On the uniform controllability of the one-dimensional transport-diffusion equation

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1. The transport-diffusion equation controlled on one side
   - Introduction
   - The transport equation
   - Uniform controllability in the vanishing viscosity limit

2. The heat equation controlled on one side
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   - Link between heat and transport-diffusion equation
   - Another approach: integral observability inequalities

3. Conclusion
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Notations

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- length $L > 0$, 
The T-D equation

The heat equation

Conclusion

Introduction

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- $y^0 \in L^2(0, L)$ initial condition (or $H^{-1}(0, L)$),
- control $v \in L^2(0, T)$,
- $Q := (0, T) \times (0, L)$. 
The transport-diffusion equation

\[ \begin{cases}
  y_t - \epsilon y_{xx} + My_x = 0 \text{ in } Q, \\
  y(.,0) = v(t) \text{ in } (0, T), \\
  y(.,L) = 0 \text{ in } (0, T), \\
  y(0,.) = y^0 \text{ in } (0, L). 
\end{cases} \quad (T-D) \]
The transport-diffusion equation

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\end{cases}
\]

(T-D)

Definition

Let \( T, L, M, \varepsilon \) be fixed. Equation (T-D) is null-controllable at time \( T \) if and only if for every \( y^0 \in L^2(0, L) \), there exists \( v \in L^2(0, T) \) such that \( y(T) = 0 \).

Null-controllability in arbitrary small time: well-known for a long time in the one-dimensional case (Fattorini-Russell‘71).
Definition

*Cost of the control for (T-D):*

\[
C_{TD}(T, L, M, \varepsilon) := \sup_{y^0 \text{ (}y, v\text{) ver. (T-D) } \& \ y(T) = 0} \frac{\|v\|_{L^2(0,T)}}{\|y^0\|_{L^2(0,L)}}.
\]
The cost of the control

**Definition**

*Cost of the control for (T-D):*

\[
C_{TD}(T, L, M, \varepsilon) := \sup_{y^0} \inf_{(y,v) \text{ ver. } (T-D) & y(T)=0} \frac{||v||_{L^2(0,T)}}{||y^0||_{L^2(0,L)}}.
\]

Equivalently, \(C_{TD}(T, L, M, \varepsilon)\) is the smallest constant \(C\) such that for every \(y^0 \in L^2(0,L)\), there exists \(v \in L^2(0,L)\) such that

\[
||v||_{L^2(0,T)} \leq C||y^0||_{L^2(0,L)}.
\]

Equivalently, \(C_{TD}(T, L, M, \varepsilon)\) is the norm of the (linear) operator \(y^0 \mapsto v_{HUM}\), where \(v_{HUM}\) is the control given by the HUM method.
Uniform controllability

**Definition**

Let $T, L, M$ be fixed. The family of equations (T-D) is **uniformly controllable** in the vanishing viscosity limit if and only if there exists $\varepsilon_0 > 0$ and $C > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$,

$$C_{TD}(T, L, M, \varepsilon) \leq C.$$
Uniform controllability

Definition

Let $T, L, M$ be fixed. The family of equations $(T-D)$ is uniformly controllable in the vanishing viscosity limit if and only if there exists $\varepsilon_0 > 0$ and $C > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$,

$$C_{TD}(T, L, M, \varepsilon) \leq C.$$ 

Of course, if $(T-D)$ is uniformly controllable at time $T_0$, it is uniformly controllable at any time $T \geq T_0$.

Question

What about the uniform controllability when $\varepsilon \rightarrow 0$, and more precisely what about the minimal time that ensures null-controllability?
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Convergence toward the transport equation (1)

Theorem (Coron-Guerrero’05)

Let \( y^0 \in L^2(0, L) \). Let \( (\varepsilon_n)_{n \in \mathbb{N}} \) be a sequence of positive numbers tending to 0. Let \( (v_n)_{n \in \mathbb{N}} \) a sequence of function of \( L^2(0, T) \) and \( v \in L^2(0, T) \) such that \( v_n \rightharpoonup v \) (in \( L^2 \)). At fixed \( n \), we denote \( y_n \) the unique solution of the problem

\[
\begin{aligned}
   y_{nt} - \varepsilon_n y_{nxx} + My_{nx} &= 0 \quad \text{in } Q, \\
   y_n(\cdot, 0) &= v_n(t) \quad \text{on } (0, T), \\
   y_n(\cdot, L) &= 0 \quad \text{on } (0, T), \\
   y_n(0, \cdot) &= y^0 \quad \text{in } (0, L).
\end{aligned}
\]
Convergence toward the transport equation (2)

**Theorem**

Then $y_n \in L^2(Q)$ and $y_n \rightharpoonup y \in L^2(Q)$ solution of

\[
\begin{cases}
  y_t + My_x = 0 & \text{in } Q, \\
  y(\cdot, 0) = v(t) & \text{on } (0, T) \text{ for } M > 0, \\
  y(\cdot, L) = 0 & \text{on } (0, T) \text{ for } M < 0, \\
  y(0, \cdot) = y^0 & \text{in } (0, L).
\end{cases}
\]
We consider (Transp).
We extend \( y^0, v \) by 0.
Unique weak solution:

\[
y^0(x - Mt) + v(t - x/M).
\]

\( x > Mt \): depends only on \( y^0 \).
\( x < Mt \): dépend only on \( v \).
We consider (Transp).
We extend $y^0, v$ by 0.
Unique weak solution:

$$y^0(x - Mt) + v(t - x/M).$$

$$T \geq L/M : x - MT < 0$$

for $x \in (0, L)$ so

$$y(T, x) \equiv 0$$

on $(0, L)$ if $v = 0$. 
We consider (Transp).
We extend \( y^0, v \) by 0.
Unique weak solution:

\[
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\( T < L/M : x - MT \in (0, L) \) and \( x \in (0, L) \)
iff \( x \in (MT, L) \) so
\( y(T, x) \) not necessary null
on \( (MT, L) \).
Corollary

*Equation (T-D) cannot be uniformly controllable in time*

\[ T < \frac{L}{|M|}. \]
Equation (T-D) cannot be uniformly controllable in time $T < L/|M|$.

Proof: by contradiction. Assume $C_{TD}(\varepsilon, T, L, M) \nrightarrow +\infty$. There exists $(\varepsilon_n)_{n \in \mathbb{N}}$ positive tending to 0 such that $(C_{TD}(\varepsilon_n, T, L, M))_{n \in \mathbb{N}}$ is bounded. Let $y^0 \in L^2(0, L)$ and let $v_n$ the optimal control driving $y^0$ to 0. Let $T_0 \in (T, L/M)$. We extend $y_n$ and $v_n$ by 0 on $(T, T_0)$. From inequality

$$\|v_n\|_{L^2(0, T)} \leq C_{TD}(\varepsilon_n)\|y^0\|_{L^2(0, T)},$$

$(\|v_n\|)_{n \in \mathbb{N}}$ is bounded in $L^2(0, T_0)$, so we extract a subsequence $(v_n)_{n \in \mathbb{N}}$ weakly converging to $v \in L^2(0, T_0)$. We deduce $y_n \rightharpoonup y$ solution of (Transp), and necessarily $y \equiv 0$ on $(T, T_0) \times (0, L)$, which gives a contradiction.
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What should we expect and what is true?

Natural conjecture concerning (T-D):

**What we expect**

\[ C_{TD}(T, L, M, \varepsilon) \to 0 \text{ if } T > \frac{L}{|M|} \quad \text{and} \quad C_{TD}(T, L, M, \varepsilon) \to +\infty \text{ if } T < \frac{L}{|M|}. \]

Conjecture (partially) false:

**Theorem (Coron-Guerrero’05)**

1. **If** \( M > 0 \), then \( C_{TD}(T, L, M, \varepsilon) \geq Ce \frac{c}{\varepsilon} \) when \( \varepsilon \to 0 \) for \( T < \frac{L}{|M|} \).
2. **If** \( M < 0 \), then \( C_{TD}(T, L, M, \varepsilon) \geq Ce \frac{c}{\varepsilon} \) when \( \varepsilon \to 0 \) for \( T < \frac{2L}{|M|} \).
On the other hand, upper bounds:

<table>
<thead>
<tr>
<th>Theorem (Coron-Guerrero’05)</th>
</tr>
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<tbody>
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<td>1. If $M &gt; 0$, then $C_{TD}(T, L, M, \varepsilon) \leq Ce^{-\frac{c}{\varepsilon}}$ when $\varepsilon \to 0$ for $T \geq 4.3L/</td>
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What should we expect and what is true? (2)

On the other hand, upper bounds:

**Theorem (Coron-Guerrero’05)**

1. If $M > 0$, then $C_{TD}(T, L, M, \varepsilon) \leq Ce \frac{-c}{\varepsilon}$ when $\varepsilon \to 0$ for $T \geq 4.3L/|M|$.

2. If $M < 0$, then $C_{TD}(T, L, M, \varepsilon) \leq Ce \frac{-c}{\varepsilon}$ when $\varepsilon \to 0$ for $T \geq 57.2L/|M|$.

Method: Carleman estimates and adapted dissipation estimate on the adjoint system.
Uniform controllability in the vanishing viscosity limit

What should we expect and what is true? (3)

Improvement:

**Theorem (Glass’09)**

1. If $M > 0$, then $C_{TD}(T, L, M, \varepsilon) \leq Ce^{-c/\varepsilon}$ when $\varepsilon \to 0$ for $T \geq 4.2L/|M|$.

2. If $M < 0$, $C_{TD}(T, L, M, \varepsilon) \leq Ce^{-c/\varepsilon}$ when $\varepsilon \to 0$ for $T \geq 6.1L/|M|$. 
What should we expect and what is true? (3)

Improvement:

Theorem (Glass’09)

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2. If $M < 0$, $C_{TD}(T, L, M, \varepsilon) \leq Ce^{-c}\varepsilon$ when $\varepsilon \to 0$ for $T \geq 6.1L/|M|$.

Method: similar to the moment method (relying essentially on the application of the Paley-Wiener Theorem).
What is conjectured

The numbers given in the previous theorems come from technical restrictions.
What is conjectured

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⇒ New “natural” conjecture concerning (T-D):

**Conjecture (Coron-Guerrero ‘05)**

1. If $M > 0$, $C_{TD}(T, L, M, \varepsilon) \to 0$ when $\varepsilon \to 0$ for $T > L/|M|$,
2. If $M < 0$, $C_{TD}(T, L, M, \varepsilon) \to 0$ when $\varepsilon \to 0$ for $T > 2L/|M|$.
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The heat equation controlled on the left side

\[\begin{cases}
z_t - z_{xx} = 0 \text{ in } Q, \\
z(., 0) = w(t) \text{ on } (0, T), \\
z(., L) = 0 \text{ on } (0, T). \end{cases}\] (Heat)

Question: what about the cost of fast controls (i.e. when $T \to 0$)?
The heat equation controlled on the left side

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(Heat)

**Definition**

Cost of the control for (Heat):

\[
C_H(T, L) = \sup_{z^0} \inf_{(z,w) \text{ ver. (Heat)} \& z(T,.) = 0} \frac{\|w\|_{L^2(0,T)}}{\|z^0\|_{L^2(0,T)}}.
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\]

Question: what about the cost of fast controls (i.e. when $T \to 0$)?
The cost of fast controls for the heat equation

We introduce

\[ \alpha^* := \limsup_{T \to 0} T \ln(C_H(T, L)). \]

\[ \alpha_* := \liminf_{T \to 0} T \ln(C_H(T, L)). \]
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- Güichal‘85: \( \alpha_* > 0 \) (i.e. for \( T \) small \( C_H(T, L) \gtrsim Ce^{C/T} \)).
- Miller‘04: \( \alpha_* \geq L^2 / 4 \) (i.e. for \( T \) small \( C_H(T, L) \gtrsim Ce^{L^2/(4T)} \)).
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- Seidman‘96: \( \alpha^* < \infty \) (i.e. for \( T \) small \( C_H(T, L) \lessgtr C e^{C/T} \)).
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- Tenenbaum-Tucsnak‘07: \( \alpha^* \leq 3L^2/4 \) (i.e. for \( T \) small \( C_H(T, L) \lesssim e^{(3/4L^2)^+}/T \)).
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Conjecture (Miller‘04, Ervedoza-Zuazua‘11, ...)

\[ \alpha^* = L^2/4, \ i.e. \ C_H(T, L) \approx e^{(L^2/4)^+}/T. \]
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3 Conclusion
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An relevant changing of unknowns

The fundamental remark is the following:

**Lemma**

\( y \) verifies (T-D) with initial condition \( y^0 \) and control \( v \) iff

\[
z(t, x) = e^{\frac{M^2 t}{4\varepsilon^2}} - \frac{Mx}{2\varepsilon} y\left(\frac{t}{\varepsilon}, x\right)
\]

verifies (Heat) posed on \((0, \varepsilon T) \times (0, L)\) with initial condition \(z^0(x) := e^{-\frac{Mx}{2\varepsilon}} y^0(x)\) and control \(w(t) := e^{M^2 t} v\left(\frac{t}{\varepsilon}\right)\).
The fundamental remark is the following:

**Lemma**

$y$ verifies (T-D) with initial condition $y^0$ and control $v$ iff

$$z(t, x) = e^{\frac{M^2 t}{4\varepsilon^2} - \frac{M x}{2\varepsilon}} y\left(\frac{t}{\varepsilon}, x\right)$$

verifies (Heat) posed on $(0, \varepsilon T) \times (0, L)$ with initial condition $z^0(x) := e^{-\frac{M x}{2\varepsilon}} y^0(x)$ and control $w(t) := e^{\frac{M^2 t}{4\varepsilon^2}} v\left(\frac{t}{\varepsilon}\right)$.

Proof: direct computations (differentiate $z$ with respect to $t$ and $x$ and compare with $y$.)
The fundamental remark is the following:

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\(z^0(x) := e^{-\frac{Mx}{2\varepsilon}} y^0(x)\) and control \(w(t) := e^{\frac{M^2 t}{4\varepsilon^2}} v\left(\frac{t}{\varepsilon}\right)\).

Proof: direct computations (differentiate \(z\) with respect to \(t\) and \(x\) and compare with \(y\).)

\(\Rightarrow\) We can estimate \(C_{TD}(T, L, M, \varepsilon)\) in the vanishing viscosity limit by estimating \(C_H(T, L)\) in small time!
The T-D equation

The heat equation

Conclusion

Link between heat and transport-diffusion equation

Estimations on the costs of controls

Link between the two costs:

Lemma

For all $T_0 < T$, one has

\[
\begin{align*}
C_{TD}(T, L, M, \varepsilon) & \leq e^{-\frac{M^2 T_0}{4\varepsilon}} C_D(\varepsilon(T - T_0), L) \text{ if } M > 0, \\
C_{TD}(T, L, M, \varepsilon) & \leq e^{-\frac{M^2 T_0}{4\varepsilon}} + \frac{|M| L}{2\varepsilon} C_D(\varepsilon(T - T_0), L) \text{ if } M < 0.
\end{align*}
\]
Estimations on the costs of controls

Link between the two costs:

**Lemma**

For all $T_0 < T$, one has

$$C_{TD}(T, L, M, \varepsilon) \leq e^{-\frac{M^2 T_0}{4\varepsilon}} C_D(\varepsilon(T - T_0), L) \text{ if } M > 0,$$

$$C_{TD}(T, L, M, \varepsilon) \leq e^{-\frac{M^2 T_0}{4\varepsilon} + \frac{|M|L}{2\varepsilon}} C_D(\varepsilon(T - T_0), L) \text{ if } M < 0.$$

Proof: use the definitions of the costs, the previous changing of unknowns and let System (T-D) evolve naturally on $(0, T_0)$ (i.e. the control is 0 on $(0, T_0)$).
Improving the constants (1)

We can deduce the following result for $M > 0$:

**Theorem (Lissy‘12, C.R.A.S.)**

Assume that $M > 0$ and $T > \frac{2 \sqrt{3L}}{M} \geq 3.47 \frac{L}{M}$. Then there exists some constant $K > 0$, there exists some constant $C > 0$ such that, for all $\varepsilon > 0$ and all $y^0 \in L^2(0, L)$, there exists a solution $(y_\varepsilon, v_\varepsilon)$ of the control problem (T-D) verifying $y_\varepsilon(T, .) = 0$ and

$$||v_\varepsilon||_{L^2(0, T)} \leq Ce^{-\frac{K}{\varepsilon}} ||y^0||_{L^2(0, L)}.$$

Moreover, if we assume that the conjecture $\alpha^* = \frac{L^2}{4}$ is verified, then one can state the same result as soon as

$$T > \frac{2L}{|M|}.$$
Improving the constants (2)

We have a similar result for $M < 0$:

**Theorem (Lissy ‘12, C.R.A.S.)**

Assume that $M < 0$ and $T > \frac{(2\sqrt{3}+2)L}{|M|} \geq 5.47 \frac{L}{M}$. Then there exists some constant $K > 0$, there exists some constant $C > 0$ such that, for all $\varepsilon > 0$ and all $y^0 \in L^2(0, L)$, there exists a solution $(y_\varepsilon, v_\varepsilon)$ of the control problem (T-D) verifying $y_\varepsilon(T, \cdot) = 0$ and

$$
\|v_\varepsilon\|_{L^2(0, T)} \leq Ce^{-\frac{K}{\varepsilon}} \|y^0\|_{L^2(0, L)}.
$$

Moreover, if we assume that the conjecture $\alpha^* = L^2/4$ is verified, then one can state the same result as soon as

$$
T > \frac{4L}{|M|}.
$$
Improving the constants (3)

Proof: use the previous Lemma and Tenenbaum-Tucsnak’07 ($\alpha^* = 3L^2/4$) and optimize $T_0$ (for $M > 0$, $T_0 = T/2$, for $M < 0$, $T_0 = 1/2 + 1/(2 + 2\sqrt{3})$). See what happens if we assume $\alpha^* = L^2/4$. 
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Remarks

- Every better upper bound on $\alpha^*$ will automatically provide a better lower bound for the time needed to ensure exponential decay of $C_{TD}(T, L, M, \varepsilon)$. 
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Remarks

1. Every better upper bound on $\alpha^*$ will automatically provide a better lower bound for the time needed to ensure exponential decay of $C_{TD}(T, L, M, \varepsilon)$.

2. Unfortunately, even with the best $\alpha^*$ possible we cannot recover the conjecture of Coron and Guerrero!
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Integral observability inequalities (1)

From the controllability-observability duality, null-controllability is equivalent to

\[ \int_0^L |\varphi(T, x)|^2 dx \leq C_H(T, L)^2 \int_0^T |\partial_x \varphi(t, 0)|^2 dt, \quad (1) \]

where \( \varphi \) satisfies

\[
\begin{cases}
\varphi_t - \varphi_{xx} = 0 & \text{in } (0, T) \times (0, L), \\
\varphi(\cdot, 0) = 0 & \text{on } (0, T), \\
\varphi(\cdot, L) = 0 & \text{on } (0, T), \\
\varphi(0, \cdot) = \varphi^0 & \text{in } (0, L).
\end{cases}
\quad (2)
\]
Integral observability inequalities (1)

From the controllability-observability duality, null-controllability is equivalent to

\[ \int_0^L |\varphi(T, x)|^2 dx \leq C_H(T, L)^2 \int_0^T |\partial_x \varphi(t, 0)|^2 dt, \quad (1) \]

where \( \varphi \) satisfies

\[
\begin{align*}
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\end{align*}
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(2)

One possible method to prove this: Carleman inequalities

\[ \int_0^T \int_0^L e^{-\frac{C_1(L)}{t}} |\varphi(t, x)|^2 dx dt \leq C_2(T, L) \int_0^T |\partial_x \varphi(t, 0)|^2 dt. \]

(Obs-Fin)
Another approach: integral observability inequalities

Integral observability inequalities (1)

(Obs-Fin) $\Rightarrow$ Observability. Moreover,

$$C_H(T, L)^2 \lesssim C_2(T, L)e^{\frac{c_1(L)^+}{T}}.$$
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Ervedoza-Zuazua’11: thanks to transmutation methods one can prove

$$\int_0^\infty \int_0^L e^{-\frac{L^2}{2t}} |\varphi(t, x)|^2 dx dt \leq C(L) \int_0^\infty |\partial_x \varphi(t, 0)|^2 dt,$$

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From (Obs-Inf) one can deduce

$$\int_0^\infty \int_0^L e^{-\frac{L^2}{2t}} |\varphi(t,x)|^2 dxdt \leq C_{int}(T,L) \int_0^T |\partial_x \varphi(t,0)|^2 dt.$$
Integral observability inequalities (3)

The method used in Ervedoza-Zuazua‘11 (argument by contradiction) cannot give any estimation on $C_{int}(T,L)$. 
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**Conjecture [Ervedoza-Zuazua’11]**

For every $\delta > 0$, one can choose $C_{int}(T,L)$ such that

$$C_{int}(T,L) = O_{T \to 0}(e^{\frac{\delta}{T}}).$$
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Link with the cost of the control?

**Proposition**

The previous conjecture implies $\alpha^* = 1/4$. 
Another approach: integral observability inequalities

Another result

**Theorem (Lissy’14, System Control Lett.)**

We assume that the Ervedoza-Zuazua conjecture is true. Then there exists some $C, K > 0$ (independent on $\varepsilon$ but maybe dependent on $T, M$ et $L$) such that for every $\varepsilon \in (0, 1)$,

$$C_{TD}(T, L, \varepsilon, M) \leq Ce^{-\frac{K}{\varepsilon}}$$

as soon as

1. $T > L/M$ if $M > 0$,
2. $T > (1 + \sqrt{2})L/|M|$ if $M < 0$. 

W e recover the Coron-Guerrero conjecture for $M > 0$, but not for $M < 0$. This result may question the validity of the Coron-Guerrero conjecture for $M < 0$.

Proof: essentially relies on the same of transformation as before.
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The T-D equation
The heat equation
Conclusion

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1. **The transport-diffusion equation controlled on one side**
   - Introduction
   - The transport equation
   - Uniform controllability in the vanishing viscosity limit

2. **The heat equation controlled on one side**
   - Introduction
   - Link between heat and transport-diffusion equation
   - Another approach: integral observability inequalities

3. **Conclusion**
   - Conclusion
Some perspectives

For the transport-diffusion equation

- We do nothing on $(0, T_0)$, which is for sure not optimal.
For the transport-diffusion equation

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  $\Rightarrow$ keep quite the same method and do something intelligent on these intervals?
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- Improve choice of the multiplier in the moment method to improve the estimate on $\alpha^*$. 
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- For the Ervedoza-Zuazua conjecture: ?

An application of a conjecture due to Ervedoza and Zuazua concerning the observability of the heat equation in small time to a conjecture due to Coron and Guerrero concerning the uniform controllability of a convection-diffusion equation in the vanishing viscosity limit, System Control Lett., 2014.
References


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Thank you for your attention!