

Diagrammatic Quantum Reasoning: Completeness and Incompleteness

Simon Perdrix

CNRS, Loria, Nancy, France

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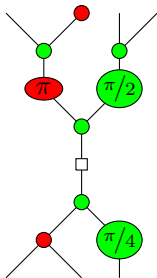


DIAGRAMMATIC LANGUAGE FOR
REASONING IN QUANTUM COMPUTING
ZX-Calculus¹

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ZX-Calculus¹

Categorical Quantum Mechanics²

- Proving properties: protocols, algorithms, models of quantum computing.
Proof assistant software: Quantomatic.

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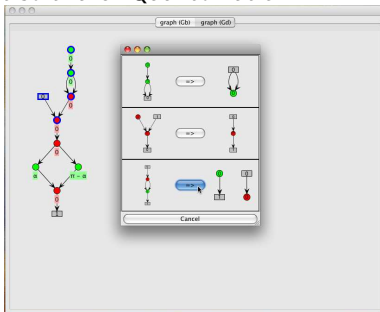
²S. Abramsky, B. Coecke. A categorical semantics for quantum protocols. LiCS'04.

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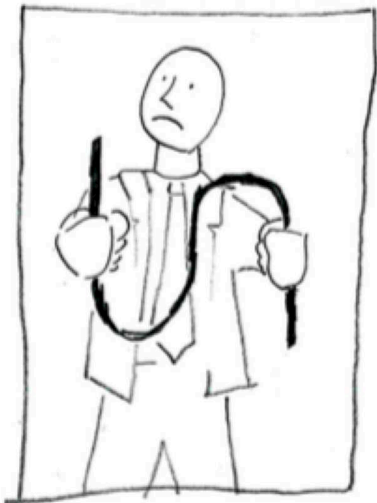
- Proving properties: protocols, algorithms, models of quantum computing.
Proof assistant software: Quantomatic.
- Foundations:
entanglement, causality
aximatisation of quantum mechanics.
- Pedagogical.

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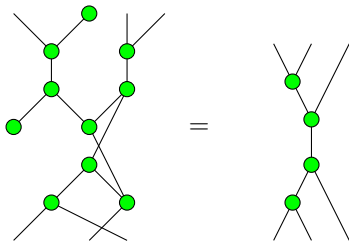
Motivating Example: Post-Selected Teleportation

Motivating Example: Post-Selected Teleportation



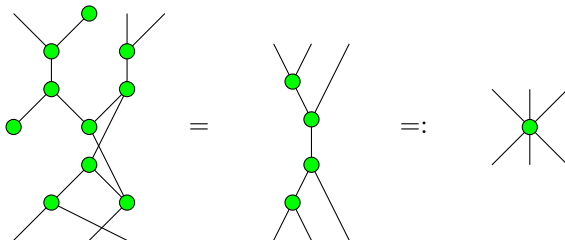
Frobenius Algebras

- (special commutative) Frobenius algebra $(\begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array}, \begin{array}{c} \bullet \\ \diagdown \\ \diagup \end{array}, \begin{array}{c} \diagdown \\ \bullet \\ \diagup \end{array}, \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array})$



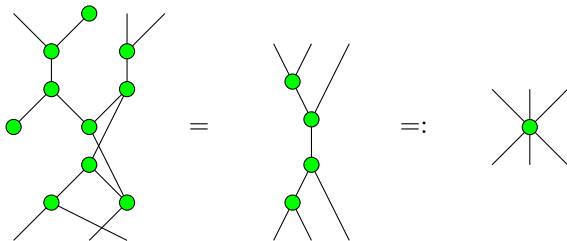
Frobenius Algebras

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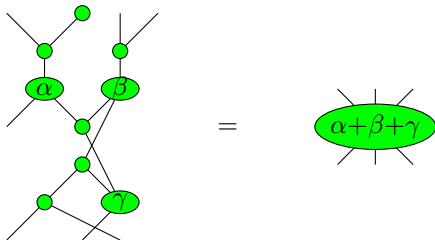
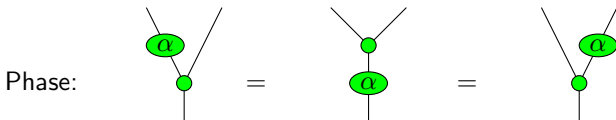
Frobenius Algebras

- (special commutative) Frobenius algebra $(\begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array}, \begin{array}{c} \bullet \\ \diagdown \\ \diagup \end{array}, \begin{array}{c} \diagdown \\ \bullet \\ \diagup \end{array}, \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array})$, in bijection with orthonormal basis in **FdHilb** [Coecke,Pavlovic,Vicary'13³]



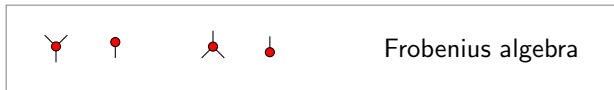
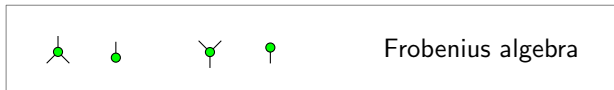
Frobenius Algebras

- (special commutative) Frobenius algebra $(\cup, \cap, \gamma, \eta)$, in bijection with orthonormal basis in **FdHilb** [Coecke,Pavlovic,Vicary'13³]
- Frobenius Algebra with Phases



³B.Coecke, D.Pavlovic, J. Vicary. A new description of orthogonal bases. MSCS 23, pp 555-567. 2013.

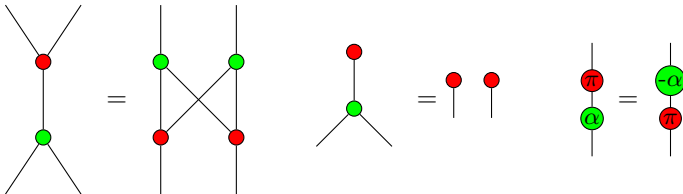
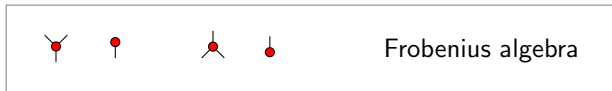
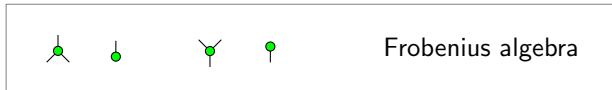
Complementary basis



³Duncan, Dunne. Interacting Frobenius Algebras are Hopf. LiCS'16.

³Bonchi, Sobocinski, Zanasi. Interacting Hopf Algebras, Journal of Pure and Applied Algebra, 2016

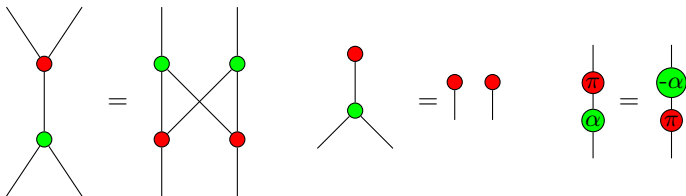
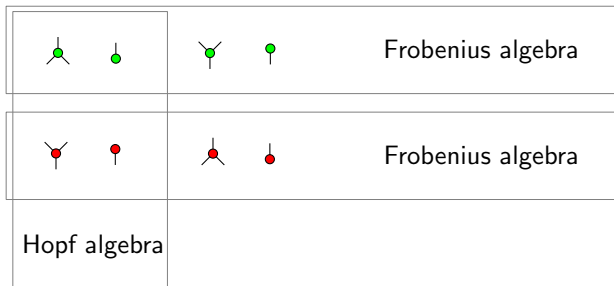
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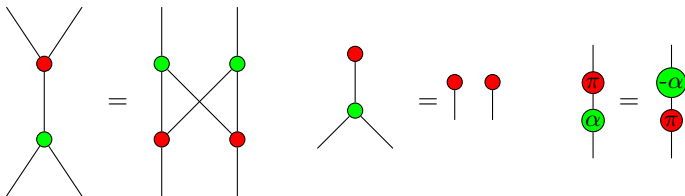


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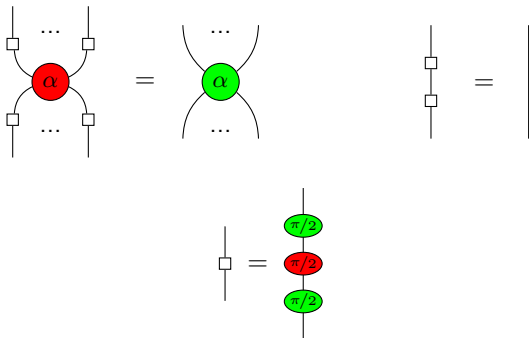
		Frobenius algebra
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Hopf algebra	Hopf algebra	



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Hadamard

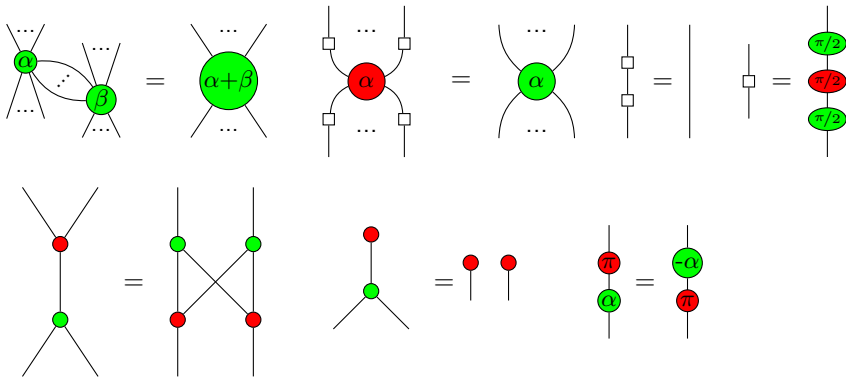


Universality, Soundness, and Completeness

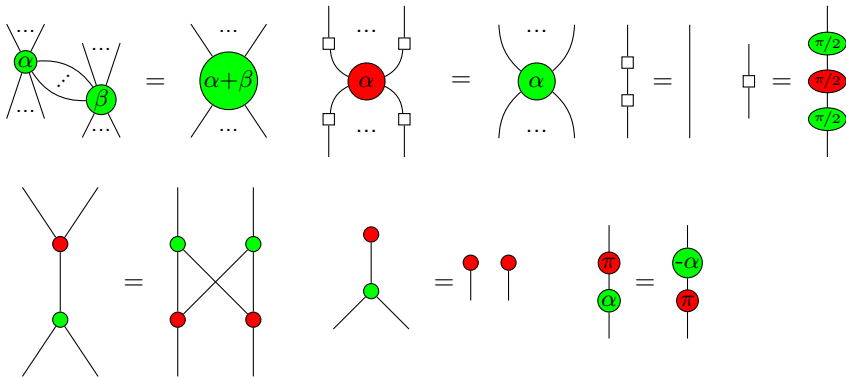
Universality

$$\begin{aligned} \left[\begin{array}{c} | \\ \square \\ | \end{array} \right] &= \begin{cases} |0\rangle \mapsto \frac{|0\rangle+|1\rangle}{\sqrt{2}} =: |+\rangle \\ |1\rangle \mapsto \frac{|0\rangle-|1\rangle}{\sqrt{2}} =: |-\rangle \end{cases} \\ \left[\begin{array}{c} \dots \\ \diagup \quad \diagdown \\ \text{\textcircled{\alpha}} \\ \diagdown \quad \diagup \\ \dots \end{array} \right] &= \begin{cases} |0\dots 0\rangle \mapsto |0\dots 0\rangle \\ |1\dots 1\rangle \mapsto e^{i\alpha} |1\dots 1\rangle \end{cases} \\ \left[\begin{array}{c} \dots \\ \diagup \quad \diagdown \\ \text{\textcircled{\alpha}} \\ \diagdown \quad \diagup \\ \dots \end{array} \right] &= \begin{cases} |+\dots+\rangle \mapsto |+\dots+\rangle \\ |-\dots-\rangle \mapsto e^{i\alpha} |-\dots-\rangle \end{cases} \end{aligned}$$

- **Universality:** for any n -qubit linear map U , $\exists D$ s.t. $\llbracket D \rrbracket = U$.
- **$\pi/4$ -fragment is approximately universal:** $\forall \epsilon > 0$ and any n -qubit linear map U , $\exists D$ with angles multiple of $\pi/4$ s.t. $\|\llbracket D \rrbracket - U\| < \epsilon$.
- **$\pi/2$ -fragment is not (approximately) universal.**



- **Soundness:** $(ZX \vdash D_1 = D_2) \Rightarrow ([D_1] \simeq [D_2])$
 where $[D_1] \simeq [D_2]$ if it exists a non zero $s \in \mathbb{C}$ s.t. $[D_1] = s [D_2]$



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- **Completeness:** $([D_1] \simeq [D_2]) \not\Rightarrow (ZX \vdash D_1 = D_2)$

“The most fundamental open problem related to the ZX-calculus is establishing its completeness properties for some of the calculus’ variants” [▶ CQM wiki](#)

Completeness of the $\pi/2$ -fragment

Theorem [Backens'12⁴] Completeness of the $\pi/2$ fragment of the ZX-calculus.

$\forall D_1, D_2$ involving angles multiple of $\pi/2$ only,

$$\llbracket D_1 \rrbracket \simeq \llbracket D_2 \rrbracket \Leftrightarrow (ZX \vdash D_1 = D_2)$$

⁴M. Backens. The ZX-calculus is complete for stabilizer quantum mechanics. New J. Phys. 16 (2014) 093021

Incompleteness of ZX-calculus

Theorem [Schröder, Zamdzhiev'14⁵]. ZX-calculus is incomplete for Qubit Quantum Mechanics.

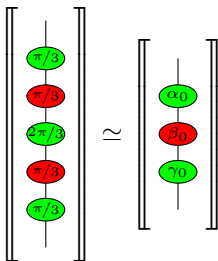
Proof.

⁵C. Schröder de Witt, V. Zamdzhiev. The ZX-calculus is incomplete for quantum mechanics. EPTCS 172, 2014

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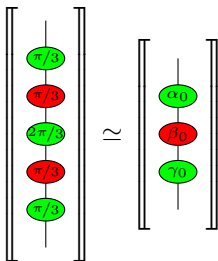
$$\alpha_0 = -\arccos\left(\frac{5}{2\sqrt{13}}\right), \beta_0 = -2\arcsin\left(\frac{\sqrt{3}}{4}\right), \gamma_0 = \arcsin\left(\frac{\sqrt{3}}{4}\right) - \alpha_0$$

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$$\llbracket \begin{array}{c} \alpha \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \rrbracket_3 := \llbracket \begin{array}{c} 3\alpha \\ | \\ \bullet \end{array} \rrbracket$$

If $ZX \vdash D_1 = D_2$ then $\llbracket D_1 \rrbracket_3 \simeq \llbracket D_2 \rrbracket_3$.

⁵C. Schröder de Witt, V. Zamdzhiev. The ZX-calculus is incomplete for quantum mechanics. EPTCS 172, 2014

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Theorem [Schröder, Zamdzhiev'14⁵]. ZX-calculus is incomplete for Qubit Quantum Mechanics.

Proof.

$$\begin{array}{c} \left[\begin{array}{c} \pi/3 \\ \pi/3 \\ 2\pi/3 \\ \pi/3 \\ \pi/3 \end{array} \right]_3 = \left[\begin{array}{c} \pi \\ \pi \\ 0 \\ \pi \\ \pi \end{array} \right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \left[\begin{array}{c} 3\alpha_0 \\ 3\beta_0 \\ 3\gamma_0 \end{array} \right] = \left[\begin{array}{c} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{array} \right]_3
 \end{array}$$

$$\alpha_0 = -\arccos\left(\frac{5}{2\sqrt{13}}\right), \beta_0 = -2\arcsin\left(\frac{\sqrt{3}}{4}\right), \gamma_0 = \arcsin\left(\frac{\sqrt{3}}{4}\right) - \alpha_0$$

$$\left[\begin{array}{c} \alpha \end{array} \right]_3 := \left[\begin{array}{c} 3\alpha \end{array} \right] \quad \text{If } ZX \vdash D_1 = D_2 \text{ then } \llbracket D_1 \rrbracket_3 \simeq \llbracket D_2 \rrbracket_3.$$

⁵C. Schröder de Witt, V. Zamdzhiev. The ZX-calculus is incomplete for quantum mechanics. EPTCS 172, 2014

(In)-completeness

- Completeness of the $\pi/2$ -fragment [Backens'12]
- Incompleteness for Qubit QM [Schröder, Zamdzhiev'14]
No obvious way to extend the ZX-calculus

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- Completeness of the 1-qubit $\pi/4$ -fragment (path diagrams) [Backens'14⁶]

⁶M. Backens. The ZX-calculus is complete for the single-qubit Clifford+T group. EPTCS 172, 2014.

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- Completeness of the 1-qubit $\pi/4$ -fragment (path diagrams) [Backens'14⁶]
- Incompleteness of the $\pi/4$ -fragment [Perdrix, Wang'16⁷]

⁶M. Backens. The ZX-calculus is complete for the single-qubit Clifford+T group. EPTCS 172, 2014.

⁷S. Perdrix, Q. Wang. Suplementarity is necessary for quantum diagram reasoning. MFCS'16

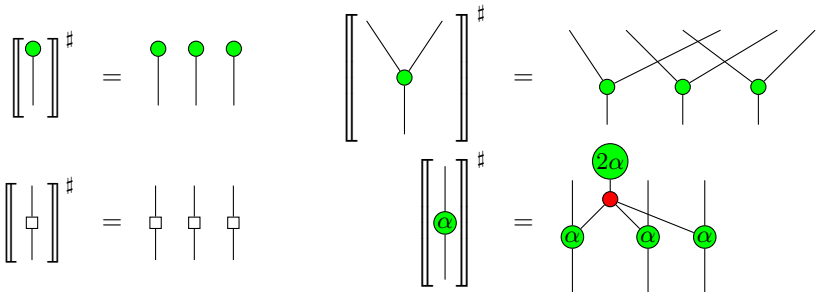
Supplementarity, a candidate for incompleteness

$$\left[\begin{array}{c} \alpha \quad \alpha + \pi \\ \text{---} \\ \bullet \\ | \end{array} \right] = \left[\begin{array}{c} 2\alpha + \pi \\ \text{---} \\ \bullet \\ | \end{array} \right]$$

- Inspired by [Coecke,Edwards'10]: supplementarity.
- Can be proven in ZX when $\alpha = \pm \frac{\pi}{2}$.

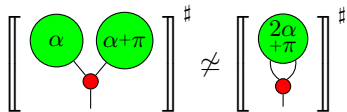
Theorem: $\left(ZX \vdash \begin{array}{c} \alpha \quad \alpha + \pi \\ \text{---} \\ \bullet \\ | \end{array} = \begin{array}{c} 2\alpha + \pi \\ \text{---} \\ \bullet \\ | \end{array} \right) \Leftrightarrow \alpha = 0 \pmod{\frac{\pi}{2}}$

Alternative interpretation

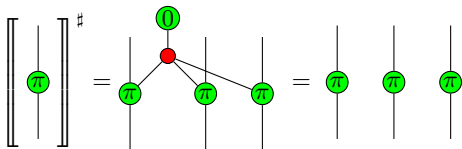


Soundness: $(ZX \vdash D_1 = D_2) \Rightarrow \llbracket D_1 \rrbracket^\sharp \simeq \llbracket D_2 \rrbracket^\sharp$

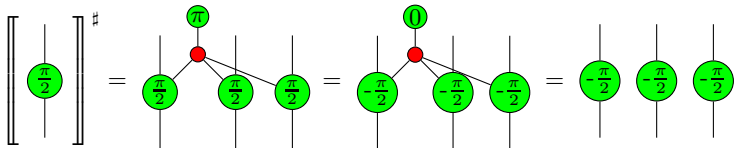
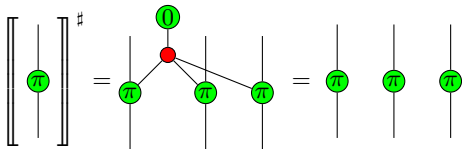
Counterexample: $\forall \alpha \neq 0 \pmod{\frac{\pi}{2}},$



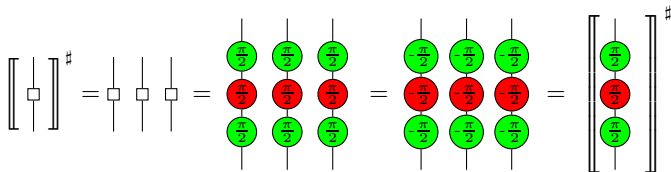
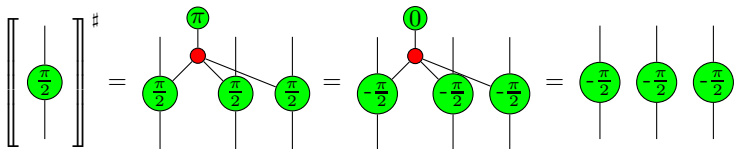
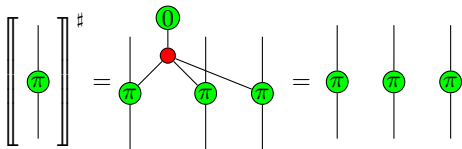
Sound interpretation (1)



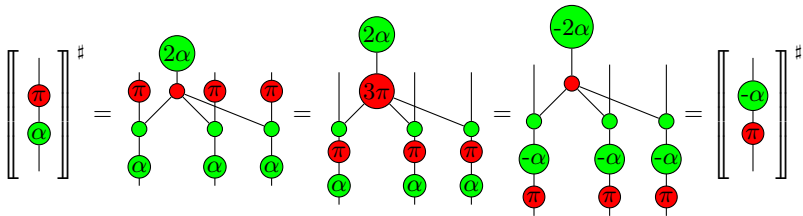
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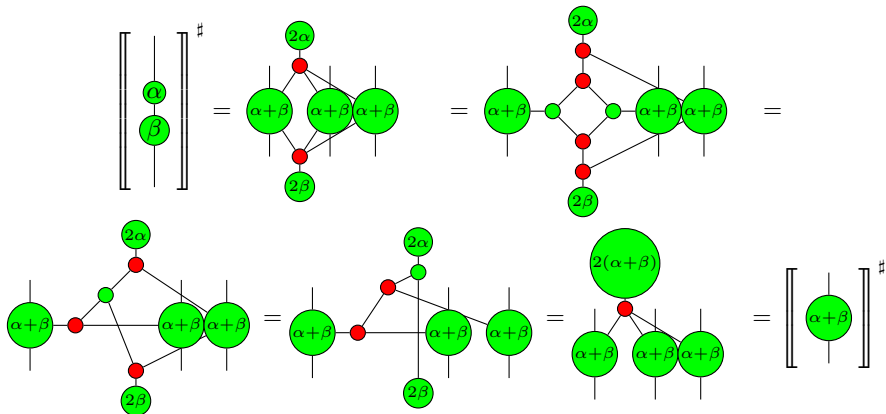
Sound interpretation (1)



Sound interpretation (2)



Sound interpretation (3)



Incompleteness

The diagram shows two configurations of green circles and a red dot, each enclosed in a double-line frame with a sharp symbol (#) at the top. The left configuration consists of two green circles, one labeled α and one labeled $\alpha + \pi$, connected by a line to a red dot below them. The right configuration consists of a single green circle labeled $2\alpha + \pi$ connected by a line to a red dot below it. An equivalence symbol \simeq is placed between the two configurations. To the right of the second configuration is the congruence condition $\Rightarrow \alpha = 0 \pmod{\frac{\pi}{2}}$.

Corollary: $\frac{\pi}{4}$ -fragment of ZX-calculus is not complete as the following equation cannot be derived:

The diagram shows an equation between two configurations. The left configuration has two green circles, one labeled $\frac{\pi}{4}$ and one labeled $-\frac{3\pi}{4}$, connected by a line to a red dot below them. The right configuration has a single green circle labeled $-\frac{\pi}{2}$ connected by a line to a red dot below it. An equals sign (=) is placed between the two configurations.

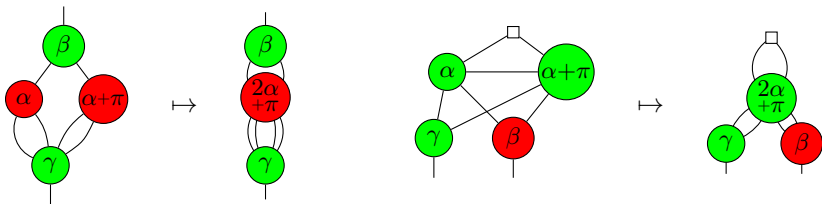
Graphical interpretation

Theorem. In ZX-calculus, antiphase twins can be merged if and only if

$$\forall \alpha, \quad \begin{array}{c} \text{green } \alpha \quad \text{green } \alpha + \pi \\ \diagdown \quad \diagup \\ \text{red dot} \\ | \end{array} = \begin{array}{c} \text{green } 2\alpha + \pi \\ \diagdown \quad \diagup \\ \text{red dot} \\ | \end{array}$$

where two dots are antiphase twins if:

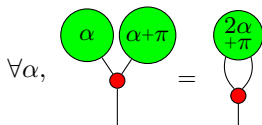
- they have the same colour;
- the difference between their angles is π ;
- they have the same neighbourhood.



Conclusion

- $\frac{\pi}{4}$ -fragment of ZX-calculus is completeness [Backens'12]
- Incompleteness in general [Schröder,Zamdzhiev'14]
No obvious way to extend the ZX-calculus
- $\frac{\pi}{4}$ -fragment is incompleteness [Perdrix,Wang'16]

Supplementarity as an axiom: ZX-calculus := ZX-calculus
+ 'Supplementarity'



Open question.

Is $\frac{\pi}{4}$ -fragment of ZX-calculus + 'Supplementarity' complete?