Models of structured populations in constant and fluctuating environments

Sepideh Mirrahimi

CNRS, IMT, Toulouse

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We study the **Darwinian** evolution of a population structured by phenotypical traits, under selection and mutation.
Example of evolution: creation of antibiotic resistance of bacteria under drug selection

Morbidostat: a selective pressure is applied continuously to the bacterial population. It automatically tunes drug concentration such that a constant growth rate is maintained.

Following evolution of bacterial antibiotic resistance in real time, Rosenthal et Elowitz, Nature Genetics 2012
The influence of fluctuating temperature on bacteria

Bacterial pathogen *Serratia marcesens* evolved in fluctuating temperature (daily variation between 24°C and 38°C, mean 31°C), outperforms the strain that evolved in constant environments (31°C):

![Bar graph showing growth rate under different conditions](image)

Figure from: Fluctuation temperature leads to evolution of thermal generalism and preadaptation to novel environments, Ketola et al. 2013
Environment fluctuation can select for generalism
⇒ It can also increase organisms’ ability to invade novel environments.
Environment fluctuation can select for **generalism**

⇒ It can also increase organisms’ ability to **invade novel environments**.

However, generalism can still be costly in terms of **reduced fitness** in some ecological contexts:

Evolution in fluctuating temperature **decreased bacterial virulence** in D. melanogoster host:

![Graph showing survival over hours with constant and fluctuating temperatures]

Figure from: Fluctuation temperature leads to evolution of thermal generalism and preadaptation to novel environments, Ketola et al. 2013
A typical model

\[
\begin{aligned}
\frac{\partial}{\partial t} n_\varepsilon - \varepsilon \Delta n_\varepsilon &= \frac{n_\varepsilon}{\varepsilon} R(x, l_\varepsilon), \\
n_\varepsilon(\cdot, t = 0) &= n^0_\varepsilon(\cdot),
\end{aligned}
\]

\[
l_\varepsilon(t) = \int_{\mathbb{R}^d} \eta(x) n_\varepsilon(x, t) \, dx.
\]

- \( x \in \mathbb{R}^d \): phenotypical trait
- \( n_\varepsilon(x, t) \): density of trait \( x \)
- \( \eta(x) \): consumption rate
- \( l_\varepsilon(t) \): total consumption
- \( R(x, l_\varepsilon) \): growth rate
- \( \varepsilon \): a small parameter
A typical model

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\begin{align*}
\frac{\partial}{\partial t} n_\varepsilon - \varepsilon \Delta n_\varepsilon &= \frac{n_\varepsilon}{\varepsilon} R(x, I_\varepsilon), \\
n_\varepsilon(\cdot, t = 0) &= n_\varepsilon^0(\cdot),
\end{align*}
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The typical behavior of the solutions

A simple typical growth rate: \[ R(x, l) = 1 - \frac{x^2}{2} - l \]

Dynamics of the dominant trait
Preliminary properties

- $-C \leq \frac{\partial R}{\partial I}(x, I) \leq -C^{-1} < 0$
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- $l_m \leq l_\varepsilon(0) \leq l_M.$
Preliminary properties

• $-C \leq \frac{\partial R}{\partial I}(x, I) \leq -C^{-1} < 0$

• $\max_{x \in \mathbb{R}^d} R(x, I_M) = 0$,

• $l_m \leq l_\varepsilon(0) \leq l_M$.

\[ \Rightarrow \quad l_m < l_\varepsilon(t) \leq l_M. \]
Preliminary properties

- \(- C \leq \frac{\partial R}{\partial l}(x, I) \leq -C^{-1} < 0\)
- \(\max_{x \in \mathbb{R}^d} R(x, I_M) = 0\),
- \(l_m \leq l_\varepsilon(0) \leq l_M\).

\[\implies l_m < l_\varepsilon(t) \leq l_M.\]

Moreover, after extraction of a subsequence, \((l_\varepsilon)_\varepsilon\) converges a.e. to \(l(t)\).
Some notations

• $n(x, t)$: weak limit of $n_\varepsilon(x, t)$ as $\varepsilon$ vanishes

• We expect $n$ to concentrate as Dirac masses

• Hopf-Cole transformation: $n_\varepsilon(x, t) = \exp\left(\frac{u_\varepsilon(x, t)}{\varepsilon}\right)$
The Hamilton-Jacobi limit

Theorem (Barles, SM, Perthame - 2009)

After extraction of a subsequence, \( u_\varepsilon \) converges locally uniformly to a continuous function \( u \), a viscosity solution to

\[
\begin{aligned}
\frac{\partial}{\partial t} u &= |\nabla u|^2 + R(x, l(t)) \\
\max_{x \in \mathbb{R}^d} u(x, t) &= 0, \\
u(0, x) &= u^0(x).
\end{aligned}
\]
Convergence to a monomorphic population

\[ n_\varepsilon(x, t) \xrightarrow[\varepsilon \to 0]{} n(x, t), \]
\[ \text{supp } n(x, t) \subset \{ u(x, t) = 0 \} \subset \{ R(x, l(t)) = 0 \}. \]
Convergence to a monomorphic population

\[ n_\varepsilon(x, t) \xrightarrow{\varepsilon \to 0} n(x, t), \]

\[ \text{supp } n(x, t) \subset \{ u(x, t) = 0 \} \subset \{ R(x, l(t)) = 0 \}. \]

In particular, if \( R : \mathbb{R} \to \mathbb{R} \) is strictly concave with respect to \( x \), then

\[ n_\varepsilon(x, t) \xrightarrow{\varepsilon \to 0} n(x, t) = \bar{\rho}(t) \delta(x - \bar{x}(t)), \]

with \( R(\bar{x}(t), l(t)) = 0 \) and \( \bar{\rho}(t) = \frac{l(t)}{\eta(\bar{x}(t))} \).
Canonical equation and long time behavior

Theorem (Lorz, SM, Perthame - 2013)

Under concavity and smoothness assumptions,

\( n_\varepsilon(x, t) \xrightarrow{\varepsilon \to 0} n(x, t) = \bar{\rho}(t) \delta(x - \bar{x}(t)) \)
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- $n_\varepsilon(x, t) \xrightarrow{\varepsilon \to 0} n(x, t) = \overline{\rho}(t) \delta(x - \overline{x}(t))$

- $\overline{x}(t)$ and $\overline{\rho}(t)$ are 'smooth'
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- \( \overline{x}(t) \) and \( \overline{\rho}(t) \) are 'smooth'
- \( R(\overline{x}(t), \overline{I}(t)) = 0, \) with \( \overline{I}(t) = \eta(\overline{x}(t)) \overline{\rho}(t). \)
Canonical equation and long time behavior

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- $\bar{x}(t)$ and $\bar{\rho}(t)$ are 'smooth'
- $R(\bar{x}(t), \bar{l}(t)) = 0$, with $\bar{l}(t) = \eta(\bar{x}(t)) \bar{\rho}(t)$.
- $\dot{\bar{x}}(t) = \left(-D^2 u(\bar{x}(t), t)\right)^{-1} \cdot \nabla_x R(\bar{x}(t), l(t))$, $\bar{x}(0) = \bar{x}^0$
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*Under concavity and smoothness assumptions,*

- \( n_\varepsilon(x, t) \xrightarrow{\varepsilon \to 0} n(x, t) = \bar{\rho}(t) \delta(x - \bar{x}(t)) \)
- \( \bar{x}(t) \) and \( \bar{\rho}(t) \) are 'smooth'
- \( R(\bar{x}(t), \bar{l}(t)) = 0, \) with \( \bar{l}(t) = \eta(\bar{x}(t)) \bar{\rho}(t) \).
- \( \dot{\bar{x}}(t) = (-D^2 u(\bar{x}(t), t))^{-1} \cdot \nabla_x R(\bar{x}(t), l(t)), \) \( \bar{x}(0) = \bar{x}^0 \)
- \( l(t) \xrightarrow{t \to \infty} l_M, \) \( \bar{x}(t) \xrightarrow{t \to \infty} x_M. \) with

\[ 0 = R(x_M, l_M) = \max_{x \in \mathbb{R}^d} R(x, l_M). \]
What if the environment is fluctuating?

\[
\begin{cases}
\varepsilon \frac{\partial}{\partial t} n_\varepsilon - \varepsilon^2 \Delta n_\varepsilon = n_\varepsilon \ R(x, \ \frac{t}{\varepsilon}, I_\varepsilon(t)), \\
n_\varepsilon(\cdot, t = 0) = n_0^\varepsilon,
\end{cases}
\]

\[I_\varepsilon(t) := \int_{\mathbb{R}^N} \eta(x)n_\varepsilon(x, t)dx,\]

with $R$ 1-periodic in the second variable.

Can we describe the limit as $\varepsilon \to 0$?
Some assumptions

- $-2K_1 \leq D_x^2 R(x, s, l) \leq -2K_2$
Some assumptions

- $-2K_1 \leq D_x^2 R(x, s, l) \leq -2K_2$
- $-K_5 \leq \frac{\partial}{\partial l} R(x, s, l) \leq -K_6$
Some assumptions

- $-2K_1 \leq D_x^2 R(x, s, I) \leq -2K_2$
- $-K_5 \leq \frac{\partial}{\partial I} R(x, s, I) \leq -K_6$
- $\max_{0 \leq s \leq 1, \, x \in \mathbb{R}^N} R(x, s, I_M) = 0$
Main result

Theorem (SM, Perthame, Souganidis - 2014)

There exist a fittest trait $\bar{x}$ and a total population size $\bar{\rho}$ such that, along subsequences $\varepsilon \to 0$,

$$n_\varepsilon(\cdot, t) \rightharpoonup \overline{\rho}(t)\delta(\cdot - \bar{x}(t)) \quad \text{weakly in the sense of measures,}$$

$$l_\varepsilon \rightharpoonup \overline{I} := \overline{\rho}\eta(\bar{x}) \quad \text{in } L^\infty(0, \infty) \text{ weak-}*,$$

and

$$R(x, t \varepsilon, l_\varepsilon(t)) \rightharpoonup \mathcal{R}(x, \bar{x}(t)) \quad \text{weakly in } t \text{ and strongly in } x.$$
Idea of the proof

• From the concavity property one can prove that, for some $\bar{x}(t) \in \mathbb{R}^N$,

$$n_\varepsilon \rightharpoonup n, \quad \text{supp } n = \{\bar{x}(t)\}.$$

• The case with constant environment $(R(x, I))$: There exists a unique constant $\bar{I}(t)$ such that

$$R(\bar{x}(t), \bar{I}(t)) = 0.$$ 

One can then prove that, as $\varepsilon \to 0$,

$$l_\varepsilon(t) \rightharpoonup \bar{I}(t), \quad n_\varepsilon \rightharpoonup \frac{\bar{I}(t)}{\eta(\bar{x}(t))} \delta(x - \bar{x}(t)).$$
• Periodic environment \((R(x, s, I))\): more work to determine the \(\text{(weak)}\) limit of \(I_\varepsilon\).

**Lemma (Cell problem)**

Let \(x = \bar{x}(t)\) for some \(t \geq 0\). There exists a unique 1-periodic positive solution \(I(x, s) : [0, 1] \rightarrow (0, I_M)\) to

\[
\begin{aligned}
\frac{d}{ds}I(x, s) &= I(x, s) \, R(x, s, I(x, s)), \\
I(x, 0) &= I(x, 1).
\end{aligned}
\]

Then

\[
l_\varepsilon(t) \xrightarrow{\varepsilon} \int_0^1 I(\bar{x}(t), s)ds,
\]

\[
R(x, \frac{t}{\varepsilon}, l_\varepsilon(t)) \xrightarrow{\varepsilon} R(x, \bar{x}(t)) := \int_0^1 R(x, s, I(\bar{x}(t), s))ds.
\]
Weak convergence of $I_\varepsilon$

Dynamics of the total population $I_\varepsilon(t)$ with

$$R(x, s, l) = (2 + \sin(2\pi s)) \frac{2 - x^2}{l + .5} - .5, \quad \eta(x) = 1, \quad \varepsilon = 0.01.$$  

The $I_\varepsilon$’s oscillate with period of order $\varepsilon$ around a monotone curve $\bar{l}$. 


The Hamilton-Jacobi limit

We use again the Hopf-Cole transformation

\[ n_\varepsilon = \exp \left( \frac{u_\varepsilon}{\varepsilon} \right). \]

**Theorem (SM, Perthame, Souganidis - 2014)**

Along subsequences \( \varepsilon \to 0 \), \( u_\varepsilon \to u \) locally uniformly in \( \mathbb{R}^N \times [0, \infty) \), where \( u \) is a viscosity solution of

\[
\begin{cases}
  u_t = \mathcal{R}(x, \bar{x}(t)) + |D_x u|^2, \\
  \max_{x \in \mathbb{R}^N} u(x, t) = 0 = u(\bar{x}(t), t), \\
  u(\cdot, 0) = u^0(\cdot).
\end{cases}
\]

Moreover, \( \bar{x} \) satisfies the canonical equation

\[
\dot{\bar{x}}(t) = ( -D_x^2 u(\bar{x}(t), t) )^{-1} \cdot D_1 \mathcal{R}(\bar{x}(t), \bar{x}(t)).
\]
In terms of adaptive dynamics...

- $\mathcal{R}(x, y)$: the **effective fitness** of a mutant $x$ in a resident population with a dominant trait $y$,

- $D_1\mathcal{R}$: the **selection gradient**, which represents the capability of invasion.

- $\left(-D^2_\bar{x} u(\bar{x}(t), t)\right)^{-1}$ is an **indicator of the diversity** around the dominant trait in the resident population.
Fluctuations may increase the population size

We consider the following example

\[ R(x, s, I) = b(x) - D_1(s)D_2(I), \quad \eta \equiv 1, \]

with \(D_1\) 1-periodic and \(D_2\) concave and increasing.

The goal is to compare the long time limit of the total population size for the model with fluctuations \((I_M^f)\) to the one obtained from the model with the “averaged rate” \((I_M^c)\):

\[ R_{av}(x, I) = b(x) - D_{1,av}D_2(I) \]

with

\[ f_{av} = \int_0^1 f \, ds. \]
Total population in constant and fluctuating environment

Let

\[ b(x_M) = \max_x b(x). \]
Total population in constant and fluctuating environment

Let

\[ b(x_M) = \max_x b(x). \]

According to the previous computations in the model with no fluctuation, the final total population \( I^c_M \) is given by

\[ b(x_M) = D_{1,av}D_2(I^c_M). \]
Total population in constant and fluctuating environment

Let

\[ b(x_M) = \max_x b(x). \]

According to the previous computations in the model with no fluctuation, the final total population \( I_M^c \) is given by

\[ b(x_M) = D_{1,av} D_2(I_M^c). \]

Equivalently in the model with fluctuation we obtain, at the final time,

\[ I_M^f = \int_0^1 I(x_M, s) \, ds \]

where, \( I \) is a periodic function which solves

\[ \frac{d}{ds} I = I \left[ b(x_M) - D_1(s) D_2(I) \right]. \]
Some computation for the fluctuating case

By integrating correspondent equations for suitable functions of $\mathcal{I}$, we obtain

$$b(x_M) = \int_0^1 D_1(s)D_2(\mathcal{I}(x_M, s))ds$$

$$\int_0^1 D_2(\mathcal{I}(x_M, s))ds \ b(x_M) = \int_0^1 D_1(s)D_2^2(\mathcal{I}(x_M, s))ds.$$
Some computation for the fluctuating case

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$$b(x_M) = \int_0^1 D_1(s) D_2(\mathcal{I}(x_M, s)) \, ds$$

$$\int_0^1 D_2(\mathcal{I}(x_M, s)) \, ds \ b(x_M) = \int_0^1 D_1(s) D_2^2(\mathcal{I}(x_M, s)) \, ds.$$ 

From the Cauchy-Schwarz inequality:

$$b(x_M)^2 < D_{1,av} \ b(x_M) \ \int_0^1 D_2(\mathcal{I}(x_M, s)) \, ds$$
Some computation for the fluctuating case

By integrating correspondent equations for suitable functions of $\mathcal{I}$, we obtain

\[
b(x_M) = \int_{0}^{1} D_1(s) D_2(\mathcal{I}(x_M, s)) ds
\]

\[
\int_{0}^{1} D_2(\mathcal{I}(x_M, s)) ds \quad b(x_M) = \int_{0}^{1} D_1(s) D_2^2(\mathcal{I}(x_M, s)) ds.
\]

From the **Cauchy-Schwarz** inequality:

\[
b(x_M)^2 < D_{1,av} b(x_M) \int_{0}^{1} D_2(\mathcal{I}(x_M, s)) ds
\]

and thus

\[
b(x_M) < D_{1,av} \int_{0}^{1} D_2(\mathcal{I}(x_M, s)) ds.
\]
Comparison between the fluctuating and the averaged model

From the computations for both cases

\[ b(x_M) = D_{1,av}D_2(I^c_M) < D_{1,av} \int_0^1 D_2(I(x_M, s))ds, \]
Comparison between the fluctuating and the averaged model

From the computations for both cases

\[ b(x_M) = D_{1,av}D_2(I_M^c) < D_{1,av} \int_0^1 D_2(I(x_M, s))ds, \]

and thus

\[ D_2(I_M^c) < \int_0^1 D_2(I(x_M, s))ds. \]
Comparison between the fluctuating and the averaged model

From the computations for both cases

\[ b(x_M) = D_{1,av}D_2(I^c_M) < D_{1,av} \int_0^1 D_2(\mathcal{I}(x_M, s))ds, \]

and thus

\[ D_2(I^c_M) < \int_0^1 D_2(\mathcal{I}(x_M, s))ds. \]

From the concavity and monotonicity of \( D_2 \) we obtain

\[ I^c_M < I^f_M = \int_0^1 \mathcal{I}(x_M, s)ds. \]

Therefore the population is more important in the fluctuating case!
Thank you for your attention!