OPTIMAL QUANTUM CLONING

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Duplication of quantum information

In quantum mechanics, the information is encoded in a quantum state ρ, a *positive* & *trace one* operator of a Hilbert space $\mathcal{H}$. A quantum state is pure if it is a rank one projector. Quantum information is transmitted by a quantum channel Φ, a *completly positive* & *trace preserving* linear map between spaces of operators.
The duplication of a pure quantum state, or quantum cloning, is a protocol:

$$\rho \mapsto \rho^{\otimes N},$$

when one wants to obtain N copies of ρ.
The no-cloning theorem states that such a quantum channel exists only for families of perfectly distinguishable pure states, that is mutually orthogonal pure states.

This scenario can be relaxed by allowing the output $\Phi(\rho)$ to be non product states. In this case, only the partial traces $\Phi_i(\rho)$ are required to be as close as possible to the input ρ. The quality of the partial traces are measured in terms of the quantum fidelity F: for a pure state ρ, and any state σ,

$$F(\rho,\sigma) = \mathrm{Tr}\left[\rho\,\sigma\right].$$

The average quantum cloning problem is defined as the optimization problem on quantum channel Φ:

$$\sup_{\Phi} \sum_{i=1}^{N} \alpha_i \cdot \mathbb{E}\left[F(\rho, \Phi_i(\rho))\right],$$

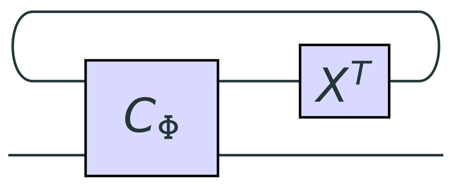
where the expectation is taken on a family of pure states ρ, and $\alpha \in [0,1]^N$.

Choi matrix

The Choi matrix C_Φ of any linear map Φ from $\mathcal{M}_d$ to $\mathcal{M}_{d'}$ is defined by:

$$C_\Phi = \left(\mathrm{id}_d \otimes \Phi\right)\left(\sum_{i,j=1}^{d} |i\rangle\langle j| \otimes |i\rangle\langle j|\right),$$

with the formula $\Phi(X) = \mathrm{Tr}_d\left[C_\Phi (X^T \otimes I_{d'})\right]$.

$\Phi(X) =$ 

A linear map $\Phi : \mathcal{M}_d \to \mathcal{M}_{d'}$ is a quantum channel if and only if

(1) $C_\Phi \geq 0$
(2) $\mathrm{Tr}_{d'}\left[C_\Phi\right] = I_d$.

Phase-covariant quantum cloning

The phase-covariant quantum cloning is the cloning of pure states of the form: $|\psi_\theta\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} e^{i\theta_j} |j\rangle,\ \theta \in [0, 2\pi)$.

$$\begin{aligned}\sum_{i=1}^{N} \alpha_i \mathop{\mathbb{E}}_{\text{phase-co.}}\left[F(\rho, \Phi_i(\rho))\right] &= \sum_{i=1}^{N} \alpha_i \cdot \mathbb{E}\left[\mathrm{Tr}\left(\rho\, \Phi_i(\rho)\right)\right] \\ &= \sum_{i=1}^{N} \alpha_i \cdot \mathbb{E}\left[\mathrm{Tr}\left(\rho_{(i)} \otimes I_d^{\otimes(N-1)}\, \Phi(\rho)\right)\right] \\ &= \sum_{i=1}^{N} \alpha_i \cdot \mathbb{E}\left[\mathrm{Tr}\left(\rho_{(1)}^T \otimes \rho_{(i)} \otimes I_d^{\otimes(N-1)}\, C_\Phi\right)\right].\end{aligned}$$

The expectation $\mathbb{E}\left[\rho^T \otimes \rho\right]$ on phase-covariant pure states is equal to

$$\begin{aligned}\mathbb{E}\left[\rho^T \otimes \rho\right] &= \frac{1}{d^2}\left(\sum_{i,j=1}^{d} |ij\rangle\langle ij| + \sum_{i,j=1}^{d} |ii\rangle\langle jj| - \sum_{i=1}^{d} |ii\rangle\langle ii|\right) \\ &= \frac{1}{d^2}\left(I_d^{\otimes 2} + \omega - X\right).\end{aligned}$$

The optimization problem is then upper bounded by

$$\begin{aligned}\sum_{i=1}^{N} \alpha_i \mathop{\mathbb{E}}_{\text{phase-co.}}\left[F(\rho, \Phi_i(\rho))\right] &= \frac{1}{d^2} \sum_{i=1}^{N} \alpha_i \cdot \mathrm{Tr}\left[\left(I_d^{\otimes 2} + \omega - X\right)_{(1i)} \otimes I_d^{\otimes(N-1)}\, C_\Phi\right] \\ &\leq \frac{\mathrm{Tr}\left[C_\Phi\right]}{d^2} \lambda_{\max}\left[\sum_{i=1}^{N} \alpha_i \cdot \left(I_d^{\otimes 2} + \omega - X\right)_{(1i)} \otimes I_d^{\otimes(N-1)}\right] \\ &\leq \frac{1}{d} \lambda_{\max}\left[\sum_{i=1}^{N} \alpha_i \cdot \left(I_d^{\otimes 2} + \omega - X\right)_{(1i)} \otimes I_d^{\otimes(N-1)}\right],\end{aligned}$$

where $\lambda_{\max}(\cdot)$ is the largest eigenvalue.

Symmetric quantum cloning of qubits $[N = 2]$

When $\alpha = (1, 1)$ we are in the case of symmetric quantum cloning. The largest eigenvalues can be found in the qubit case ($d = 2$):

$$\lambda_{\max}\left[\sum_{i=1}^{2} \alpha_i \cdot \left(I_2^{\otimes 2} + \omega - X\right)_{(1i)} \otimes I_2\right] = 1 + \frac{1}{\sqrt{2}},$$

$$\lambda_{\max}\left[\sum_{i=1}^{2} \alpha_i \cdot \left(I_2^{\otimes 2} + \omega\right)_{(1i)} \otimes I_2\right] = \frac{5}{2}.$$

We find the well-known bounds on the fidelity of phase-covariant quantum cloning (PQC) and universal quantum cloning (UQC):

(PQC) $\dfrac{1+\sqrt{2}}{2\sqrt{2}} \approx 0.85,$

(UQC) $\dfrac{5}{6} \approx 0.83.$

Universal quantum cloning

In the case of the universal quantum cloning, the expectation is taken on all the pure states ρ.

$$\sum_{i=1}^{N} \alpha_i \mathop{\mathbb{E}}_{\text{pure}}\left[F(\rho, \Phi_i(\rho))\right] = \sum_{i=1}^{N} \alpha_i \cdot \mathbb{E}\left[\mathrm{Tr}\left(\rho_{(1)}^T \otimes \rho_{(i)} \otimes I_d^{\otimes(N-1)}\, C_\Phi\right)\right].$$

The expectation $\mathbb{E}\left[\rho^T \otimes \rho\right]$ on all pure states becomes

$$\mathbb{E}\left[\rho^T \otimes \rho\right] = \frac{1}{d(d+1)}\left(I_d^{\otimes 2} + \omega\right).$$

Similarly the optimization problem is then upper bounded by

$$\sum_{i=1}^{N} \alpha_i \mathop{\mathbb{E}}_{\text{phase-co.}}\left[F(\rho, \Phi_i(\rho))\right] \leq \frac{1}{d+1} \lambda_{\max}\left[\sum_{i=1}^{N} \alpha_i \cdot \left(I_d^{\otimes 2} + \omega\right)_{(1i)} \otimes I_d^{\otimes(N-1)}\right].$$

Expectations on families of pure states

The expectation taken on all the pure states ρ is defined by

$$\mathop{\mathbb{E}}_{\text{pure}}\left[f(\rho)\right] = \int f\left(|x\rangle\langle x|\right) \mathrm{d}\nu(x),$$

where ν is the uniform measure on the set of pure states. Equivalently we can write

$$\mathop{\mathbb{E}}_{\text{pure}}\left[f(\rho)\right] = \int_{\mathcal{U}_d} f\left(U|+\rangle\langle+|U^*\right) \mathrm{d}U,$$

where the integral is taken with respect to the normalized Haar measure on the unitary group $\mathcal{U}_d$, and $|+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle$. Similarly the expectation taken on phase-covariant pure states is

$$\mathop{\mathbb{E}}_{\text{phase-co.}}\left[f(\rho)\right] = \int_{\mathcal{U}_d \text{ diag.}} f\left(U|+\rangle\langle+|U^*\right) \mathrm{d}U,$$

where the integral is taken this time on diagonal unitaries.
The two previous formulas

$$\mathop{\mathbb{E}}_{\text{pure}}\left[\rho^T \otimes \rho\right] \quad \text{and} \quad \mathop{\mathbb{E}}_{\text{phase-co.}}\left[\rho^T \otimes \rho\right]$$

can be found using Weingarten calulus and integration over random diagonal unitary matrices [I. Nechita, S. Singh].

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