UNIVERSAL PROPERTY

of the category of dg-categories

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Introduction

There have been quite a few models for $(\infty, 1)$-categories (∞-categories where every n-morphism is invertible for $n > 1$): quasi-categories, simplicial categories, complete Segal spaces, Segal categories... These models, arising each from different motivations and fields of research, have different strong points and disadvantages, but they are all equivalent. Indeed, in [Toe05] Toën gave an axiomatization of a "theory of $(\infty, 1)$-categories" such that every model that fulfills the axioms is Quillen equivalent to the complete Segal spaces.

Definition
A Reedy fibrant simplicial space W is a **complete Segal space** if the morphism $s_0 : W_0 \to W_{hoequiv}$ is a weak equivalence of simplicial sets and the Segal condition is fullfilled, i.e. for every $n \geq 2$

$$\varphi : W_k \to W_1 \times_{W_0} \ldots \times_{W_0} W_1$$

is also a weak equivalence.

But we don't really need to get into the details of what all those terms mean: it suffices to know that it is some kind of simplicial space, i.e. an element of $\mathrm{Fun}(\Delta^{op}, \mathbf{sSet})$, such that some special morphisms are weak equivalences.

Theorem [Toe05]

Let (M, C) be a theory of $(\infty, 1)$-categories. Then M is Quillen equivalent to CSS the category of complete Segal spaces.

In comparison to the extensive work done on ∞-categories, a lot less has been said about the linear version of them. Consequently, k-linear ∞-categories are a lot less well known, and a lot more complicated to manipulate. Here, we will concentrate on the analogous question for k-linear ∞-categories. Thankfully, those have already been proven by Cohn to be Quillen equivalent to dg-categories [Coh13], so we will restrain ourselves to that context.

Differential graded categories

Let k be a commutative ring.

Definition

We define T a **dg-category** (differential graded category) to be a category enriched over $\mathbf{C(k)}$ the category of chain complexes. Equivalently, T is a category in which for all $a, b \in \mathrm{Obj}(T)$, $\mathrm{Hom}_T(a, b)$ is also a chain complex.
A morphism of dg-categories is a functor which is also a morphism of chain complexes on every $\mathrm{Hom}(a, b)$, for all $a, b \in \mathrm{Obj}(T)$.
We construct the category of dg-categories, and we call it $\mathbf{dg-cat}$.

Examples [Toe11]

- With a bit of tweaking on the morphism spaces, the category $\mathbf{C(k)}$ itself is a dg-category. Indeed, we can define a dg-category, $\underline{\mathbf{C(k)}}$, in which
  - ⋆ $\mathrm{Obj}(\underline{\mathbf{C(k)}})$ = the chain complexes over k.
  - ⋆ Let E, F be chain complexes. We define the complex of morphisms from E to F to be

$$\mathrm{Hom}^n(E, F) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}(E^i, F^{i+n})$$

  for all $n \in \mathbb{Z}$, where the differential is given by the following formula:

$$\{f^i\} \longmapsto \{d_F \circ f^i - (-1)^n f^{i+1} \circ d_E\}.$$

- As for an exemple that arises from geometry, let $k = \mathbb{R}$ and X a differential manifold (say C^∞). We define A_{DR} to be the following dg-category:
  - ⋆ $\mathrm{Obj}(A_{DR})$ = the flat vector bundles on X, i.e. smooth vector bundles V over X coupled with a connexion

$$\triangledown : A^0(X, V) \to A^1(X, V),$$

  where $A^n(X, V)$ are the smooth n-forms on X with coefficients on V, such that $\triangledown^2 = 0$.
  - ⋆ Let $(V, \triangledown_V), (W, \triangledown_W)$ be two such flat vector bundles. We define the complex $A^*_{DR}(V, W)$ to be

$$A^n_{DR}(V, W) = A^n(X, \underline{\mathrm{Hom}(V, W)})$$

  where $\underline{\mathrm{Hom}(V, W)}$ is the vector bundle of morphisms from V to W. The differential is given by

$$\omega \otimes f \mapsto f(\omega) \otimes f + (-1)^n \omega \wedge \triangledown(f).$$

  By the Riemann-Hilbert correspondence, A_{DR} is equivalent to the category of finite dimensional linear representations of the fundamental group of X.

But $\mathbf{dg-cat}$ is not just a category. It has been proven that it is also a model category.

Theorem [Tab10]
Let $\Delta_k(0)$ be the dg-category with one object and k as its complex of automorphisms; $\Delta_k(1, s, 1)$ the dg-category with two objects 0 and 1 where $\mathrm{Hom}(0, 1)$ is the chain complex that is valued 0 on every degree but for degree s where it is k; and $\Delta^c_k(1, s, 1)$ the dg-category with two objects where $\mathrm{Hom}(0, 1)$ is the chain complex that is valued 0 on every degree but for degrees s and $s + 1$, where it is k.
Then $\mathbf{dg-cat}$ has a cofibrantly generated model structure where the weak equivalences are the Morita equivalences and the generating cofibrations are $\emptyset \to \Delta_k(0)$ and $\Delta_k(1, s, 1) \to \Delta^c_k(1, s, 1)$ for all $s \in \mathbb{Z}$.

Even with that, dg-categories are still not really very manageable. For example, they have a monoidal structure and a model structure, but they are not compatible, which makes some notions a lot harder to define. As a consequence, we will be trying to follow the exemple set in ∞-categories, to find a description of dg-categories in terms of generators and relations, in a way that is actually easy to use.

Our project

For this, we are going to take inspiration from the aforementioned result by Tabuada and from the construction of ∞-categories as functors from Δ^{op} to $\mathbf{sSet}$.

Definition

We call Δ_k the category defined as follows:

- $\mathrm{Obj}(\Delta_k)$ consists of the $\Delta_k(n, s, d)$ with $n \in \mathbb{N}$, $s \in \mathbb{Z}^n$, $d \in \mathbb{N}^n$ where $\Delta_k(n, s, d)$ is the dg-category with $n + 1$ objects, and where for each $0 < i \leq n$, $\mathrm{Hom}(i - 1, i)$ is the chain complex which is 0 everywhere and k^{d_i} on degree s_i.
- The morphisms of Δ_k are the dg-morphisms between elements of the form $\Delta_k(n, s, d)$.

We already have a nice definition of what a morphism from a $\Delta_k(n, s, d)$ to a dg-category T looks like.

Proposition
Let $\Delta_k(n, s, d)$ be an object of Δ_k and T be a dg-category. Then, if we fix $n + 1$ objects in T, $(x_0, \ldots, x_n) \in \mathrm{Obj}(T)$,

$$\mathrm{Map}_{\mathbf{dg-cat}}(\Delta_k(n, s, d), T) = \mathrm{Map}_{\mathbf{C(k)}}(k, T(x_i, x_{i+1})[-s_i])^{d_i} = \prod Z^{-s_i}(T(x_i, x_{i+1}))^{d_i}.$$

To have a description of $\mathbf{dg-cat}$ equivalent to the one of complete Segal spaces we would need to make a Quillen adjunction like this:

$$F : \mathbf{dg-cat} \leftrightarrows \mathrm{Fun}(\Delta^{op}_k, \mathbf{sSet}) : Q.$$

But unfortunately, we would need for Δ_k to be discrete for it to work, and it isn't. More precisely, we would need to be able to say that $\mathrm{Map}(-, \Delta_k(n, s, d)) = \mathrm{Hom}(-, \Delta_k(n, s, d))$, and it is not: the mapping space is a groupoid, not a set. We will try to get around this problem by seeing both $\mathbf{dg-cat}$ and Δ_k as simplicial categories, or, in particular, (2,1)-categories.

Theorem (in progress)

There exists a chain of Quillen adjunctions and equivalences

$$F : \mathbf{dg-cat} \leftrightarrows \ldots \leftrightarrows \mathrm{Fun}^{\mathbb{S}}(\Delta^{op}_k, \mathbf{sSet}) : Q,$$

where we see Δ_k and $\mathbf{dg-cat}$ as (2,1)-categories, and where $\mathrm{Fun}^{\mathbb{S}}(\Delta^{op}_k, \mathbf{sSet})$ is the model category of simplicial functors from Δ_k to $\mathbf{sSet}$.
In particular, the functor $\mathrm{Ho}(\mathbf{dg-cat}) \to \mathrm{Ho}(\mathrm{Fun}^{\mathbb{S}}(\Delta^{op}_k, \mathbf{sSet}))$ is fully faithful.

And finally, if we are able to describe the image of $\mathbf{dg-cat}$ by F as some kind of k-linear complete Segal spaces, we will have a model for k-linear ∞-categories by generators and relations. We will clearly need the Segal condition, that says that

$$F(\Delta_k(n, s, d)) \times_{F([0])} F(\Delta_k(n', s', d')) \to F(\Delta_k(n + n', s \coprod s', d \coprod d'))$$

is a weak equivalence. In practice, this encodes the concatenation of $\Delta_k(n, s, d)$, and lets us work solely with the case $n = 1$, as the rest can be recovered by product of objects of the form $f(\Delta_k(1, s, d))$. But in this case we will have the linear structure to work through too. That means, in particular, finding a way to encode the sum on modules, and more importantly, the shift of complexes. This may be our biggest challenge, because although we can get all the negative shifts just by pushouts, in the case of positive shifts we only have a formula using homotopy pushouts.

Possible applications

- As always when dealing with presentations by generators and relations, the first and more obvious application would be to calculate explicitly functors from $\mathbf{dg-cat}$ to other spaces. Indeed, we now only need to compute the image of the elements of Δ_k, and make sure that they respect the weak equivalences.

  In that direction, [Toe05] uses its axiomatization of ∞-categories to calculate the group of automorphisms of ∞-categories, and finds

$$\mathrm{Aut}(CSS) = \mathbb{Z}/2\mathbb{Z}.$$

  We expect the group of automorphisms of $\mathbf{dg-cat}$ to also be interesting, and we think we should be able to compute it from our construction. That said, it won't be as simple as the one of CSS: we will obviously need the automorphisms of k, but probably also the Brauer group, and maybe even more things.

- Also, as we will have proven that every dg-category can be viewed as a simplicial functor from Δ_k to $\mathbf{sSet}$, we should be able to actually find the functors that give us those on the more prominent cases. Coming back to one of our examples, we should be able to define the dg-category of connections over a variety X, A_{DR}, as a simplicial functor from Δ^{op}_k to $\mathbf{sSet}$ that would code for us the representations of the fundamental group of X.
- On a completely different field of research, dg-categories appear naturally in superstring theory, in Physics, for example the dg-category of graded D-branes (graded matrix factorizations over schemes). A presentation by generators and relations of those could be useful as a way to simplify calculations.

References

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