On estimating the derivatives of symmetric diffusions in stationary random environment, with applications to $\nabla\phi$ interface model.

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Abstract. *We consider diffusions on $\mathbb{R}^d$ or random walks on $\mathbb{Z}^d$ in a random environment which is stationary in space and in time and with symmetric and uniformly elliptic coefficients. We show existence and Hölder continuity of second space derivatives and time derivatives for the annealed kernels of such diffusions and give estimates for these derivatives. In the case of random walks, these estimates are applied to the Ginzburg-Landau $\nabla\phi$ interface model.*

Classically it is known that the kernels of symmetric diffusions have some Hölder regularity. By "classically" we refer to the theory developed by E. De Giorgi, J. Nash and J. Moser in the fifties and sixties for uniformly elliptic second-order operators in divergence form.

Now if we consider annealed kernels with respect to a stationary random environment, we obtain more regularity, namely Hölder regularity for the second space derivatives and the first time derivative. In fact this comes from a result of Hölder regularity obtained for the first space derivatives when they are "L^2-annealed".

The idea to search for better regularity of elliptic diffusions in random environment thanks to stationarity of the law of the environment has already appeared in [CN], completed by [C]. We answer positively to their question about the existence of second space derivatives (page 155 in [CN]).

After the presentation of the main results in §1 for kernels and Green functions with time-dependence, the basic argument leading to "L^2-annealed" first space derivatives is presented in §2 for simplicity in the case of Green functions without time-dependence. This way no time is involved at all but we need to work in $\mathbb{R}^d$ with $d \geq 3$, so that these Green functions exist.

In §3, the time variable is added and we apply the basic argument to the the case of the heat kernel on $\mathbb{R}^d$.

The argument also applies in different settings, as long as there is some translation notion giving sense to stationarity. We develop carefully in §4 the case of random walks on $\mathbb{Z}^d$ with continuous time (jumps), since it is used in §6.

In §5, using semi-groups techniques, we deduce the results for second space derivatives and time derivatives of the kernels, either of the diffusion or of the random walk.

Finally in §6 we give an application of our results to the estimates of space-time covariances for $\nabla\phi$ interface models. This is a model of locally interacting diffusions with a strictly convex gradient interaction, cf. [F]. Our main result describes the decay of the correlations for the invariant measure and a dimension dependent L^2 algebraic convergence for the semigroup. In particular we apply Theorem 1.1 to show that the field of the gradient, which is orthogonal to the preserved mean, has a faster decaying rate, cf. Theorem 6.2 below.

§1 Estimates.

Let us introduce continuous-time random walks X_t on $\mathbb{Z}^d$, with transition rates from x to a neighbor $z \neq x$

$$\mathbf{P}(X_{t+h} = z \mid X_t = x) \sim a(t,x,z)h, \qquad h \to 0.$$

These transition rates a are the environment, which will be assumed to belong to a certain class $\mathcal{A}$ of symmetric and uniformly elliptic environments. Let Γ be an irreducible finite symmetric subset of $\mathbb{Z}^d$. Symmetric means that $\forall y \in \Gamma, -y \in \Gamma$. And irreducible means that $\forall x \in \mathbb{Z}^d$,

$$\|x\|_\Gamma = \inf\{n \in \mathbb{N}, \quad \exists y_1, \ldots, y_n \in \Gamma,\ x = y_1 + \ldots + y_n\} < \infty.$$

In this paper we will rather use this Γ distance, but of course since Γ is finite, one can easily replace it by any equivalent usual distance on $\mathbb{Z}^d$.

We say that $a \in \mathcal{A}(C_-, C_+, d, \Gamma)$ for $0 < C_- \le C_+$ if
a is measurable (with respect to t),
a is symmetric, $\qquad a(t,x,z) = a(t,z,x), \qquad t \in \mathbb{R}, x, z \in \mathbb{Z}^d$,
a is supported by Γ, $\qquad y \notin \Gamma \Longrightarrow a(t,x,x+y) = 0, \qquad t \in \mathbb{R}, x \in \mathbb{Z}^d$,
a is Γ-elliptic, $\qquad y \in \Gamma \Longrightarrow C_- \le a(t,x,x+y) \le C_+, \qquad t \in \mathbb{R}, x \in \mathbb{Z}^d$.

In the spirit of the classical theory of E. De Giorgi, J. Nash and J. Moser for uniformly elliptic second-order operators in divergence form, it is possible to handle the discrete operator

$$L_a u(t,x) = \sum_{y\in\Gamma} a(t,x,x+y)\big(u(t,x+y) - u(t,x)\big) \tag{1.1}$$

when $a \in \mathcal{A}(C_-, C_+, d, \Gamma)$. As a consequence, it is now known that, despite non uniform transition rates, X_t behaves roughly like a standard random walk with uniform transitions. To make this precise, denote $p_a(s,x,t,z) = \mathbf{P}(X_t = z \mid X_s = x)$, for $s \le t$. It is the fundamental solution of the parabolic equation

$$\begin{aligned} p_a(s,x_0,s,\cdot) &= \delta_{x_0}(\cdot), \\ \frac{\partial p_a(s,x_0,t,x)}{\partial t} &= L_a p_a(s,x_0,t,x). \end{aligned}$$

The basic estimate[1] is

$$c_1 p^*(k_1 t, x) \le p_a(0,0,t,x) \le c_2 p^*(k_2 t, x), \tag{1.2}$$

where p^* is used as a reference and is defined by $p^*(t,x) = p_{a^*}(0,0,t,x)$ for $a^*(t,x,z) = 1_{z-x\in\Gamma}$, but could as well be defined using $\Gamma = \{y \in \mathbb{Z}^d, \|y\| = 1\}$ or defined as an explicit function $\mathcal{E}(t,x)/(1\vee t)^{d/2}$. This is explained in §4. For now we shall postpone the difficulty of the $\mathcal{E}$ function which behaves differently from $e^{-\|x\|^2/2t}$ when $\|x\| >> t$.

As far as integrability in t over $[0,+\infty[$ is concerned, p_a has the same qualitative behavior as p^* so that for $d \ge 3$, $x \ne 0$, the Green function

$$c_1\|x\|^{2-d} \le g_a(0,x) = \int_0^\infty p_a(0,0,t,x)dt \le c_2\|x\|^{2-d}.$$

[1]Here and below adopt the following **convention**: $c, k, c_1, c_2, \ldots$ are positive constants depending only on C_-, C_+, d and Γ. They may take different values at different occurrences but the inequalities always apply uniformly with respect to the other variables in the inequality, like x and t. Note also that the statements usually written with starting points at 0 in space or time for clarity may immediately be written from any starting point through a simple translation.

Inequality (1.2) was in particular useful in [DGI] and in [GOS] about the so-called $\nabla\phi$ interface model [F], since a central role is played by a random walk representation going back to [HS]. Another crucial idea for this model is that its dynamic preserves constants, so that processes for gradients should have faster rates of convergence. This leads to the introduction of gradients of the kernel p, $\nabla_x^y p(0,0,t,\cdot) = p(0,0,t,x+y) - p(0,0,t,x)$ for $y \in \Gamma$. Now sadly one cannot expect a better estimate than

$$\left|\nabla_x^y p_a(0,0,t,\cdot)\right| \leq c_1 \frac{p^*(kt,x)}{(1\vee t)^{c_2}}, \tag{1.3}$$

where c_2 might be very small, although for uniform transitions one would obtain $c_2 = 1/2$. This is related to the fact that in the classical theory for uniformly elliptic second-order operators in divergence form, the weak solutions turn out to have some Hölder regularity but not more, without regularity assumptions for a. See the end of the section for more details.

Our typical result about "L^2-annealed" space derivatives, second space derivatives and time derivatives is the following theorem. We will assume that the environment a is random with a stationary law $\mathbb{P}$ and denote p_a^ω the associated random kernel. Let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be endowed with a measurable group of translations $\tau_{t,x} : \Omega \to \Omega$, where $t \in \mathbb{R}$ and $x \in \mathbb{Z}^d$, so that stationarity will be characterized by the fact that $\mathbb{P}$ is preserved by all translations $\tau_{t,x}$ (which means $\forall t \in \mathbb{R}, \forall x \in \mathbb{R}^d, \mathbb{P} \circ \tau_{t,x} = \mathbb{P}$).

We denote

$$\bar{p}_a(0,0,t,x) = \mathbb{E}\big(p_a^\omega(0,0,t,x)\big) \qquad \text{and for } d \geq 3, \quad \bar{g}_a(0,x) = \mathbb{E}\big(g_a(0,x)\big).$$

Note also that the notation ∇ may be iterated in the following way, like a second derivative:

$$\begin{aligned}
&\nabla_x^y \nabla_\cdot^{y'} \bar{p}_a(0,0,t,\cdot)\\
&= \Big(\bar{p}_a(0,0,t,x+y+y') - \bar{p}_a(0,0,t,x+y)\Big) - \Big(\bar{p}_a(0,0,t,x+y') - \bar{p}_a(0,0,t,x)\Big),
\end{aligned}$$

or in this other way, like a mixed derivative:

$$\begin{aligned}
&\nabla_0^{2,y} \nabla_x^{4,y'} p_a^\omega(0,\cdot,t,\cdot)\\
&= \Big(p_a^\omega(0,y,t,x+y') - p_a^\omega(0,y,t,x)\Big) - \Big(p_a^\omega(0,0,t,x+y') - p_a^\omega(0,0,t,x)\Big).
\end{aligned}$$

Theorem 1.1. *Let a^ω be a random environment always in $\mathcal{A}(C_-, C_+, d, \Gamma)$ which can be written*

$$a^\omega(t,x,z) = a(z-x, \tau_{t,x}\omega).$$

If $\mathbb{P}$ is preserved by all translations $\tau_{t,x}$, then $\bar{p}_a(0,0,t,x)$ is differentiable in t with a Lipschitz derivative and for $y, y' \in \Gamma$,

$$\sqrt{\mathbb{E}\Big(\big(\nabla_x^y p_a^\omega(0,0,t,\cdot)\big)^2\Big)} \leq c_1 \frac{p^*(k_1 t,x)}{\sqrt{1\vee t}}, \tag{1.4}$$

$$\mathbb{E}\Big(\big|\nabla_0^{2,y} \nabla_x^{4,y'} p_a^\omega(0,\cdot,t,\cdot)\big|\Big) \leq c_2 \frac{p^*(k_2 t,x)}{1\vee t}, \tag{1.5a}$$

$$\left|\nabla_x^y \nabla_\cdot^{y'} \bar{p}_a(0,0,t,\cdot)\right| \leq c_2 \frac{p^*(k_2 t,x)}{1\vee t}, \tag{1.5b}$$

$$\left|\frac{\partial}{\partial t} \bar{p}_a(0,0,t,x)\right| \leq c_3 \frac{p^*(k_3 t,x)}{1\vee t}. \tag{1.6}$$

As a corollary for $d \geq 3$,

$$\left|\nabla_x^y \bar{g}_a(0,\cdot)\right| \leq c_4 \|x\|^{1-d}, \tag{1.7}$$

$$\left|\nabla_x^y \nabla_\cdot^{y'} \bar{g}_a(0,\cdot)\right| \leq c_5 \|x\|^{-d}. \tag{1.8}$$

For $d = 2$*, we can define for any* ω *a "*∇g_a*" as an integral* $\int_0^\infty \nabla p_a$ *thanks to (1.3), although* g_a *itself is infinite. If its expected value plays the role of* $\nabla \bar{g}_a$*, then (1.7) and (1.8) still hold.*
For $d = 1$*, we can similarly define a "*$\nabla\nabla\bar{g}_a$*" thanks to (1.5b) and this way (1.8) still holds.*

In this paper we shall start with the setting of symmetric diffusions on $\mathbb{R}^d$, that of what we called De Giorgi-Nash-Moser theory. We say that an environment $b : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d} \in \mathcal{B}(C_-, C_+, d)$ if
b is a measurable field of matrices,
b is bounded, $\quad b_{ij}(t,x) \leq C_+, \quad t \in \mathbb{R}, x \in \mathbb{R}^d, 1 \leq i,j \leq d,$
b is elliptic, $\quad C_-\|\xi\|^2 \leq \sum_{i,j} b_{ij}(t,x)\xi_i\xi_j \leq C_+\|\xi\|^2, \quad t \in \mathbb{R}^+, x, \xi \in \mathbb{R}^d.$

The kernel p_b of the diffusion associated to such an environment is the fundamental solution of the formal parabolic equation

$$p_b(s,x_0,s,\cdot) = \delta_{x_0}(\cdot),$$
$$\frac{\partial p_b(s,x_0,t,x)}{\partial t} = L_b p_b(s,x_0,t,x),$$

with

$$L_b u(t,x) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\Big(b_{ij}(t,x)\frac{\partial}{\partial x_j}u(t,x)\Big).$$

We precise "formal" because this should be understood in a weak form, we give later short details for the elliptic equation, see (2.3).

For p_b when $b \in \mathcal{B}(C_-, C_+, d)$, we have the following Aronson estimates, see Theorem 1 in [A1] and [A2] where existence in the weak form is also discussed.

$$\frac{c_1}{t^{d/2}} e^{-\frac{\|x\|^2}{k_1 t}} \leq p_b(0,0,t,x) \leq \frac{c_2}{t^{d/2}} e^{-\frac{\|x\|^2}{k_2 t}}. \tag{1.9}$$

And for $d \geq 3$,

$$c_1\|x\|^{2-d} \leq g_b(0,x) = \int_0^\infty p_b(0,0,t,x)dt \leq c_2\|x\|^{2-d}.$$

Now here is our result for derivatives in a stationary random environment. The probability space is still endowed with a group of translations $\tau_{t,x}$, now $x \in \mathbb{R}^d$. We keep the same notations p_b^ω, $\bar{p}_b$ for the random and annealed kernels, $\bar{g}_b$ for the annealed Green function.

Theorem 1.2. *Let* b^ω *be a random environment always in* $\mathcal{B}(C_-, C_+, d)$ *which can be written*

$$b^\omega(t,x) = b(\tau_{t,x}\omega).$$

If $\mathbb{P}$ is preserved by all translations $\tau_{t,x}$, then $\bar{p}_b(0,0,t,x)$ is twice differentiable in x and once differentiable in t with Hölder continuous derivatives. Moreover

$$\sqrt{\mathbb{E}\big(\|\nabla p_b^\omega(0,0,t,x)\|^2\big)} \le \frac{c_1}{t^{d/2+1/2}} e^{-\frac{\|x\|^2}{k_1 t}}, \tag{1.10}$$

$$\mathbb{E}\big(\big\|\nabla^2\nabla^4 p_b(0,0,t,x)\big\|\big) \le \frac{c_2}{t^{d/2+1}} e^{-\frac{\|x\|^2}{k_2 t}}, \tag{1.11a}$$

$$\big\|\nabla\nabla\bar{p}_b(0,0,t,x)\big\| \le \frac{c_2}{t^{d/2+1}} e^{-\frac{\|x\|^2}{k_2 t}}, \tag{1.11b}$$

$$\Big|\frac{\partial}{\partial t}\bar{p}_b(0,0,t,x)\Big| \le \frac{c_3}{t^{d/2+1}} e^{-\frac{\|x\|^2}{k_3 t}}. \tag{1.12}$$

As a corollary for $d \ge 3$,

$$\big\|\nabla\bar{g}_b(0,x)\big\| \le c_4\|x\|^{1-d}, \tag{1.13}$$

$$\big\|\nabla\nabla\bar{g}_b(0,x)\big\| \le c_5\|x\|^{-d}. \tag{1.14}$$

For $d = 2$ we can define a "$\nabla\bar{g}_b$" thanks to (1.10), for which (1.13) and (1.14) still hold.
For $d = 1$ we can define a "$\nabla\nabla\bar{g}_b$" thanks to (1.11), for which (1.14) still holds.

We think our key argument in the proofs of Theorems 1.1 and 1.2 will be more transparent if we present it for the Green function of a time-independent environment which is the fundamental solution of an elliptic equation with no time variable. Let $c : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d} \in \mathcal{C}(C_-, C_+, d)$, where $\mathcal{C}$ is the obvious analog of $\mathcal{B}$ without the time variable. For $d \ge 3$, the Green function is the formal solution of

$$\begin{aligned} &L_c g_c(x_0,\cdot) = -\delta_{x_0}(\cdot),\\ &g_c(x_0,x) \longrightarrow 0, \quad \text{for } \|x\| \longrightarrow \infty, \end{aligned}$$

with

$$L_c u(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(c_{ij}(x)\frac{\partial}{\partial x_j}u(x)\Big).$$

The estimates for g_c when $c \in \mathcal{C}(C_-, C_+, d)$, see for instance Remark 5 in [A1], are

$$c_1\|x\|^{2-d} \le g_c(0,x) \le c_2\|x\|^{2-d}, \qquad x \ne 0. \tag{1.15}$$

Consider now a group $\tau_x, x \in \mathbb{R}^d$,

Proposition 1.1. *Let $d \ge 3$ and c^ω be a random environment always in $\mathcal{C}(C_-, C_+, d)$ which can be written*

$$c^\omega(x) = c(\tau_x\omega).$$

If $\mathbb{P}$ is preserved by all translations τ_x, then $\mathbb{E}(\|\nabla g_c^\omega(0,x)\|^2)$ is a Hölder continuous function satisfying

$$\mathbb{E}\big(\|\nabla g_c^\omega(0,x)\|^2\big) \le c_1\|x\|^{2-2d}. \tag{1.16}$$

This proposition is proved in §2 and serves as a proof model. The first estimates of both theorems (1.10) and (1.4) are proved respectively in §3 and §4. Then the remains of both theorems are deduced in §5 with semi-group techniques. As far as the Green functions are concerned, the results are immediate integration of the estimates for the kernels.

To have a concrete picture of this regularity issue, especially in the space variable, one may take the example in $\mathbb{R}^2$ of the following matrix field c given in polar coordinates around x,

$$c\big(x + (r\cos\theta, r\sin\theta)\big) \cdot e_r = e_r$$
$$c\big(x + (r\cos\theta, r\sin\theta)\big) \cdot e_\theta = \eta^2 e_\theta$$

for some $0 < \eta < 1$. One may check that $u(x + (r\cos\theta, r\sin\theta)) = r^\eta \cos\theta$ is an harmonic function, $L_c u \equiv 0$ in $\mathbb{R}^2$. Thus, although c is symmetric and uniformly elliptic ($C_- = \eta^2$, $C_+ = 1$), we have an example of harmonic function which is no more than Hölder continuous in x. One can easily imagine that a similar singularity $\approx r^\eta$ could appear in x for the kernel associated to $b(t,\cdot) = c(\cdot)$ so that $p_b(0,0,t,\cdot)$ is not differentiable at x.

It should also be possible to construct a periodic matrix field c which has around each integer point the same local behavior as around x in the previous example. Then considering the uniform probability $\mathbb{P}$ on $\Omega = [0,1[^2$ and $b^\omega(t,x) = c(\tau_{t,x}\omega) = c(\omega + x)$, it is clear that a quenched value of $p_b^\omega(0,0,t,\cdot)$ may not have a space derivative. But the effect of taking the expected value can be approximated as an integration in space of this derivative. And the existence of "L^2-annealed" first space derivatives is related to the fact that $(r^{\eta-1})^2 r\ dr$ is integrable near 0. Whereas the existence of annealed second space derivatives is related to the fact that $r^{\eta-2} r\ dr$ is integrable near 0.

§2 Green function on $\mathbb{R}^d$, with $d \geq 3$.

In this section we prove Proposition 1.1. We shall first recall some classical results about L_c for $c \in \mathcal{C}(C_-, C_+, d)$. We denote in the following $B(x,r)$ the ball centered at x with radius r, and $V(r)$ the volume of a ball of radius r.

References can be found in [Mo1] where the symmetry of a is not essential. Caccioppoli inequality is Lemma 4, Mean-value inequality is Theorem 2 and Harnack inequality (3.3).

Proposition 2.1 (Caccioppoli inequality). *If u is a solution of*

$$L_c u \equiv 0 \qquad \text{in } B(x,2r), \tag{2.1}$$

where $c \in \mathcal{C}(C_-, C_+, d)$, then

$$\int_{B(x,r)} \|\nabla u\|^2 \leq \frac{c_1}{r^2} \int_{B(x,2r)} u^2. \tag{2.2}$$

Note that "u is a solution" means first that u is in the Sobolev space $H^{1,2}(B(x,2r))$, i.e. is in L^2 and admits a weak derivative in L^2, so that it is possible to give a sense to (2.1) with the help of test functions:

$$\int_{B(x,2r)} \sum_{i,j=1}^{d} c_{ij} \partial_i u \partial_j \phi = 0, \qquad \phi \in C_0^\infty\big(B(x,2r)\big), \tag{2.3}$$

and also it is possible to extend (2.3) and replace ϕ by u itself multiplied by a smooth cut-off function, which is the standard proof of Proposition 2.1.

Recall also the convention adopted in the introduction, c_1 depends only on C_-, C_+ and d and not on x, r, neither on u.

Proposition 2.2 (Mean-value inequality). *If u is a solution of*

$$L_c u \equiv 0 \qquad \text{in } B(x,2r),$$

where $c \in \mathcal{C}(C_-, C_+, d)$, then

$$\sup_{B(x,r)} u^2 \leq \frac{c_2}{V(2r)} \int_{B(x,2r)} u^2.$$

Proposition 2.3 (Harnack inequality). *If u is a non-negative solution of*

$$L_c u \equiv 0 \qquad \text{in } B(x,2r),$$

where $c \in \mathcal{C}(C_-, C_+, d)$, then

$$\sup_{B(x,r)} u \leq c_3 \inf_{B(x,r)} u.$$

This Harnack inequality for uniformly elliptic second-order operators in divergence form with a uniform constant c_3 has been introduced in [Mo1]. It states some regularity in the smaller ball, using that u is a non-negative solution in the bigger ball. The next proposition states it in the other way: if there is some oscillation in the smaller ball, there should be even more in the bigger ball.

Proposition 2.5 (increase of oscillation). *There exists $\lambda(C_-, C_+, d) > 1$ such that, if u is a solution of*

$$L_c u \equiv 0 \qquad \text{in } B(x,2r),$$

where $c \in \mathcal{C}(C_-, C_+, d)$, then

$$osc_{B(x,2r)} u \geq \lambda osc_{B(x,r)} u.$$

The notation osc means $\sup - \inf$.

Proof. We show that this a straightforward consequence of the Harnack inequality. Note that $u - \inf_{B(x,2r)} u$ and $\sup_{B(x,2r)} u - u$ are two non-negative solutions, therefore

$$\sup_{B(x,r)} u - \inf_{B(x,2r)} u \leq c_3 (\inf_{B(x,r)} u - \inf_{B(x,2r)} u),$$

and

$$\sup_{B(x,2r)} u - \inf_{B(x,r)} u \leq c_3 (\sup_{B(x,2r)} u - \sup_{B(x,r)} u).$$

Adding both inequalities, we get

$$\mathrm{osc}_{B(x,2r)} u + \mathrm{osc}_{B(x,r)} u \leq c_3 \big(\mathrm{osc}_{B(x,2r)} u - \mathrm{osc}_{B(x,r)} u \big),$$

that is $\lambda = (c_3+1)/(c_3-1)$ works. $\square$

Note that the Hölder regularity of u with exponent $\eta = \log \lambda / \log 2$ is obtained by iterating this increase of oscillation. Let $y \in B(x,r)$,

$$\begin{aligned} \big| u(x) - u(y) \big| &\leq \mathrm{osc}_{B(x, \|x-y\|)} u \leq \lambda^{-\frac{\log \frac{r}{\|x-y\|}}{\log 2}} \mathrm{osc}_{B(x,2r)} u \\ &= \Big(\frac{\|x-y\|}{r} \Big)^{\frac{\log \lambda}{\log 2}} \mathrm{osc}_{B(x,2r)} u. \end{aligned} \tag{2.4.}$$

We now state an integral version of this increase of oscillation. The matrix field c will be random but without stationarity assumption.

Lemma 2.1. *There exists $A(C_-, C_+, d) > 1$ such that, if the v^ω's are solutions of $L_c^\omega v^\omega \equiv 0$ in $B(x, Ar)$, where all c^ω's are in $\mathcal{C}(C_-, C_+, d)$, and*

$$v(y)^2 \leq \frac{9}{10} v(x)^2 \quad \textit{for some } y \in B(x,r) \qquad \textit{a.s.},$$

then

$$\sup_{B(x,Ar)} \mathbb{E}(v^2) \geq 10\mathbb{E}(v(x)^2).$$

We shall not use it later but y may be random.

Proof. The only delicate point when integrating the increase of oscillation for all ω's is that the supremum is not necessary reached at the same point. We propose to integrate the L^2 norm and only after to extract the supremum thanks to the mean-value inequality.

For any ω, mean-value inequality yields

$$\frac{1}{V(Ar)} \int_{B(x,Ar)} (v^\omega)^2 \geq \frac{1}{c_2} \sup_{B(x,Ar/2)} (v^\omega)^2.$$

We give a lower bound of the right hand side by iterating the increase of oscillation n times, as long as $2^n \leq A/2$.

$$\begin{aligned} \sup_{B(x,Ar/2)} |v^\omega| &\geq \frac{1}{2} \mathrm{osc}_{B(x,Ar/2)} v^\omega \\ &\geq \frac{\lambda^n}{2} \mathrm{osc}_{B(x,r)} v^\omega \\ &\geq \frac{\lambda^n}{2} \big(1 - \sqrt{9/10}\big) |v^\omega(x)| \quad \text{a.s.} \end{aligned}$$

It is then possible to integrate in ω.

$$\frac{1}{V(Ar)} \int_{B(x,Ar)} \mathbb{E}(v^2) \geq \frac{\lambda^{2n}}{4c_2} \big(1 - \sqrt{9/10}\big)^2 \mathbb{E}(v(x)^2).$$

Since we can choose A so that n is big enough, this proves the lemma. $\square$

Lemma 2.2. *Let c^ω be a random environment always in $\mathcal{C}(C_-, C_+, d)$ which can be written*

$$c^\omega(x) = c(\tau_x \omega).$$

Denote

$$\Phi(\xi) = \mathbb{E}\Big(\big(\partial_{2,k} g_c(0,\xi)\big)^2\Big), \tag{2.5}$$

where $\partial_{2,k}$ is the partial derivative with respect to the k-th coordinate of the second variable.

If $\mathbb{P}$ is preserved by all translations τ_x, then Φ is a Hölder continuous function on $\mathbb{R}^d \setminus \{0\}$. Moreover, for any $x \neq 0$, denote $R = \|x\|$, we prove that there exist $l \in \mathbb{N}$ and $x' \in B(x, R/2)$ such that

$$\Phi \geq \frac{4^l}{2} \Phi(x) \qquad \textit{on } B(x', 2^{-l-2} A^{-1} R). \tag{2.6}$$

We need the continuity of Φ so that assertion (2.6) makes sense. Indeed, Φ is defined in (2.5) with weak derivatives. It may also be expressed in another way:

$$\begin{aligned}\Phi(\xi) &= \mathbb{E}\big((\partial_{2,k} g_c^\omega(0,\xi))^2\big)\\ &= \mathbb{E}\big((\partial_{2,k} g_c^{\tau_\xi\omega}(-\xi,0))^2\big)\\ &= \mathbb{E}\big((\partial_{2,k} g_c^\omega(-\xi,0))^2\big) = \mathbb{E}\big(v^\omega(\xi)^2\big),\end{aligned} \tag{2.7}$$

where $v^\omega(\xi) = \partial_{2,k} g_c^\omega(-\xi,0)$. The crucial point is that those v^ω's are almost surely solutions of some $L_{\tilde{c}}^\omega$ in $\mathbb{R}^d \setminus \{0\}$. Constructing the environment $\tilde{c}$ taking the minus sign into account is rather easy, the only difficulty here is whether v^ω is in $H^{1,2}_{loc}(\mathbb{R}^d \setminus \{0\})$. Then the famous result of de Giorgi [DG] and Nash [N] tells that v^ω is Hölder continuous.

As a consequence there remain three arguments in this section: proof of $v^\omega \in H^{1,2}$ *a.s.*, proof of (2.6) and why Lemma 2.2 yields Proposition 1.1. The reader may consider them in any order. The first one is rather technical, the second is the core of our argument and the third may enlighten on the meaning of (2.6).

Proof of $v^\omega \in H^{1,2}_{loc}(\mathbb{R}^d \setminus \{0\})$ *a.s.*. We will work on a ball B of radius r such that the ball $2B$ with same center and radius $2r$ stays in $\mathbb{R}^d \setminus \{0\}$. For instance we prove that v^ω is almost surely in $L^2(B)$ since we know that the Green functions g_c^ω are uniformly (with respect to ω) in $H^{1,2}(B)$. This is due to Proposition 2.1 and uniform estimates of g_c.

$$\begin{aligned}\infty &> \int_B \mathbb{E}\big((\partial_{2,k} g_c^\omega(0,\xi))^2\big)\ d\xi\\ &= \int_B \mathbb{E}\big(v^\omega(\xi)^2\big)\ d\xi.\end{aligned}$$

More intricate is the fact that v^ω almost surely admits a weak derivative in $L^2(B)$. We shall approach the gradient with differences, it is indeed sufficient to prove that for h small and e_i a unit vector in $\mathbb{R}^d$,

$$\int_B \mathbb{E}\Big(\big(\frac{v^\omega(\xi+he_i) - v^\omega(\xi)}{h}\big)^2\Big)\ d\xi < K$$

where K does not depend on h. Then we can use Lemma 7.24 in [GT] that can be adapted to a sequence $h_n \to 0$, it says, denoting $\delta(h) = \int_B((v^\omega(\xi+he_i) - v^\omega(\xi))/h)^2\ d\xi$, that $\delta(h_n)$ bounded implies v^ω admits a weak derivative in $L^2(B)$. In our case we may prove for instance that $\mathbb{P}(\inf\{h : \delta(h) \leq \lambda K\} > 0) \leq 1/\lambda$ and conclude.

$$\begin{aligned}
&\int_B \mathbb{E}\Big(\big(\frac{v^\omega(\xi+he_i)-v^\omega(\xi)}{h}\big)^2\Big)\ d\xi\\
&=\int_B \mathbb{E}\Big(\big(\frac{\partial_{2,k}g_c^\omega(-\xi-he_i,0)-\partial_{2,k}g_c^\omega(-\xi,0)}{h}\big)^2\Big)\ d\xi\\
&=\int_B \mathbb{E}\Big(\big(\partial_{2,k}\frac{g_c^\omega(-he_i,\xi)-g_c^\omega(0,\xi)}{h}\big)^2\Big)\ d\xi\\
&\le \frac{c_1}{r^2}\int_{2B} \mathbb{E}\Big(\big(\frac{g_c^\omega(-he_i,\xi)-g_c^\omega(0,\xi)}{h}\big)^2\Big)\ d\xi\\
&=\frac{c_1}{r^2}\int_{2B} \mathbb{E}\Big(\big(\frac{g_c^\omega(-\xi-he_i,0)-g_c^\omega(-\xi,0)}{h}\big)^2\Big)\ d\xi\\
&\le \frac{c_1}{r^2}\int_{2B+B(0,h)} \mathbb{E}\Big(\big(\nabla_1 g_c^\omega(-\xi,0)\big)^2\Big)\ d\xi<K.
\end{aligned}$$

The last arguments are a reverse result bounding norms of differences with norms of derivatives, Lemma 7.23 in [GT], and the fact that the Green functions are in $H^{1,2}(2B)$, uniformly with respect to ω. $\square$

Proof of (2.6). The expression (2.7) of Φ involving the v^ω's will also be useful since we will use Lemma 2.1.

We shall construct a sequence $x=x_0,x_1,\dots$ satisfying

$$\|x_{i+1}-x_i\|\le 2^{-i-2}R \qquad\text{and}\qquad \Phi(x_i)\ge 4^i\Phi(x_0).$$

The point is that, as long as $l=i$ and $x'=x_i$ do not satisfy (2.6), it is possible to construct x_{i+1}.

Indeed, assume there exists $y\in B(x_i,2^{-i-2}A^{-1}R)$ such that $\Phi(y)\le\Phi(x_i)/2$ and denote F the event

$$F=\big\{\omega\in\Omega\ ;\ v^\omega\in H^{1,2}_{loc}\text{ and } v^\omega(y)^2\le\frac{9}{10}v^\omega(x_i)^2\big\}.$$

Lemma 2.1 yields

$$\sup_{B(x_i,2^{-i-2}R)}\mathbb{E}(v^2\mid F)\ge 10\mathbb{E}(v(x_i)^2\mid F). \tag{2.8}$$

To construct x_{i+1}, we want to prove

$$\sup_{B(x_i,2^{-i-2}R)}\mathbb{E}(v^2)\ge 4\mathbb{E}(v(x_i)^2). \tag{2.9}$$

We can check that one goes from (2.8) to (2.9) through an elementary conditional decomposition. Of course we can estimate

$$\mathbb{E}(v^2)\ge\mathbb{P}(F)\cdot\mathbb{E}(v^2\mid F)$$

but we can also estimate

$$\mathbb{P}(F)\cdot\mathbb{E}(v(x_i)^2\mid F)\ge\frac{4}{9}\mathbb{E}(v(x_i)^2).$$

This is a consequence of

$$\begin{aligned}\mathbb{E}(v(x_i)^2) &= \mathbb{P}(F)\cdot\mathbb{E}(v(x_i)^2 \mid F) + \mathbb{P}(F^{\complement})\cdot\mathbb{E}(v(x_i)^2 \mid F^{\complement})\\ &\le \mathbb{P}(F)\cdot\mathbb{E}(v(x_i)^2 \mid F) + \mathbb{P}(F^{\complement})\cdot\frac{10}{9}\mathbb{E}(v(y)^2 \mid F^{\complement})\\ &\le \mathbb{P}(F)\cdot\mathbb{E}(v(x_i)^2 \mid F) + \frac{10}{9}\mathbb{E}(v(y)^2)\\ &\le \mathbb{P}(F)\cdot\mathbb{E}(v(x_i)^2 \mid F) + \frac{5}{9}\mathbb{E}(v(x_i)^2).\end{aligned}$$

Since the sequence $x_0, x_1, \ldots$ may not be infinite because of the Hölder continuity, (2.6) follows. The reader may find in [Lan] an approach of the classical theory with similar technic iterating some increase of oscillation on a sequence of points that should end. It has inspired our argument. □

Proof of Proposition 1.1. Denote $R = \|x\|$, we prove for any index k that

$$\mathbb{E}\big((\partial_{2,k} g_c(0,x))^2\big) \le cR^{2-2d}.$$

The idea is (1.15) gives upper estimates for g_c, Caccioppoli inequality tends to improve them for ∇g_c, at least in a L^2 sense, thanks to the factor r^2 in (2.2), and Lemma 2.2 gives a ball $B(x', 2^{-l-1}A^{-1}R)$ where the L^2 sense yields an L^∞ estimate.

We start with Lemma 2.2 giving l and x'. We use in the following that if $\xi \in B(x', 2^{-l-1}A^{-1}R)$, then $\|\xi\| \ge R/4$ (just ensure that $A \ge 2$).

$$\begin{aligned}\frac{4^l V(2^{-l-2}A^{-1}R)}{2}\mathbb{E}\big((\partial_{2,k} g_c(0,x))^2\big) &\le \int_{B(x',2^{-l-2}A^{-1}R)} \mathbb{E}\big((\partial_{2,k} g_c(0,\cdot))^2\big)\\ &\le \int_{B(x',2^{-l-2}A^{-1}R)} \mathbb{E}\big(\|\nabla_2 g_c(0,\cdot)\|^2\big)\\ &\le \frac{c_1}{(2^{-l-2}A^{-1}R)^2}\int_{B(x',2^{-l-1}A^{-1}R)} \mathbb{E}\big(g_c(0,\cdot)^2\big)\\ &\le c_1 4^{l+2}A^2R^{-2}V(2^{-l-1}A^{-1}R)c_2^2\big(\frac{R}{4}\big)^{4-2d},\end{aligned}$$

The constant c_2 comes from (1.15). Now (1.16) follows after simplifications. □

§3 Heat kernel on $\mathbb{R}^d$.

In this section we prove (1.10), that is we give a parabolic version of the method of §2. The notations will be more intricate since balls will be replaced by cylinders with a time scale like the square of the distance.

Let us recall the parabolic versions of the classic results that we already stated without the time variable. Caccioppoli inequality is (3.3) and (3.4) in [Mo2], Mean-value inequality is Theorem 3 and Harnack inequality is Theorem 1.

Proposition 3.1 (Caccioppoli inequality). *If u is a solution of*

$$\partial_t u \equiv L_b u \qquad \textit{in } [0, 2r^2]\times B(x, 2r),$$

where $b \in \mathcal{B}(C_-, C_+, d)$, *then*

$$\sup_{t\in[r^2,2r^2]} \int_{B(x,r)} u(t,\cdot)^2 + \int_{[r^2,2r^2]\times B(x,r)} \|\nabla u\|^2 \leq \frac{c_1}{r^2} \int_{[0,2r^2]\times B(x,2r)} u^2.$$

The regularity required for "solutions" u in the parabolic setting is a quite technical point. In many papers it is avoided by assuming more regularity than necessary. It is useful for us to have the weakest conditions since we will have to prove that the v^ω's are almost surely solutions. From [Lad, Chapter II, §3] we conclude that it suffices if u is in $H^{1,2}([0,2r^2] \times B(x,2r))$, that is

$$\int_{[0,2r^2]\times B(x,2r)} u^2 + \|\nabla u\|^2 < \infty. \tag{3.1}$$

Proposition 3.2 (Mean-value inequality). *If u is a solution of*

$$\partial_t u \equiv L_b u \qquad in\ [0,4r^2] \times B(x,2r),$$

where $b \in \mathcal{B}(C_-, C_+, d)$, *then*

$$\sup_{[3r^2,4r^2]\times B(x,r)} u^2 \leq \frac{c_2}{4r^2 V(2r)} \int_{[0,4r^2]\times B(x,2r)} u^2.$$

Proposition 3.3 (Harnack inequality). *If u is a non-negative solution of*

$$\partial_t u \equiv L_b u \qquad in\ [0,4r^2] \times B(x,2r),$$

where $b \in \mathcal{B}(C_-, C_+, d)$, *then*

$$\sup_{[r^2,2r^2]\times B(x,r)} u \leq c_3 \inf_{[3r^2,4r^2]\times B(x,r)} u.$$

Proposition 3.4 (increase of oscillation). *There exists $\lambda(C_-, C_+, d) > 1$ such that, if u is a solution of*

$$\partial_t u \equiv L_b u \qquad in\ Q = [0,4r^2] \times B(x,2r),$$

where $b \in \mathcal{B}(C_-, C_+, d)$, *then*

$$osc_Q u \geq \lambda osc_{Q'} u,$$

where $Q' = [3r^2, 4r^2] \times B(x,r)$.

Proof. Use that $u - \inf_Q u$ and $\sup_Q u - u$ are two non-negative solutions, therefore

$$u(2r^2,x) - \inf_Q u \leq c_3(\inf_{Q'} u - \inf_Q u),$$

and

$$\sup_Q u - u(2r^2,x) \leq c_3(\sup_Q u - \sup_{Q'} u).$$

Adding both inequalities, we have

$$\mathrm{osc}_Q u \leq c_3\big(\mathrm{osc}_Q u - \mathrm{osc}_{Q'} u\big),$$

that is $\lambda = (c_3)/(c_3 - 1)$ works. $\square$

Lemma 3.1. *There exists $A(C_-, C_+, d)$ such that, if the v^ω's are solutions of $L_b^\omega v^\omega \equiv 0$ in $[-A^2r^2, 0] \times B(x, Ar)$ and*

$$v(t,y)^2 \le \frac{9}{10} v(0,x)^2 \quad \text{for some } (t,y) \in [-r^2, 0] \times B(x,r) \qquad a.s.,$$

then

$$\sup_{[-A^2r^2,0]\times B(x,Ar)} \mathbb{E}(v^2) \ge 10\mathbb{E}(v(0,x)^2).$$

Proof. See Lemma 2.1, for any ω,

$$\frac{1}{A^2r^2V(Ar)} \int_{[-A^2r^2,0]\times B(x,Ar)} (v^\omega)^2 \ge \frac{1}{c_2} \sup_{[-A^2r^2/4,0]\times B(x,Ar/2)} (v^\omega)^2.$$

As long as $2^n \le A/2$.

$$\begin{aligned}
\sup_{[-A^2r^2/4,0]\times B(x,Ar/2)} |v^\omega| &\ge \frac{1}{2} \mathrm{osc}_{[-A^2r^2/4,0]\times B(x,Ar/2)} v^\omega \\
&\ge \frac{\lambda^n}{2} \mathrm{osc}_{[-r^2,0]\times B(x,r)} v^\omega \\
&\ge \frac{\lambda^n}{2} \big(1 - \sqrt{9/10}\big) |v^\omega(0,x)| \quad \text{a.s.}
\end{aligned}$$

$$\frac{1}{A^2r^2V(Ar)} \int_{[-A^2r^2,0]\times B(x,Ar)} \mathbb{E}(v^2) \ge \frac{\lambda^{2n}}{4c_2} \big(1 - \sqrt{9/10}\big)^2 \mathbb{E}(v(0,x)^2).$$

Since we can choose A so that n is big enough, this proves the lemma. $\square$

Lemma 3.2. *Let b^ω be a random environment always in $\mathcal{B}(C_-, C_+, d)$ which can be written*

$$b^\omega(t,x) = b(\tau_{t,x}\omega).$$

Denote

$$\Phi(\theta,\xi) = \mathbb{E}\Big(\big(\partial_{4,k} p_b(0,0,\theta,\xi)\big)^2\Big).$$

If $\mathbb{P}$ is preserved by all translations $\tau_{t,x}$, then Φ is a Hölder continuous function on $\mathbb{R}^+ \times \mathbb{R}^d \setminus \{(0,0)\}$. Moreover, for any (t,x), we prove that there exist $l \in \mathbb{N}$, $t' \in [t/2, t]$ and $x' \in B(x, \sqrt{t})$ such that

$$\Phi \ge \frac{4^l}{2} \Phi(t,x) \qquad \text{on } [(1 - 2^{-2l-2}A^{-2})t', t'] \times B(x', 2^{-l-1}A^{-1}\sqrt{t'}). \tag{3.2}$$

Similarly to Lemma 2.2, we may express Φ differently,

$$\begin{aligned}
\Phi(\theta,\xi) &= \mathbb{E}\big((\partial_{4,k} p_b^\omega(0,0,\theta,\xi))^2\big) \\
&= \mathbb{E}\big((\partial_{4,k} p_b^{\tau_{\theta,\xi}\omega}(-\theta,-\xi,0,0))^2\big) \\
&= \mathbb{E}\big((\partial_{4,k} p_b^\omega(-\theta,-\xi,0,0))^2\big) = \mathbb{E}\big(v^\omega(\theta,\xi)^2\big).
\end{aligned}$$

Now we check that almost surely v^ω is a solution of some $L_{\tilde{b}}^\omega$. For that we check (3.1) the same way as Lemma 2.2 *Proof of* $v^\omega \in H^{1,2}_{loc}(\mathbb{R}^d \setminus \{0\})$. As a consequence, Φ is Hölder continuous and (3.2) makes sense.

Proof. We shall construct two sequences $t = t_0, t_1, \dots$ and $x = x_0, x_1, \dots$ satisfying

$$t_{i+1} \in [(1 - 2^{-2i-2})t_i, t_i], \quad x_{i+1} \in B(x_i, 2^{-i-1}\sqrt{t_i}) \quad \text{and} \quad \Phi(t_i, x_i) \geq 4^i \Phi(t_0, x_0).$$

The point is that, as long as $l = i$, $t' = t_i$ and $x' = x_i$ do not satisfy (3.2), it is possible to construct (t_{i+1}, x_{i+1}).

Indeed, assume there exist $s \in [(1-2^{-2i-2}A^{-2})t_i, t_i]$ and $y \in B(x_i, 2^{-i-1}A^{-1}\sqrt{t_i})$ such that $\Phi(s, y) \leq \Phi(t_i, x_i)/2$ and denote F the event

$$F = \big\{\omega \in \Omega\ ;\ v^\omega(s, y)^2 \leq \frac{9}{10} v^\omega(t_i, x_i)^2\big\}.$$

Lemma 3.1 yields

$$\sup_{[(1-2^{-2i-2})t_i, t_i] \times B(x_i, 2^{-i-1}\sqrt{t_i})} \mathbb{E}(v^2 \mid F) \geq 10 \mathbb{E}(v(t_i, x_i)^2 \mid F).$$

To construct x_{i+1}, we want to prove

$$\sup_{[(1-2^{-2i-2})t_i, t_i] \times B(x_i, 2^{-i-1}\sqrt{t_i})} \mathbb{E}(v^2) \geq 4 \mathbb{E}(v(t_i, x_i)^2).$$

We can check that one goes from the previous line to this inequality through the same elementary conditional decomposition as in the proof of Lemma 2.2. □

Proof of (1.10). We prove for any index j that

$$\mathbb{E}\big((\partial_{4,j} p_b(0,0,t,x))^2\big) \leq \frac{c_1}{t^{d+1}} e^{-k\frac{\|x\|^2}{t}}.$$

We start with Lemma 3.2 giving l, t' and x'.

$$\begin{aligned}
&\frac{4^l 2^{-2l-2} A^{-2} (t/2) V(2^{-l-1}A^{-1}\sqrt{t/2})}{2} \mathbb{E}\big((\partial_{4,j} p_b(0,0,x,t))^2\big) \\
&\leq \frac{4^l 2^{-2l-2} A^{-2} t' V(2^{-l-1}A^{-1}\sqrt{t'})}{2} \mathbb{E}\big((\partial_{4,j} p_b(0,0,x,t))^2\big) \\
&\leq \int_{[(1-2^{-2l-2}A^{-2})t', t'] \times B(x', 2^{-l-1}A^{-1}\sqrt{t'})} \mathbb{E}\big((\partial_{4,j} p_b(0,0,\cdot,\cdot))^2\big) \\
&\leq \int_{[(1-2^{-2l-2}A^{-2})t', t'] \times B(x', 2^{-l-1}A^{-1}\sqrt{t'})} \mathbb{E}\big(\|\nabla_4 p_b(0,0,\cdot,\cdot)\|^2\big) \\
&\leq \frac{c_1}{2^{-2l-2}A^{-2}t'} \int_{[(1-2^{-2l-1}A^{-2})t', t'] \times B(x', 2^{-l}A^{-1}\sqrt{t'})} \mathbb{E}\big(p_b(0,0,\cdot,\cdot)^2\big) \\
&\leq c_1 4^{l+2} A^2 \frac{2}{t} 2^{-2l-1} A^{-2} t V(2^{-l}A^{-1}\sqrt{t}) \Big(\frac{c_4}{(t/2)^{d/2}} e^{-\frac{\sup\{0, \|x\| - 2^{-l}A^{-1}\sqrt{t}\}^2}{k_4 t}}\Big)^2,
\end{aligned}$$

so that it is possible to extract c_1 and k. □

§4 Random walks on $\mathbb{Z}^d$.

We start with some references for (1.2) then we explain how we adapt the proof in §3 to prove (1.4). An extension of the classical theory of E. De Giorgi, J. Nash and J. Moser to discrete operators L_a when $a \in \mathcal{A}(C_-, C_+, d, \Gamma)$ may be found in [SZ] with an extension to time dependence in Appendix B of [GOS] and in [Del]. Here is for instance one part of the analog of Caccioppoli inequality.

Proposition 4.1 (Caccioppoli inequality). *If u is a solution of*

$$\partial_t u \equiv L_a u \qquad \text{in } [0, 2r^2] \times B_\Gamma(x, 2r),$$

where $\in \mathcal{A}(C_-, C_+, d, \Gamma)$, then

$$\int_{[r^2,2r^2]} \sum_{z,z' \in B_\Gamma(x,2r), z'-z \in \Gamma} [u(t,z') - u(t,z)]^2 \, dt \le \frac{c_1}{r^2} \int_{[0,2r^2]} \sum_{z \in B_\Gamma(x,r)} u(t,z)^2 \, dt.$$

The proof of this Caccioppoli inequality and of other statements analogous to those stated in the previous section may be extracted from Section 2 of [Del], with the following changes: $m(x)$ should be replaced by 1 and μ_{xy} by $a(t,x,y)$. The dependence in t for a does not create additional difficulties since only norms weighted with m are differentiated in t along the proof. Because of ellipticity, the standard Poincaré inequality in $\mathbb{Z}^d$ yields a Poincaré inequality that holds uniformly in t with the weights $a(t,x,y)$.

It may be extracted from Proposition 3.4 in [Del] that

Proposition 4.2. *If $a \in \mathcal{A}(C_-, C_+, d, \Gamma)$,*

$$p_a(0,0,t,x) \le \frac{c}{(1 \vee t)^{d/2}} \mathcal{E}(kt, \|x\|_\Gamma),$$

where

$$\begin{aligned}\mathcal{E}(t,D) &= \exp\Big(- \max_\lambda \{\lambda D - t(\cosh \lambda - 1)\}\Big) \\ &= \exp\Big(- D \operatorname{arg\,sinh} \frac{D}{t} - t\Big(\sqrt{1 + D^2/t^2} - 1\Big)\Big),\end{aligned}$$

since $\lambda = \operatorname{arg\,sinh}(D/t)$ gives the maximum.

One point needs some comments: in the last sentence page 212 in [Del], *Since the weights...*, it is used that $m(x) = \sum_y \mu_{xy}$, we shall note instead that since $\partial K / \partial t \le 0$,

$$\begin{aligned}&\sum_{x \in \mathbb{Z}^d} u^2(t,x) \frac{\partial K}{\partial t}(t,x) \\ &\le \sum_{x,y \in \mathbb{Z}^d} \frac{a(t,x,y)}{|\Gamma| C_+} u^2(t,x) \frac{\partial K}{\partial t}(t,x) \\ &= \sum_{x,y \in \mathbb{Z}^d} \frac{a(t,x,y)}{2} \Big(u^2(t,x) \frac{1}{|\Gamma| C_+} \frac{\partial K}{\partial t}(t,x) + u^2(t,y) \frac{1}{|\Gamma| C_+} \frac{\partial K}{\partial t}(t,y)\Big),\end{aligned}$$

therefore k depends on C_+ and Γ.

Now (1.2) will be proved if we have the following lower estimate (which can also be applied to p^*).

Proposition 4.3. *If $a \in \mathcal{A}(C_-, C_+, d, \Gamma)$,*

$$p_a(0,0,t,x) \geq \frac{c}{(1 \vee t)^{d/2}} \mathcal{E}(kt, \|x\|_\Gamma).$$

Proof. For $\|x\|_\Gamma \leq t$ this is quite clear because you can prove a classical lower bound

$$p_a(0,0,t,x) \geq \frac{c_1}{(1 \vee t)^{d/2}} e^{-\frac{\|x\|_\Gamma^2}{k_1 t}},$$

the same way as for discrete time (Theorem 3.8 page 221 in [Del]) and we can check that, since $\|x\|_\Gamma \leq \frac{1}{k}(kt)$, $\mathcal{E}(kt, \|x\|_\Gamma) \leq \exp(-\operatorname{arg\,sinh}(1/k) \cdot \|x\|_\Gamma^2/(2t))$, see (3.18) page 214 in [Del]. Thus we can choose k small enough to obtain a coefficient under the exponential which is bigger than $1/k_1$.

Now for $\|x\|_\Gamma \geq t$ a similar decomposition method can be used with the help of the following claim: if $t \leq 1$ and $y_i \in \Gamma$ then

$$p_a(0,0,t,y_i) \geq c_2 t.$$

This claim is a consequence of (for $0 \leq s \leq t \leq 1$)

$$\begin{aligned} p_a(0,0,s,0) &\geq \exp(-C_+|\Gamma|s) \geq \exp(-C_+|\Gamma|) \\ \text{and } \frac{\partial p_a(0,0,s,y_i)}{\partial s} &\geq C_- p_a(0,0,s,0) - C_+|\Gamma| p_a(0,0,s,y_i), \end{aligned}$$

which means, the diffusion stays a certain quantity at 0 and this is sufficient to prove the claim that the diffusion also reaches the neighbor y_i. Here is the decomposition for $D = \|x\|_\Gamma \geq t$, denoting $x = y_1 + y_2 + \ldots + y_D$,

$$\begin{aligned} p(0,0,t,x) &\geq p(0,0,\frac{t}{D},y_1) p(\frac{t}{D}, y_1, \frac{2t}{D}, y_1 + y_2) \\ &\quad \ldots p(\frac{(D-1)t}{D}, y_1 + y_2 + \ldots + y_{D-1}, t, x) \\ &\geq \left(c_2 \frac{t}{D}\right)^D \end{aligned}$$

which may be bounded below by some $\mathcal{E}(kt, D)$, just by using $\operatorname{arg\,sinh} r \geq \ln(2r)$. □

Proof of (1.4). The proof is the same as for the continuous time kernel in the previous section. In the analogous of Lemma 3.2 we consider

$$\begin{aligned} \Phi(\theta, \xi) &= \mathbb{E}\big((\nabla_\xi^y p_a^\omega(0,0,t,\cdot))^2\big) \\ &= \mathbb{E}\big((p_a^\omega(0,0,t,\xi+y) - p_a^\omega(0,0,t,\xi))^2\big) \\ &= \mathbb{E}\big((p_a^{\tau_{t,\xi}\omega}(-t,-\xi,0,y) - p_a^{\tau_{t,\xi}\omega}(-t,-\xi,0,0))^2\big) \\ &= \mathbb{E}\big((p_a^\omega(-t,-\xi,0,y) - p_a^\omega(-t,-\xi,0,0))^2\big) = \mathbb{E}\big(v^\omega(t,\xi)^2\big). \end{aligned}$$

Thus we get exactly like in §3,

$$\mathbb{E}\Big(\big(\nabla_x^y p_a(0,0,t,\cdot)\big)^2\Big) \leq c_1 \frac{p^*(k_1 t, x)^2}{t}.$$

This estimate is rather poor when $x = 0$ and t is small since the left hand side is at least known to be smaller than 1. For this reason we improve it lightly in the exact form of (1.4)

$$\mathbb{E}\Big(\big(\nabla_x^y p_a(0,0,t,\cdot)\big)^2\Big) \le c_2 \frac{p^*(k_2 t, x)^2}{1 \vee t}$$

by the following argument:

If $x = 0$ and $t < 1$, we know from Proposition 4.3 that $p^*(kt, x) \ge c_3$, so that we just have to ensure that $c_2 \ge c_3^{-1}$. Now for $x \ne 0$, one should note that $\mathcal{E}(kt, D)$ with $D \ge 1$ is much faster decreasing than t when $t \to 0$. As a consequence playing with the constant k_2 allows to remove t. □

§5 Second space derivatives and time derivatives.

Thanks to a semi-group argument the estimates (1.4) and (1.10) which are L^2 in probability for the first space derivatives, yield L^1 estimates (1.5) and (1.11) for the second space derivatives and (1.6) and (1.12) for the time derivatives. We begin with the second space derivatives (which are differences) in the setting of random walks, so that there will not be any existence/regularity issue for the derivatives.

Proof of (1.5). For the first line (1.5a) we introduce the position at time $t/2$, then by Cauchy-Schwarz inequality two estimates like (1.4) appear.

$$\begin{aligned}
&\mathbb{E}\Big|\Big(p_a^\omega(0,y,t,x+y') - p_a^\omega(0,y,t,x)\Big) - \Big(p_a^\omega(0,0,t,x+y') - p_a^\omega(0,0,t,x)\Big)\Big| \\
&= \mathbb{E}\Big|\sum_{\xi\in\mathbb{Z}^d} \Big(p_a(0,y,\frac{t}{2},\xi) - p_a(0,0,\frac{t}{2},\xi)\Big)\Big(p_a(\frac{t}{2},\xi,t,x+y') - p_a(\frac{t}{2},\xi,t,x)\Big)\Big| \\
&\le \sum_{\xi\in\mathbb{Z}^d} \sqrt{\mathbb{E}\Big(\big(p_a(0,y,\frac{t}{2},\xi) - p_a(0,0,\frac{t}{2},\xi)\big)^2\Big)} \\
&\qquad\qquad \times \sqrt{\mathbb{E}\Big(\big(p_a(\frac{t}{2},\xi,t,x+y') - p_a(\frac{t}{2},\xi,t,x)\big)^2\Big)} \\
&\le \sum_{\xi\in\mathbb{Z}^d} \frac{c_1^2}{1\vee\frac{t}{2}} p^*(k_1\frac{t}{2},\xi) p^*(k_1\frac{t}{2},x-\xi) \\
&= \frac{c_1^2}{1\vee\frac{t}{2}} p^*(k_1 t, x) \le \frac{c_2}{1\vee t} p^*(k_1 t, x). \qquad \square
\end{aligned}$$

For the second line (1.5b) we use in this final argument stationarity just in space and we write τ_y for $\tau_{0,y}$.

$$\begin{aligned}
&\mathbb{E}\Big(\big(p_a(0,0,t,x+y+y') - p_a(0,0,t,x+y)\big) - \big(p_a(0,0,t,x+y') - p_a(0,0,t,x)\big)\Big) \\
&= \mathbb{E}\Big(p_a^{\tau_y\omega}(0,-y,t,x+y') - p_a^{\tau_y\omega}(0,-y,t,x)\Big) - \mathbb{E}\Big(p_a^\omega(0,0,t,x+y') - p_a^\omega(0,0,t,x)\Big) \\
&= \mathbb{E}\Big(p_a^\omega(0,-y,t,x+y') - p_a^\omega(0,-y,t,x)\Big) - \mathbb{E}\Big(p_a^\omega(0,0,t,x+y') - p_a^\omega(0,0,t,x)\Big) \\
&= \mathbb{E}\Big(\nabla_0^{2,-y}\nabla_x^{4,y'} p_a(0,\cdot,t,\cdot)\Big). \qquad \blacksquare
\end{aligned}$$

As a consequence (1.5a) yields (1.5b). $\square$

Proof of (1.11) and Hölder continuity.
We prove (1.11a) like (1.5a) since

$$\partial_{2,i}\partial_{4,j}p_b(0,0,t,x) = \int \partial_{2,i}p_b(0,0,t/2,\xi)\partial_{4,j}p_b(t/2,\xi,t,x)\,d\xi.$$

In fact this is the reason why the left-hand side in (1.5a) makes sense. Then we use Cauchy-Schwarz inequality and (1.10). The end of the computation is the same as the one for (1.11b) that we shall do now more carefully.

We discuss first why $\bar{p}(0,0,t,x) = \mathbb{E}(p_b(0,0,t,x))$ is C^1 in x. We can reproduce the argument for Φ in Lemma 3.2 without putting the derivative to the square, so that we prove that the a priori a weak derivative of $\bar{p}$, $\mathbb{E}(\partial_{4,j}p_b(0,0,t,x)) = \mathbb{E}(v^\omega(t,x))$ is continuous (at least is almost surely equal to a continuous function). Now as a continuous function admitting a continuous weak derivative, $\bar{p}$ is in fact C^1. We will denote $\bar{p}_j = \partial\bar{p}/\partial x^j$ the usual derivative. Again because of the representation $\bar{p}_j = \mathbb{E}v$, $\bar{p}_j$ has a weak derivative, say with respect to x^i. There remains to prove that this weak derivative is Hölder continuous.

Let us be more precise about the Hölder regularity of Φ obtained at the beginning of Lemma 3.2. Recall that $\partial_{2,i}p_b(0,0,\cdot,\cdot)$ is almost surely a solution of some heat equation, as such it satisfies mean-value inequality of Proposition 3.2 and Hölder regularity in the spirit of (2.4), that is

$$|u(t,y) - u(t,z)| \leq \Big(\frac{\|y-z\|}{r}\Big)^{c_1} \mathrm{osc}_{[t-4r^2,t]\times B(y,2r)} u,$$

for $z \in B(y,r)$ and $20r^2 \leq t$. Such a $cr^2 \leq t$ condition ensures that it is possible to iterate Proposition 3.4 in cylinders to mimic the proof of (2.4). Now we can prove the following preliminary with mean-value inequality and (1.10).

$$\begin{aligned}
&\sqrt{\mathbb{E}\Big(\big(\partial_{2,i}p_b(0,0,t,y) - \partial_{2,i}p_b(0,0,t,z)\big)^2\Big)} \\
&\leq \sqrt{\mathbb{E}\Big(\Big(\frac{\|y-z\|}{r}\Big)^{2c_1} \mathrm{osc}^2_{[t-4r^2,t]\times B(y,2r)} \partial_{2,i}p_b(0,0,\cdot,\cdot)\Big)} \\
&\leq \Big(\frac{\|y-z\|}{r}\Big)^{c_1} \sqrt{\mathbb{E}\Big(4\frac{c_2}{16r^2V(4r)} \int_{[t-16r^2,t]\times B(y,4r)} \partial_{2,i}p_b(0,0,\cdot,\cdot)^2\Big)} \\
&\leq \Big(\frac{\|y-z\|}{r}\Big)^{c_1} \frac{c_3}{t^{d/2+1/2}} e^{-\frac{\|y\|^2}{k_1 t}}.
\end{aligned}$$

To study the regularity of a a priori weak derivative of $\bar{p}_j$, we integrate it against a smooth ϕ_ϵ tending to $\delta_y - \delta_z$ when $\epsilon \to 0$, with $z \in B(y,r)$ and $10r^2 \leq t/2$. Then we have the previous preliminary appear as a limit when $\epsilon \to 0$. Thanks to our quantitative estimates we have dominated convergence and we can apply the limit under the integral.

$$\begin{aligned}
&\limsup_{\epsilon\to 0}\Big|\int \frac{\partial}{\partial x^i}\bar p_j(0,0,t,x)\phi_\epsilon(x)\ dx\Big| \\
&= \limsup_{\epsilon\to 0}\Big|\int \bar p_j(0,0,t,x)\frac{\partial}{\partial x^i}\phi_\epsilon(x)\ dx\Big| \\
&= \limsup_{\epsilon\to 0}\Big|\int\int \mathbb{E}\big(p_b(0,-x,t/2,\xi)\partial_{4,j}p_b(t/2,\xi,t,0)\big)\ d\xi\ \frac{\partial}{\partial x^i}\phi_\epsilon(x)\ dx\Big| \\
&= \limsup_{\epsilon\to 0}\Big|\int \mathbb{E}\Big(\int \partial_{2,i}p_b(0,-x,t/2,\xi)\phi_\epsilon(x)\ dx\ \cdot\ \partial_{4,j}p_b(t/2,\xi,t,0)\Big)\ d\xi\Big| \\
&\le \limsup_{\epsilon\to 0}\int\sqrt{\mathbb{E}\Big(\Big(\int \partial_{2,i}p_b(0,-x,t/2,\xi)\phi_\epsilon(x)\ dx\Big)^2\Big)}\sqrt{\mathbb{E}\big((\partial_{4,j}p_b(t/2,\xi,t,0))^2\big)}\ d\xi \\
&\le \int\Big(\frac{\|y-z\|}{r}\Big)^{c_1}\frac{c_3}{(t/2)^{d/2+1/2}}e^{-\frac{\|y+\xi\|^2}{k_1t/2}}\frac{c_4}{(t/2)^{d/2+1/2}}e^{-\frac{\|\xi\|^2}{k_2t/2}} \\
&\le \frac{c_5}{t^{d/2+1}}e^{-\frac{\|y\|^2}{k_3t}}\Big(\frac{\|y-z\|}{r}\Big)^{c_1}.
\end{aligned}$$

The last inequality may either be interpreted as a convolution of two heat kernels, or directly computed using for $\|y\|^2\le t$ that the integral is essentially supported by $\xi\in B(-y/2,c\sqrt{t})$.

Now the a priori weak derivative of $\bar p_j$ is Hölder continuous, that yields $\bar p$ is C^2 in x with Hölder continuous second derivatives. Furthermore if we formally replace the ϕ_ϵ by a δ_x in the computation, we obtain (1.11b). $\square$

Proof of (1.12) and Hölder continuity.

$$\begin{aligned}
&\frac{\partial}{\partial t}\bar p_b(0,0,t,x) \\
&= \mathbb{E}\Big(\int_{\mathbb{R}^d} p_b(0,0,t/2,z)\partial_3 p_b(t/2,z,t,x)\ dz\Big) \\
&= \mathbb{E}\Big(\int_{\mathbb{R}^d} p_b(0,-z,t/2,0)\partial_3 p_b(t/2,0,t,x-z)\ dz\Big) \\
&= \mathbb{E}\Big(\int_{\mathbb{R}^d} p_b(0,-z,t/2,0)\sum_{i,j}\frac{-\partial}{\partial z_i}\big(b_{i,j}(t,x-z)\frac{-\partial}{\partial z_j}p_b(t/2,0,t,x-z)\big)\ dz\Big) \\
&= -\sum_{i,j}\mathbb{E}\Big(\int_{\mathbb{R}^d}\frac{\partial}{\partial z_i}p_b(0,-z,t/2,0)b_{i,j}(t,x-z)\frac{\partial}{\partial z_j}p_b(t/2,0,t,x-z)\ dz\Big).
\end{aligned}$$

We then use Hölder inequality, to finish like for (1.11).

$$\begin{aligned}
&\Big|\frac{\partial}{\partial t}\bar p_b(0,0,t,x)\Big| \\
&\le c_1\sum_{i,j}\int_{\mathbb{R}^d}\sqrt{\mathbb{E}\Big(\big(\frac{\partial}{\partial z_i}p_b(0,-z,t/2,0)\big)^2\Big)\mathbb{E}\Big(\big(\frac{\partial}{\partial z_j}p_b(t/2,0,t,x-z)\big)^2\Big)}\ dz \\
&\le \frac{c_2}{t^{d/2+1}}e^{\frac{\|x\|^2}{kt}}.
\end{aligned}$$

As far as Hölder continuity is concerned, it follows from the usual trick in the spirit of (2.7):

$$\mathbb{E}\Big(\frac{\partial}{\partial t} p_b(0,0,t,x)\Big) = \mathbb{E}\big(-\partial_1 p_b(0,0,t,x)\big).$$

Now the regularity requirements for solutions (3.1) are immediate since we have directly proved (1.12). $\square$

Proof of (1.6) and Lipschitz regularity of the derivative. For a random walk we show how we adapt the integration by parts in the previous computation for the diffusion. By a change of variable $(y,z) \leftrightarrow (-y, z-y)$,

$$\begin{aligned}
&\sum_{z\in\mathbb{Z}^d, y\in\Gamma} p_a(0,-z,t/2,0)a(t,x-z,x-z+y) \\
&\qquad\qquad \big(p_a(t/2,0,t,x-z+y) - p_a(t/2,0,t,x-z)\big) \\
&= \sum_{z\in\mathbb{Z}^d, y\in\Gamma} p_a(0,-z+y,t/2,0)a(t,x-z+y,x-z) \\
&\qquad\qquad \big(p_a(t/2,0,t,x-z) - p_a(t/2,0,t,x-z+y)\big) \\
&= -\sum_{z\in\mathbb{Z}^d, y\in\Gamma} p_a(0,-z+y,t/2,0)a(t,x-z,x-z+y) \\
&\qquad\qquad \big(p_a(t/2,0,t,x-z+y) - p_a(t/2,0,t,x-z)\big).
\end{aligned}$$

So that in a similar computation as for (1.12) we have

$$\begin{aligned}
&\mathbb{E}\Big(\sum_{z\in\mathbb{Z}^d} p_a(0,-z,t/2,0)\partial_3 p_a(t/2,0,t,x-z)\Big) \\
&= \mathbb{E}\Big(\sum_{z\in\mathbb{Z}^d} p_a(0,-z,t/2,0)\sum_{y\in\Gamma} a(t,x-z,x-z+y) \\
&\qquad\qquad \big(p_a(t/2,0,t,x-z+y) - p_a(t/2,0,t,x-z)\big)\Big) \\
&= \frac{1}{2}\mathbb{E}\Big(\sum_{z\in\mathbb{Z}^d, y\in\Gamma} \big(p_a(0,-z,t/2,0) - p_a(0,-z+y,t/2,0)\big)a(t,x-z,x-z+y) \\
&\qquad\qquad \big(p_a(t/2,0,t,x-z+y) - p_a(t/2,0,t,x-z)\big).
\end{aligned}$$

And otherwise the proof of (1.6) is identical to (1.12). As far as regularity is concerned (in t) a solution of $\partial_t \equiv L_a$ is Lipschitz. $\square$

§6 Estimates of space-time covariances for $\nabla\phi$ interface models.

In this section we apply the previous estimates to the stochastic dynamic of an interface describing the separation of two pure thermodynamical phases, known in the literature as the Ginzburg Landau $\nabla\phi$ model. The lecture of Funaki [F] at Saint-Flour in 2003 was mostly devoted to this interface model. The interface is described by a continuous scalar field $\Phi(t) : \mathbb{Z}^d \longrightarrow \mathbb{R}$, with discrete basis $\mathbb{Z}^d$ and continuous time $t \geq 0$. Thus, $\Phi_x(t) \in \mathbb{R}$ is the height of the interface at site $x \in \mathbb{Z}^d$ and time $t \geq 0$.

The stochastic dynamic is then given by an infinite system of stochastic differential equations (SDE) where neighboring sites interact through a gradient interaction by an elastic potential V:

$$\Phi_x(t) = \phi_x - \int_0^t \sum_{e:|e|=1} V'(\Phi_x(s) - \Phi_{x+e}(s))\, ds + \sqrt{2} W_x(t), \qquad x \in \mathbb{Z}^d, \tag{6.1}$$

where ϕ is the interface at time 0 and $\{W_x, x \in \mathbb{Z}^d\}$ is a collection of independent Wiener processes. The potential $V \in C^2(\mathbb{R}; \mathbb{R}^+)$ is even and strictly convex:

$$C_- \leq V'' \leq C_+, \tag{6.2}$$

for some $0 < C_- \leq C_+ < \infty$. Introducing for each $r > 0$ the subset

$$E_r = \{\phi \in E \equiv \mathbb{R}^{\mathbb{Z}^d} : \sum_x |\phi_x|^2 e^{-2r|x|} < \infty\}$$

of tempered configurations, the SDE (6.1) has in view of (6.2) for every initial $\phi \in E_r$ a unique strong solution $\Phi(t) = \Phi(t, \phi) \in E_r, \forall t > 0$, cf. [FS]. The special case of quadratic potential V corresponds to the well known Gaussian situation, also called critical Ornstein Uhlenbeck process in [Deu].

Consider the formal Hamiltonian

$$\mathcal{H}(\phi) = \frac{1}{2} \sum_{x,y:|x-y|=1} V(\phi_x - \phi_y)$$

and the partial derivatives $\partial_x = \frac{\partial}{\partial \phi_x}, x \in \mathbb{Z}^d$, then we see that the drift can be expressed as a partial derivative:

$$-\sum_{e:|e|=1} V'(\phi_x - \phi_{x+e}) = -\partial_x \mathcal{H}(\phi).$$

This implies that the formal Gibbs measure

$$\mu(d\phi) = \frac{1}{Z} \exp(-\mathcal{H}(\phi)) \prod_{x \in \mathbb{Z}^d} d\phi_x$$

is a reversible equilibrium for the dynamic. In fact, this can be made rigorous for the corresponding dynamic on a finite box. However since the formal Hamiltonian has a continuous symmetry $\mathcal{H}(\phi) = \mathcal{H}(\phi + \text{constant})$, one cannot expect Gibbs measures on the whole lattice $\mathbb{Z}^d$ in the recurrent lattice dimensions $d = 1, 2$, cf. [FP]. In transient dimensions $d \geq 3$, using the Brascamp-Lieb inequality, cf. [BL], [F], one can show for each $m \in \mathbb{R}$ the existence of extremal translation invariant Gibbs measures μ_m with

$$\langle \phi_x \rangle_{\mu_m} \equiv \int \phi_x \mu_m(d\phi) = m, \qquad x \in \mathbb{Z}^d.$$

Thus we have a phase transition with a one dimensional order parameter $m \in \mathbb{R}$. These measures are extremal reversible equilibrium for the SDE.

Since the drift depends only on the discrete gradient $\nabla_b \phi = \phi(b_2) - \phi(b_1), b \in \mathbb{Z}^d_*$, the space of oriented edges:

$$-\sum_{e:|e|=1} V'(\phi_x - \phi_{x+e}) = \sum_{b:b_1=x} V'(\nabla_b \phi),$$

we see that the gradient process $\{\nabla_b \Phi(t), b \in \mathbb{Z}^d_*, t \geq 0\}$ is again a diffusion. More precisely, let for $r > 0$

$$\mathcal{X}_r = \{\eta \in \mathbb{R}^{\mathbb{Z}^d_*} : \eta_b = \nabla_b \phi \quad \text{for some} \quad \phi \in E_r\}$$

be the subset of tempered gradients. Then for each $\nabla \phi \in \mathcal{X}_r$, $\nabla \Phi$ is the unique strong solution of the SDE

$$\begin{aligned} \nabla_b \Phi(t) = \nabla_b \phi(0) + \int_0^t \left(\sum_{b':b'_1=b_2} V'(\nabla_{b'} \Phi(s)) - \sum_{b':b'_1=b_1} V'(\nabla_{b'} \Phi(s)) \right) ds \\ + \sqrt{2} \nabla_b W(t), \qquad b \in \mathbb{Z}^d_*. \end{aligned} \tag{6.3}$$

In a recent paper [FS], Funaki and Spohn have shown in any lattice dimensions $d \geq 1$, for each $u \in \mathbb{R}^d$, the existence and uniqueness of extremal translation invariant reversible measures $\tilde{\mu}_u$ with

$$\langle \nabla_b \phi \rangle_{\tilde{\mu}_u} = u, \quad b \in \mathbb{Z}^d_*.$$

Next, let $C^1_b(E_r)$ and $C^1_b(\mathcal{X}_r)$ be the set of differentiable functions with bounded derivatives depending on finitely many coordinates. For $\tilde{F} \in C^1_b(\mathcal{X}_r)$, we write $\partial_b \tilde{F}(\nabla \phi) \equiv \frac{\partial \tilde{F}(\nabla \phi)}{\partial \nabla_b \phi}$. Also $C^{1,+}_b(E_r)$ will denote the set of monotone increasing functions with nonnegative derivatives. We will see now how the results of the previous sections can be applied to obtain estimates for the covariances $\mathrm{cov}_{\mu_m}(F(\Phi(0)), J(\Phi(t)))$ and $\mathrm{cov}_{\tilde{\mu}_u}(\tilde{F}(\nabla \Phi(0)), \tilde{J}(\nabla \Phi(t)))$.

More precisely, let $\{P_t, t \geq 0\}$ be the Markov semigroup associated with Φ. Then using Itô's formula and the extremallity of μ_m we get

$$\begin{aligned} \mathrm{cov}_{\mu_m}(F(\Phi(0)), J(\Phi(t))) &= \langle F, P_t J \rangle_{\mu_m} - \langle F \rangle_{\mu_m} \langle J \rangle_{\mu_m} \\ &= \int_0^\infty \sum_x \langle \partial_x F, \partial_x (P_{t+s} J) \rangle_{\mu_m} ds. \end{aligned} \tag{6.4}$$

The idea, due to Hellfer and Sjöstrand cf. [HS], [GOS], is to use a random walk representation in order to express $\partial_x (P_{t+s} J)$. The role of the dynamical random environment is played by the gradient process $\{\nabla \Phi(t), t \geq 0\}$ starting from the invariant measure $\tilde{\mu}_u$. Note that the law of the environment is space-time invariant like the $\mathbb{P}$ in Theorem 1.1. Given the environment $\{\nabla \Phi\}$, $\{X_t, t \geq 0\}$ will denote the random walk on $\mathbb{Z}^d$ with time dependent jump rates along the bond $b = (b_1, b_2) \in \mathbb{Z}^d_*$ given by

$$a^{\nabla \Phi}(t, b_1, b_2) = V''(\nabla_b \Phi(t)).$$

Since the function V is even, we have symmetric jump rates: $a^{\nabla\Phi}(t,b_1,b_2) = a^{\nabla\Phi}(t,b_2,b_1)$. Moreover the condition (6.2) guarantees ellipticity. We write

$$\mathbf{p}^{\nabla\Phi}(s,x,t,z) = \mathbf{P}^{\nabla\Phi}(X_t = z \big| X_s = x),$$

for the solution of the forward equation

$$\frac{\partial}{\partial t}\mathbf{p}^{\nabla\Phi}(s,x_0,t,x) = \sum_{b:b_1=x} V''(\nabla_b\Phi(t))\nabla_b\mathbf{p}^{\nabla\Phi}(s,x_0,t,\cdot)$$

with initial condition $\mathbf{p}^{\nabla\Phi}(s,x_0,s,\cdot) = \delta_{x_0}(\cdot)$. Then in view of the SDE (6.1), $\mathbf{p}^{\nabla\Phi}(0,x,t,y)$ is precisely the partial derivative of $\Phi_y(t,\phi)$ with respect to the initial ϕ_x: $\mathbf{p}^{\nabla\Phi}(0,x,t,y) = \frac{\partial\Phi_y(t,\phi)}{\partial\phi_x}$. So that by the chain rule

$$\begin{aligned}\partial_x[P_tJ](\phi) &= \frac{\partial}{\partial\phi_x}\mathbb{E}[J(\Phi(t,\phi))] \\ &= \sum_y \mathbb{E}[\partial_y J(\Phi(t,\phi))\frac{\partial}{\partial\phi_x}\Phi_y(t,\phi)] \\ &= \sum_y \mathbb{E}[\partial_y J(\Phi(t,\phi))\mathbf{p}^{\nabla\Phi}(0,x,t,y)].\end{aligned} \tag{6.5}$$

Putting things together, we get

$$\big\langle\partial_x F, \partial_x[P_{t+s}J]\big\rangle_{\mu_m} = \sum_y \mathbb{E}_{\mu_m}[\partial_x F(\Phi(0))\partial_y J(\Phi(t+s))\mathbf{p}^{\nabla\Phi}(0,x,t+s,y)], \tag{6.6}$$

so that we finally obtain the following expression for the covariances

$$\begin{aligned}&\mathrm{cov}_{\mu_m}(F(\phi(0)),J(\phi(t))) \\ &= \int_0^\infty \sum_{x,y} \mathbb{E}_{\mu_m}[\partial_x F(\Phi(0))\partial_y J(\Phi(t+s))\mathbf{p}^{\nabla\Phi}(0,x,t+s,y)]\,ds.\end{aligned} \tag{6.7}$$

As an immediate consequence, using (1.2), we have

Theorem 6.1. *Let $d \geq 3$ and consider any extremal reversible μ_m. Then there exist constants $c_1, k_1, \ldots$ such that for all $F, J \in C_b^1(E_r)$,*

$$|cov_{\mu_m}(F(\phi(0)),J(\phi(t)))| \leq c_2 \sum_{x,y} \|\partial_x F\|_{L^2(\mu_m)} G^*(k_2 t, y-x)\|\partial_y J\|_{L^2(\mu_m)} \tag{6.8}$$

where

$$\frac{c_3}{\big(\|z\|^2 \vee t \vee 1\big)^{d/2-1}} \leq G^*(t,z) = \int_t^\infty p^*(s,z)\,ds \leq \frac{c_4}{\big(\|z\|^2 \vee t \vee 1\big)^{d/2-1}}. \tag{6.9}$$

In particular we have

$$\|P_tF - \big\langle F\big\rangle_{\mu_m}\|^2_{L^2(\mu_m)} \leq c_5(1\vee t)^{-d/2+1}\big(\sum_x \|\partial_x F\|_{L^2(\mu_m)}\big)^2, \qquad t \geq 0. \tag{6.10}$$

Moreover

$$\log \big\langle \exp \big[P_t F - \langle F \rangle_{\mu_m}\big] \big\rangle_{\mu_m} \le c_2 \sum_{x,y} \|\partial_x F\|_\infty G^*(2k_2 t, y-x) \|\partial_y F\|_\infty. \tag{6.11}$$

Also for $F > 0$ with $F^{1/2} \in C_b^1(E)$ and $\langle F \rangle_\mu = 1$

$$\big\langle P_t F \log P_t F \big\rangle_{\mu_m} \le 4c_2^2 \sum_{x,y} \|\partial_x F^{1/2}\|_{L^2(\mu_m)} G^*(2k_2 t, y-x) \|\partial_y F^{1/2}\|_{L^2(\mu_m)}. \tag{6.12}$$

Finally for monotone $F \in C_b^{1,+}(E)$ we have

$$cov_{\mu_m}(F(\phi(0)), F(\phi(t)) \ge c_1 \sum_{x,y} \|\partial_x F\|_{L^1(\mu_m)} G_{k_1 t}(y-x) \|\partial_y F\|_{L^1(\mu_m)} \tag{6.13}$$

Proof. The first inequality (6.8) follows from (6.7), the upperbound in (1.2) and

$$\big\langle |\partial_x F|, P_t(|\partial_y J|) \big\rangle_{\mu_m} \le \|\partial_x F\|_{L^2(\mu_m)} \|\partial_y J\|_{L^2(\mu_m)}$$

since μ_m is reversible and by Cauchy-Schwarz and Jensen's inequalities. (6.9) and (6.10) are simple consequences of estimates for p^*, that can be found in Section 4 but that are well-known since p^* has uniform transitions. For the lower bound (6.13), we use the lower bound in (1.2), the reversibility and Cauchy-Schwarz to show

$$\begin{aligned}
&\sum_{x,y} \mathbb{E}_\mu[\partial_x F(\Phi(0)) \partial_y F(\Phi(t)) \mathbf{p}^{\nabla\Phi}(0,x,t,y)] \\
&\quad \ge c_1 \sum_{x,y} \mathbb{E}_{\mu_m}[\partial_x F(\phi(0)) \partial_y F(\phi(t))] \mathbf{p}^*(k_1 t, y-x) \\
&\quad = c_1 \sum_{x,y,z} \big\langle \partial_x F, P_t[\partial_y F] \big\rangle_{\mu_m} \mathbf{p}^*(k_1 t/2, y-z) \mathbf{p}^*(k_1 t/2, z-x) \\
&\quad = c_1 \sum_z \left\langle \left(\sum_x \mathbf{p}^*(k_1 t/2, z-x) P_{t/2}[\partial_x F] \right)^2 \right\rangle_{\mu_m} \\
&\quad \ge c_1 \sum_z \left(\sum_x \mathbf{p}^*(k_1 t/2, z-x) \big\langle P_{t/2}[\partial_x F] \big\rangle_{\mu_m} \right)^2 \\
&\quad = c_1 \sum_{x,y} \mathbf{p}^*(k_1 t, y-x) \big\langle \partial_x F \big\rangle_{\mu_m} \big\langle \partial_y F \big\rangle_{\mu_m}
\end{aligned}$$

The result then follows from (6.7).

Take now $F \in C_b^1(E)$ with $\langle F \rangle_{\mu_m} = 0$. An application of Itô's formula yields

$$-\frac{d}{dt} \big\langle \exp(P_t F) \big\rangle_{\mu_m} = \sum_x \big\langle \exp(P_t F) \big(\partial_x [P_t F]\big)^2 \big\rangle_{\mu_m}.$$

Thus in view of the extremallity of μ_m, we get

$$\log \big\langle \exp[P_t F] \big\rangle_{\mu_m} \le \int_t^\infty \sum_x \|\partial_x [P_s F]\|_\infty^2 \, ds.$$

Now (6.11) follows from (6.6) and the upperbound in (1.2).

For the inequality (6.12), we use first Itô's formula and the extremallity of μ_m to show

$$\left\langle P_tF\log P_tF\right\rangle_{\mu_m} = \int_0^\infty \sum_x \left\langle \frac{\partial_x[P_{t+s}F]\cdot\partial_x[P_{t+s}F]}{P_{t+s}F}\right\rangle_{\mu_m} ds. \tag{6.14}$$

Next the random walk representation (6.5), the upperbound in (1.2) and Cauchy-Schwarz and Jensen's inequalities yield

$$\begin{aligned}
&\sum_x \left\langle \frac{\partial_x P_tF\cdot\partial_x P_tF}{P_tF}\right\rangle_{\mu_m}\\
&\le c_2^2 \sum_{x,y,z} p^*(k_2t,y-x)p^*(k_2t,z-x)\left\langle \frac{P_t|\partial_yF|\cdot P_t|\partial_zF|}{P_tF}\right\rangle_{\mu_m}\\
&= 4\sum_{x,y,z} p^*(k_2t,y-x)p^*(k_2t,z-x)\left\langle \frac{P_t|F^{1/2}\partial_yF^{1/2}|\cdot P_t|F^{1/2}\partial_zF^{1/2}|}{P_tF}\right\rangle_{\mu_m}\\
&\le 4\sum_{x,y,z} p^*(k_2t,y-x)p^*(k_2t,z-x)\left\langle \left(P_t[(\partial_yF^{1/2})^2]\right)^{1/2}\cdot\left(P_t[(\partial_zF^{1/2})^2]\right)^{1/2}\right\rangle_{\mu_m}\\
&\le 4\sum_{x,y,z} p^*(k_2t,y-x)p^*(k_2t,z-x)\|\partial_yF^{1/2}\|_{L^2(\mu_m)}\cdot\|\partial_zF^{1/2}\|_{L^2(\mu_m)}
\end{aligned}$$

and we get (6.12) from (6.14). □

Remark 6.1.

1. The inequality (6.10) is an example of algebraic decay to equilibrium, which is in contrast with the usual exponential decay

$$\mathrm{var}_{\mu_m}(P_tF)\le \exp(-\lambda t)\mathrm{var}_{\mu_m}(F)$$

in the presence of a spectral gap or Poincaré inequality of the type

$$\mathrm{var}_{\mu_m}(F)\le \frac{1}{\lambda}\sum_{x\in\mathbb{Z}^d}\|\partial_xF\|^2_{L^2(\mu_m)},$$

for some spectral gap $\lambda>0$. However for this model on the whole lattice the spectral gap vanishes. The Poincaré inequality is replaced by a generalized Nash inequality of the form

$$\mathrm{var}_{\mu_m}(F)\le c_1\left(\sum_{x\in\mathbb{Z}^d}\|\partial_xF\|^2_{L^2(\mu_m)}\right)^{1-2/d}\left(\sum_{x\in\mathbb{Z}^d}\|\partial_xF\|_{L^2(\mu_m)}\right)^{4/d}, \tag{6.15}$$

which via differential inequalities and monotonicity implies the algebraic decay (6.10), cf. also [Deu], [BZ], [JLQY], [Ca], [Li] for similar results on algebraic convergence in different situations. The inequality (6.15) follows from the Brascamp-Lieb inequality, cf. [DGI], and the classical Nash inequality, [SZ].

2. The estimate (6.11) implies Gaussian concentrations for the semigroup: for $F \in C_b^1(E_r)$,

$$\log \mu_m\big(|P_tF - \langle F\rangle_{\mu_m}| \geq L\big) \leq -c\frac{t^{d/2-1}L^2}{(\sum_x \|\partial_x F\|_\infty)^2}, \qquad L > 0,$$

for some positive constant c.

3. The estimate (6.12) gives the decay of the relative entropy, more precisely, let $\nu \ll \mu_m$ be the initial distribution of the process with $\nu(d\phi) = F(\phi)\mu(d\phi)$ and let ν_t be the distribution at time $t \geq 0$. Then by reversibility, $\nu_t(d\phi) = P_tF(\phi)\mu_m(d\phi)$ and if $H(\nu_t|\mu_m)$ denotes the relative entropy of ν_t with respect to μ_m we have

$$H(\nu_t|\mu_m) = \langle P_tF \log P_tF\rangle_{\mu_m} \leq ct^{-d/2+1}\Big(\sum_x \|\partial_x F^{1/2}\|_{L^2(\mu_m)}\Big)^2$$
$$= 4ct^{d/2-1}\Big(\sum_x \|\partial_x F\|_{L^2(\nu)}\Big)^2.$$

In particular, if the support of F is concentrated on a finite box $V_N = [-N,N]^d \cap \mathbb{Z}^d$, then the process needs a relaxation time t of the order $N^{4d/d-2}$ to get to equilibrium.

The lower bound (6.13) shows that we have the correct order of decay for *monotone* functions F. As we will see below functions of the discrete gradient $\nabla\Phi$ have a faster decay to equilibrium:

Theorem 6.2. *Let $d \geq 1$ and let $\tilde\mu_u$ be any translation invariant extremal Gibbs state, then there exist constants $c_1, c_2, k_2, \dots$ such that for all $\tilde F, \tilde J \in C_b^1(\mathcal{X}_r)$,*

$$|cov_{\tilde\mu_u}(\tilde F(\nabla\Phi(0)), \tilde J(\nabla\Phi(t))| \leq c_2 \sum_{b,b' \in \mathbb{Z}^d_*} \|\partial_b \tilde F\|_\infty R^*(k_2t, b_1' - b_1)\|\partial_{b'}\tilde J\|_\infty \tag{6.16}$$

where

$$R^*(t,z) = \int_t^\infty \frac{p^*(s,z)}{s \vee 1}, ds \leq \frac{c_1}{\big(|z|^2 \vee t \vee 1\big)^{d/2}}. \tag{6.17}$$

In particular we have

$$\|P_t\tilde F - \langle \tilde F\rangle_{\tilde\mu_u}\|^2_{L^2(\tilde\mu_u)} \leq c_3(t\vee 1)^{-d/2}\Big(\sum_b \|\partial_b \tilde F\|_\infty\Big)^2. \tag{6.18}$$

Proof. Using

$$\partial_x\tilde F(\phi) = \operatorname{div}(\tilde\partial\tilde F(\nabla\phi))_x = \sum_{b:b_2=x}\partial_b\tilde F(\nabla\phi) - \sum_{b:b_1=x}\partial_b\tilde F(\nabla\phi)$$

and summation by part, we get in view of (6.7)

$$\begin{aligned}
&\mathrm{cov}_{\tilde{\mu}_u}(\tilde{F}(\nabla\phi(0)), \tilde{J}(\nabla\phi(t))) \\
&= \int_0^\infty \sum_{x,y} \mathbb{E}_{\tilde{\mu}_u}\Big[\partial_x \tilde{F}(\nabla\phi(0))\partial_y \tilde{J}(\nabla\phi(t+s))\mathbf{p}^{\nabla\Phi}(0,x,t+s,y)\Big]\,ds \\
&= \int_0^\infty \sum_{b,b'} \mathbb{E}_{\tilde{\mu}_u}\Big[\partial_b \tilde{F}(\nabla\phi(0))\partial_{b'} \tilde{J}(\nabla\phi(t+s))\nabla_b^2\nabla_{b'}^4\mathbf{p}^{\nabla\Phi}(0,\cdot,t+s,\cdot)\Big]\,ds
\end{aligned}$$

where

$$\begin{aligned}
\nabla_b^2\nabla_{b'}^4\mathbf{p}^{\nabla\Phi}(0,\cdot,t+s,\cdot)] =& [\mathbf{p}^{\nabla\Phi}(0,b_2,t+s,b_2') - \mathbf{p}^{\nabla\Phi}(0,b_2,t+s,b_1')] \\
&- [\mathbf{p}^{\nabla\Phi}(0,b_1,t+s,b_2') - \mathbf{p}^{\nabla\Phi}(0,b_1,t+s,b_1')].
\end{aligned}$$

Next, we use (1.5a) and show

$$\begin{aligned}
&\left|\mathbb{E}_{\tilde{\mu}_u}\Big[\partial_b \tilde{F}(\nabla\Phi(0))\partial_{b'}\tilde{J}(\nabla\Phi(t+s))\nabla_b^2\nabla_{b'}^4\mathbf{p}^{\nabla\Phi}(0,\cdot,t+s,\cdot)\Big]\right| \\
&\le \|\partial_b\tilde{F}\|_\infty \|\partial_{b'}\tilde{J}\|_\infty \mathbb{E}_{\tilde{\mu}_u}[|\nabla_b^2\nabla_{b'}^4\mathbf{p}^{\nabla\Phi}(0,\cdot,t+s,\cdot)|] \\
&\le c_2\|\partial_b F\|_\infty \|\partial_{b'}\tilde{J}\|_\infty \frac{p^*(k_2(t+s), b_1'-b_1)}{(t+s)\vee 1}. \quad \square
\end{aligned}$$

Remark 6.2.

1. For the invariant measure $\tilde{\mu}_u$, (6.16) states in particular that

$$|\mathrm{cov}_{\tilde{\mu}_u}(\nabla_b\phi, \nabla_{b'}\phi)| \le \frac{c}{(1\vee|b_1'-b_1|)^d}, \qquad b,b'\in\mathbb{Z}_*^d \tag{6.18}$$

   Unlike (6.13) we do not have the corresponding lower bound. However this is the correct order as one can easily verify in the Gaussian case, cf. [Du] for a related estimate.

2. The faster algebraic convergence rate (6.16) is also predicted by the following generalized Nash inequality:
   (6.19)

$$\mathrm{var}_{\tilde{\mu}_u}(\tilde{F}) \le c_1 \left(\sum_{x\in\mathbb{Z}^d} \|\partial_x\tilde{F}\|^2_{L^2(\tilde{\mu}_u)}\right)^{1-2/(d+2)} \left(\sum_{b\in\mathbb{Z}_*^d} \|\partial_b\tilde{F}\|_{L^2(\tilde{\mu}_u)}\right)^{4/(d+2)},$$

   which is also a consequence of the Brascamp-Lieb inequality and a Nash inequality for gradients. However a proof of (6.16) based on (6.19) would require monotonicity of the second term on the RHS for the semigroup P_t applied to $\tilde{F}(\nabla\phi)$ which seems out of reach at this point !

We conclude this section with a simple application of our estimates in the derivation of CLT for the time average. More precisely, for given $F \in C_b^1(E_r)$ with $\langle F\rangle_{\mu_m} = 0$, set

$$Y_t^{(T)}(F) = T^{-1/2}\int_0^{T\cdot t} F(\Phi(s))\,ds, \qquad t\in[0,1].$$

Corollary 6.1. *Let $d \geq 5$ and assume that the process starts with the invariant extremal measure μ_m. Then for each $F \in C_b^1(E_r)$, as $T \to \infty$, the law of $\{Y_t^{(T)}(F), t \in [0,1]\}$ converges weakly to a Brownian motion with variance*

$$\sigma^2(F) = 2\int_0^\infty \left\langle F, P_t F\right\rangle_{\mu_m} dt.$$

In case $F(\phi) = \tilde{F}(\nabla\phi)$ with $\tilde{F} \in C_b^1(\mathcal{X}_r)$ the same result holds for any $d \geq 3$.

Proof. The main idea is to apply the very general CLT of Kipnis and Varadhan for reversible processes, cf. [KV]. Then the main point is to verify that

$$\begin{aligned}\lim_{T\to\infty} \operatorname{var}_{\mu_m}(Y_1^{(T)}(F)) &= \lim_{T\to\infty} \iint_0^{T\cdot t} \langle F, P_{|s-u|}F)\, ds\, du \\ &= \lim_{T\to\infty} \iint_0^{T\cdot t} \operatorname{var}_{\mu_m}(P_{|s-u|/2})\, ds\, du \\ &= \sigma^2(F) < \infty.\end{aligned}$$

But this follows simply from (6.10) for $d \geq 5$ in case $F \in C_b^1(E_r)$, resp. from (6.18) for $d \geq 3$ for $F(\phi) = \tilde{F}(\nabla\phi)$ in case $\tilde{F} \in C_b^1(\mathcal{X}_r)$. $\square$

Remark 6.3.

The lower dimensional case $d = 3, 4$ remains open. Here we can see from our lower and upper bounds (6.13) and (6.10) that the right scaling for a monotone $F \in C_b^{1,+}(E_r)$ should be $Y_t^{(T)}(F) = T^{-3/4}\int_0^{T\cdot t} F(\Phi(s))\, ds$ in $d = 3$ and $Y_1^{(T)}(F) = (T\log T)^{-1/2}\int_0^{T\cdot t} F(\Phi(s))\, ds$ in $d = 4$, and one expects then a fractional Brownian motion as limiting process but apart from the explicit Gaussian case, we do not see how to prove this.

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