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Decompositions of high-frequency Helmholtz solutions and application to the finite element method

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(joint work with Jeffrey Galkowski, Euan A. Spence, Jared Wunsch)

Motivation and informal statement of our results. We are interested in the Helmholtz equation in the exterior of an obstacle $\mathcal{O}$, with Dirichlet boundary condition and Sommerfeld radiation condition at infinity (corresponding to the fact that we are looking for an *outgoing* wave)

$$\begin{cases} \Delta u + k^2 u = f & \text{in } \mathbb{R}^d \backslash \mathcal{O}, \\ u = 0 & \text{on } \partial\mathcal{O}, \\ \partial_r u - iku = o(r^{-(d-1)/2}) & \text{as } r \to \infty. \end{cases}$$

A popular choice to solve numerically such an equation is the hp-finite element method (hp-FEM), where one decreases the meshsize h and increases the polynomial degree p of the approximation, both depending on the frequency k of the solution, to obtain accuracy. A natural question in this framework is the following: *what is a condition on h, p, and k for these methods to converge?* As the solution oscillates at scale k^{-1}, we should need at least a number of degrees of freedom $\#\text{DOF} \gtrsim k^d$. Is it enough ?

Melenk and Sauter [MS10, MS11] gave a positive answer to this question when the obstacle is *analytic* (see also [MPS13] and [EM12] for the interior impedance problem). They have shown that, under the conditions

$$\frac{hk}{p} \leq C_1, \qquad p \geq C_2 \log k,$$

the solution to the discrete problem exists, is unique, and is quasi-optimal (that is, it is the best possible approximation of the solution by a piecewise polynomial, up to a multiplicative constant). In particular, under these conditions, one can construct h and p so that the number of degrees of freedom of the problem verifies

$$\#\text{DOF} \simeq \left(\frac{p}{h}\right)^d \lesssim k^d.$$

In other words, hp-FEM applied to this setting does not suffer from the *pollution effect* that plagues the h-FEM (where p is left constant), for which one needs strictly more degrees of freedom than k^d to maintain accuracy [BS00].

The proof of Melenk and Sauter [MS10, MS11] is based on a decomposition of the Helmholtz solutions

$$(\star) \qquad u = u_{H^2} + u_{\mathcal{A}},$$

where u_{H^2} verifies *better* estimates in the frequency k than u, and $u_{\mathcal{A}}$ verifies the same estimates in k as u but is *analytic*. The idea is that u_{H^2} contains the high frequencies ($\gtrsim k$) of the solution, and $u_{\mathcal{A}}$ the low frequencies ($\lesssim k$). Their proof of the decomposition ($\star$) is based on explicit computations that cannot be generalised in a straightforward way to more general problems, such as the Helmholtz equation with variables coefficients, despite the large interest for such a problem.

In the works [LSW22, GLSW21, GLSW22], we tackled the question of *understanding the frequency-decomposition ($\star$) in the most general possible situation.* We obtained the following results.

(1) In [LSW22], we obtained the decomposition ($\star$) for the variable C^∞ coefficients equation in $\mathbb{R}^d$.
(2) Then, in [GLSW21], we have shown such a decomposition in the very general *black-box scattering* framework of Sjöstrand-Zworski.
(3) Finally, in [GLSW22], we extended this result to the problem truncated with a Perfectly Matched Layer (PML).

In particular, one can apply our results to show that *hp*-FEM applied to the equation

(a) without obstacle and with variable C^∞ coefficients,
(b) posed in the exterior of an analytic obstacle and with variable C^∞ coefficients which are analytic near the obstacle,

does not suffer from the pollution effect, both for the outgoing problem and PML.

Some ideas behind the proofs of the results [LSW22, GLSW21, GLSW22]**.** The decomposition in $\mathbb{R}^d$ for the C^∞ variable-coefficients equation [LSW22] is obtained by projecting the solution u, spatially truncated in the ball $B(0,R)$ where we seek to obtain the decomposition, on its high ($\gtrsim k$) and low ($\lesssim k$) Fourier modes. In other words, we define

$$u_{H^2} := \Pi_{\mathrm{High}}(\varphi u), \quad u_{\mathcal{A}} := \Pi_{\mathrm{Low}}(\varphi u),$$

where $\varphi \in C_c^\infty$ is equal to one in $B(0,R)$, Π_{Low} is defined as a Fourier multiplier truncating in Fourier variables $\leq \mu k$ for some $\mu \gg 1$, and $\Pi_{\mathrm{High}} := I - \Pi_{\mathrm{Low}}$. Thanks to its Fourier localisation, it is immediate to see that $u_{\mathcal{A}}$ is analytic, and even entire, using Parseval identity. On the other hand, the bound on u_{H^2} is obtained using *semiclassical ellipticity*: for μ large enough, u_{H^2} lives in phase-space where the equation is invertible modulo negligible terms.

Attempting to generalize this method [LSW22] to setups including boundaries, we run into technical issues involving the extension of solutions to the whole space when trying to use frequency projections defined from Fourier multipliers. Instead, we have another idea: rather use frequency projections defined *through the functional calculus.* In other words, we define

$$\Pi_{\mathrm{High}} = (1-\psi)(P), \quad \Pi_{\mathrm{Low}} := \psi(P),$$

where P is the operator associated with our Helmholtz equation $Pu + k^2u = f$ and $\psi \in C_c^\infty(\mathbb{R})$, and u_{H^2} and $u_{\mathcal{A}}$ will be defined as previously. This idea has two immediate advantages: these projections commute with the equation, and we can now try to work with an operator P as general as possible. Taking advantage of the later, we will work in the very general *black-box scattering* framework of Sjöstrand and Zworski [SZ91], where in addition to some suitable compatibility conditions, P is only assumed to be a self-adjoint operator coinciding with the Laplacian outside "the black-box" $B(0, R_0)$, where it is left unspecified. Following this idea, we were able to show a very general, albeit abstract decomposition result, reading in an informal way:

Theorem 1 (Main abstract decomposition from [GLSW21], informal version)**.** *Let P be a black-box scattering operator of Sjöstrand-Zworski. We make the following assumptions.*

- (H1) *The solution operator associated with the Helmholtz equation is polynomially bounded in the frequency k.*
- (H2) *One has an estimate quantifying the regularity of P "inside the black-box" $B(0, R_0)$.*

Then, any solution u of the Helmholtz equation $(P+k^2)u = k^2 f$ can be decomposed as

$$u = u_{H^2} + u_{\mathcal{A}}.$$

Where

- *u_{H^2} verifies a black-box version of the estimate*
$$\|u\|_{L^2} + k^{-m}\|u\|_{\dot{H}^m} \lesssim \|f\|_{L^2}.$$
- *$u_{\mathcal{A}}$ verifies the same estimates in k as u but is* regular*. This regularity is dictated by the regularity of the underlying problem as measured by (H2).*

The bound on u_{H^2} relies once again on ellipticity : near the black-box, we are able to show an abstract ellipticity result from functional-calculus abstract manipulations; whereas away from the black-box, the functional calculus coincides with the semiclassical pseudo-differential calculus up to negligible terms as observed by Sjöstrand [Sj97], and we are able to use genuine semiclassical ellipticity as in [GLSW21]. On the other hand, the regularity bounds on $u_{\mathcal{A}}$ follow from the morphism property of the functional calculus together with the estimate (H2).

As the assumption (H1) (arising similarly in [LSW22]) always holds outside a set of frequencies of arbitrarily small measure [LSW21], the key to apply such a result to concrete Helmholtz problems is to find a suitable estimate of type (H2). For example, outside an analytic Dirichlet obstacle for the equation with C^∞ variable-coefficients which are analytic near the obstacle, we are able to use as (H2) an heat-flow estimate (more precisely, we combine a folklore estimate tracing back to [Fri69] with the more recent [EMZ17]) to obtain a decomposition in this set-up, allowing us to show the sharp convergence result for *hp*-FEM.

Whereas we first obtained these results for the outgoing Helmholtz solutions, the corresponding PML problems have the substantial additional difficulty that the scaled Laplacian is a non self-adjoint operator. In [GLSW22], building on the outgoing case and the recent progress [GLS21] on PML accuracy, we were able to obtain strictly analogous results in such a setup, using frequency cut-offs defined via the *non-scaled* calculus as in [GLSW21] together with the (semiclassical) ellipticity of PML in the scaling region.

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