Selected topics in statistics
Spatial Statistics
Final exam

Lecturer: François Bachoc, PhD

Specifications

- No documents, no calculators.
- You can use the results of the lectures without reproving them.
- Tentative notation: 8/6/6.

Reminder of the Gaussian conditioning theorem

We write $\mathcal{N}(m, S)$ as a shorthand for the multidimensional Gaussian distribution with mean vector m and covariance matrix S.

Let

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right),$$

where Y_1 and m_1 are of size $n_1 \times 1$, Y_2 and m_2 are of size $n_2 \times 1$, Σ_{11} is of size $n_1 \times n_1$, Σ_{12} is of size $n_1 \times n_2$, $\Sigma_{21} = \Sigma_{12}^t$ and Σ_{22} is of size $n_2 \times n_2$. Then, conditionally to $Y_1 = y_1$, we have

$$\mathcal{L}(Y_2 | Y_1 = y_1) = \mathcal{N}\left(m_2 + \Sigma_{21}\Sigma_{11}^{-1}(y_1 - m_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \right).$$

Exercise 1

a) Let

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

be a 2-dimensional Gaussian vector, with mean vector

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and covariance matrix

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

What is the distribution of the Gaussian variable Y_2, conditionally to $Y_1 = 3$? *Indication:* Let A be a 1×1 matrix with $A_{11} \neq 0$. Then A^{-1} is the 1×1 matrix so that $\left(A^{-1}\right)_{11} = \frac{1}{A_{11}}$.

b) Prove that

$$\begin{pmatrix} Y_1 + Y_2 \\ Y_1 \\ Y_2 \end{pmatrix}$$

is a Gaussian vector and give its mean vector and its covariance matrix.

c) What it the distribution of the Gaussian vector

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

conditionally to $Y_1 + Y_2 = 0$?

Exercise 2

Are the following functions, $\mathbb{R}^2 \to \mathbb{R}$, symmetric non-negative definite? Prove your answers.

a) $(x, y) \to 1 + (x - y)^2$.

b) $(x, y) \to (1 + x^2)(1 + y^2)$.

Exercise 3

Let $(X_i)_{i \in \mathbb{Z}}$ be a sequence of *iid* random variables, with mean 0 and variance 1. For $i \in \mathbb{Z}$, let

$$Z(i) = X_i - X_{i-1}.$$

a) Show that Z is a stochastic process on $\mathcal{D} = \mathbb{Z}$.

b) Calculate the mean function m and the covariance function K of Z. *Suggestion:* you can write the covariance function in the following form:

$$K(i, j) = \mathbf{1}_{j=i} f_1(i) + \mathbf{1}_{j=i+1} f_2(i) + \mathbf{1}_{j=i-1} f_3(i) + \mathbf{1}_{|i-j| \geq 2} f_4(i).$$