The word problem in $\mathbb{Z}^2$ and formal language theory

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Outline

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The group language of $\mathbb{Z}^2$

A similar problem in computational linguistics

Multiple Context Free Grammars (MCFGs)

A grammar for O_2

Proof of the Theorem

A Theorem on Jordan curves

Conjectures

Group languages

Group finite presentation:

- a finite set of generators Σ
- a finite set of defining equations E

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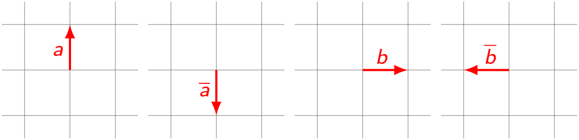
Example: a group language is context free iff its underlying group is virtually free (Muller Schupp 1983)

A simple presentation of $\mathbb{Z}^2$

- Generators: $\{a; \overline{a}; b; \overline{b}\}$
- Defining equations: $a^{-1} = \overline{a}$, $b^{-1} = \overline{b}$, $xy = yx$



The associated group language is

$$O_2 = \{w \in \{a; \overline{a}; b; \overline{b}\}^* \mid |w|_a = |w|_{\overline{a}} \wedge |w|_b = |w|_{\overline{b}}\}$$

O_2 and computational group theory

- Gilman (2005)

is indexed but not context free seems to have been open for several years. It does not even seem to be known whether or not the word problem of $Z \times Z$ is indexed.

Outline

MIX

$$MIX = \{w \in \{a; b; c\}^* || w|_a = |w|_b = |w|_c\}$$

MIX and O_2 are rationally equivalent

The Bach language

- Bach (1981)

Exercise 2: Let $L = \{X | X = (abc)^n\}$. L is CF (in fact regular). But Scramble (L) is not CF. For let $L' = \{X | X = a^n b^m c^k\}$ then $L' \cap L = \{X | X = a^n b^n c^n\}$ is not CF, but since the intersection of a CF language and a regular language is CF, L can't be CF.

Wikipedia entry:
`http://en.wikipedia.org/wiki/Bach_language`

The *MIX* language

- Marsh (1985)

Conjecture: *MIX* is not an indexed language.

Proof. Consider the language MIX = SCRAMBLE($(abc)^+$) (the names 'mix' and 'MIX' — pronounced 'little mix' and 'big mix' were the happy invention of Bill Marsh; 'little mix' is the scramble of $(ab)^+$).

MIX and Tree Adjoining Grammars

- Joshi (1985)

[*MIX*] represents the extreme case of the degree of free word order permitted in a language. This extreme case is linguistically not relevant. [...] TAGs also cannot generate this language although for TAGs the proof is not in hand yet.

MIX and Tree Adjoining Grammars

- Vijay Shanker, Weir, Joshi (1991)

of strings of equal number of a's, b's, and c's in any order. MIX can be regarded as the extreme case of free word order. It is not known yet whether TAG, HG, CCG and LIG can generate MIX. This has turned out to be a very difficult problem. In fact, it is not even known whether an IG can generate MIX.

MIX and mildly context sensitive languages

- Joshi, Vijay Shanker, Weir (1991)

in MCSL; 2) languages in MCSL can be parsed in polynomial time; 3) MCSGs capture only certain kinds of dependencies, e.g., nested dependencies and certain limited kinds of crossing dependencies (e.g., in the subordinate clause constructions in Dutch or some variations of them, but perhaps not in the so-called MIX (or Bach) language, which consists of equal numbers of a's, b's, and c's in any order 4) languages in MCSL have constant growth property, i.e., if the strings of a language

Outline

Original paper

Fundamental Study

On multiple context-free grammars*

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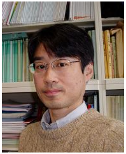

Communicated by M. Takahashi
Received July 1989
Revised January 1990

A generalization of context-free grammars

Rule of a context free grammar:

$$A \to w_1 B_1 \dots w_n B_n w_{n+1}$$

with A, B_1, ..., B_n non-terminals and w_1, $\dots w_{n+1}$ string of terminals.

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A bottom-up view:

$$A(w_1 x_1 \dots w_n x_n w_{n+1}) \leftarrow B_1(x_1), \dots, B_n(x_n)$$

A generalization of context-free grammars

Replace strings by tuple of strings:

$$B(s_1, \ldots, s_m) \leftarrow B_1(x^1_1, \ldots, x^1_{k_1}), \ldots, B_n(x^n_1, \ldots, x^n_{k_n})$$

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- the strings s_i are made of terminals and of the variables x_j^i,
- the variables x_j^i are pairwise distinct (otherwise we get Groenink's Literal Movement Grammars),
- each variable x_j^i has at most one occurrence in the string $s_1 \ldots s_m$ (otherwise we get Parallel Multiple Context-Free Grammars).

Formal definition

A m-MCFG(r) is a 4-tuple (N, T, P, S) such that:

- N is a ranked alphabet of *non-terminals* of max. rank m.

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- P is a set of rules of the form:

$$A(s_1, \ldots, s_k) \leftarrow B_1(x_1^1, \ldots, x_{k_1}^1), \ldots, B_n(x_1^n, \ldots, x_{k_n}^n)$$

  where:

  - A is a non-terminal of rank k, B_i is non-terminal of rank k_i, $n \leq r$,
  - the variables x_j^i are pairwise distinct,
  - the strings s_i are in $(T \cup X)^*$ with $X = \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_i} \{x_j^i\}$,
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  - each variable x_j^i has at most one occurrence in $s_1 \ldots s_k$
- S is a non-terminal of rank 1, *the starting symbol*.

The language generated by an MCFG

Given an MCFG $G = (N, T, P, S)$, if the following conditions holds:

- $B_1(s^1_1, \ldots, s^1_{k_1})$, $\ldots$, $B_n(s^n_1, \ldots, s^n_{k_n})$ are derivable,
- $A(s_1, \ldots, s_k) \leftarrow B_1(x^1_1, \ldots, x^1_{k_1}), \ldots, B_n(x^n_1, \ldots, x^n_{k_n})$ is a rule in P

then $A(t_1, \ldots, t_k)$ with $t_i = s_i[x^i_j \leftarrow s^i_j]_{i \in [1;n], j \in [1;k_i]}$ is derivable.

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The language define by G, $L(G)$ is:

$$\{w \mid S(w) \text{ is derivable}\}$$

An example

$$
\begin{aligned}
&S(x_1 y_1 x_2 y_2) \leftarrow P(x_1, x_2),\ Q(y_1, y_2)\\
&P(ax_1, bx_2) \leftarrow P(x_1, x_2)\\
&P(\epsilon, \epsilon) \leftarrow\\
&Q(cx_1, dx_2) \leftarrow Q(x_1, x_2)\\
&Q(\epsilon, \epsilon) \leftarrow
\end{aligned}
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$$

$$S(a^n c^m b^n d^m) \leftarrow P(a^n, b^n),\ Q(c^m, d^m)$$

The language is: $\{a^n c^m b^n d^m \mid n \in \mathbb{N} \land m \in \mathbb{N}\}$

The well-nestedness constraint

$$I(x_1 y_1, y_2 x_2) \leftarrow J(x_1, x_2),\ K(y_1, y_2)$$

$$I(x_1 y_1, x_2 y_2) \leftarrow J(x_1, x_2),\ K(y_1, y_2)$$

$$A(x_1 z_1, z_2 x_2 y_1, y_2 y_3 x_3) \leftarrow B(x_1, x_2, x_3)\ C(y_1, y_2, y_3)\ D(z_1, z_2)$$

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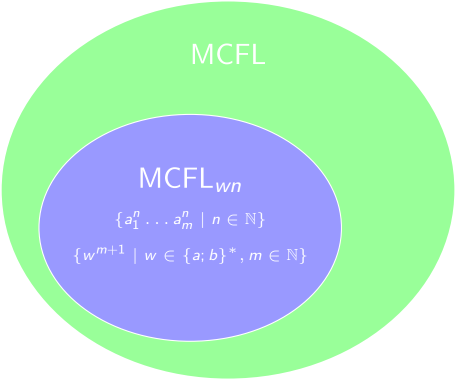
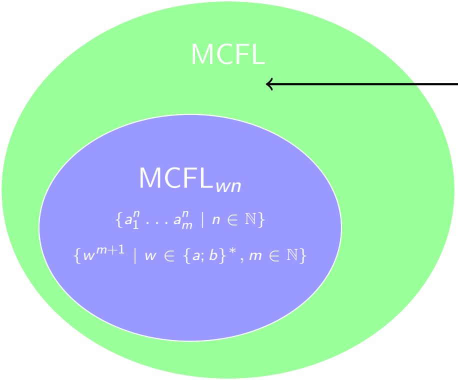
MCFL_{wn} and MCFL



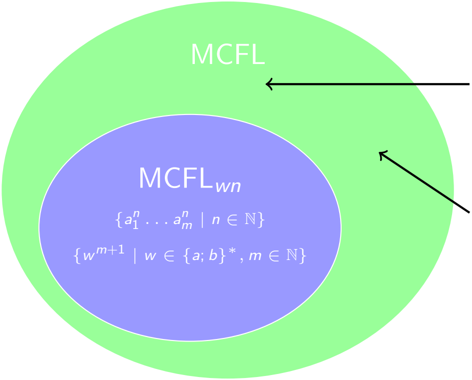
MCFL_{wn} and MCFL



$\{w_1 \ldots w_n z_n w_n z_{n-1} \ldots z_1 w_1 z_0 w_1^r \ldots w_n^r \mid n \in \mathbb{N}, w_i \in \{c;d\}^+, z_0, \ldots, z_n \in D_1^*\}$

Staudacher 1993
Michaelis 2005

MCFL_{wn} and MCFL



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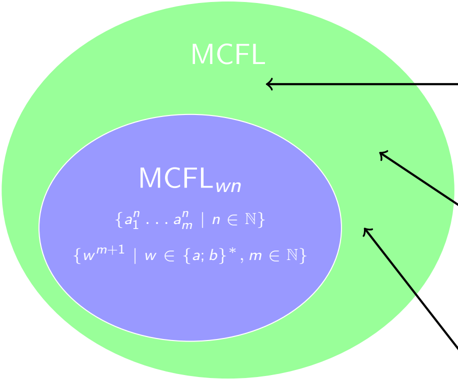

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Kanazawa, S. 2010

Outline

A 2-MCFG for O_2

$$
\begin{array}{rcl}
S(xy) & \leftarrow & Inv(x, y) \\
\hline
Inv(x_1 y_1, y_2 x_2) & \leftarrow & Inv(x_1, x_2), Inv(y_1, y_2) \\
Inv(x_1 y_1 y_2, x_2) & \leftarrow & Inv(x_1, x_2), Inv(y_1, y_2) \\
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well-nested binary rules

non well-nested rules

where $\alpha \in \{a; b\}$

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well-nested binary rules

rules for constants

non well-nested rules

where $\alpha \in \{a; b\}$

A 2-MCFG for O_2

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\begin{array}{rcll}
S(xy) & \leftarrow & Inv(x,y) & \} \text{ terminal rule} \\
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Inv(x_1y_1, y_2x_2) & \leftarrow & Inv(x_1,x_2), Inv(y_1,y_2) & \\
Inv(x_1y_1y_2, x_2) & \leftarrow & Inv(x_1,x_2), Inv(y_1,y_2) & \\
Inv(x_1, y_1y_2x_2) & \leftarrow & Inv(x_1,x_2), Inv(y_1,y_2) & \text{well-nested binary rules} \\
Inv(x_1x_2y_1, y_2) & \leftarrow & Inv(x_1,x_2), Inv(y_1,y_2) & \\
Inv(x_1, x_2y_1y_2) & \leftarrow & Inv(x_1,x_2), Inv(y_1,y_2) & \\
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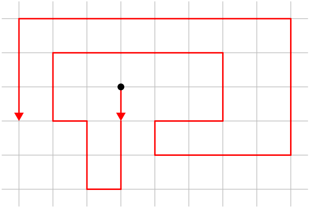
Inv(\alpha x_1\overline{\alpha}, x_2) & \leftarrow & Inv(x_1,x_2) & \\
Inv(\alpha x_1, \overline{\alpha}x_2) & \leftarrow & Inv(x_1,x_2) & \\
Inv(\alpha x_1, x_2\overline{\alpha}) & \leftarrow & Inv(x_1,x_2) & \text{rules for constants} \\
Inv(x_1\alpha, \overline{\alpha}x_2) & \leftarrow & Inv(x_1,x_2) & \\
Inv(x_1\alpha, x_2\overline{\alpha}) & \leftarrow & Inv(x_1,x_2) & \\
Inv(x_1, \alpha x_2\overline{\alpha}) & \leftarrow & Inv(x_1,x_2) & \\
\hline
Inv(x_1y_1x_2, y_2) & \leftarrow & Inv(x_1,x_2), Inv(y_1,y_2) & \text{non well-nested rules} \\
Inv(x_1, y_1x_2y_2) & \leftarrow & Inv(x_1,x_2), Inv(y_1,y_2) & \\
\hline
Inv(\epsilon,\epsilon) & \leftarrow & & \} \text{ initial rule}
\end{array}
$$

where $\alpha \in \{a; b\}$
Theorem: Given w_1 and w_2 such that $w_1w_2 \in O_2$, $Inv(w_1, w_2)$ is derivable.

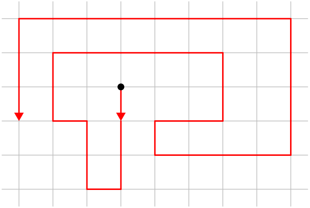
A graphical interpretation of O_2.

Graphical interpretation of the word $\overline{a}\overline{a}\overline{a}\overline{b}aa\overline{b}aabbbbb\overline{a}\overline{a}\overline{b}\overline{b}\overline{a}bbbbaaaa\overline{b}\overline{b}\overline{b}\overline{b}\overline{b}\overline{b}\overline{b}\overline{b}\overline{a}\overline{a}\overline{a}$:

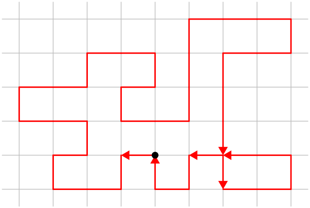


A graphical interpretation of O_2.

Graphical interpretation of the word $\bar{a}\bar{a}\bar{a}\bar{b}aa\bar{b}aabbbbb\bar{a}\bar{a}\bar{b}\bar{b}\bar{a}bbbbaaaa\bar{b}\bar{b}\bar{b}\bar{b}\bar{b}\bar{b}\bar{b}\bar{b}\bar{a}\bar{a}\bar{a}$:

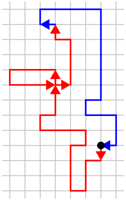


The words in O_2 are precisely the words that are represented as closed curves:
$\bar{b}\bar{a}\bar{b}\bar{b}aba\bar{b}\bar{b}abbabb\bar{a}\bar{b}\bar{a}bbaaabbb\bar{a}\bar{b}\bar{b}\bar{a}\bar{a}\bar{a}\bar{a}bba\bar{b}\bar{b}\bar{b}\bar{a}\bar{b}a$

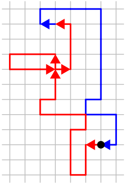


Parsing with the grammar

Rule $Inv(\overline{a}x_1 a, x_2) \leftarrow Inv(x_1, x_2)$



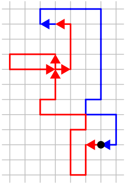
$Inv(\overline{a}\overline{b}\overline{a}\overline{a}\overline{b}aaaba\overline{b}\overline{b}\overline{b}abaaa\overline{b}\overline{b}\overline{b}\overline{a}bbbbaaa\overline{b}a, \overline{b}abbbb\overline{a}\overline{a}\overline{a}\overline{a}\overline{a}\overline{a}\overline{b}\overline{a}bb\overline{a}\overline{a}\overline{b})$



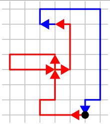
$Inv(\overline{b}\overline{a}\overline{a}\overline{b}aaaba\overline{b}\overline{b}\overline{b}abaaa\overline{b}\overline{b}\overline{b}\overline{a}bbbbaaa\overline{b}, \overline{b}abbbb\overline{a}\overline{a}\overline{a}\overline{a}\overline{a}\overline{a}\overline{b}\overline{a}bb\overline{a}\overline{a}\overline{b})$

Parsing with the grammar

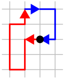
Rule: $Inv(x_1 y_1, y_2 x_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2)$



$Inv(\overline{b}\overline{aa}\overline{b}aaaba\overline{bbb}abaaa\overline{bbb}\overline{a}bbbbaaa\overline{b}, \overline{b}abbbb\overline{aaaaaa}\overline{b}\overline{a}bb\overline{aa}\overline{b})$



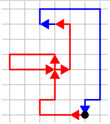
$Inv(\overline{bbb}abaaa\overline{bbb}\overline{a}bbbbaaa\overline{b}, \overline{b}abbbb\overline{aaaaaa}\overline{b}\overline{a})$



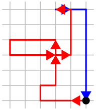
$Inv(\overline{b}\overline{aa}\overline{b}aaaba, bb\overline{aa}\overline{b})$

Parsing with the grammar

Rule $Inv(x_1, y_1 x_2 y_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2)$



$Inv(\overline{bbb}abaaa\overline{bbb}\overline{a}bbbbaaa\overline{b}, \overline{b}abbbb\overline{aaaaaa}\overline{b}\overline{a})$

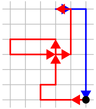


$Inv(\overline{bbb}abaaa\overline{bbb}\overline{a}bbbbaaa\overline{b}, bb\overline{aaaaaa})$

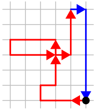
$Inv(\overline{b}abb, \overline{b}\overline{a})$

Parsing with the grammar

Rule: $Inv(x_1\overline{b}, bx_2) \leftarrow Inv(x_1, x_2)$



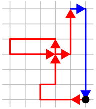
$Inv(\overline{bbb}abaaa\overline{bbba}bbbbaaa\overline{b}, bb\overline{aaaaaa})$



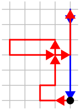
$Inv(\overline{bbb}abaaa\overline{bbba}bbbbaaa, b\overline{aaaaaa})$

Parsing with the grammar

Rule: $Inv(\bar{b}x_1, bx_2) \leftarrow Inv(x_1, x_2)$



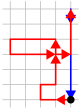
$Inv(\bar{b}\bar{b}\bar{b}abaaa\bar{b}\bar{b}\bar{b}\bar{a}bbbbaaa,\ b\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\bar{a})$



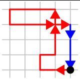
$Inv(\bar{b}\bar{b}abaaa\bar{b}\bar{b}\bar{b}\bar{a}bbbbaaa,\ \bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\bar{a})$

Parsing with the grammar

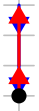
Rule: $Inv(x_1 y_1, y_2 x_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2)$



$Inv(\overline{bb}abaaa\overline{bbb}\overline{a}bbbbaaa, \overline{aaaaaa})$



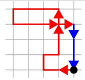
$Inv(\overline{bb}abaaa\overline{bbb}\overline{a}bbbb, \overline{aaa})$



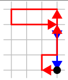
$Inv(aaa, \overline{aaa})$

Parsing with the grammar

Rule: $Inv(\overline{b}x_1 b, x_2) \leftarrow Inv(x_1, x_2)$



$Inv(\overline{b}\overline{b}abaaa\overline{b}\overline{b}\overline{b}\overline{a}bbbb, \overline{a}\overline{a}\overline{a})$



$Inv(\overline{b}abaaa\overline{b}\overline{b}\overline{b}\overline{a}bbb, \overline{a}\overline{a}\overline{a})$

Outline

The proof of the Theorem

Theorem: Given w_1 and w_2 such that $w_1 w_2 \in O_2$, $Inv(w_1, w_2)$ is derivable.

The proof is done by induction on the lexicographically ordered pairs $(|w_1 w_2|, \max(|w_1|, |w_2|))$.
There are five cases:

Case 1: w_1 or w_2 equal ϵ:

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There are five cases:

Case 1: w_1 or w_2 equal ϵ:
w.l.o.g., $w_1 \neq \epsilon$, then by induction hypothesis, for any v_1 and v_2 different from ϵ such that $w_1 = v_1 v_2$, $Inv(v_1, v_2)$ is derivable then:

$$\frac{Inv(v_1, v_2) \quad Inv(\epsilon, \epsilon)}{Inv(v_1 v_2 = w_1, \epsilon)} \; Inv(x_1 x_2, y_1 y_2) \leftarrow Inv(x_1, x_2),\ Inv(y_1, y_2)$$

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There are five cases:

Case 2: $w_1 = s_1 w_1' s_2$ and $w_2 = s_3 w_2' s_4$ and for $i, j \in \{1; 2; 3; 4\}$, s.t. $i \neq j$, $\{s_i; s_j\} \in \{\{a; \overline{a}\}; \{b; \overline{b}\}\}$:

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Case 2: $w_1 = s_1 w_1' s_2$ and $w_2 = s_3 w_2' s_4$ and for $i, j \in \{1; 2; 3; 4\}$, s.t. $i \neq j$, $\{s_i; s_j\} \in \{\{a; \overline{a}\}; \{b; \overline{b}\}\}$:
e.g., if $i = 1$, $j = 2$, $s_1 = a$ and $s_2 = \overline{a}$ then by induction hypothesis $Inv(w_1', w_2)$ is derivable and:

$$\frac{Inv(w_1', w_2)}{Inv(a w_1' \overline{a}, w_2)} \; Inv(a x_1 \overline{a}, x_2) \leftarrow Inv(x_1, x_2)$$

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Case 3: the curves representing w_1 and w_2 have a non-trivial intersection point:

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Theorem: Given w_1 and w_2 such that $w_1 w_2 \in O_2$, $Inv(w_1, w_2)$ is derivable.

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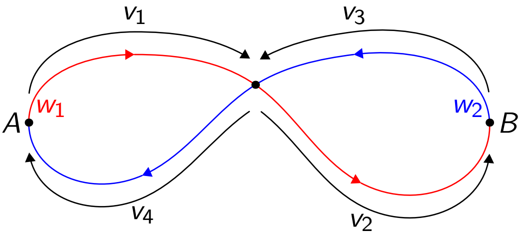
$(|w_1 w_2|, \max(|w_1|, |w_2|))$.
There are five cases:

Case 3: the curves representing w_1 and w_2 have a non-trivial intersection point:



$$\frac{Inv(v_1, v_4) \quad Inv(v_2, v_3)}{Inv(v_1 v_2 = w_1, v_3 v_4 = w_2)}$$

The proof of the Theorem

Theorem: Given w_1 and w_2 such that $w_1 w_2 \in O_2$, $Inv(w_1, w_2)$ is derivable.

The proof is done by induction on the lexicographically ordered pairs $(|w_1 w_2|, \max(|w_1|, |w_2|))$.
There are five cases:

Case 4: the curve representing w_1 or w_2 starts or ends with a loop:

The proof of the Theorem

Theorem: Given w_1 and w_2 such that $w_1 w_2 \in O_2$, $Inv(w_1, w_2)$ is derivable.

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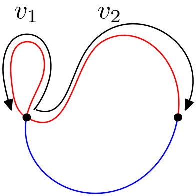
Case 4: the curve representing w_1 or w_2 starts or ends with a loop:



$$\frac{Inv(v_1, \epsilon) \quad Inv(v_2, w_2)}{Inv(v_1 v_2 = w_1, w_2)}$$

The proof of the Theorem

Theorem: Given w_1 and w_2 such that $w_1 w_2 \in O_2$, $Inv(w_1, w_2)$ is derivable.

The proof is done by induction on the lexicographically ordered pairs $(|w_1 w_2|, \max(|w_1|, |w_2|))$.
There are five cases:

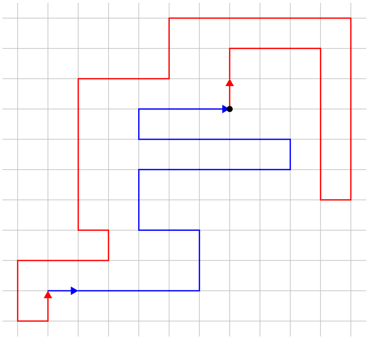
Case 5: w_1 and w_2 do not start or end with compatible letters, the curve representing then do not intersect and do not start or end with a loop.

Case 5

No rule other than

$$\begin{array}{rcl} Inv(x_1 y_1 x_2, y_2) & \leftarrow & Inv(x_1, x_2), Inv(y_1, y_2) \\ Inv(x_1, y_1 x_2 y_2) & \leftarrow & Inv(x_1, x_2), Inv(y_1, y_2) \end{array}$$

can be used.

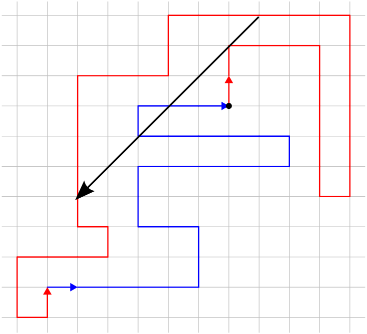


Case 5

No rule other than

$$
\begin{aligned}
Inv(x_1y_1x_2, y_2) &\leftarrow Inv(x_1, x_2), Inv(y_1, y_2) \\
Inv(x_1, y_1x_2y_2) &\leftarrow Inv(x_1, x_2), Inv(y_1, y_2)
\end{aligned}
$$

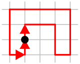
can be used.



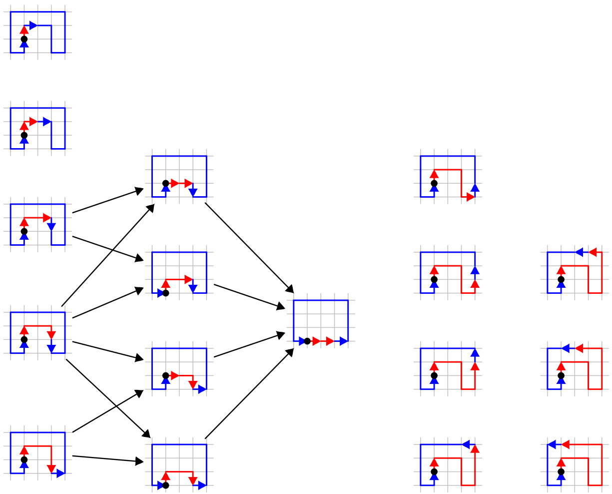
The relevance of case 5

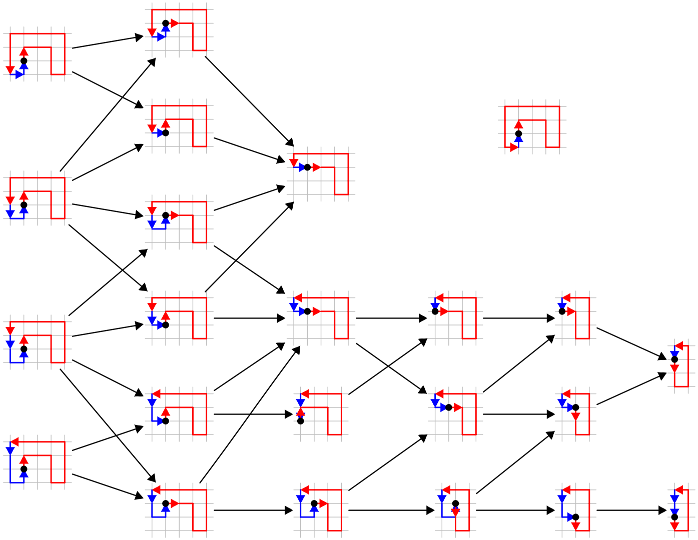
The word

$$ abb\overline{a}\overline{a}baaa\overline{b}\overline{b}\overline{b}\overline{b}\overline{a}\overline{a}\overline{a}ba $$



is not in the language of the grammar only containing the well-nested rules.





The relevance of case 5: a proof is now in hand

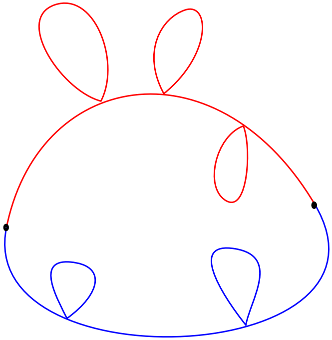
- Joshi (1985)

[*MIX*] represents the extreme case of the degree of free word order permitted in a language. This extreme case is linguistically not relevant. [. . .] TAGs also cannot generate this language although for TAGs the proof is not in hand yet.

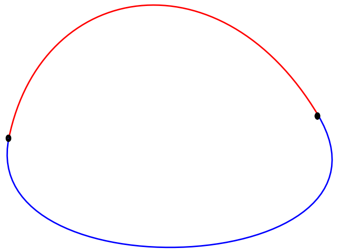
Theorem (Kanazawa, S. 12)

There is no 2-$MCFL_{wn}$ (or TAG) generating MIX or O_2.

Solving case 5: towards geometry



Solving case 5: towards geometry



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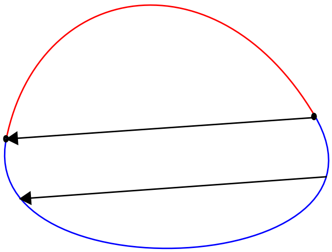
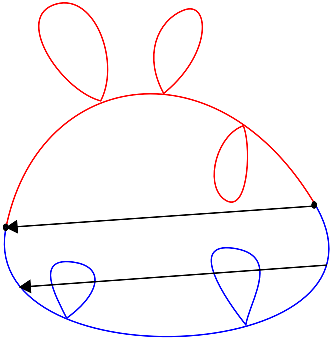
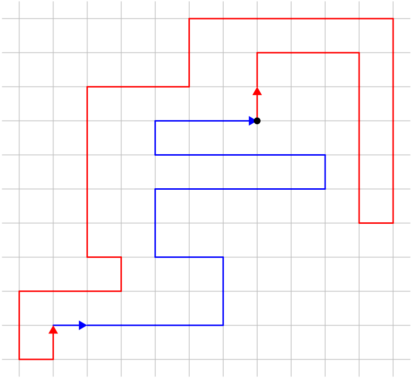
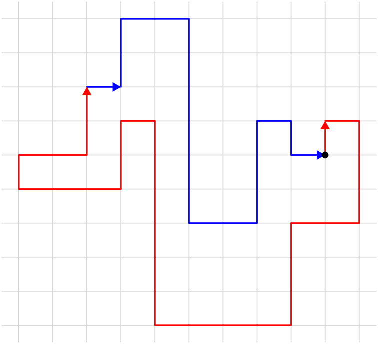


Solving case 5: towards geometry

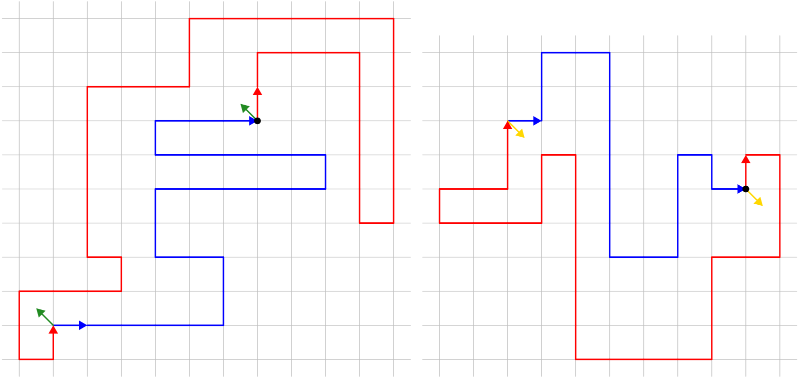


Solving case 5: a geometrical invariant



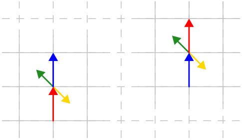
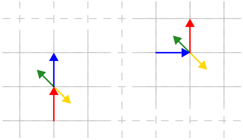
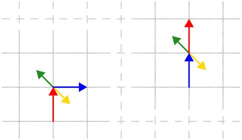
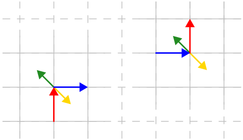


Solving case 5: a geometrical invariant



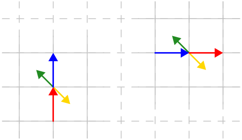
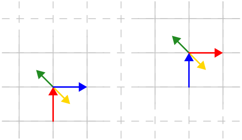
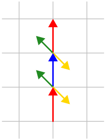
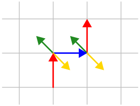
Solving case 5: a geometrical invariant

An invariant on the Jordan curve representing $w_1'w_2'$:

 $w_1' = aw_1''a$ and $w_2' = aw_2''a$	 $w_1' = aw_1''a$ and $w_2' = aw_2''b$
 $w_1' = aw_1''a$ and $w_2' = bw_2''a$	 $w_1' = aw_1''a$ and $w_2' = bw_2''b$

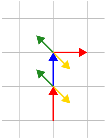
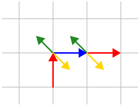
Solving case 5: a geometrical invariant

An invariant on the Jordan curve representing $w'_1 w'_2$:

 $w'_1 = aw''_1 b$ and $w'_2 = aw''_2 b$	 $w'_1 = aw''_1 b$ and $w'_2 = bw''_2 a$
 $w'_1 = aw''_1 a$ and $w'_2 = a$	 $w'_1 = aw''_1 a$ and $w'_2 = b$

Solving case 5: a geometrical invariant

An invariant on the Jordan curve representing $w_1' w_2'$:

	
$w_1' = a w_1'' b$ and $w_2' = a$	$w_1' = a w_1'' b$ and $w_2' = b$

Outline

Jordan curves

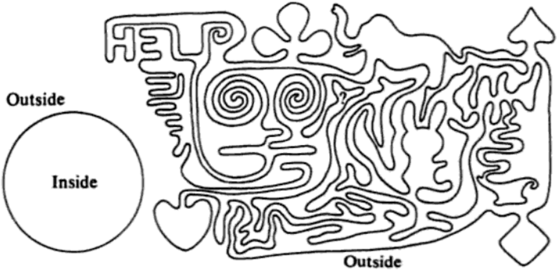
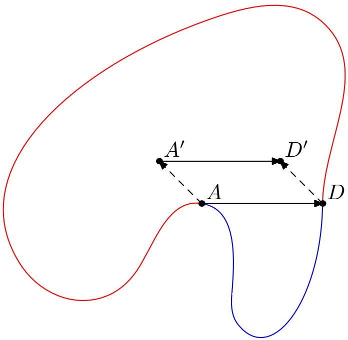


Figure 13.1 Two Jordan curves.

illustration from: A combinatorial introduction to topology by Michael Henle (Dover Publications).

A theorem on Jordan curves

Theorem: If A and D are two points on a Jordan curve J such that there are two points A' and D' inside J such that $\overrightarrow{AD} = \overrightarrow{A'D'}$, then there are two points B and C pairwise distinct from A and D such that A, B, C, and D appear in that order on one of the arcs going from A to D and $\overrightarrow{AD} = \overrightarrow{BC}$.



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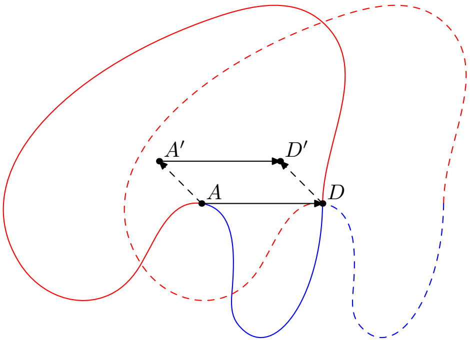

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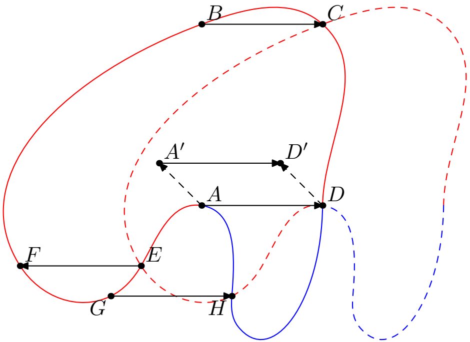

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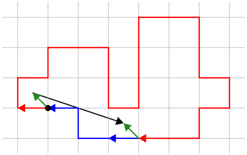
Applying this Theorem solves case 5.



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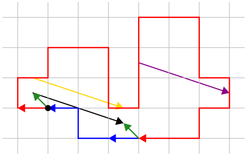

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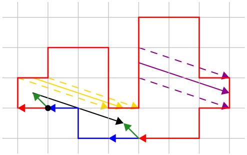


Winding number

Let $wn(J, z)$ be the winding number of a closed curve around z.

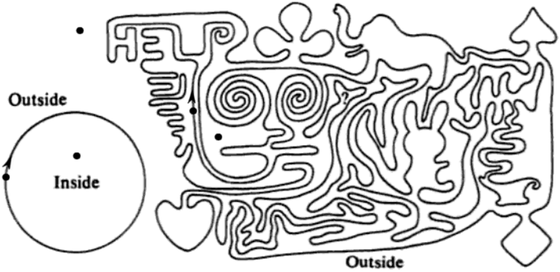


Figure 13.1 Two Jordan curves.

illustration from: A combinatorial introduction to topology by Michael Henle (Dover Publications).

An interesting Lemma

Let $\exp : \left\{ \begin{array}{ccc} \mathbb{C} & \rightarrow & \mathbb{C} - \{0\} \\ z & \rightarrow & e^{2i\pi z} \end{array} \right.$.

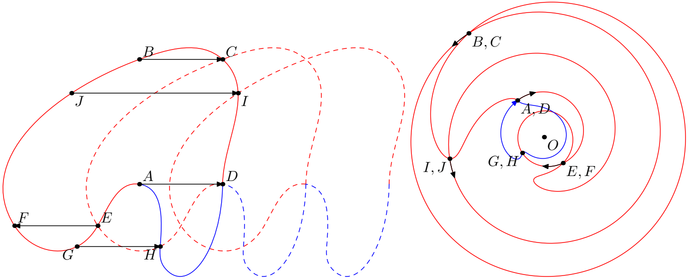
Lemma

Given an simple arc $\overset{\curvearrowright}{AB}$ *such that* $\overrightarrow{AB} = k \in \mathbb{N}$, *we have:*

$$wn(\exp(\overset{\curvearrowright}{AB}), 0) = k$$

Translation becomes rotation

$$\exp : \left\{ \begin{array}{ccc} \mathbb{C} & \rightarrow & \mathbb{C} - \{0\} \\ z & \rightarrow & e^{2i\pi z} \end{array} \right. .$$



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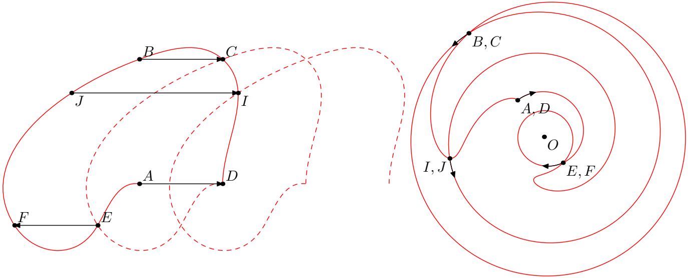

An interesting characterization

Lemma

Given an simple arc $\overset{\curvearrowright}{AD}$ such that $\overrightarrow{AD} = 1$, we have:

- $\overset{\curvearrowright}{AD}$ *contains a proper subarc* $\overset{\curvearrowright}{BC}$ *such that* $\overrightarrow{AD} = \overrightarrow{BC}$ *iff* $\exp(\overset{\curvearrowright}{AD})$ *is not a Jordan curve.*

Jordan curves and winding numbers

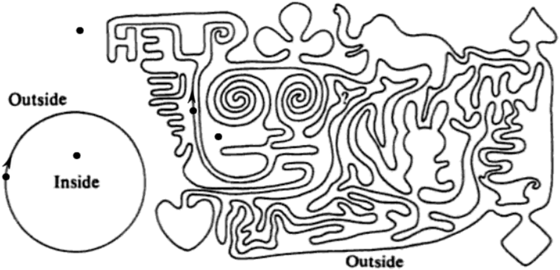

Figure 13.1 Two Jordan curves.

illustration from: A combinatorial introduction to topology by Michael Henle (Dover Publications).

Theorem: There is $k \in \{-1; 1\}$ such that the winding number of Jordan curve around a point in its interior is k, its winding number around a point in its exterior is 0.

Proving the characterization

Lemma

Given an simple arc $\overset{\curvearrowright}{AD}$ such that $\overrightarrow{AD} = 1$, we have:

- *$\overset{\curvearrowright}{AD}$ contains a proper subarc $\overset{\curvearrowright}{BC}$ such that $\overrightarrow{AD} = \overrightarrow{BC}$ iff $\exp(\overset{\curvearrowright}{AD})$ is not a Jordan curve.*

Proof

Proving the characterization

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Given an simple arc $\overset{\curvearrowright}{AD}$ *such that* $\overrightarrow{AD} = 1$, *we have:*

- $\overset{\curvearrowright}{AD}$ *contains a proper subarc* $\overset{\curvearrowright}{BC}$ *such that* $\overrightarrow{AD} = \overrightarrow{BC}$ *iff* $\exp(\overset{\curvearrowright}{AD})$ *is not a Jordan curve.*

Proof

- by 1-periodicity of $\exp$, if $\overset{\curvearrowright}{AD}$ contains a proper subarc $\overset{\curvearrowright}{BC}$ such that $\overrightarrow{AD} = \overrightarrow{BC}$, then $\exp(\overset{\curvearrowright}{AD})$ is not a Jordan curve,

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- if $\exp(\overset{\curvearrowright}{AD})$ is not a Jordan curve:
  - take the closed curve $\mathcal{C}$ obtained by removing the closed subcurves of $\exp(\overset{\curvearrowright}{AD})$ that have a negative winding number,

Proving the characterization

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Given an simple arc $\overset{\frown}{AD}$ *such that* $\overrightarrow{AD} = 1$, *we have:*

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Proof

- by 1-periodicity of $\exp$, if $\overset{\frown}{AD}$ contains a proper subarc $\overset{\frown}{BC}$ such that $\overrightarrow{AD} = \overrightarrow{BC}$, then $\exp(\overset{\frown}{AD})$ is not a Jordan curve,
- if $\exp(\overset{\frown}{AD})$ is not a Jordan curve:
  - take the closed curve $\mathcal{C}$ obtained by removing the closed subcurves of $\exp(\overset{\frown}{AD})$ that have a negative winding number,
  - take a proper closed subcurve $\mathcal{D}$ of $\mathcal{C}$ that is minimal for inclusion,

Proving the characterization

Lemma

Given an simple arc $\overset{\curvearrowright}{AD}$ *such that* $\overrightarrow{AD} = 1$*, we have:*

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Proof

- by 1-periodicity of $\exp$, if $\overset{\curvearrowright}{AD}$ contains a proper subarc $\overset{\curvearrowright}{BC}$ such that $\overrightarrow{AD} = \overrightarrow{BC}$, then $\exp(\overset{\curvearrowright}{AD})$ is not a Jordan curve,
- if $\exp(\overset{\curvearrowright}{AD})$ is not a Jordan curve:
  - take the closed curve $\mathcal{C}$ obtained by removing the closed subcurves of $\exp(\overset{\curvearrowright}{AD})$ that have a negative winding number,
  - take a proper closed subcurve $\mathcal{D}$ of $\mathcal{C}$ that is minimal for inclusion,
  - $\mathcal{D}$ is a Jordan curve winding positively (*i.e.* once) around 0,

Proving the characterization

Lemma

Given an simple arc $\overset{\curvearrowright}{AD}$ such that $\overrightarrow{AD} = 1$, we have:

- *$\overset{\curvearrowright}{AD}$ contains a proper subarc $\overset{\curvearrowright}{BC}$ such that $\overrightarrow{AD} = \overrightarrow{BC}$ iff $\exp(\overset{\curvearrowright}{AD})$ is not a Jordan curve.*

Proof

- by 1-periodicity of $\exp$, if $\overset{\curvearrowright}{AD}$ contains a proper subarc $\overset{\curvearrowright}{BC}$ such that $\overrightarrow{AD} = \overrightarrow{BC}$, then $\exp(\overset{\curvearrowright}{AD})$ is not a Jordan curve,
- if $\exp(\overset{\curvearrowright}{AD})$ is not a Jordan curve:
  - take the closed curve $\mathcal{C}$ obtained by removing the closed subcurves of $\exp(\overset{\curvearrowright}{AD})$ that have a negative winding number,
  - take a proper closed subcurve $\mathcal{D}$ of $\mathcal{C}$ that is minimal for inclusion,
  - $\mathcal{D}$ is a Jordan curve winding positively (*i.e.* once) around 0,
  - $\mathcal{D}$ induces a proper subcurve $\mathcal{J}$ of $\exp(\overset{\curvearrowright}{AD})$ whose winding number is 1,

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Proof

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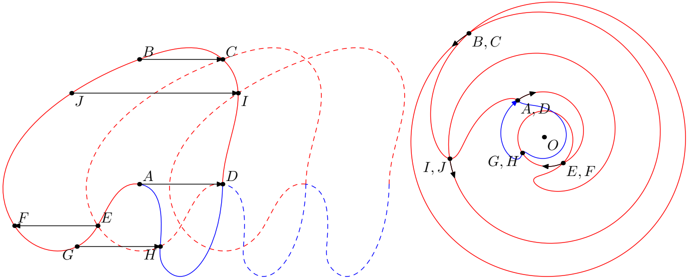
- if $\exp(\overset{\curvearrowright}{AD})$ is not a Jordan curve:
  - take the closed curve $\mathcal{C}$ obtained by removing the closed subcurves of $\exp(\overset{\curvearrowright}{AD})$ that have a negative winding number,
  - take a proper closed subcurve $\mathcal{D}$ of $\mathcal{C}$ that is minimal for inclusion,
  - $\mathcal{D}$ is a Jordan curve winding positively (*i.e.* once) around 0,
  - $\mathcal{D}$ induces a proper subcurve $\mathcal{J}$ of $\exp(\overset{\curvearrowright}{AD})$ whose winding number is 1,
  - $\mathcal{J}$ induces a proper subarc $\overset{\curvearrowright}{BC}$ of $\exp(\overset{\curvearrowright}{AD})$ such that $\overrightarrow{AD} = \overrightarrow{BC}$.

The characterization on the example

$$\exp : \left\{ \begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} - \{0\} \\ z & \to & e^{2i\pi z} \end{array} \right. .$$



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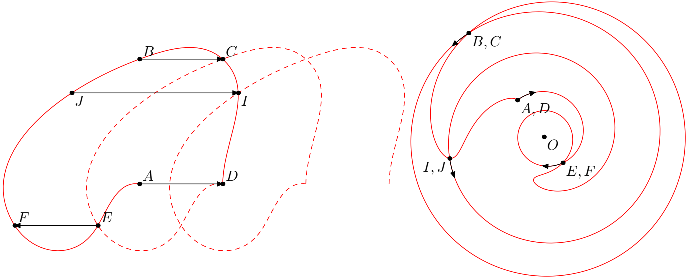
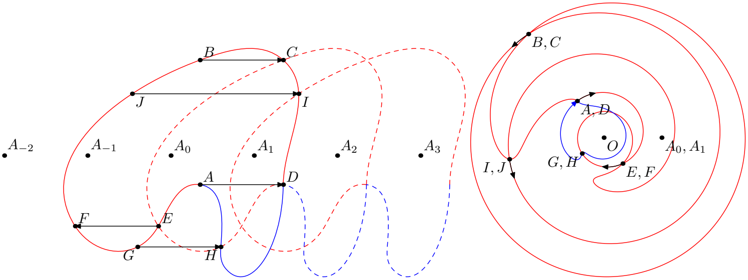


Yet another observation from algebraic topology

Let's suppose that $\overrightarrow{AD} = 1$ and that $A_0 = A' = 0$, $A_1 = D' = 1, \ldots, A_k = k$

let $\exp : \begin{cases} \mathbb{C} & \rightarrow & \mathbb{C} - \{0\} \\ z & \rightarrow & e^{2i\pi z} \end{cases}$.



$\exp$ sums up the winding number of a Jordan curve around the A_i's as the winding number around $\exp(A_0) = \exp(0) = 1$.

Proving the Theorem

Let's suppose that $\overrightarrow{AD} = 1$,

Lemma

Given an simple arc $\overset{\curvearrowright}{AD}$ *we have:*

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Corollary: a simple path J from A to D (*resp.* D to A) does not contain B and C as required in the Theorem iff $\varphi(J)$ is a Jordan curve of $\mathbb{C} - \{1\}$ that winding 0 or 1 (*resp.* or -1) time around 1.

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Given an simple arc $\overset{\curvearrowright}{AD}$ we have:

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Corollary: if J is a simple closed curve of $\mathbb{C}$ composed with two curves J_1 and J_2 respectively going from A to D and D to A which do not contain points B and C as required in the Theorem then $|wn(\exp(J), 1)| = |wn(\exp(J_1), 1) + wn(\varphi(J_2), 1)| \leq 1$.

Proving the Theorem

Let's suppose that $\overrightarrow{AD} = 1$,

Lemma

Given an simple arc $\overset{\curvearrowright}{AD}$ we have:

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The Theorem follows by contradiction.

Outline

The group language of $\mathbb{Z}^2$

A similar problem in computational linguistics

Multiple Context Free Grammars (MCFGs)

A grammar for O_2

Proof of the Theorem

A Theorem on Jordan curves

Conjectures

Nederhof's conjecture

- Nederhof (2016)

Conjecture: for every k,

$$O_k = \{w \in \{a_1, \overline{a_1} \ldots, a_k, \overline{a_k}\}^* \mid \forall 1 \leq i \leq k, |w|_{a_i} = |w|_{\overline{a_i}}\}$$

is generated by the grammar with rules of the form:

$$\begin{array}{rl}
S(x_1 \ldots x_k) \leftarrow & Inv(x_1, \ldots, x_k) \\
\hline
Inv(s_1, \ldots, s_k) \leftarrow & Inv(x_1, \ldots, x_k),\ Inv(y_1, \ldots, y_k) \\
& s_1 \ldots s_k \in \texttt{perm}(x_1 \ldots x_k y_1 \ldots y_k) \\
\hline
Inv(x_1, \ldots, \alpha x_i, \ldots, \overline{\alpha} x_j, \ldots, x_k) \leftarrow & Inv(x_1, \ldots, x_k) \\
& \vdots \\
\hline
Inv(\epsilon, \ldots, \epsilon) \leftarrow &
\end{array}$$

Status of the conjecture

Positive arguments

- The conjecture has been tested on millions of examples

Status of the conjecture

Positive arguments

- The conjecture has been tested on millions of examples
- In the case of $\mathbb{Z}^3$, some cases can be solved using braiding arguments

Negative argument

Status of the conjecture

Positive arguments

- The conjecture has been tested on millions of examples
- In the case of $\mathbb{Z}^3$, some cases can be solved using braiding arguments

Negative argument

- for the case of $\mathbb{Z}^2$ many arguments are strongly related to planarity $\rightarrow$ no clear way of generalizing to higher dimensions