Cohomologie Equivariante II

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exemple $S^1 \circlearrowright S^3$ V_1 vecteur tangent aux orbites

Hopf $\swarrow$ $\downarrow \pi$

action diag. std. S^2

métrique Riem. invariante $\Rightarrow$

V_1^* duale de V_1 sur S^3

$V_1^*(V_1) = 1$

Formes invariantes

- $d^{eg} = 0$ $\pi^* f$ $f \in C^\infty(S^2)$
- $d^{eg} = 1$ V_1^*, $\pi^*\alpha$ où $\alpha \in A^1(S^2)$
- $\deg = 2$: $\pi^*(g \, vol_{S^2}) = \pi^*\beta$, $g \in C^\infty(S^2)$, $\beta \in A^2(S^2)$
  et $V_1^* \wedge \pi^*\alpha$, $\alpha \in A^1(S^2)$
- $\deg 3$ $(\pi^* h)\, vol_{S^3}$, $h \in C^\infty(S^2)$.

$d_{\mathfrak{g}} = d - x\, \iota_{V_1}$ $d^{eg} = 0$ $(d - x\iota_{V_1})\pi^* f = \pi^* df$

$\deg = 1$ $(d - x\iota_{V_1}) V_1^* = dV_1^* - x \, cst \not\equiv 0$

$$(d - x\iota_{V_1})\pi^*\alpha = \pi^* d\alpha - x\,\underbrace{\alpha(\pi_* V_1)}_{=0}$$

$= \pi^* d\alpha$. $\equiv 0 \iff d\alpha \equiv 0$

$\Rightarrow H^1_{\mathfrak{g}}(S^3) = 0$ pque $H^1_{\mathfrak{g}}(S^2) = 0$

$\deg = 2$ $(d - x\iota_{V_1})(\pi^*(g\, vol_{S^2})) = \pi^* d(\underbrace{g\, vol_{S^2}}_{\deg 3 \text{ sur } S^2 \to 0}) \equiv 0$ singlet

$~~= \pi^*(dg \wedge vol_{S^2})~~$

$(d - x\iota_{V_1})(V_1^* \wedge \pi^*\alpha) = -x \cdot cst \times \pi^*\alpha + cst\, \cancel{\pi^* vol_{S^2} \wedge \pi^*\alpha}_{\to 0} - V_1^* \pi^* d\alpha = 0$ ssi $d\alpha = 0$

$$\pi^* \mathrm{vol}_{S^2} = \frac{1}{cst}\, d V_1^*$$

$$(d - x L_{V_1})\, V_1^* = cst \cdot \pi^* \mathrm{vol}_{S^2} - x\, cst$$

$$\Rightarrow \quad H^2_{\mathfrak{g}}(S^3) = \mathbb{C}[\pi^* \mathrm{vol}_{S^2}] = \mathbb{C}[x]$$

$\deg = 3$
$$(d - x L_{V_1})(\pi^* f)\, \mathrm{vol}_{S^3} = -x\, \pi^* f\, cst''\, \pi^* \mathrm{vol}_{S^2} \neq 0$$

$$\mathrm{vol}_{S^3} = cst''\, V_1^* \wedge \pi^* \mathrm{vol}_{S^2}$$

sauf si $f = 0$

oublie : $\deg 2$
$$(d - x L_{V_1})(x \pi^* f) = x\, \pi^* df$$

$*$

$$(d - x L_{V_1})(x \pi^* f + V_1^* \wedge \pi^* \alpha) = x(\pi^* df - cst \cdot \pi^* \alpha) - V_1^* \wedge \pi^* d\alpha$$

on a encore $H^2_{\mathfrak{g}}(S^3) = \mathbb{C}[x]$

$$= 0 \quad \text{si} \quad \alpha = \frac{1}{cst}\, \pi^* df.$$

$$x \pi^* f + V_1^* \wedge \frac{1}{cst}\, \pi^* df = (d - x L_{V_1})\left(\frac{-f V_1^*}{cst}\right) - \frac{\pi^* f}{cst}\, \pi^* \mathrm{vol}_{S^2}$$

$$x \pi^* f + V_1^* \wedge \pi^* df \quad - x \cdot cst$$

$\to$ tq. $\int (f - cst)\, \mathrm{vol}_{S^2} = 0$

Rem : β d-exacte sur S^2 ssi $\int_{S^2} \beta = 0$

deg=3

$$(d - x\,\iota_{V_1})\left((\pi^* g)\,\mathrm{Vol}_{S^3} + \lambda x\, V_1^* + \mu\, x\, \pi^* \alpha\right)$$

$$= -x\,\pi^*(\mathrm{cst}''\, g\,\mathrm{Vol}_{S^2}) + \lambda x\, \mathrm{cst}'\, \pi^*\mathrm{Vol}_{S^2} - x^2 \lambda\, \mathrm{cst}$$
$$+ \mu\, x\, \pi^* d\alpha$$

$\equiv 0 \quad \Rightarrow \quad \lambda = 0 \quad$ et $\quad \pi^* d\alpha = \frac{\mathrm{cst}''}{\mu} \pi^* g\,\mathrm{vol}_{S^2}$

donc $d\alpha = \frac{\mathrm{cst}''}{\mu} g\,\mathrm{Vol}_{S^2}$

<u>Condition</u> $\int_{S^2} g\,\mathrm{vol}_{S^2} = 0$

auquel cas elle est d-exacte parceque :

$$(d - x\,\iota_{V_1})\left(\frac{\mu}{(\mathrm{cst}'')^2}\, V_1^* \wedge \pi^*\alpha\right) = (\pi^* g)\,\mathrm{vol}_{S^3}$$
$$- x\,\mu\left(\frac{\mathrm{cst}}{(\mathrm{cst}'')^2}\right) \pi^*\alpha.$$

on peut choisir cst tq

et $\quad \mathrm{cst}'' = 1$

Rappel : $\quad \mathcal{L}_{V_1} V_1^* = \mathrm{cst}$

$dV_1^* = \mathrm{cst}'\, \pi^*\mathrm{Vol}_{S^2}$

$V_1^* \wedge \pi^*\mathrm{vol}_{S^2} = \mathrm{cst}''\, \mathrm{vol}_{S^3}$

<u>Rem</u>: $\pi^*\mathrm{vol}_{S^2}$ engendre une classe de cohomologie (non-nulle)

$x\pi^*\mathrm{vol}_{S^2}$ est d_S-exacte $\qquad x\,\pi^*\mathrm{vol}_{S^2} = (d - x\,\iota_{V_1})\left(\frac{\mathrm{cst}}{\mathrm{cst}''}\mathrm{Vol}_{S^3}\right)$

et x^2 est aussi exacte.

<u>Rem</u> : $\{1\} \subset H \subset G$

$$A_G(M) \longrightarrow A_H(M)$$

et $$H_G(M) \longrightarrow H_H(M)$$

$$H_{\{1\}}(M) = H_{dR}(M)$$

$$\begin{tikzcd} H_G(M) \arrow[r] \arrow[dr, "\text{evaluation en } 0 \in \mathfrak{g}"'] & H_H(M) \arrow[d] \\ & H_{dR}(M) \end{tikzcd}$$