A GEOMETRICAL DESCRIPTION OF THE 1 → 2 QUANTUM CLONING REGION

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Is there a perfect 1 → 2 quantum cloning map?

A perfect 1 → 2 quantum cloning map is a quantum channel $T : \mathcal{M}_d \to \mathcal{M}_d \otimes \mathcal{M}_d$, compatible with the identity, i.e. for all pure states ρ

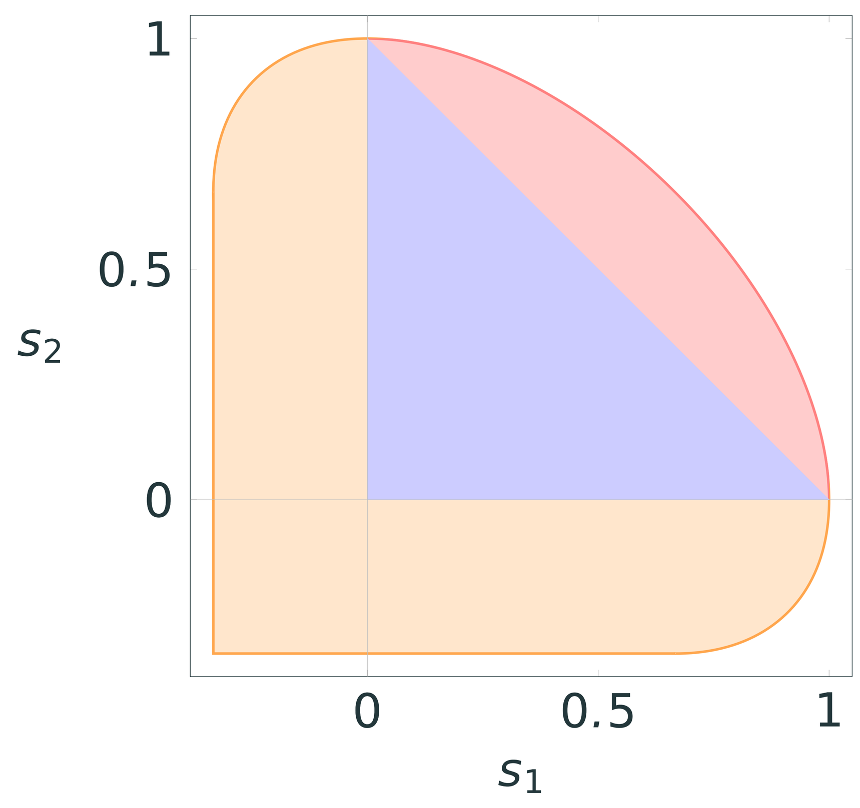
$$T_i(\rho) = \rho, \qquad \forall i \in \{1, 2\}.$$

The no-cloning theorem states that such a quantum channel cannot exist. Perfect quantum cloning being possible only for families of perfectly distinguishable quantum states, that is mutually orthogonal quantum states.

The asymmetric 1 → 2 quantum cloning problem consists in determining the set of $s \in [0,1]^2$ such that there exists a 1 → 2 quantum cloning map T with marginals equal on all pure states ρ to

$$T_i(\rho) = s_i \cdot \rho + (1 - s_i)\frac{I_d}{d}, \qquad \forall i \in \{1, 2\}.$$

The admissible region contains the trivial convex set ■, a quantum advantage set ■ that decreases with the dimension d, as well as a negative part ■.

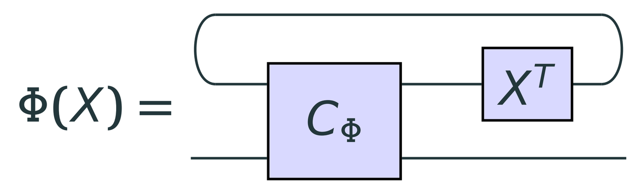


Permutation operators

The Choi matrix C_Φ of a linear map $\Phi : \mathcal{M}_d \to \mathcal{M}_{d'}$ is defined by

$$C_\Phi = (\mathrm{id}_d \otimes \Phi)\left(\sum_{i,j=1}^{d} |i\rangle\langle j| \otimes |i\rangle\langle j|\right)$$

One can recover the linear map Φ from the Choi matrix C_Φ by the formula $\Phi(X) = \mathrm{Tr}_d\left[C_\Phi(X^T \otimes I_{d'})\right]$.



A linear map $\Phi : \mathcal{M}_d \to \mathcal{M}_{d'}$ is a quantum channel if and only if

(1) $C_\Phi \geq 0$

(2) $\mathrm{Tr}_{d'}\left[C_\Phi\right] = I_d$.

The symmetrization of a 1 → 2 quantum cloning map T is a quantum cloning map defined by

$$\widetilde{T}(\rho) = \int_{\mathcal{U}_d} (U \otimes U)\, T(U^* \rho\, U)\, (U^* \otimes U^*)\, \mathrm{d}U,$$

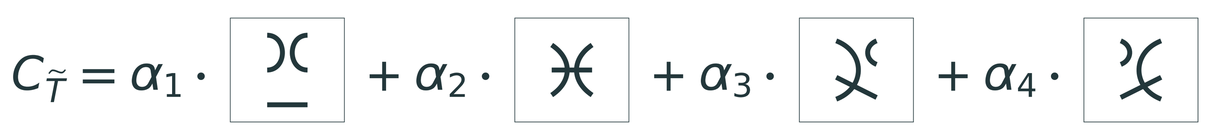
where the integral is taken with respect to the normalized Haar measure on the unitary group $\mathcal{U}_d$. The admissible region is covered by the symmetrized quantum cloning maps. The Choi matrix of a symmetrized 1 → 2 quantum cloning map $\widetilde{T}$ is a complex linear combination of partially transposed permutation operators, i.e.

$$C_{\widetilde{T}} = \sum_{\pi \in \mathfrak{S}_3} \alpha_\pi \cdot V^\Gamma(\pi),$$

where V is the natural representation of the symmetric group $\mathfrak{S}_3$ acting on $(\mathbb{C}^d)^{\otimes 3}$, and the partial transposition acts on the first copy of $\mathcal{M}_d$.

Restricted quantum cloning maps

We consider the restricted case when the Choi matrix $C_{\widetilde{T}}$ is a linear combination of only 4 partially transposed permutation operators of $\mathfrak{S}_3$, i.e.

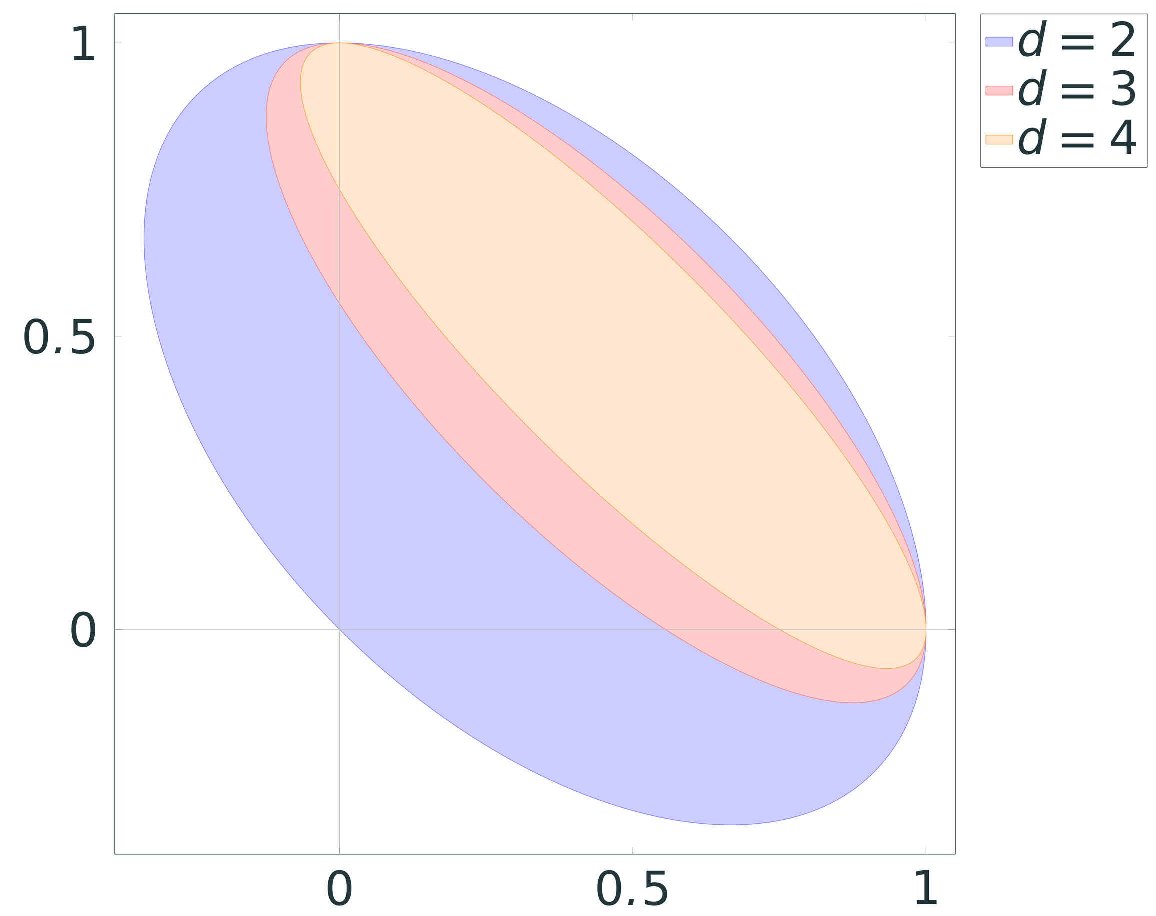


We want to know when conditions (1) and (2) are satisfied. The 4 partially transposed permutation operators can be block diagonalized in the basis of the $2d$ vectors

The block diagonal decomposition of the Choi matrix $C_{\widetilde{T}}$ becomes

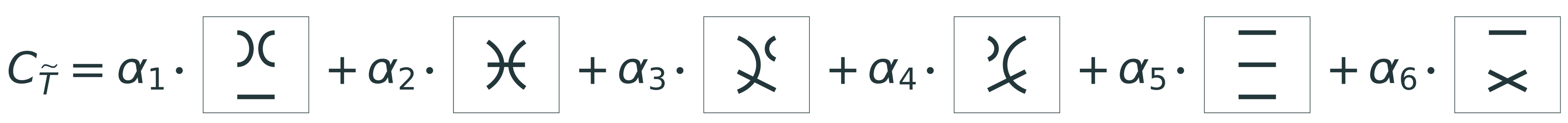
$$C_{\widetilde{T}} = \begin{pmatrix} d\,\alpha_1 + \alpha_4 & \alpha_1 + d\,\alpha_4 \\ \alpha_2 + d\,\alpha_3 & d\,\alpha_2 + \alpha_3 \end{pmatrix}^{\oplus d} \bigoplus 0^{\oplus (d^3 - 2d)}$$

The admissible region for the restricted 1 → 2 asymmetric quantum cloning is the ellipse given by $\left\{(s_1, s_2) \in \left[\frac{-1}{d^2-1}, 1\right]^2 \;:\; \frac{(1-s_1)(1-s_2)}{d^2} \geq \left(\frac{s_1+s_2-1}{2}\right)^2\right\}$.



General quantum cloning maps

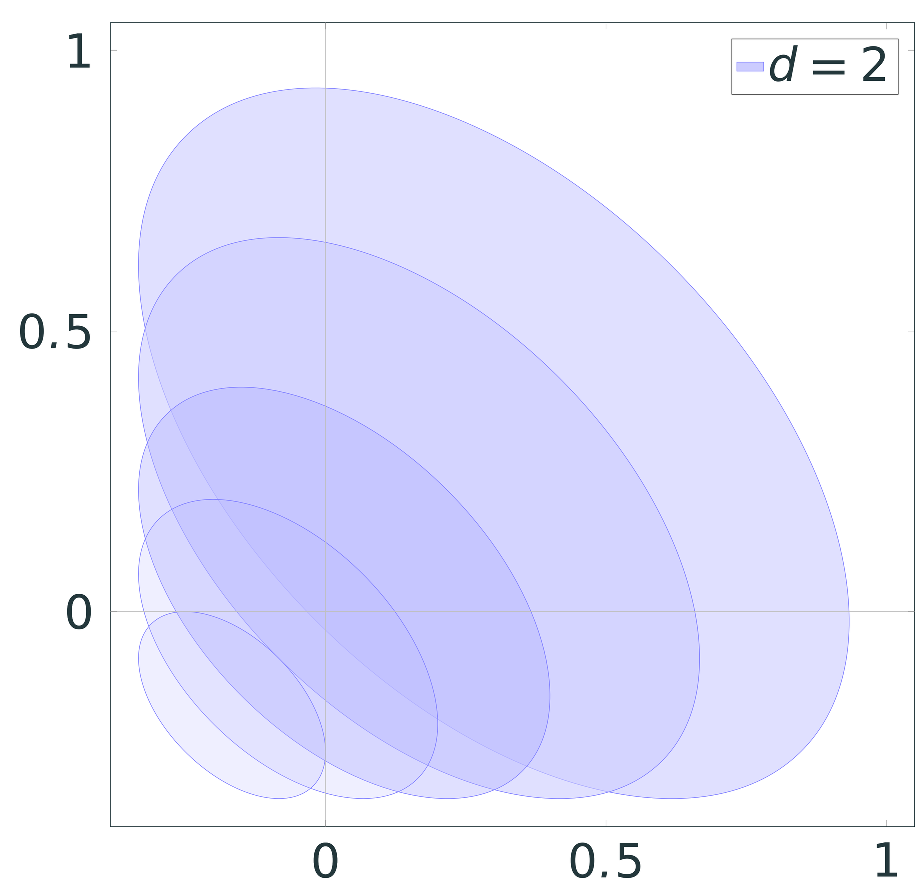
In the general case, the Choi matrix $C_{\widetilde{T}}$ is a linear combination of the 6 partially transposed permutation operators of $\mathfrak{S}_3$, i.e.



The admissible region of the 1 → 2 asymmetric quantum cloning is the union of a family of ellipses given by

$$\frac{(s_1 - s_2)^2}{a^2} + \frac{\left((s_1 + s_2) - c\right)^2}{b^2} \leq \lambda^2,$$

with $a = \frac{1}{\sqrt{d^2-1}}$, $b = \frac{1}{d^2-1}$, $c = \frac{\lambda d - 2}{d^2-1}$ and $\lambda \in [0, d]$.



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