_k|\cdots|A_k^{kn-1}B_k) = kn, \quad \forall k \geq 1,$$

with

$$A_k = \begin{pmatrix} A-\lambda_1 I & & & \\ & A-\lambda_2 I & & \\ & & \ddots & \\ & & & A-\lambda_k I \end{pmatrix}, \quad \text{and } B_k = \begin{pmatrix} B \\ B \\ \vdots \\ B \end{pmatrix}.$$

- Kalman condition $\mathrm{rank}(B|AB|\ldots|A^{n-1}B) = n$ is necessary but not sufficient.
- **Example for** $n=2$ **:** Let $\mathrm{Sp}(A^*) = \{\mu_1, \mu_2\}$.

$$(\star) \text{ is null-controllable} \Leftrightarrow \begin{cases} \text{Kalman condition} \\ \lambda_k - \lambda_l \neq \mu_1 - \mu_2, \quad \forall k \neq l. \end{cases}$$

DISTRIBUTED CONTROL

$$(S_D)\begin{cases} \partial_t y - \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{d} \end{pmatrix} \partial_x^2 y + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y = \mathbf{1}_\omega \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, & \text{in }]0,\pi[\\ y(t,0) = y(t,\pi) = 0, \end{cases}$$

RESULTS FOR $d = 1$

- (S_D) is app. controllable at T for any $T > 0$.
- (S_D) is null-controllable at T for any $T > 0$.

BOUNDARY CONTROL

$$(S_B)\begin{cases} \partial_t y - \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{d} \end{pmatrix} \partial_x^2 y + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y = 0, & \text{in }]0,\pi[\\ y(t,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, \quad y(t,\pi) = 0, \end{cases}$$

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$$(S_D)\begin{cases}\partial_t y - \begin{pmatrix}1 & 0\\ 0 & \mathbf{d}\end{pmatrix}\partial_x^2 y + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} y = \mathbf{1}_\omega \begin{pmatrix}1\\ 0\end{pmatrix} v, \ \text{in }]0,\pi[\\ y(t,0) = y(t,\pi) = 0,\end{cases}$$

RESULTS FOR $d = 1$

- (S_D) is app. controllable at T for any $T > 0$.
- (S_D) is null-controllable at T for any $T > 0$.

RESULTS FOR $d \neq 1$

- (S_D) is app. controllable at T for any $T > 0$.
- (S_D) is null-controllable at T for any $T > 0$.

BOUNDARY CONTROL

$$(S_B)\begin{cases}\partial_t y - \begin{pmatrix}1 & 0\\ 0 & \mathbf{d}\end{pmatrix}\partial_x^2 y + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} y = 0, \ \text{in }]0,\pi[\\ y(t,0) = \begin{pmatrix}1\\ 0\end{pmatrix} v, \ y(t,\pi) = 0,\end{cases}$$

RESULTS FOR $d = 1$

- (S_B) is app. controllable at T for any $T > 0$.
- (S_B) is null-controllable at T for any $T > 0$.

RESULTS FOR $d \neq 1$

- (S_B) is app. controllable at T if and only if

$$\sqrt{d} \notin \mathbb{Q}.$$

- (S_B) is null-controllable at T if and only if

$$T \underset{?}{>} \begin{cases} +\infty, & \text{if } \sqrt{d} \in \mathbb{Q},\\ c_0(\Lambda), & \text{if } \sqrt{d} \notin \mathbb{Q},\end{cases}$$

$\Lambda = \{k^2, dk^2\}_k$, $c_0 =$ condensation index.

Main issue :

The biorthogonal families of $(e^{-\Lambda_p t})_p$ satisfy $\|q_p\|_{L^2(0,T)} \leq C_{\varepsilon,T} e^{(\varepsilon + c_0(\Lambda)) \operatorname{Re}(\Lambda_p)}$.

(Olive, '14)

$$(S_B)\begin{cases}\partial_t y - \Delta y + Ay = 0, & \text{in } \Omega \subset \mathbb{R}^d \\ y = \mathbf{1}_{\Gamma_0} Bv, & \text{on } \partial\Omega.\end{cases}$$

THEOREM

Let $(\lambda_k)_k$ the eigenvalues of $-\Delta$ and $(\mu_i)_{1\leq i\leq n}$ the eigenvalues of A^*. Assume that

$$\lambda_k + \mu_i = \lambda_l + \mu_j \iff \begin{cases}\lambda_k = \lambda_l \\ \mu_i = \mu_j\end{cases}. \tag{C}$$

System (S_B) is approximately controllable at time $T > 0$ if and only if

$$\text{rank}(B|AB|\cdots|A^{n-1}B) = n.$$

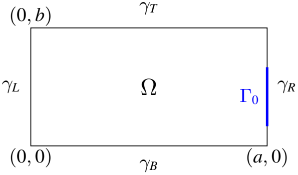
REMARK 1 : If A has only one eigenvalue (in particular in the cascade form), condition (C) holds.

REMARK 2 : In 1D : condition (C) is necessary if $m = 1$.

(Olive, '14)

$$(S_B)\begin{cases} \partial_t y - \Delta y + Ay = 0, & \text{in } \Omega \subset \mathbb{R}^d \\ y = \mathbf{1}_{\Gamma_0} Bv, & \text{on } \partial\Omega. \end{cases}$$

EXAMPLE ON A 2D RECTANGLE DOMAIN



THEOREM (CASE $\Gamma_0 \subset \gamma_R$)

The 2D system (S_B) is approximately controllable if and only if so is the 1D system

$$\begin{cases} \partial_t y - \partial_x^2 y + Ay = 0, \quad \text{in }]0,a[\\ y(t,0) = 0, \ y(t,a) = Bv. \end{cases}$$

(Olive, '14)

$$(S_B)\begin{cases}\partial_t y - \Delta y + Ay = 0, & \text{in } \Omega \subset \mathbb{R}^d\\ y = \mathbf{1}_{\Gamma_0} Bv, & \text{on } \partial\Omega.\end{cases}$$

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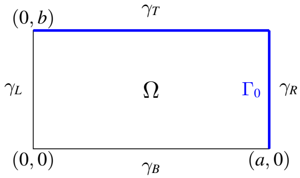


THEOREM (CASE $\Gamma_0 = \gamma_R \cup \gamma_T$)

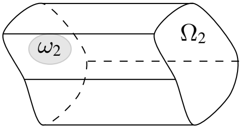
- *If $n = 2$:*
  The 2D system (S_B) is approximately controllable if and only if

$$\text{rank}(B|AB) = 2.$$

- *For $n \geq 4$:*
  There exists a system (S_B) satisfying the Kalman condition and which is **not** *approximately controllable.*

(Benabdallah, B., González-Burgos, Olive, '14)

$$(S_B)\begin{cases}\partial_t y - \Delta y + Ay = 0, & \text{in } \Omega =]0,\pi[\times\Omega_2\\ y = \mathbf{1}_{\{0\}\times\omega_2}Bv, & \text{on } \partial\Omega.\end{cases}$$



$\Omega_1 =]0,\pi[$

THEOREM

System (S_B) is null-controllable at time $T>0$ if and only if

$$\text{rank}(B_k|A_kB_k|\cdots|A_k^{nk-1}B_k) = nk, \quad \forall k\geq 1.$$

REMARKS

- Same condition as for the 1D case.
- The controllability is independent of T.

MAIN IDEAS OF THE PROOF

- Infinite dimensional variant of the Lebeau-Robbiano strategy in the variable $x_2\in\Omega_2$ to deal with the **subdomain** ω_2.
- Each stage of the LR method requires to solve a **1D boundary control problem** in the variable $x_1\in\Omega_1$ whose cost C_T needs to be estimated.

$$(S_D)\begin{cases}\partial_t y - \partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \mathbf{1}_\omega \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, & \text{in }]0,1[\\ y(t,0) = y(t,1) = 0, \end{cases}$$

- If $\mathrm{Supp}(a_{21}) \cap \omega \neq \emptyset$ then (S_D) is null-controllable at any time $T > 0$.

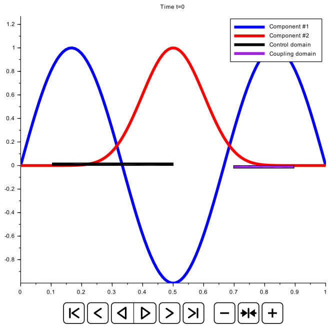
$$(S_D)\begin{cases}\partial_t y - \partial_x^2 y + \begin{pmatrix}0 & 0\\ a_{21}(x) & 0\end{pmatrix} y = \mathbf{1}_\omega \begin{pmatrix}1\\0\end{pmatrix} v, \ \text{in }]0,1[\\ y(t,0)=y(t,1)=0,\end{cases}$$

- If $\mathrm{Supp}(a_{21})\cap\omega=\emptyset$ and $a_{21}\geq 0$, $a_{21}\neq 0$ then (S_D) is null-controllable at any time $T>0$. **(Rosier, de Teresa, '11)**

$$a_{21}(x) = \mathbf{1}_{]0.7,0.9[}(x),$$

$$\omega =]0.1, 0.5[,$$

$$y_0(x) = \begin{pmatrix}\sin(3\pi x)\\ \sin(\pi x)^{10}\end{pmatrix}.$$



$$(S_D)\begin{cases} \partial_t y - \partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \mathbf{1}_\omega \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, \quad \text{in }]0,1[\\ y(t,0) = y(t,1) = 0, \end{cases}$$

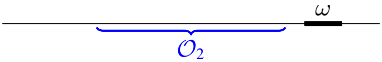
- If $\mathrm{Supp}(a_{21}) \cap \omega = \emptyset$ and a_{21} changes its sign
  - There are some cases (depending on a_{21} and ω) that are **not approximately controllable**.

    (B., Olive, '13)
  - It may exist a minimal time $T_0 > 0$ for the null-controllability.

    (Ammar-Khodja, Benabdallah, González-Burgos, de Teresa, '14)

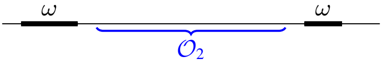
- Take $a_{21} = (x-\alpha)\mathbf{1}_{\mathcal{O}_2}$ **(B.-Olive, '13)**

  - Case 1 :



$$\boxed{(S_D) \text{ is approximately controllable} \Leftrightarrow \alpha \notin \{\alpha_k\}_k,} \text{ with } \alpha_k = \frac{\int_{\mathcal{O}_2} x\phi_k^2\,dx}{\int_{\mathcal{O}_2} \phi_k^2\,dx}.$$

  - Case 2 :



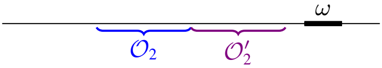
$$\boxed{(S_D) \text{ is approximately controllable} \Leftrightarrow \int_{\mathcal{O}_2} (x-\alpha)\phi_k\tilde{\phi}_k \neq 0,\ \forall k, \text{s.t. } \alpha_k = \alpha.}$$

Here $\tilde{\phi}_k$ is the other solution of $(-\partial_x^2 - \lambda_k)\tilde{\phi}_k = 0$.

- Take $a_{21} = \mathbf{1}_{\mathcal{O}_2} - \mathbf{1}_{\mathcal{O}_2'}$ **(B.-Olive, '13)**

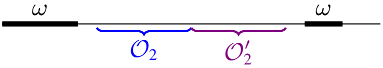
$$\mathcal{O}_2 =]\alpha - d, \alpha[, \mathcal{O}_2' =]\alpha, \alpha + d[,$$

- Case 1 :



(S_D) is approximately controllable $\Leftrightarrow d \notin \mathbb{Q}$ and $\alpha \notin \mathbb{Q}$

- Case 2 :

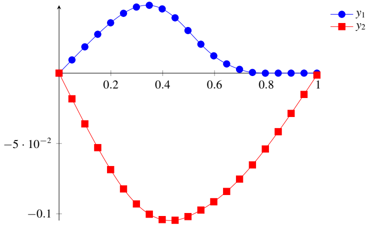


(S_D) is approximately controllable $\Leftrightarrow d \notin \mathbb{Q}$

$$a_{21}(x) = \mathbf{1}_{]1/2-1/2\sqrt{3},1/2[}(x) - \mathbf{1}_{]1/2,1/2+1/2\sqrt{3}[}(x),$$

$$\omega =]0.8, 1.0[,$$

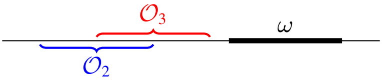
$$y_0(x) = \begin{pmatrix} \sin(\pi x)^{10} \\ -2\sin(2\pi x)^{10} \end{pmatrix}.$$



$$(S_D)\begin{cases} \partial_t y - \partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ a_{31}(x) & 0 & 0 \end{pmatrix} y = \mathbf{1}_\omega \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} v, \quad \text{in }]0,1[\\ y(t,0) = y(t,1) = 0, \end{cases}$$

- Take $a_{21} = 1_{\mathcal{O}_2}$, $a_{31} = 1_{\mathcal{O}_3}$ **(B., Olive, '13)**

- Case 1 :

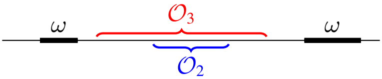


(S_D) is **not** approximately controllable

- Case 2 :

(S_D) is approximately controllable

- Case 3 :



$\mathcal{O}_2 =]1/2 - \delta_2, 1/2 + \delta_2[$
$\mathcal{O}_3 =]\alpha_3 - \delta_3, \alpha_3 + \delta_3[$

(S_D) is approximately controllable $\Leftrightarrow \alpha_3 \notin \mathbb{Q}$ and $\delta_3 \notin \mathbb{Q}$

$$(S_D)\begin{cases}\partial_t y - \partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ 0 & a_{32}(x) & 0 \end{pmatrix} y = \mathbf{1}_\omega \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} v, \quad \text{in }]0,1[\\ y(t,0) = y(t,1) = 0, \end{cases}$$

- Take $\omega =]1/2, 1[$
  - Case 1 : $a_{21} = \mathbf{1}_{]0,1/2[}$ and $a_{31} = \mathbf{1}$

    (S_D) is approximately controllable

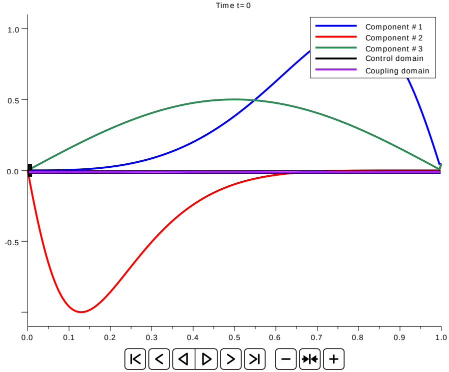
  - Case 2 : $a_{21} = \mathbf{1}_{]0,1/2[}$ and $a_{31} = x - 1/2$

    (S_D) is **not** approximately controllable

(Olive, '14)

$$\partial_t y - \partial_x^2 y + \begin{pmatrix} -2 & -6 & -2 \\ -4 & 0 & 2 \\ -2 & 3/2 & -2 \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \\ \mathbf{1}_\omega(x) \end{pmatrix}$$ ~~v_1~~, $$y(t,0) = \begin{pmatrix} v_2 \\ 0 \\ 0 \end{pmatrix}$$

$$y_0(x) = \left(\sin(\pi x^3), -\sin(\pi(1-x)^5), 0.5\sin(\pi x)\right)^t, \ \omega =]0.3, 0.7[,$$



(Olive, '14)

$$\partial_t y - \partial_x^2 y + \begin{pmatrix} -2 & -6 & -2 \\ -4 & 0 & 2 \\ -2 & 3/2 & -2 \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \\ \mathbf{1}_\omega(x) \end{pmatrix} v_1, \quad y(t,0) = \begin{pmatrix} v_2 \\ 0 \\ 0 \end{pmatrix}$$

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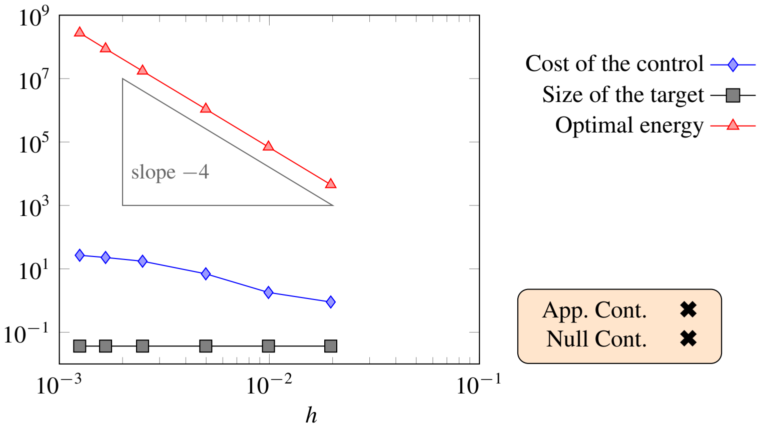
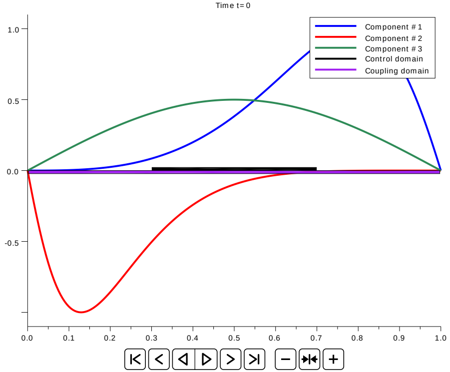


App. Cont.	✖
Null Cont.	✖

(Olive, '14)

$$\partial_t y - \partial_x^2 y + \begin{pmatrix} -2 & -6 & -2 \\ -4 & 0 & 2 \\ -2 & 3/2 & -2 \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \\ \mathbf{1}_\omega(x) \end{pmatrix} v_1 \ , \ \ y(t,0) = \begin{pmatrix} \cancel{v_2} \\ 0 \\ 0 \end{pmatrix}$$

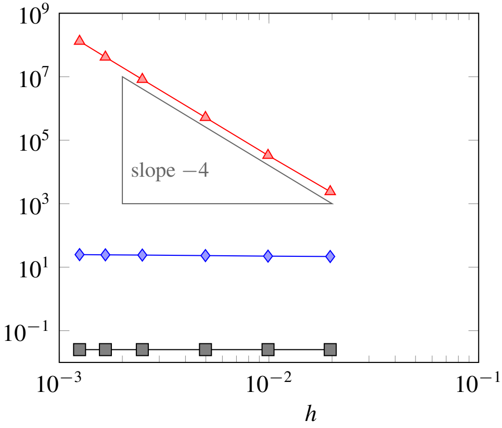
$$y_0(x) = \Big(\sin(\pi x^3), -\sin(\pi(1-x)^5), 0.5\sin(\pi x)\Big)^t, \ \omega =]0.3, 0.7[,$$



(Olive, '14)

$$\partial_t y - \partial_x^2 y + \begin{pmatrix} -2 & -6 & -2 \\ -4 & 0 & 2 \\ -2 & 3/2 & -2 \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \\ \mathbf{1}_\omega(x) \end{pmatrix} v_1 \ , \quad y(t,0) = \begin{pmatrix} \cancel{v_2} \\ 0 \\ 0 \end{pmatrix}$$

$$y_0(x) = \Big(\sin(\pi x^3), -\sin(\pi(1-x)^5), 0.5\sin(\pi x)\Big)^t, \ \omega =]0.3, 0.7[,$$

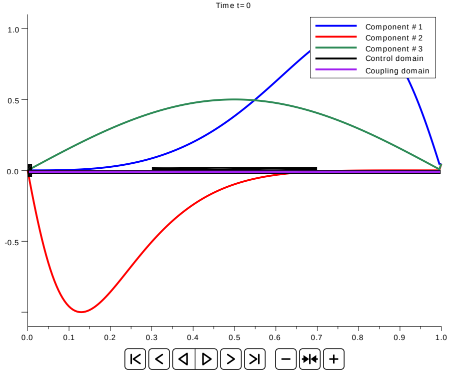


App. Cont.	✖
Null Cont.	✖

(Olive, ’14)

$$\partial_t y - \partial_x^2 y + \begin{pmatrix} -2 & -6 & -2 \\ -4 & 0 & 2 \\ -2 & 3/2 & -2 \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \\ \mathbf{1}_\omega(x) \end{pmatrix} v_1 \;,\; y(t,0) = \begin{pmatrix} v_2 \\ 0 \\ 0 \end{pmatrix}$$

$$y_0(x) = \left(\sin(\pi x^3), -\sin(\pi(1-x)^5), 0.5\sin(\pi x)\right)^t, \; \omega =]0.3, 0.7[,$$



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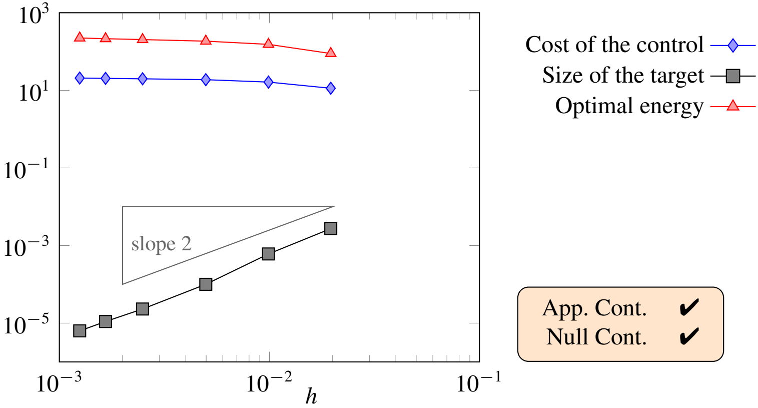

App. Cont. ✔
Null Cont. ✔

Various available methods with different strengths and weakness

- Moment methods (need precise spectral estimates)
- Carleman methods
  - Parabolic Carleman $\Rightarrow$ direct proof of observability
  - Elliptic Carleman $\Rightarrow$ Lebeau-Robbiano strategy
- Transmutation methods
- Multiplier methods
- ...

Various results

- Boundary and Distributed control problems may not be equivalent.
- Unconditional approximate or null controllability.
- Minimal null-control time (even $T_0 = +\infty$!) could appear.

No general controllability criterion available even for linear systems

NUMEROUS OPEN PROBLEMS

- Boundary control of parabolic systems in multi-D
- Time-space dependent coupling coefficients (even in 1D)
- Different diffusion coefficients

$$\begin{cases} \partial_t y - \begin{pmatrix} \partial_x(\gamma_1(x)\partial_x \cdot) & 0 \\ 0 & \partial_x(\gamma_2(x)\partial_x \cdot) \end{pmatrix} y + Ay = \mathbf{1}_\omega Bv, & \text{in }]0,\pi[\\ y(t,0) = y(t,\pi) = 0, \end{cases}$$

- Higher-order coupling terms. Cross diffusions.

$$\begin{cases} \partial_t y - \begin{pmatrix} \Delta & \alpha\Delta \\ 0 & \Delta \end{pmatrix} y + Ay = \mathbf{1}_\omega Bv, & \text{in } \Omega \\ y = 0, & \text{on } \partial\Omega \end{cases}$$

OTHER KIND OF PARABOLIC MODELS

- Nonlinear systems
- Navier-Stokes
- Degenerate parabolic equations

OTHER KIND OF QUESTIONS

- More detailed numerical analysis and adapted algorithms.
- Optimal control / Constrained control.
- Stabilization.