Handsome Proof-Nets for Cyclic Linear Logic

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Organisation

Logic and Natural Language

Combining Words

John *sleeps* is a (English) sentence

$NP \quad \wedge \quad (NP \Rightarrow S) \qquad \vdash \quad S$

John *loves* *Mary* is a (English) sentence

$NP \quad \wedge \quad (NP \Rightarrow (NP \Rightarrow S)) \quad \wedge \quad NP \qquad \vdash \quad S$

loves *John* *Mary* is not a (English) sentence

$(NP \Rightarrow (NP \Rightarrow S)) \quad \wedge \quad NP \quad \wedge \quad NP \qquad \vdash \quad \not\vdash \quad S$

Non commutativity

John *loves* is not a (English) sentence

$NP \quad \wedge \quad (NP \Rightarrow (NP \Rightarrow S)) \qquad \vdash \quad \not\vdash \quad S$

Resource sensitivity

Principle

A sequence of worlds is a sentence if, from the sequence of the associated propositions, we can prove S

Sequent Calculus

Propositional Logic

Definition (Sequent)

A *sequent* is a triple, denoted by $\Gamma \vdash [\Delta'] \, \Delta$, where Γ, Δ, and Δ' are multi-sets of formulas.

Ex.: $NP, NP \Rightarrow S \vdash S$, $\qquad NP \Rightarrow S \vdash NP^{\perp}, S$

Definition (Inference rules for LK)

Identities

$$\frac{}{\vdash_{\mathrm{LK}} A, A^{\perp}}\,\mathrm{Id} \qquad \frac{\vdash_{\mathrm{LK}} \Gamma, A \quad \vdash_{\mathrm{LK}} A^{\perp}, \Delta}{\vdash_{\mathrm{LK}} [A \bullet A^{\perp}]\ \Gamma, \Delta}\,\mathrm{Cut}$$

Logical rules

$$\frac{\vdash_{\mathrm{LK}} \Gamma, A \quad \vdash_{\mathrm{LK}} B, \Delta}{\vdash_{\mathrm{LK}} \Gamma, A \wedge B, \Delta}\,\wedge \qquad \frac{\vdash_{\mathrm{LK}} \Gamma, A, B, \Gamma'}{\vdash_{\mathrm{LK}} \Gamma, A \vee B, \Gamma'}\,\vee$$

Structural rules

$$\frac{\vdash_{\mathrm{LK}} \Gamma}{\vdash_{\mathrm{LK}} A, \Gamma}\,\mathrm{W} \qquad \frac{\vdash_{\mathrm{LK}} A, A, \Gamma}{\vdash_{\mathrm{LK}} A, \Gamma}\,\mathrm{C}$$

$$\frac{\vdash_{\mathrm{LK}} \Gamma, A}{\vdash_{\mathrm{LK}} A, \Gamma}\,\mathrm{CEx} \qquad \frac{\vdash_{\mathrm{LK}} \Gamma, A, B}{\vdash_{\mathrm{LK}} \Gamma, B, A}\,\mathrm{Ex}$$

Sequent Calculus

Proofs

Example ($\vdash \; P \Rightarrow (P \Rightarrow Q) \Rightarrow Q$)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_{\mathrm{LK}} \; Q, Q^{\perp}}\,\mathrm{Id} \quad \dfrac{}{\vdash_{\mathrm{LK}} \; P, P^{\perp}}\,\mathrm{Id}}{\vdash_{\mathrm{LK}} \; Q, Q^{\perp} \wedge P, P^{\perp}}\,\wedge}{\vdash_{\mathrm{LK}} \; P^{\perp}, Q, Q^{\perp} \wedge P}\,\mathrm{CEx}}{\vdash_{\mathrm{LK}} \; Q^{\perp} \wedge P, P^{\perp}, Q}\,\mathrm{CEx}}{\vdash_{\mathrm{LK}} \; Q^{\perp} \wedge P, Q, P^{\perp}}\,\mathrm{Ex}}{\vdash_{\mathrm{LK}} \; P^{\perp}, Q^{\perp} \wedge P, Q}\,\mathrm{CEx}}{\vdash_{\mathrm{LK}} \; P^{\perp} \vee (Q^{\perp} \wedge P) \vee Q}\,\vee}{\vdash_{\mathrm{LK}} \; P \Rightarrow (P \Rightarrow Q) \Rightarrow Q}\;\text{By definition of } \Rightarrow$$

Theorem

A sequent is valid ($\vDash \Gamma$) if and only if $\vdash_{LK} \; \Gamma$ is provable.

Sequent Calculus

Linear Logic (Girard 1987)

Definition (Inference rules for LKLLCyLL)

Identities

$$\frac{}{\vdash\ A, A^{\perp}}\,\mathrm{Id} \qquad \frac{\vdash\ \Gamma, A \quad \vdash\ A^{\perp}, \Delta}{\vdash\ \Gamma, \Delta}\,\mathrm{Cut}$$

Logical rules

$$\frac{\vdash\ \Gamma, A \quad \vdash\ B, \Delta}{\vdash\ \Gamma, A \wedge B, \Delta}\,\wedge \qquad \frac{\vdash\ \Gamma, A, B, \Gamma'}{\vdash\ \Gamma, A \vee B, \Gamma'}\,\vee$$

Structural rules

$$\frac{\vdash\ \Gamma}{\vdash\ A, \Gamma}\,\mathrm{W} \qquad \frac{\vdash\ A, A, \Gamma}{\vdash\ A, \Gamma}\,\mathrm{C}$$

$$\frac{\vdash\ \Gamma, A}{\vdash\ A, \Gamma}\,\mathrm{CEx} \qquad \frac{\vdash\ \Gamma, A, B}{\vdash\ \Gamma, B, A}\,\mathrm{Ex}$$

- Resource sensitivity: Linear Logic (LL, Girard (1987))
- Non commutativity: Cyclic Linear Logic (CyLL, Yetter (1990))
- For all proofs of $\vdash\ \Gamma$, there exists a proof of $\vdash\ \Gamma$ that does not use Cut (Gentzen 1935; Gentzen 1955, for LK), (Girard 1995, for LL).
- Intuitionnistic fragment of CyLL: Lambek Calculus (Lambek 1958)

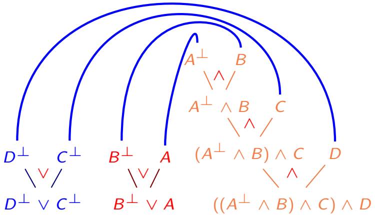
Proof Trees

Spurious Ambiguities

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}\text{Id} \quad \dfrac{}{\vdash B, B^{\perp}}\text{Id}}{\vdash A, A^{\perp}\wedge B, B^{\perp}}\wedge}{\vdash B^{\perp}, A, A^{\perp}\wedge B}\text{CEx} \quad \dfrac{}{\vdash C, C^{\perp}}\text{Id}}{\vdash B^{\perp}, A, (A^{\perp}\wedge B)\wedge C, C^{\perp}}\wedge}{\vdash C^{\perp}, B^{\perp}, A, (A^{\perp}\wedge B)\wedge C}\text{CEX} \quad \dfrac{}{\vdash D, D^{\perp}}\text{Id}}{\vdash C^{\perp}, B^{\perp}, A, ((A^{\perp}\wedge B)\wedge C)\wedge D, D^{\perp}}\wedge}{\vdash D^{\perp}, C^{\perp}, B^{\perp}, A, ((A^{\perp}\wedge B)\wedge C)\wedge D}\text{CEx}}{\vdash D^{\perp}, C^{\perp}, B^{\perp}\vee A, ((A^{\perp}\wedge B)\wedge C)\wedge D}\vee}{\vdash D^{\perp}\vee C^{\perp}, B^{\perp}\vee A, ((A^{\perp}\wedge B)\wedge C)\wedge D}\vee$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}\text{Id} \quad \dfrac{}{\vdash B, B^{\perp}}\text{Id}}{\vdash A, A^{\perp}\wedge B, B^{\perp}}\wedge}{\vdash B^{\perp}, A, A^{\perp}\wedge B}\text{CEx} \quad \dfrac{}{\vdash C, C^{\perp}}\text{Id}}{\vdash B^{\perp}, A, (A^{\perp}\wedge B)\wedge C, C^{\perp}}\wedge}{\vdash C^{\perp}, B^{\perp}, A, (A^{\perp}\wedge B)\wedge C}\text{CEX} \quad \dfrac{}{\vdash D, D^{\perp}}\text{Id}}{\vdash C^{\perp}, B^{\perp}, A, ((A^{\perp}\wedge B)\wedge C)\wedge D, D^{\perp}}\wedge}{\vdash D^{\perp}, C^{\perp}, B^{\perp}, A, ((A^{\perp}\wedge B)\wedge C)\wedge D}\text{CEx}}{\vdash D^{\perp}\vee C^{\perp}, B^{\perp}, A, ((A^{\perp}\wedge B)\wedge C)\wedge D}\vee}{\vdash D^{\perp}\vee C^{\perp}, B^{\perp}\vee A, ((A^{\perp}\wedge B)\wedge C)\wedge D}\vee$$

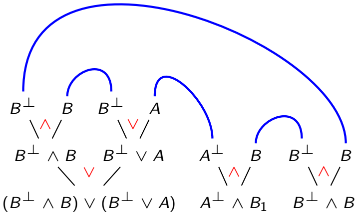
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}\text{Id} \quad \dfrac{}{\vdash B, B^{\perp}}\text{Id}}{\vdash A, A^{\perp}\wedge B, B^{\perp}}\wedge}{\vdash B^{\perp}, A, A^{\perp}\wedge B}\text{CEx} \quad \dfrac{}{\vdash C, C^{\perp}}\text{Id}}{\vdash B^{\perp}, A, (A^{\perp}\wedge B)\wedge C, C^{\perp}}\wedge}{\vdash C^{\perp}, B^{\perp}, A, (A^{\perp}\wedge B)\wedge C}\text{CEX} \quad \dfrac{}{\vdash D, D^{\perp}}\text{Id}}{\vdash C^{\perp}, B^{\perp}, A, ((A^{\perp}\wedge B)\wedge C)\wedge D, D^{\perp}}\wedge}{\vdash C^{\perp}, B^{\perp}\vee A, ((A^{\perp}\wedge B)\wedge C)\wedge D, D^{\perp}}\vee}{\vdash D^{\perp}, C^{\perp}, B^{\perp}\vee A, ((A^{\perp}\wedge B)\wedge C)\wedge D}\text{CEx}}{\vdash D^{\perp}\vee C^{\perp}, B^{\perp}\vee A, ((A^{\perp}\wedge B)\wedge C)\wedge D}\vee$$



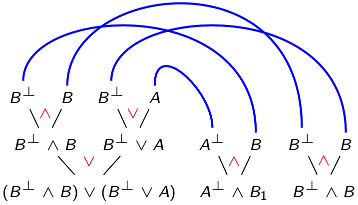
Proof Trees

Actual Ambiguities

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$\vdash A, A^\perp$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$\vdash B_1, B_1^\perp$}
\RightLabel{$\wedge$}
\BinaryInfC{$\vdash A, A^\perp \wedge B_1, B_1^\perp$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$\vdash B_2, B_2^\perp$}
\RightLabel{$\wedge$}
\BinaryInfC{$\vdash A, A^\perp \wedge B_1, B_1^\perp \wedge B_2, B_2^\perp$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$\vdash B_3, B_3^\perp$}
\RightLabel{$\wedge$}
\BinaryInfC{$\vdash A, A^\perp \wedge B_1, B_1^\perp \wedge B_2, B_2^\perp \wedge B_3, B_3^\perp$}
\RightLabel{CEx}
\UnaryInfC{$\vdash B_3^\perp, A, A^\perp \wedge B_1, B_1^\perp \wedge B_2, B_2^\perp \wedge B_3$}
\RightLabel{CEx}
\UnaryInfC{$\vdash B_2^\perp \wedge B_3, B_3^\perp, A, A^\perp \wedge B_1, B_1^\perp \wedge B_2$}
\RightLabel{$\vee$}
\UnaryInfC{$\vdash B_2^\perp \wedge B_3, B_3^\perp \vee A, A^\perp \wedge B_1, B_1^\perp \wedge B_2$}
\RightLabel{$\vee$}
\UnaryInfC{$\vdash (B_2^\perp \wedge B_3) \vee (B_3^\perp \vee A), A^\perp \wedge B_1, B_1^\perp \wedge B_2$}
\end{prooftree}
$$



$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$\vdash A, A^\perp$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$\vdash B_1, B_1^\perp$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$\vdash B_2, B_2^\perp$}
\RightLabel{$\wedge$}
\BinaryInfC{$\vdash B_1, B_1^\perp \wedge B_2, B_2^\perp$}
\RightLabel{$\wedge$}
\BinaryInfC{$\vdash A, A^\perp \wedge B_1, B_1^\perp \wedge B_2, B_2^\perp$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$\vdash B_3, B_3^\perp$}
\RightLabel{$\wedge$}
\BinaryInfC{$\vdash A, A^\perp \wedge B_1, B_1^\perp \wedge B_2, B_2^\perp \wedge B_3, B_3^\perp$}
\RightLabel{CEx}
\UnaryInfC{$\vdash B_3^\perp, A, A^\perp \wedge B_1, B_1^\perp \wedge B_2, B_2^\perp \wedge B_3$}
\RightLabel{$\vee$}
\UnaryInfC{$\vdash B_3^\perp \vee A, A^\perp \wedge B_1, B_1^\perp \wedge B_2, B_2^\perp \wedge B_3$}
\RightLabel{Ex}
\UnaryInfC{$\vdash B_3^\perp \vee A, A^\perp \wedge B_1, B_2^\perp \wedge B_3, B_1^\perp \wedge B_2$}
\RightLabel{CEx}
\UnaryInfC{$\vdash B_1^\perp \wedge B_2, B_3^\perp \vee A, A^\perp \wedge B_1, B_2^\perp \wedge B_3$}
\RightLabel{$\vee$}
\UnaryInfC{$\vdash (B_1^\perp \wedge B_2) \vee (B_3^\perp \vee A), A^\perp \wedge B_1, B_2^\perp \wedge B_3$}
\end{prooftree}
$$



R&B-Graphs

Definition (Matching)

A set of edges in a graph G is a *matching* if no two edges are adjacent.
A matching is *perfect* if every vertex is incident to an edge of the matching.

Definition (R&B-Graphs)

A *R&B-graph* G is a triple (V, B, R) such that:

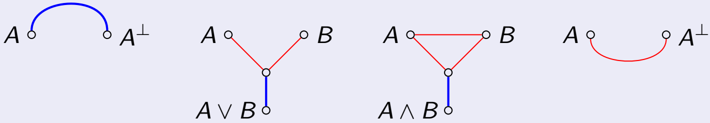
- V is a set of vertices;
- (V, B) and (V, R) are simple graphs;
- B is a perfect matching of the underlying (multi-) graph $\underline{G} = (V, B \cup R)$.

Proof-Structures and Proof-Nets of Linear Logic

Definition (Links (Girard 1987; Retoré 1999; Retoré 2003))

Link	*Axiom*	*Disjunction*	*Conjunction*	*Cut*

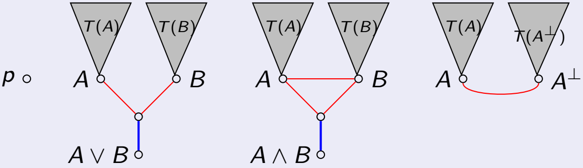
R&B-graphs



Definition (R&B-Tree of a Formula)

Formula C $\quad p \quad A \vee B \quad A \wedge B \quad A \bullet A^\perp$

R&B-tree $T(C)$



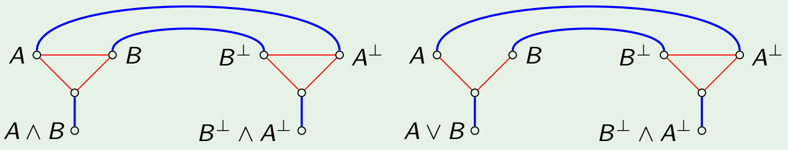
Proof-Structures and Proof-Nets

Examples

Definition (Proof-Structure)

A *proof-structure* for $C_1, \ldots, C_n$ is a sequence of R&B-trees $T(C_1), \cdots, T(C_n)$ with B-edges between each pair of dual atoms A and $A^\perp$.

Example



Definition (Proof-Net (Correctness Criterion))

A *proof-net* is a proof-structure that does not contain any æ-cycle. Moreover, there is an æ-path between any two vertices.

Proof-Nets and Sequent Calculus

Theorem (From Sequents to Proof-Nets)

For any LL proof π *of* $\vdash\ [\Delta]\ \Gamma$*, there is a proof-net with conclusions* Γ *and cuts* Δ*.*

Theorem (From Proof-Nets to Sequents)

For any proof-net Π *with conclusions* Γ *and cuts* Δ*, there exists a LL proof of* $\vdash\ [\Delta]\ \Gamma$*.*

Remark

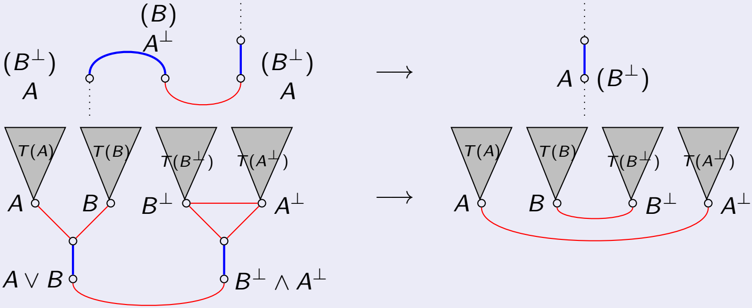
The correctness criterion also applies to proofs and proof-nets *with* Cuts.

Proof-Nets and Cut-Elimintation

Cut-Elimination (reminder)

A proof π of $\vdash\ [\Delta]\ \Gamma$ can be turned into a proof π' of $\vdash\ \Gamma$ that doesn't use Cut.

Cut-Elimination on Proof-Nets



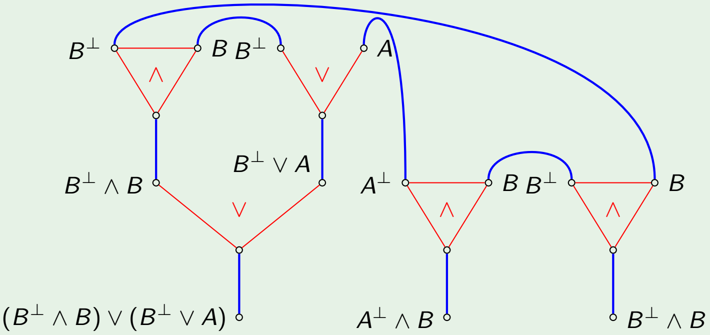
Theorem (Retoré (1993))

Cut-elimination on proof-nets is confluent and strongly normalizing.

Proof-Nets

Examples

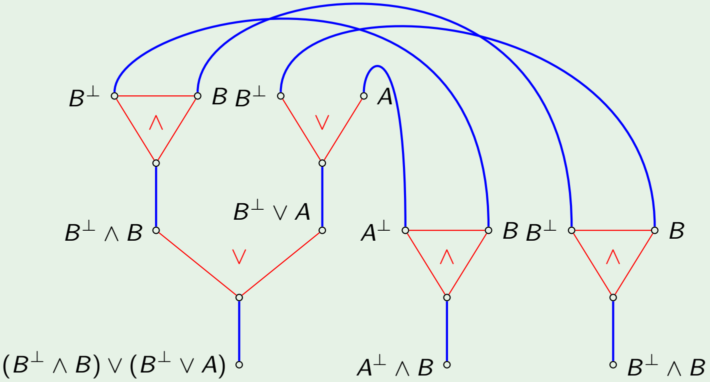
Example ($\vdash\ (B^\perp \wedge B) \vee (B^\perp \vee A), A^\perp \wedge B, B^\perp \wedge B$)



Proof-Nets

Examples

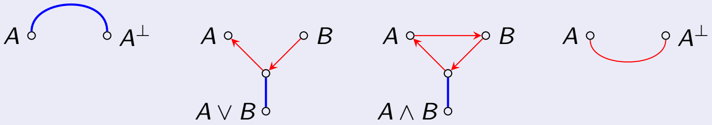
Example ($\vdash (B^\perp \wedge B) \vee (B^\perp \vee A), A^\perp \wedge B, B^\perp \wedge B$)



Proof-Nets for CyLL

Definition (Links (Abrusci and Maringelli 1998))

Link	*Axiom*	*Disjunction*	*Conjunction*	*Cut*
R&B-graphs				



Definition (Cyclic Proof-Net)

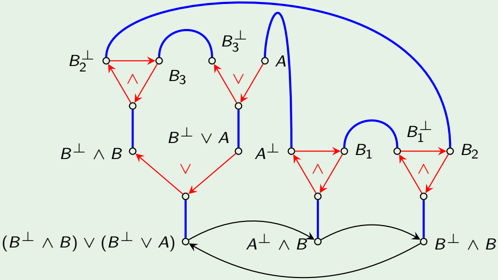
A cyclic proof-structure π of conclusion Γ is a *cyclic proof-net* if

- the undirected proof-structure is a proof-net (of commutative LL);
- for each $\vee$-link, there is a æ-path from the right premise to the left one;
- the graph $Conc(\pi)$ is a cycle, where $c_1 c_2 \in Conc(\pi)$ iff $c_1, c_2 \in \Gamma$ and there exists a æ-path from c_2 to c_1 .

Proof-Net for CyLL

Example

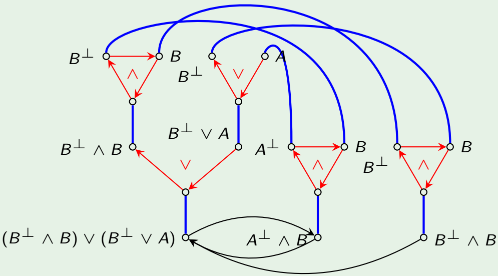
Example



Proof-Nets for CyLL

Example

Example



Remark

The bracketing induced by the axiom links have to be *compatible* with a cyclic order of the conclusions

Proof-Nets for CyLL (Abrusci and Maringelli 1998)

Theorem (From Sequents to Proof-Nets)

For any CyLL proof π of $\vdash \Gamma$, there is a cyclic proof-net with conclusions Γ.

Theorem (From Proof-Nets to Sequents)

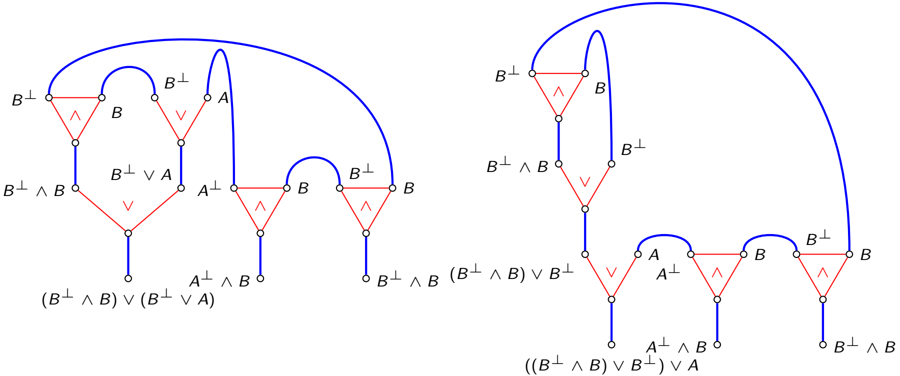
For any cyclic proof-net Π with conclusions Γ, there exists a LL proof of $\vdash \Gamma$.

Remark

Apply to proof-nets *without* Cut.

Handsome (Commutative) Proof-Nets

Identifying Proofs up to Associativity and Commutativity



Handsome proof-nets (or SP-R&B-proof-nets)

Cographs (SP-Graphs)

Definition (Complement Graph)

Let $G = (V, R)$ be a simple graph. Its *complement graph* is $G^c = (V, R^C)$ where $xy \in R^c$ iff $x \neq y$ and $xy \notin R$.

Definition (Parallel and Series Composition)

Let $G = (V, R)$ and $G' = (V', R')$ two simple graphs such that $V \cap V' = \emptyset$.

- Their *parallel composition* (or *sum*): $G + G' = (V \cup V', R \cup R')$
- Their *series composition* $G * G' = (G^c + G'^c)^c = (V \cup V', R \cup R' \cup (V \times V') \cup (V' \times V))$

Definition (Cographs (Series-Parallel Graphs))

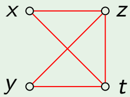
The class of *cographs* (or *SP-graphs*) is the smallest class of simple graphs containing $(\{x\}, \emptyset)$ and closed under series and parallel composition.

Cographs

Example

Let $X = (\{x\}, \emptyset)$, $Y = (\{y\}, \emptyset)$, $Z = (\{z\}, \emptyset)$, and $T = (\{t\}, \emptyset)$.

$$(X + Y) * (Z * T) =$$



SP-R&B-Proof-Structures and SP-R&B-Proof-Nets

Definition (SP-R&B-Graphs)

A *SP-R&B-graph* is a R&B-graph $G = (V, B, R)$ such that the simple graph (V, R) is a SP-graph.
It is said to be

- *chorded* whenever every æ-cycle contains a chord;
- *critically chorded* whenever any pair of distinct vertices are joined by a chordless æ-path.

Definition (SP-R&B-Proof-Structure)

A *SP-R&B-proof-structure* is any SP-R&B-graph such that the complement of the perfect matching is a series-parallel graph.

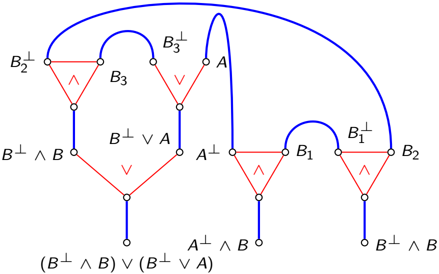
Definition (SP-R&B-Proof-Net)

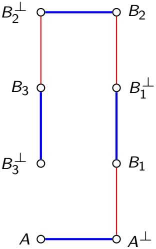
A *SP-R&B-proof-net* is a critically chorded SP-R&B-proof-structure.

A formula F is turned into an SP-term by replacing every $\wedge$ by $*$ and every $\vee$ by $+$.

SP-R&B-Proof-Nets

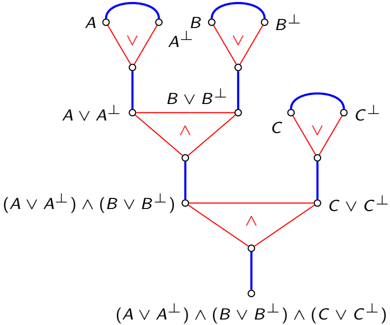
Example

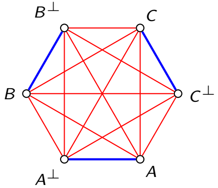




SP-R&B-Proof-Nets

Example





Proof-Nets and Sequent Calculus (Retoré 1999; Retoré 2003)

Theorem (From Sequents to Proof-Nets)

For any LL proof π of $\vdash\ [\Delta]\ \Gamma$, there is a SP-R&B-proof-net with conclusions Γ and cuts Δ.

Theorem (From Proof-Nets to Sequents)

For any SP-R&B-proof-net Π with conclusions Γ and cuts Δ, there exists a LL proof of $\vdash\ [\Delta]\ \Gamma$.

Remark

The correctness criterion also applies to proofs and proof-nets *with* Cuts.

Cyclic Cographs (SSP-Graphs)

Definition (Parallel, Series, and Symmetric Composition)

Let $G = (V, R, N)$ and $G' = (V', R', N')$ two simple graphs such that $V \cap V' = \emptyset$.

- Their *parallel composition* (or *sum*): $G + G' = (V \cup V', R \cup R', N \cup N' \cup (N \times N'))$
- Their *series composition* $G * G' = (V \cup C', R \cup R' \cup (V \times V'), N \cup N')$
- Their *symmetric composition* $G * G' = (V \cup C', R \cup R' \cup (V \times V') \cup (V' \times V), N \cup N')$

Definition (Symmetric-Series-Parallel Graphs)

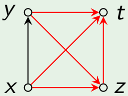
The class of *cographs* (or *SP-graphs*) is the smallest class of simple graphs containing $(\{x\}, \emptyset)$ and closed under (directed and undirected) series and parallel composition.

Directed Cographs

Example (K-graph of a formula)

Let $X = (\{x\}, \emptyset)$, $Y = (\{y\}, \emptyset)$, $Z = (\{z\}, \emptyset)$, and $T = (\{t\}, \emptyset)$.

$$(X + Y) * (Z * T) =$$



Such a formula defines a linear order, called a *segment*. This is an *Hamiltonian* path.

Cyclic SSP-R&B-Proof-Structures and CyclicSSP-R&B-Proof-Nets

A formula F is turned into an SP-term by replacing every $\wedge$ by $*$ and every $\vee$ by $+$.

Definition (Cyclic K-graph)

A *cyclic K-graph* is a finite set $K_0, \dots, K_{n-1}$ $[K_n, \dots, K_{n+m}]$ of K-graphs such that

- the main series composition of the K_i, $i \leq n-1$ are R series composition (directed)
- the main series composition of the K_i, $n \leq i \leq n+m$ are R symmetric series composition (directed)
- the K_i, $i \leq n-1$ are endowed with a cyclic order.

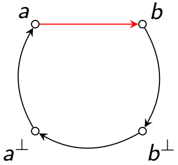
A cyclic K-graph contains an Hamiltonian circuit that consists in:

- the hamiltonian path of each K_i, $i \leq n-1$
- all the N-arcs from the last vertex of $K_{i[n]}$ to the first of $K_{i+1[n]}$

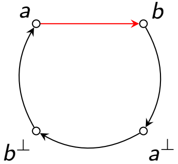
Cyclic K-Graphs

Examples

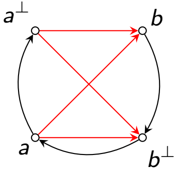
$a \wedge b, b^{\perp}, a^{\perp}$



$a \wedge b, a^{\perp}, b^{\perp}$



$(a \vee a^{\perp}) \wedge (b \vee b^{\perp})$



Cyclic Proof-Structure and Cyclic Proof-Nets

Definition (Cyclic Proof-Structure)

A *cyclic proof-structure* is a cyclic K-graph enriched with a perfect matching of B-edges linking vertices with dual names (a and $a^{\perp}$).

Definition (B^*)

We define B^* such that xB^*y iff there is an axiom-cut path from x to y.
Moreover, if x belongs to a Cut, y is the last vertex on this path that do not belong to this Cut or that is not part of a Cut.

Definition (SP-R&B-Proof-Net)

A cyclic proof-structure π is a *cyclic proof-net* if:

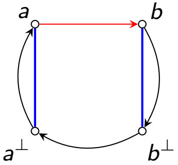
- the underlyng R&B-graph is a commutative proof-net;
- B^* is compatible with the hamiltonian circuit of π
- B^* is decreasing on Cuts (i.e., for any Cut $\phi \bullet \phi^{\perp}$, if $x <_{\phi} y$, then $B^*(y) < B^*(x)$.

$B^* = B$ if there is not Cut.

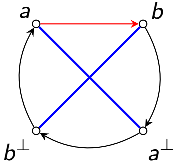
Cyclic Proof-Structures

Examples

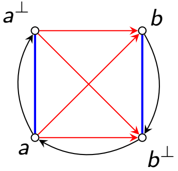
$\vdash a \wedge b, b^\perp, a^\perp$



$\not\vdash \; a \wedge b, a^\perp, b^\perp$



$\vdash \; (a \vee a^\perp) \wedge (b \vee b^\perp)$



Proof-Nets and Sequent Calculus

Theorem (From Sequents to Proof-Nets Pogodalla and Retoré (2004))

For any LL proof π of $\vdash \Gamma$, there is a cyclic proof-net with conclusions Γ.

Theorem (From Proof-Nets to Sequents Pogodalla and Retoré (2004))

For any cyclic proof-net Π with conclusions Γ, there exists a LL proof of $\vdash \Gamma$.

Theorem (From Sequents to Proof-Nets)

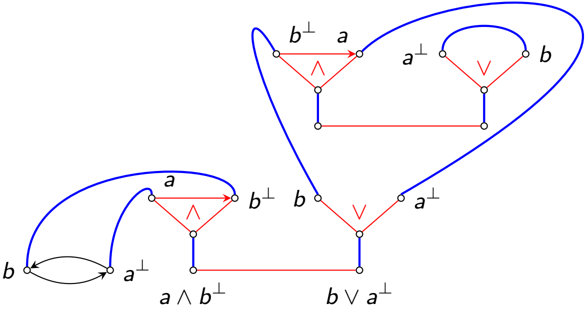
For any LL proof π of $\vdash [\Delta]\ \Gamma$, there is a SP-R&B-proof-net with conclusions Γ and cuts Δ.

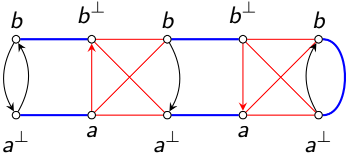
Theorem (From Proof-Nets to Sequents)

For any SP-R&B-proof-net Π with conclusions Γ and cuts Δ, there exists a LL proof of $\vdash [\Delta]\ \Gamma$.

Cyclic Proof-Nets with Cuts

Pathological Cases





Conclusion

- Natural language to cyclic linear logic (Lambek grammars)
- Proof-nets for linear logic
- Proof-nets for cyclic linear logic (without Cuts)
- Handsome proof-nets for linear logic
- Handsome proof-nets for cyclic linear logic (with Cuts):
  - Symmetric-Series-Parallel graph endowed with a perfect matching
  - Hamiltonian circuit on the conclusions
  - Adequacy of the (extended) perfect matching to the Hamiltonian circuit

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