Selected topics in statistics
Spatial Statistics
Mid-term exam

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Specifications

- No documents, no calculators.
- You can use the results of the lectures without reproving them.
- You must prove your answers
- Clarity and readability will be taken into account in the final mark
- Tentative notation: 10/20/10.

Exercise 1

Let A be a uniform random variable on $[-1, 1]$ and B be a standard Gaussian random variable. Assume that A and B are independent. Let, for $t \in \mathbb{R}$, $Y(t) = t^2 A + tB$.

a) Show that Y is a stochastic process with domain $\mathcal{D} = \mathbb{R}$.

b) Calculate the mean and covariance functions of Y. Indication: You can use, without proof, that $Var(A) = 1/3$.

c) Is Y a stationary process?

Exercise 2

Let Y be a Gaussian process on $\mathbb{R}$, with zero mean function and with covariance function $K(x, y) = e^{-|x-y|}$.

a) Prove that Y is stationary

b) Prove that Y is mean square continuous on $\mathbb{R}$.

Let U a random variable, independent of Y[1], so that $P(U = -1) = P(U = 1) = 1/2$. Let $Z(t) = UY(t)$. Then Z is a stochastic process (you do not need to prove it).

c) Prove that Z is a Gaussian process, with same mean and covariance function as Y. Indication: you can use the following fact without proving it: If V is a Gaussian vector with zero mean vector, then $-V$ and V follow the same distribution.

Let now

$$W(t) = \begin{cases} Y(t) & \text{if } t \leq 0 \\ Z(t) & \text{if } t > 0 \end{cases}.$$

d) Prove that W is not mean square continuous. [Hint: you can consider $\lim_{t \to 0; t > 0} \mathbb{E}\left[(W(t) - W(0))^2\right]$].

e) Prove that W is not a Gaussian process. [Hint: you can show that $(W(1), W(0))$ is not a Gaussian vector.] Indication: The characteristic function of a $k \times 1$ Gaussian vector V with mean vector m and covariance matrix Σ is $\mathbb{E}\left(e^{iu'V}\right) = e^{iu'm-(1/2)u'\Sigma u}$, where $i^2 = -1$ and where u is a $k \times 1$ deterministic vector.

[1] That is, for any $n \in \mathbb{N}^*$, for any $x_1, ..., x_n \in \mathbb{R}$, the random vector $(Y(x_1), ..., Y(x_n))$ is independent of U.

Exercise 3

Let Z be a stationary stochastic process on $[0,1]$, with (stationary) covariance function K satisfying $|K(t) - K(0)| \leq (1/2)|t|$, for all $t \in [0,1]$. Let, for $n \in \mathbb{N}^*$

$$W_n = \frac{1}{2^n} \sum_{k=0}^{2^n-1} Z\left(\frac{k}{2^n}\right).$$

a) [Difficult, you can consider doing b) first.] Prove that, for any $n, p \in \mathbb{N}^*$

$$\mathbb{E}\left[(W_n - W_{n+p})^2\right] = \frac{1}{(2^{n+p})^2}\mathbb{E}\left(\left[\sum_{k=0}^{2^n-1}\sum_{l=0}^{2^p-1}\left\{Z\left(\frac{k}{2^n}\right) - Z\left(\frac{k}{2^n} + \frac{l}{2^{n+p}}\right)\right\}\right]^2\right).$$

b) Prove that, for any $n, p \in \mathbb{N}^*$

$$\mathbb{E}\left[(W_n - W_{n+p})^2\right] \leq \frac{1}{2^n}.$$

Indication (1): You can use the result of question a) even if you did not prove it.

Indication (2): You can use the following Cauchy-Schwarz inequality without proving it: For any $n, p \in \mathbb{N}^*$, for any finite set of real numbers of the form $(t_{k,l})_{k=0,\ldots,2^n-1;l=0,\ldots,2^p-1}$: $(\sum_{k=0}^{2^n-1}\sum_{l=0}^{2^p-1} t_{k,l})^2 \leq 2^{n+p}\left(\sum_{k=0}^{2^n-1}\sum_{l=0}^{2^p-1} t_{k,l}^2\right)$.