Math. Proc. Camb. Phil. Soc. (2009), **147**, 593 © *Cambridge Philosophical Society* 2009
doi:10.1017/S030500410900262X
First published online 10 June 2009

Locally constant functors

BY DENIS–CHARLES CISINSKI
LAGA, CNRS (UMR 7539), Université Paris 13, Avenue Jean-Baptiste Clément, 93430 Villetaneuse France.
e-mail: cisinski@math.univ-paris13.fr

(*Received* 31 *July* 2008; *revised* 10 *March* 2009)

Abstract

We study locally constant coefficients. We first study the theory of homotopy Kan extensions with locally constant coefficients in model categories, and explain how it characterizes the homotopy theory of small categories. We explain how to interpret this in terms of left Bousfield localization of categories of diagrams with values in a combinatorial model category. Finally, we explain how the theory of homotopy Kan extensions in derivators can be used to understand locally constant functors.

1. *Homology with locally constant coefficients*

1·1. Given a model category[1] $\mathcal{V}$ with small colimits, and a small category A, we will write $[A, \mathcal{V}]$ for the category of functors from A to $\mathcal{V}$. Weak equivalences in $[A, \mathcal{V}]$ are the termwise weak equivalences. We denote by $\mathbf{Ho}([A, \mathcal{V}])$ the localization of $[A, \mathcal{V}]$ by the class of weak equivalences.

1·2. We denote by $\mathbf{LC}(A, \mathcal{V})$ the full subcategory of the category $\mathbf{Ho}([A, \mathcal{V}])$ whose objects are the locally constant functors, i.e. the functors

$$F: A \longrightarrow \mathcal{V}$$

such that for any map $a \to a'$ in A, the map

$$F_a \longrightarrow F_{a'}$$

is a weak equivalence in $\mathcal{V}$, or equivalently, an isomorphism in $\mathbf{Ho}(\mathcal{V})$ (where F_a is the evaluation of F at the object a).

Note that for any functor $u: A \to B$, the inverse image functor

$$u^*: [B, \mathcal{V}] \longrightarrow [A, \mathcal{V}] \tag{1·2·1}$$

preserves weak equivalences, so that it induces a functor

$$u^*: \mathbf{Ho}([B, \mathcal{V}]) \longrightarrow \mathbf{Ho}([A, \mathcal{V}]). \tag{1·2·2}$$

[1] We mean a Quillen closed model category [**Qui67**]. However, we could take any kind of model category giving rise to a good theory of homotopy colimits (i.e. to a Grothendieck derivator); see the work of Andrei Rădulescu–Banu [**RB06**] for more general examples.

The functor u^* obviously preserves locally constant functors, so that it induces a functor

$$u^*\colon \mathbf{LC}(B, \mathcal{V}) \longrightarrow \mathbf{LC}(A, \mathcal{V}). \tag{1·2·3}$$

In terms of ∞-categories, we can think intuitively of this construction in the following way.[2] Let us denote by $\underline{\mathcal{V}}$ the weak ∞-category associated to $\mathcal{V}$ (obtained from $\mathcal{V}$ by formally inverting weak equivalences), and write $\pi_\infty(A)$ for the weak ∞-groupoid obtained from A by inverting all its arrows. Then we can think of the category $\mathbf{LC}(A, \mathcal{V})$ as the homotopy category of the ∞-category of functors from $\pi_\infty(A)$ to $\underline{\mathcal{V}}$. In particular, the category $\mathbf{LC}(A, \mathcal{V})$ only depends on the homotopy type of the classifying space of A. The aim of this paper is to explain this assertion in an elementary way (using model categories and, eventually, the theory of derivators), and to address the question of the functorialities arising from such a construction. We formulate the first instance of this principle as follows.

THEOREM 1·3. *Let $u\colon A \to B$ be a functor whose nerve is a simplicial weak equivalence. Then the functor* (1·2·3) *is an equivalence of categories.*

1·4. Note that Theorem 1·3 has a converse: if a functor $u\colon A \to B$ induces an equivalence of category of type (1·2·3) for any model category $\mathcal{V}$ (it is even sufficient to consider this property in the case where $\mathcal{V}$ is the usual model category of simplicial sets), then the nerve of u is a simplicial weak equivalence; see Corollary 1·28. Before going into the technicalities of the proof, we propose to examine some examples which gives the combinatorial and geometrical meaning of such a result.[3]

Example 1·5. Theorem 1·3 can be used to strictify some combinatorial situation. For instance, let us consider the small category A associated to the following graph:

$$x \overset{f}{\underset{g}{\rightrightarrows}} y\,.$$

A functor from A to $\mathcal{V}$ is thus the data of parallel pair of arrows in $\mathcal{V}$. Let B be the free groupoid with a single object z endowed with a non-trivial automorphism $h\colon z \to z$. The functor

$$u\colon A \longrightarrow B$$

defined by sending f to 1_z and g to h induces a simplicial weak equivalence after applying the nerve functor: the geometric realization of A is homeomorphic to the circle S^1, and the geometric realization of B is the classifying space of the group $\mathbf{Z}$, so that the geometric realization of u is a homotopy equivalence from S^1 to $B\mathbf{Z}$ (it defines a generator of $\pi_1(B\mathbf{Z}) = \mathbf{Z}$). The equivalence of categories (1·2·3) states that any parallel pair of weak equivalences in $\mathcal{V}$ can be replaced canonically (up to a zig zig of weak equivalences in $[A, \mathcal{V}]$) by a pair of the form $h, 1_Z\colon Z \to Z$, where h is an actual isomorphism in $\mathcal{V}$.

Example 1·6. The categories $\mathbf{LC}(A, \mathcal{V})$ allows us to study "local systems" of spaces. Given a simplicial set X, we can consider the category Δ/X of simplices of X. The objects of Δ/X are the pairs (n, x), where $n \geqslant 0$ is an integer, and x is an n-simplex of X. A

[2] This is really to be considered as a heurisitic point of view: we won't explicitly use the theory of ∞-categories in these notes.

[3] These examples are given here as illustrations and are not used during this paper.

morphism $\varphi\colon (m, x) \to (n, y)$ is a morphism $\varphi\colon \Delta_m \to \Delta_n$ such that $\varphi^*(y) = x$, or, equivalently, such that the following diagram of simplicial sets commutes.

$$\begin{tikzcd} \Delta_m \arrow[rr, "\varphi"] \arrow[dr, "x"'] & & \Delta_n \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

Given a morphism of simplicial sets $f\colon X \to Y$, we get a functor, which will write abusively again $f\colon \Delta/X \to \Delta/Y$, defined by $f(n, x) = (n, f(x))$. This turns this construction into a functor from the category of simplicial sets to the category of small categories. A folklore result attributed to Quillen states that there exists a functorial weak equivalence from the nerve of Δ/X to X; see for instance [**Cis06**, proposition 4·3·15]. In particular, a morphism of simplicial sets $f\colon X \to Y$ is a simplicial weak equivalence if and only if the nerve of the functor $f\colon \Delta/X \to \Delta/Y$ is a simplicial weak equivalence. We define, for any simplicial set X, the category $\mathbf{LC}(X, \mathcal{V})$ by the formula

$$\mathbf{LC}(X, \mathcal{V}) = \mathbf{LC}(\Delta/X, \mathcal{V}). \tag{1·6·1}$$

Any morphism of simplicial sets $f\colon X \to Y$ induces a functor

$$f^*\colon \mathbf{LC}(Y, \mathcal{V}) \longrightarrow \mathbf{LC}(X, \mathcal{V})\,. \tag{1·6·2}$$

Assuming Theorem 1·3, the functors (1·6·2) are equivalences of categories whenever f is a simplicial weak equivalence.

Note that this simplicial point of view is compatible with the categorical point of view introduced before in the following sense. If A is a small category, its nerve being denoted by $N(A)$, then we have a natural functor

$$p_A\colon \Delta/N(A) \longrightarrow A$$

defined by $p(n, x) = x(n)$ (interpreting x as a functor from the ordered set $[n] = \{0, \dots, n\}$ to A). The nerve of p_A is a simplicial weak equivalence, so that, assuming Theorem 1·3, the functor

$$p_A^*\colon \mathbf{LC}(A, \mathcal{V}) \longrightarrow \mathbf{LC}(\Delta/N(A), \mathcal{V}) = \mathbf{LC}(N(A), \mathcal{V}) \tag{1·6·3}$$

is an equivalence of categories.

Example 1·7. In the case where $\mathcal{V} = sSet$ is the model category of simplicial sets, the category $\mathbf{LC}(A, sSet)$ is canonically equivalent to the homotopy category of spaces over the nerve of $A \times \Delta$; see [**Cis06**, propositions 4·4·15, 4·4·18, 4·4·21 and corollary 6·4·27]. As the category of simplices Δ has a terminal object, its nerve is contractible, so that the projection $N(A) \times N(\Delta) \to N(A)$ induces an equivalence of categories $\mathbf{Ho}(sSet/N(A) \times N(\Delta)) \simeq \mathbf{Ho}(sSet/N(A))$. Finally we have a canonical equivalence of categories

$$\mathbf{LC}(A, sSet) \simeq \mathbf{Ho}(sSet/N(A)). \tag{1·7·1}$$

Under these identifications, given a functor $u\colon A \to B$, the functor (1·2·3) corresponds to the homotopy pullback functor

$$\begin{gathered} \mathbf{Ho}(sSet/N(B)) \longrightarrow \mathbf{Ho}(sSet/N(A)) \\ (Y \longrightarrow N(B)) \longmapsto (Y \times^h_{N(B)} N(A) \longrightarrow N(A))\,. \end{gathered} \tag{1·7·2}$$

This shows that Theorem 1·3 is true for $\mathcal{V} = sSet$ (using the fact that *sSet* is right proper). However the proof of Theorem 1·3 will not use this fact.[4]

In the simplicial setting, using the natural weak equivalence $N(\Delta/X) \to X$, we get an equivalence of categories

$$\mathbf{Ho}(sSet/N(\Delta/X)) \simeq \mathbf{Ho}(sSet/X), \tag{1·7·3}$$

hence a natural equivalence of categories

$$\mathbf{LC}(X, sSet) \simeq \mathbf{Ho}(sSet/X) \tag{1·7·4}$$

for any simplicial set X.

Example 1·8. Consider the left Quillen equivalence

$$|-|\colon sSet \longrightarrow Top \tag{1·8·1}$$

induced by the geometric realization functor from simplicial sets to topological spaces. Denoting by *Sing* its right adjoint, it induces, for any topological space X, a left Quillen equivalence

$$|-|\colon sSet/Sing(X) \longrightarrow Top/X \tag{1·8·2}$$

defined by sending $Y \to Sing(X)$ to the its geometric realization composed with the counit map from $|Sing(X)|$ to X.

If X is locally contractible (with some other mild assumptions which are fulfilled for instance when X is a locally compact CW-complex), the category $\mathbf{Ho}(Top/X)$ is equivalent to the homotopy category of locally constant ∞-stacks on X. This is proven for example by Shulman [**Shu08**]. Locally constant stacks on X also admit the following description.

Let $O_c(X)$ be the set of contractible open subsets of X, ordered by inclusion. The category of simplicial presheaves on $O_c(X)$ is endowed with the Joyal-Heller model structure: the cofibrations (resp. the weak equivalences) are the termwise cofibrations (resp. weak equivalences) of simplicial sets. Write $L[O_c(X)^{op}, sSet]$ for the left Bousfield localization of $[O_c(X)^{op}, sSet]$ by the inclusions between contractible open subsets of X. We then have an equivalence of categories

$$\mathbf{Ho}(L[(O_c(X)^{op}, sSet]) \simeq \mathbf{LC}(O_c(X)^{op}, sSet) \tag{1·8·3}$$

(the local objects of this Bousfield localization are exactly the locally constant functors on $O_c(X)^{op}$; see Proposition 2·3 below). Let us write $sSh(X)$ for the category of sheaves of simplicial sets on X. It is endowed with the Joyal model structure: the cofibrations are the monomorphisms, and the weak equivalences are the maps $F \to G$ such that, for any point x of X, the morphism of simplicial sets $F_x \to G_x$ is a simplicial weak equivalence. This corresponds to the homotopy theory of stacks on X. Define $sSh_{lc}(X)$ as the left Bousfield localization of $sSh(X)$ by the inclusions between contractible open subsets of X. The model category $sSh_{lc}(X)$ corresponds to the homotopy theory of locally constant stacks on X.

The sheafification functor

$$a\colon L[O_c(X)^{op}, sSet] \longrightarrow sSh_{lc}(X) \tag{1·8·4}$$

[4] But it is possible to prove it this way: this follows from Theorem 3·6 below, and from the identification (3·7·2).

is then a left Quillen equivalence. It is obviously a left Quillen functor. Consider an hypercovering $\mathcal{U}^\bullet \longrightarrow U$ of a contractible open subspace U of X, such that, for any $n \geqslant 0$, $\mathcal{U}^n$ is a sum of contractible open subspaces of U. Then the map $\mathcal{U}^\bullet \longrightarrow U$ is a weak equivalence of $L[O_c(X)^{op}, sSet]$ (this follows formally from [**Cis03a**, example 3·4·23, proposition 4·3·14 and corollary 5·3·18]). The fact (1·8·4) is a left Quillen equivalence thus follows from [**Cis03a**, theorem 3·3·1] or [**DHI04**, theorem 1·2]. In other words: any locally constant prestack on $O_c(X)$ satisfies descent (hence is a stack on X).

In conclusion, when X is nice enough (e.g. a locally compact CW-complex) we have the following equivalences of categories

$$\mathbf{LC}(O_c(X)^{op}, sSet) \simeq \mathbf{Ho}(sSh_{lc}(X)) \simeq \mathbf{Ho}(Top/X) \simeq \mathbf{LC}(Sing(X), sSet), \tag{1·8·5}$$

where the first equivalence comes from the Quillen equivalence (1·8·4), the second equivalence comes from Shulman's Theorem[5] [**Shu08**], and the third one comes from the left Quillen equivalence (1·8·2) and from (1·7·4).

1·9. We will now go to the proof of Theorem 1·3, which will need a little preparation.

Define a functor between small categories $u\colon A \longrightarrow B$ to be a *weak equivalence* if the functor (1·2·3) is an equivalence of categories (we assume the cocomplete model category $\mathcal{V}$ has been fixed).

A small category A is *aspherical* if the map $A \longrightarrow e$ is a weak equivalence (where e denotes the terminal category). A functor u: $A \longrightarrow B$ will be said to be *aspherical* if, for any object b of B, the functor $A/b \longrightarrow B/b$ is a weak equivalence (where $A/b = (u \downarrow b)$ is the comma category of objects of A over b, and $B/b = (1_B \downarrow b)$).

Theorem 1·3 asserts that any functor whose nerve is a simplicial weak equivalence is a weak equivalence in the sense defined above. This will follow from the result below.

THEOREM 1·10. *Let $\mathcal{W}$ be a class of functors between small categories. We assume that $\mathcal{W}$ is a weak basic localizer in the sense of Grothendieck* [**Mal05a**], *which means that the following properties are satisfied.*

- La *Any identity is in $\mathcal{W}$. The class $\mathcal{W}$ satisfies the two out of three property. If a map i: $A \longrightarrow B$ has a retraction r: $B \longrightarrow A$ such that ir is in $\mathcal{W}$, then i is in $\mathcal{W}$.*
- Lb *If a small category A has a terminal object, then the map from A to the terminal category is in $\mathcal{W}$.*
- Lc *Given a functor u: $A \longrightarrow B$, if for any object b of B, the induced functor $A/b \longrightarrow B/b$ is in $\mathcal{W}$, then u is in $\mathcal{W}$.*

Then any functor between small categories whose nerve is a simplicial weak equivalence is in $\mathcal{W}$.

Proof. See [**Cis06**, theorem 6·1·18].

Remark 1·11. The class of functors between small categories whose nerve is a simplicial weak equivalence is known to be a weak basic localizer: property La comes from the

[5] Shulman does not consider the Joyal model structure on $sSh(X)$, but rather the Lurie model structure, which has *less* weak equivalences. However, the Quillen equivalence (1·8·4) shows in particular that, as far as we are concerned with locally constant ∞-stacks on a locally contractible space, there is no difference between the classical point of view and Lurie's version.

stability of simplicial weak equivalences by retracts, property Lb comes from the fact the nerve of category with a terminal object is contractible, and property Lc is nothing but Quillen's Theorem A [**Qui73**]. Hence Theorem 1·3 can be reformulated by saying that the class of functors between small categories whose nerve is a simplicial weak equivalence is the smallest weak basic localizer.

1·12. To prove Theorem 1·3, we will prove that the class of weak equivalences satisfies the properties listed in the previous theorem. Property La is easy to check. It thus remains to prove properties Lb and Lc. In other words, we have to prove that any category with a terminal object is aspherical, and that any aspherical functor is a weak equivalence. We will use the theory of homotopy Kan extensions in $\mathcal{V}$.

Given a functor $u\colon A \to B$, the functor (1·2·1) has a left adjoint

$$u_!\colon [A, \mathcal{V}] \longrightarrow [B, \mathcal{V}] \tag{1·12·1}$$

which admits a total left derived functor

$$\mathbf{L}u_!\colon \mathbf{Ho}([A, \mathcal{V}]) \longrightarrow \mathbf{Ho}([B, \mathcal{V}]). \tag{1·12·2}$$

The functor $\mathbf{L}u_!$ is also a left adjoint of the functor (1·2·2); see [**Cis03b**, **RB06**]. When B is the terminal category, we will write $\mathbf{L}\varinjlim_A = \mathbf{L}u_!$ for the corresponding homotopy colimit functor.

1·13. For each object b of B, we have the following pullback square of categories

$$\begin{tikzcd} A/b \arrow[r, "v"] \arrow[d, "u/b"'] & A \arrow[d, "u"] \\ B/b \arrow[r, "w"'] & B \end{tikzcd} \tag{1·13·1}$$

(where w is the obvious forgetful functor). Given a functor F from A (resp. B) to $\mathcal{V}$, we will write

$$F/b = v^*(F) \quad (\text{resp. } F/b = w^*(F)). \tag{1·13·2}$$

This gives the following formula for a functor $F\colon B \to \mathcal{V}$

$$u^*(F)/b = (u/b)^*(F/b). \tag{1·13·3}$$

It is a fact that left homotopy Kan extensions can be computed pointwise (like in ordinary category theory), which can be formulated like this:

PROPOSITION 1·14. *For any functor $F\colon A \to \mathcal{V}$, and any object b of B, the base change map*

$$\mathbf{L}\varinjlim_{A/b} F/b \longrightarrow \mathbf{L}u_!(F)_b$$

is an isomorphism in $\mathbf{Ho}(\mathcal{V})$.

Proof. See [**Cis03b**] or, in a more general context, [**RB06**, theorem 9·6·5].

PROPOSITION 1·15. *Let I be a small category. A morphism $F \to G$ in* $\mathbf{Ho}([I, \mathcal{V}])$ *is an isomorphism if and only for any object i of I, the map $F_i \to G_i$ is an isomorphism in* $\mathbf{Ho}(\mathcal{V})$.

Proof. See [**Cis03b**] or [**RB06**, theorem 9·7·1].

Remark 1·16. It is obvious, by construction, that the evaluation at i of any isomorphism of $\mathbf{Ho}([I, \mathcal{V}])$ induces an isomorphism of $\mathbf{Ho}(\mathcal{V})$. The purpose of the proof of Proposition 1·15 is to ensure that we have inverted enough weak equivalences of $[I, \mathcal{V}]$ for the converse to hold.

PROPOSITION 1·17. *Let $u\colon A \to B$ be a weak equivalence of small categories. Then, for any locally constant functor $F\colon B \to \mathcal{V}$, the map*

$$\mathbf{L}\varinjlim_A u^*(F) \longrightarrow \mathbf{L}\varinjlim_B F$$

is an isomorphism in $\mathbf{Ho}(\mathcal{V})$.

Proof. Given a small category I and an object X, denote by X_I the constant functor from I to $\mathcal{V}$ with value X. Let $F\colon B \to \mathcal{V}$ be a locally constant functor. Using the fact that (1·2·3) is fully faithful, we see that

$$\mathrm{Hom}_{\mathbf{Ho}([B,\mathcal{V}])}(F, X_B) \longrightarrow \mathrm{Hom}_{\mathbf{Ho}([A,\mathcal{V}])}(u^*(F), u^*(X_B))$$

is bijective. As $u^*(X_B) = X_A$, the identifications

$$\mathrm{Hom}_{\mathbf{Ho}([B,\mathcal{V}])}(F, X_B) \simeq \mathrm{Hom}_{\mathbf{Ho}(\mathcal{V})}(\mathbf{L}\varinjlim_B F, X)$$

$$\mathrm{Hom}_{\mathbf{Ho}([A,\mathcal{V}])}(u^*(F), X_A) \simeq \mathrm{Hom}_{\mathbf{Ho}(\mathcal{V})}(\mathbf{L}\varinjlim_A u^*(F), X)$$

and the Yoneda Lemma applied to $\mathbf{Ho}(\mathcal{V})$ completes the proof.

COROLLARY 1·18. *Let I be an aspherical category, and $F\colon I \to \mathcal{V}$ be a locally constant functor. Then for any object i of I, the map*

$$F_i \longrightarrow \mathbf{L}\varinjlim_I F$$

is an isomorphism in $\mathbf{Ho}(\mathcal{V})$.

Proof. Apply Proposition 1·17 to the functor from the terminal category to I defined by i.

LEMMA 1·19. *Let $u\colon A \to B$ be a functor between small categories which has a left or a right adjoint $v\colon B \to A$. Then u and v are weak equivalences.*

Proof. It is sufficient to prove that the functors

$$u^*\colon \mathbf{LC}(B, \mathcal{V}) \longrightarrow \mathbf{LC}(A, \mathcal{V}) \quad \text{and} \quad v^*\colon \mathbf{LC}(A, \mathcal{V}) \longrightarrow \mathbf{LC}(B, \mathcal{V})$$

are quasi-inverse to each other. Assume for instance that v is right adjoint to u. Then, as

$$A \longmapsto \mathbf{Ho}([A, \mathcal{V}])$$

is obviously a 2-functor, we see immediately that u^* is right adjoint to v^*, the unit $1 \to u^*v^*$ (resp. the counit $v^*u^* \to 1$) being induced by the unit $1 \to vu$ (resp. by the counit $uv \to 1$). To conclude the proof, it remains to see that, in the case where F is locally constant, the maps

$$F \longrightarrow u^*v^*(F) \quad \text{and} \quad v^*u^*(F) \longrightarrow F$$

are termwise weak equivalences, which is obvious.

PROPOSITION 1·20. *Any small category which has a terminal object is aspherical.*

Proof. A small category I has a terminal object if and only if the functor from I to the terminal category has a right adjoint. This proposition thus follows immediately from the preceding lemma.

COROLLARY 1·21. *A functor $u\colon A \to B$ is aspherical if and only if for any object b of B, the category A/b is aspherical.*

Proof. As the class of weak equivalences satifies the two out of three property, this follows from the fact that the category B/b has a terminal object (namely $(b, 1_b)$).

1·22. A functor $u\colon A \to B$ is *locally constant* if for any map $b \to b'$ in B, the functor $A/b \to A/b'$ is a weak equivalence. For example, by virtue of Corollary 1·21, any aspherical functor is locally constant.

Example 1·23. Let $p\colon X \to Y$ a Kan fibration. Then, assuming Theorem 1·3, the functor $p\colon \Delta/X \to \Delta/Y$ is locally constant: a morphism $(m, y) \to (n, z)$ in Δ/Y correspond to a diagram of simplicial sets of shape

$$\Delta_m \longrightarrow \Delta_n \longrightarrow Y\,,$$

so that one can form the following pullbacks.

$$\begin{tikzcd}
X \times_Y \Delta_m \arrow[r] \arrow[d] & X \times_Y \Delta_n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta_m \arrow[r] & \Delta_n \arrow[r] & Y.
\end{tikzcd}$$

These induce pullback squares of categories

$$\begin{tikzcd}
\Delta/(X \times_Y \Delta_m) \arrow[r] \arrow[d] & \Delta/(X \times_Y \Delta_n) \arrow[r] \arrow[d] & \Delta/X \arrow[d, "p"] \\
\Delta/\Delta_m \arrow[r] & \Delta/\Delta_n \arrow[r] & \Delta/Y
\end{tikzcd}$$

from which we deduce that the functor

$$(\Delta/X)/(y, m) \longrightarrow (\Delta/X)/(z, n)$$

is isomorphic to the functor

$$\Delta/(X \times_Y \Delta_m) \longrightarrow \Delta/(X \times_Y \Delta_n)\,.$$

The fact that p is a Kan fibration implies that the nerve of the latter is a simplicial weak equivalence (using the right properness of the model category of simplicial sets).

PROPOSITION 1·24 (Formal Serre spectral sequence). *If $u\colon A \to B$ is locally constant, then the functor (*1·12·2*) preserves locally constant functors. In particular, it induces a functor*

$$\mathbf{L}u_!\colon \mathbf{LC}(A, \mathcal{V}) \longrightarrow \mathbf{LC}(B, \mathcal{V})$$

*which is a left adjoint to the functor (*1·2·3*).*

Proof. Let F be a locally constant functor, and $\beta\colon b \to b'$ be a map in B, We have to show that the induced map

$$\mathbf{L}u_!(F)_b \longrightarrow \mathbf{L}u_!(F)_{b'}$$

is an isomorphism in $\mathbf{Ho}(\mathcal{V})$. Denote by $j_\beta\colon A/b \to A/b'$ the functor induced by β (which is a weak equivalence by assumption on u). With the notations (1·13·2), we have

$$j^*_\beta(F/b') = F/b\,.$$

This corollary thus follows immediately from Proposition 1·14 and from Proposition 1·17.

Remark 1·25. It can be proved that the functor (1·2·3) always has a left adjoint; see Corollary 3·11. The purpose of Proposition 1·24 is really to compute this adjoint.

Scholium 1·26. Proposition 1·24 is related with the classical Serre spectral sequence as follows. Consider the case where $\mathcal{V} = \mathit{Comp}(\mathit{Ab})$ is the category of complexes of abelian groups (considered with the projective model structure, i.e. the weak equivalences are the quasi-isomorphisms, and the fibrations are the degreewise surjective maps; see [**Hov99**, Theorem 2·3·11]). Then, for any small category A, the category $\mathbf{Ho}([A, \mathcal{V}])$ is nothing but the (unbounded) derived category $D([A, \mathit{Ab}])$ of the abelian category of functors from A to abelian groups. Given a functor F from A to the category of complexes of abelian groups, the homology of A with value in F is defined by

$$H_i(A, F) = H^{-i}(\mathbf{L}\varinjlim_A F)\,.$$

Let L be a local system on A (i.e. a functor from $\pi_1(A)$ to Ab). It can be considered as an object of $\mathbf{LC}(A, \mathit{Comp}(\mathit{Ab}))$. For a given functor $u\colon A \to B$, we have a canonical isomorphism

$$\mathbf{L}\varinjlim_B \mathbf{L}u_!(L) \simeq \mathbf{L}\varinjlim_A L\,.$$

Proposition 1·24 ensures that, if u is locally constant, then $\mathbf{L}u_!(L)$ is locally constant, which means in particular that the objects $H^{-i}(\mathbf{L}u_!(L)) = \mathbf{L}_i u_!(L)$ are local systems on B (i.e. presheaves of abelian groups on $\pi_1(B)$) for any $i \geqslant 0$. As a consequence, the corresponding Leray-Grothendieck spectral sequence

$$E^2_{p,q} = H_p(B, \mathbf{L}_q u_!(L)) \Longrightarrow H_{p+q}(A, L)$$

is a spectral sequence which involves only homology groups with coefficients in local systems. Note that the local system $\mathbf{L}_i u_!(L)$ can be described explicitly: for any object b of B, we have, by Proposition 1·14,

$$\mathbf{L}_i u_!(L)_b \simeq H_i(A/b, L/b)\,.$$

Assume furthermore that, for any map $b \to b'$ in B the map $N(A/b) \to N(A/b')$ is a simplicial weak equivalence (which implies that u is locally constant, at least if we admit Theorem 1·3). Then, by virtue of Quillen's Theorem B [**Qui73**], for any object b of B, A/b is the homotopy fiber at b of the map $u\colon A \to B$. Hence the local system $\mathbf{L}_i u_!(L)$ is locally the homology of the homotopy fiber of u with coefficients in the local system L.

In the case $u\colon A \to B$ is the functor $\Delta/X \to \Delta/Y$ associated to a Kan fibration (see Example 1·23), we obtain the spectral sequence of a Kan fibration in homology (compare with [**GZ67**, appendix II]). The classical Serre spectral sequence in homology is obtained from the latter using Example 1·8.

The same arguments in the case where $\mathcal{V} = \mathit{Comp}(\mathit{Ab})^{op}$ gives the Serre spectral sequence in cohomology.

PROPOSITION 1·27 (Formal Quillen Theorem A). *Any aspherical functor is a weak equivalence.*

Proof. Let $u\colon A \to B$ be an aspherical functor. Then u is in particular locally constant, so that, by virtue of Proposition 1·24, the functor $\mathbf{L}u_!$ preserves locally constant functor, and induces a left adjoint to the functor (1·2·3). Let $F\colon B \to \mathcal{V}$ be a locally constant functor. We will prove that the co-unit map

$$\mathbf{L}u_! u^*(F) \longrightarrow F$$

is an isomorphism. According to Proposition 1·15, it is sufficient to prove that for any object b of B, the map

$$(\mathbf{L}u_! u^*(F))_b \longrightarrow F_b$$

is an isomorphism in $\mathbf{Ho}(\mathcal{V})$. This follows immediately from the computations below:

$$\begin{aligned}
(\mathbf{L}u_! u^*(F))_b &\simeq \mathbf{L}\varinjlim_{A/b} u^*(F)/b \quad \text{(Proposition 1·14)}\\
&= \mathbf{L}\varinjlim_{A/b} (u/b)^*(F/b) \quad \text{(Formula (1·13·3))}\\
&\simeq \mathbf{L}\varinjlim_{B/b} F/b \quad \text{(Proposition 1·17 applied to } u/b\text{)}\\
&\simeq F_b \quad \text{(because } (b, 1_b) \text{ is a terminal object of } B/b\text{).}
\end{aligned}$$

Consider now a locally constant functor F: $A \to \mathcal{V}$. We will show that the unit map

$$F \longrightarrow u^*\mathbf{L}u_!(F)$$

is an isomorphism. By virtue of Proposition 1·15, we are reduced to prove that, for any object a of A, the map

$$F_a \longrightarrow (u^*\mathbf{L}u_!(F))_a$$

is an isomorphism. We compute again

$$\begin{aligned}
F_a &\simeq \mathbf{L}\varinjlim_{A/u(a)} F/u(a) \quad \text{(Corollary 1·18 for } I = A/u(a) \text{ and } i = (a, 1_{u(a)})\text{)}\\
&\simeq \mathbf{L}u_!(F)_{u(a)} \quad \text{(Proposition 1·14)}\\
&= (u^*\mathbf{L}u_!(F))_a
\end{aligned}$$

and this ends the proof.

Proof of Theorem 1·3. It is sufficient to check that the class of weak equivalences satisfies the properties listed in Theorem 1·10. The class of weak equivalences obviously satifies property La. Property Lb follows from Proposition 1·20, and property Lc from Proposition 1·27.

COROLLARY 1·28. *Let u: $A \to B$ a functor between small categories. The nerve of u is a simplicial weak equivalence if and only if for any model category $\mathcal{V}$, the functor*

$$u^*: \mathbf{LC}(B, \mathcal{V}) \longrightarrow \mathbf{LC}(A, \mathcal{V})$$

is an equivalence of categories.

Proof. Theorem 1·3 asserts this is a necessary condition. It is very easy to check that this is also sufficient: we can either use Proposition 1·17 and [**Cis06**, 6·5·11], or we can use the fact that, for a given small category A, the homotopy colimit of the constant functor indexed by A whose value is the terminal simplicial set is precisely the nerve of A (which is completely obvious if we consider for example the Bousfield–Kan construction of homotopy colimits).

2. *Model structures for locally constant functors*

2·1. We consider now a left proper combinatorial model category $\mathcal{V}$ and a small category A (see [**Bek00**] for the definition of a combinatorial model category). The category of functors $[A, \mathcal{V}]$ has two canonical model structures. The *projective model structure* on $[A, \mathcal{V}]$ is defined as follows: the weak equivalences are the termwise weak equivalences, and the fibrations are the termwise fibrations. The *injective model structure* on $[A, \mathcal{V}]$ is defined dually: the weak equivalences are the termwise weak equivalences, and the cofibrations are the termwise cofibrations. One can check that the identity functor is a left Quillen equivalence from the projective model structure to the injective model structure (this is just an abstract way to say that all the cofibrations of the projective model structure are termwise cofibrations, which is easy to check; see for example [**Cis06**, lemma 3·1·12]). These two model structures are left proper. We refer to Barwick's paper [**Bar07**] for an account on the theory of left Bousfield localization of combinatorial model categories.

2·2. We fix a (regular) cardinal α with the following properties (see [**Dug01**]).

(a) Any object of $\mathcal{V}$ is a α-filtered colimit of α-small objects.
(b) The class of weak equivalences of $\mathcal{V}$ is stable by α-filtered colimits.
(c) There exists a cofibrant resolution functor Q which preserves α-filtered colimits.

Given an object a of A, we denote by

$$a_!: \mathcal{V} \longrightarrow [A, \mathcal{V}]$$

the left adjoint to the evaluation functor at a. We define S as the (essentially small) set of maps of shape

$$a_!(QX) \longrightarrow a'_!(QX) \tag{2·2·1}$$

associated to each map $a \to a'$ in A and each α-small object X (and Q is some fixed cofibrant resolution functor satisfying the condition (c) above).

We define the *projective local model structure* on $[A, \mathcal{V}]$ as the left Bousfield localization of the projective model structure on $[A, \mathcal{V}]$ by S. The *injective local model structure* on $[A, \mathcal{V}]$ is the left Bousfield localization of the injective model structure on $[A, \mathcal{V}]$ by S. It is clear that the identity functor is still a left Quillen equivalence from the projective local model structure to the injective local model structure. The weak equivalences of these two model structures will be called the *local weak equivalences*. Note that, despite the appearences, the projective and injective local model structures do not depend on the chosen cardinal α: indeed, the following proposition shows that the corresponding local objects can be described independently of α.

PROPOSITION 2·3. *A functor $F\colon A \to \mathcal{V}$ is fibrant in the projective (resp. injective) local model structure if and only if it is fibrant for the projective (resp. injective) model structure and if it is locally constant.*

Proof. Note first that, thanks to condition (b), for any α-filtered category I and any functor F from I to $[A, \mathcal{V}]$, the natural map

$$\mathbf{L}\varinjlim_I F \longrightarrow \varinjlim_I F$$

is an isomorphism in $\mathbf{Ho}([A, \mathcal{V}])$. Hence it remains an isomorphism in the homotopy category of the projective (resp. injective) local model structure. This implies that local weak equivalences are stable by α-filtered colimits. Conditions (a) and (b) thus imply that for any object X of $\mathcal{V}$, and any arrow $a \to a'$ in A, the map

$$a_!(QX) \longrightarrow a'_!(QX)$$

is a local weak equivalence. For any object a of A, the functor $a_!$ is a left Quillen functor from $\mathcal{V}$ to the projective model structure (hence also to the injective model structure, as any cofibration of the projective model structure is a termwise cofibration). As a consequence, if a is an object of A, X is an object of $\mathcal{V}$, and F a functor from A to $\mathcal{V}$, then

$$\mathrm{Hom}_{\mathbf{Ho}(\mathcal{V})}(X, F_a) \simeq \mathrm{Hom}_{\mathbf{Ho}([A,\mathcal{V}])}(\mathbf{L}a_!(X), F) \simeq \mathrm{Hom}_{\mathbf{Ho}([A,\mathcal{V}])}(a_!(QX), F).$$

It is now easy to see that, if moreover F is fibrant for the projective (resp. injective) model structure, then it is fibrant for the projective (resp. injective) local model structure if and only if it is locally constant.

The general properties of left Bousfield localization leads to the two following corollaries; see [**Hir03**].

COROLLARY 2·4. *The localization of $[A, \mathcal{V}]$ by the class of local weak equivalences is* $\mathbf{LC}(A, \mathcal{V})$.

COROLLARY 2·5. *The inclusion functor $\mathbf{LC}(A, \mathcal{V}) \to \mathbf{Ho}([A, \mathcal{V}])$ has a left adjoint.*

PROPOSITION 2·6. *Let $u\colon A \to B$ be a functor between small categories. Then the functor*

$$u^*\colon [B, \mathcal{V}] \longrightarrow [A, \mathcal{V}]$$

is a right Quillen functor for the projective local model structures.

If moreover the nerve of u is a simplicial weak equivalence, then the functor u^* is a right Quillen equivalence.

Proof. The left adjoint $u_!$ of u^* preserves cofibrations: this is obviously a left Quillen functor for the projective model structures. It is thus sufficient to check that u^* preserves fibrations between fibrant objects; see [**JT07**, proposition 7·15]. It follows from Proposition 2·3 that fibrations between fibrant objects are just fibrations of the projective model structure between fibrant objects of the projective model structure which are locally constant. It is clear that u^* preserves this property. This proves that u^* is a right Quillen functor. The last assertion follows from Theorem 1·3.

Remark 2·7. According to the preceding proposition, the functor $u_!$ has a total left derived functor

$$\mathbf{L}u_!\colon \mathbf{LC}(A, \mathcal{V}) \longrightarrow \mathbf{LC}(B, \mathcal{V}).$$

It also has a total left derived functor

$$\mathbf{L}u_!\colon \mathbf{Ho}([A, \mathcal{V}]) \longrightarrow \mathbf{Ho}([A, \mathcal{V}]).$$

but, in general, the diagram (in which i_A and i_B denote the inclusion functors)

$$\begin{tikzcd} \mathbf{LC}(A, \mathcal{V}) \arrow[r, "\mathbf{L}u_!"] \arrow[d, "i_A"'] & \mathbf{LC}(B, \mathcal{V}) \arrow[d, "i_B"] \\ \mathbf{Ho}([A, \mathcal{V}]) \arrow[r, "\mathbf{L}u_!"'] & \mathbf{Ho}([B, \mathcal{V}]) \end{tikzcd}$$

does not (even essentially) commute. There is only a natural map

$$\mathbf{L}u_! i_A(F) \longrightarrow i_B \mathbf{L}u_!(F).$$

However, Proposition 1·24 asserts that this natural map is an isomorphism whenever u is locally constant.

PROPOSITION 2·8. *Let $u\colon A \longrightarrow B$ be a functor between small categories. Assume that the functor $u^{op}\colon A^{op} \longrightarrow B^{op}$ is locally constant. Then the functor*

$$u^*\colon [B, \mathcal{V}] \longrightarrow [A, \mathcal{V}]$$

is a left Quillen functor for the injective local model structures.

If moreover the nerve of u is a simplicial weak equivalence, then the functor u^* is a left Quillen equivalence.

Proof. We know that u^* is a left Quillen functor for the injective model structure. Hence, by virtue of Proposition 2·3, it is sufficient to prove that the total right derived functor

$$\mathbf{R}u_*\colon \mathbf{Ho}([A, \mathcal{V}]) \longrightarrow \mathbf{Ho}([B, \mathcal{V}])$$

preserves locally constant functors. But this latter property is just Proposition 1·24 applied to $\mathcal{V}^{op}$. The last assertion follows again from Theorem 1·3.

3. *Locally constant coefficients in Grothendieck derivators*

3·1. We start this section by fixing some notations.

Let A be a small category. We will consider the category $[A^{op}, sSet]$ of simplicial presheaves on A endowed with the projective model structure. Given a subcategory S of A, we denote by $L_S[A^{op}, sSet]$ the left Bousfield localization of the projective model structure on $[A^{op}, sSet]$ by S (where S is seen as a set of maps in $[A^{op}, sSet]$ via the Yoneda embedding). The fibrant objects of $L_S[A^{op}, sSet]$ are the simplicial presheaves F on A which are termwise Kan complexes and which sends the maps of S to simplicial homotopy equivalences. In particular, in case $A = S$, $L_A[A^{op}, sSet]$ is the projective local model structure on $[A^{op}, sSet]$ studied in the previous section.

3·2. We refer to [**Mal01**, **Cis03b**, **Cis08b**] for the definition of derivators and of the basic notions in this setting. We give here only a small sketch of the definition.

A prederivator is a (strict) contravariant[6] 2-functor $\mathbb{D}$ from the 2-category of small categories to the 2-category of (possibly large) categories. If $u\colon A \longrightarrow B$ is a functor between small categories the induced functor is denoted by

$$u^*\colon \mathbb{D}(B) \longrightarrow \mathbb{D}(A)\,. \tag{3·2·1}$$

A derivator is roughly a prederivator $\mathbb{D}$ which has small (homotopy) colimits: for any functor $u\colon A \longrightarrow B$ between small categories, the functor (3·2·1) has a left adjoint

$$u_!\colon \mathbb{D}(A) \longrightarrow \mathbb{D}(B)\,, \tag{3·2·2}$$

called the homological direct image functor associated to u, or the left homotopy Kan extension of u, and a right adjoint

$$u_*\colon \mathbb{D}(A) \longrightarrow \mathbb{D}(B)\,, \tag{3·2·3}$$

called the cohomological direct image functor associated to u, or the right homotopy Kan extension of u. These data are asked to satisfy some natural conditions. The main examples of derivators come from model categories: if $\mathcal{V}$ is a cocomplete model category, then we get a derivator $\mathbb{H}\mathrm{o}(\mathcal{V})$ defined by

$$\mathbb{H}\mathrm{o}(\mathcal{V})(A) = \mathbf{Ho}([A^{op}, \mathcal{V}]) \tag{3·2·4}$$

(see [**Cis03b**, theorem 6·11]). In the case where $\mathbb{D} = \mathbb{H}\mathrm{o}(\mathcal{V})$, the main axioms a derivator should satisfy are expressed by Proposition 1·14 (as well as its dual version) and Proposition 1·15. It is very possible to understand the statements which will follow by considering only derivators of shape $\mathbb{H}\mathrm{o}(\mathcal{V})$ for a cocomplete model category $\mathcal{V}$.

Morphisms of (pre)derivators are just pseudo-natural transformations. Prederivators naturally form a 2-category.

3·3. Given two prederivators $\mathbb{D}$ and $\mathbb{D}'$, we denote by $\mathcal{H}om(\mathbb{D}, \mathbb{D}')$ the category of morphisms of derivators; see [**Cis03b**]. If $\mathbb{D}$ and $\mathbb{D}'$ are derivators, we denote by $\mathcal{H}om_!(\mathbb{D}, \mathbb{D}')$ the full subcategory of $\mathcal{H}om(\mathbb{D}, \mathbb{D}')$ whose objects are the morphisms of prederivators which commute with left homotopy Kan extensions (which are called cocontinuous morphisms in [**Cis03b**]).

Given a (small) category, we denote by $\underline{A}$ the prederivator which associates to each small category I the category $[I^{op}, A]$ of presheaves on I with values in A. This defines a 2-functor from the 2-category of small categories to the 2-category of prederivators. Note that we have a Yoneda Lemma for prederivators: given a small category A and a prederivator $\mathbb{D}$, the functor

$$\mathcal{H}om(\underline{A}, \mathbb{D}) \longrightarrow \mathbb{D}(A^{op}), \qquad F \longmapsto F(1_A) \tag{3·3·1}$$

is an equivalence of categories.

THEOREM 3·4. *For any derivator $\mathbb{D}$, the composition by the Yoneda embedding h : $\underline{A} \longrightarrow \mathbb{H}\mathrm{o}([A^{op}, sSet])$ induces an equivalence of categories*

$$\mathcal{H}om_!(\mathbb{H}\mathrm{o}([A^{op}, sSet]), \mathbb{D}) \simeq \mathcal{H}om(\underline{A}, \mathbb{D})\,.$$

Proof. This is a translation of [**Cis08b**, corollary 3·26] using (3·3·1).

[6] By which we mean that the 1-cells and the 2-cells are inverted.

3·5. We denote by $\mathcal{H}om_S(\underline{A}, \mathbb{D})$ the full subcategory of morphisms $\underline{A} \longrightarrow \mathbb{D}$ such that the induced functor $A \longrightarrow \mathbb{D}(e)$ sends the maps of S to isomorphims (where e denotes the terminal category). A formal consequence of Theorem 3·4 is:

THEOREM 3·6. *For any derivator $\mathbb{D}$, the composition by the Yoneda morphism h: $\underline{A} \longrightarrow \mathbb{H}\mathbb{o}(L_S[A^{op}, sSet])$ induces an equivalence of categories*

$$\mathcal{H}om_!(\mathbb{H}\mathbb{o}(L_S[A^{op}, sSet]), \mathbb{D}) \simeq \mathcal{H}om_S(\underline{A}, \mathbb{D}) .$$

Proof. This follows immediately from Theorem 3·4 and from the universal property of left Bousfield localization for derivators; see [**Tab08**, theorem 5·4].

3·7. Given a small category A and a derivator $\mathbb{D}$, we define

$$\mathbf{LC}(A, \mathbb{D}) = \mathcal{H}om_A(\underline{A}, \mathbb{D}) . \qquad (3·7·1)$$

It is clear that for a model category $\mathcal{V}$, we have by definition

$$\mathbf{LC}(A, \mathcal{V}) = \mathbf{LC}(A, \mathbb{H}\mathbb{o}(\mathcal{V})) . \qquad (3·7·2)$$

COROLLARY 3·8. *Let u: $A \longrightarrow B$ be a functor between small categories. Then the nerve of u is a simplicial weak equivalence if and only if for any derivator $\mathbb{D}$, the functor*

$$u^*: \mathbf{LC}(B, \mathbb{D}) \longrightarrow \mathbf{LC}(A, \mathbb{D})$$

is an equivalence of categories.

Proof. As any model category gives rise to a derivator, this is certainly a sufficient condition, by virtue of Corollary 1·28. It thus remains to prove that this is a necessary condition. The nerve of the functor u is a simplicial weak equivalence if and only if the nerve of u^{op}: $A^{op} \longrightarrow B^{op}$ is so. This result is thus a consequence of Proposition 2·6, of Theorem 3·6, and of the fact that any Quillen equivalence induces an equivalence of derivators.

LEMMA 3·9. *Let A be a small category. The inclusion morphism*

$$i: \mathbb{H}\mathbb{o}(L_A[A^{op}, sSet]) \longrightarrow \mathbb{H}\mathbb{o}([A^{op}, sSet])$$

(defined as the right adjoint of the localization morphism) preserves left homotopy Kan extensions.

Proof. It is sufficient to check that it preserves homotopy colimits; see [**Cis08b**, proposition 2·6]. This reduces to check that locally constant functors are stable by homotopy colimits in the model category of simplicial presheaves on a small category, which is obvious.

PROPOSITION 3·10. *For any derivator $\mathbb{D}$ and any small category A, the inclusion functor*

$$\mathbf{LC}(A, \mathbb{D}) \longrightarrow \mathcal{H}om(\underline{A}, \mathbb{D})$$

has a left adjoint and a right adjoint.

Proof. We have a localization morphism

$$\gamma: \mathbb{H}\mathbb{o}([A^{op}, sSet]) \longrightarrow \mathbb{H}\mathbb{o}(L_A[A^{op}, sSet])$$

which has a right adjoint in the 2-category of prederivators

$$i\colon \mathbb{H}\mathrm{o}(L_A[A^{op}, sSet]) \longrightarrow \mathbb{H}\mathrm{o}([A^{op}, sSet])\,.$$

We know that γ is cocontinuous (as it comes from a left Quillen functor; see [**Cis03b**, proposition 6·2]). The previous lemma asserts that i is cocontinuous as well. It thus follows from the fact $\mathcal{H}om_!(-, \mathbb{D})$ is 2-functor and from Theorem 3·6 that the inclusion functor

$$\mathbf{LC}(A, \mathbb{D}) \longrightarrow \mathcal{H}om(\underline{A}, \mathbb{D})$$

(which is induced by γ) has a left adjoint (which is induced by i). Applying this to the opposite derivator $\mathbb{D}^{op}$ (and replacing A by A^{op}) also gives a right adjoint.

COROLLARY 3·11. *Let $u\colon A \longrightarrow B$ be a functor between small categories. For any derivator $\mathbb{D}$, the inverse image functor*

$$u^*\colon \mathbf{LC}(B, \mathbb{D}) \longrightarrow \mathbf{LC}(A, \mathbb{D})$$

has a left adjoint and a right adjoint.

3·12. It is possible to construct a prederivator $\mathbb{L}\mathbb{C}\,(A, \mathbb{D})$ such that

$$\mathbb{L}\mathbb{C}\,(A, \mathbb{D})(e) = \mathbf{LC}(A, \mathbb{D}) \tag{3·12·1}$$

(where e still denotes the terminal category). If $\mathbb{D}$ is a derivator, and A is a small category, then we define a derivator $\mathbb{D}^A$ by the formula

$$\mathbb{D}^A(I) = \mathbb{D}(A^{op} \times I)\,. \tag{3·12·2}$$

It is easy to see that $\mathbb{D}^A$ is again a derivator. Moreover, the homotopy colimits in $\mathbb{D}^A$ can be computed termwise; see [**Cis08b**, proposition 2·8]. In the case where $\mathbb{D} = \mathbb{H}\mathrm{o}(\mathcal{V})$ for a model category $\mathcal{V}$, we get the formula

$$\mathbb{H}\mathrm{o}(\mathcal{V})^A(I) = \mathbf{Ho}([A \times I^{op}, \mathcal{V}])\,. \tag{3·12·3}$$

The prederivator $\mathbb{L}\mathbb{C}\,(A, \mathbb{D})$ is the full subprederivator of $\mathbb{D}^A$ defined by the formula

$$\mathbb{L}\mathbb{C}\,(A, \mathbb{D})(I) = \mathbf{LC}(A, \mathbb{D}^{I^{op}})\,. \tag{3·12·4}$$

In other words, $\mathbb{L}\mathbb{C}\,(A, \mathbb{D})(I)$ is the full subcategory of $\mathbb{D}(A^{op} \times I)$ whose objects are the objects F of $\mathbb{D}(A^{op} \times I)$ such that the induced functor

$$\mathsf{dia}(F)\colon A \longrightarrow [I^{op}, \mathbb{D}(e)]$$

sends any morphism of A to isomorphisms.

THEOREM 3·13. *For any small category A, and any derivator $\mathbb{D}$, the prederivator $\mathbb{L}\mathbb{C}\,(A, \mathbb{D})$ is a derivator, and the full inclusion*

$$\mathbb{L}\mathbb{C}\,(A, \mathbb{D}) \longrightarrow \mathbb{D}^A$$

has a left adjoint and a right adjoint.

Proof. The proof will follow essentially the same lines as the proof of Proposition 3·10.

Recall that there is an internal Hom for prederivators: if $\mathbb{D}$ and $\mathbb{D}'$ are prederivators, we define a prederivator $\mathbb{H}\mathrm{om}\,(\mathbb{D}, \mathbb{D}')$ by the formula

$$\mathbb{H}\mathrm{om}\,(\mathbb{D}, \mathbb{D}')(I) = \mathcal{H}om(\mathbb{D}, \mathbb{D}'^{I^{op}})$$

for any small category I; see [**Cis08b**, corollary 5·3]. If moreover $\mathbb{D}$ and $\mathbb{D}'$ are derivators, we define a prederivator $\mathbb{H}\mathrm{om}_!(\mathbb{D}, \mathbb{D}')$ as a full subprederivator of $\mathbb{H}\mathrm{om}(\mathbb{D}, \mathbb{D}')$ as follows: for each small category I, we put

$$\mathbb{H}\mathrm{om}_!(\mathbb{D}, \mathbb{D}')(I) = \mathcal{H}om_!(\mathbb{D}, \mathbb{D}'^{I^{op}})\,.$$

Then $\mathbb{H}\mathrm{om}_!(\mathbb{D}, \mathbb{D}')$ is again a derivator; see [**Cis08b**, proposition 5·8]. Theorem 3·4 gives the following result. If A is a small category, then for any derivator $\mathbb{D}$, the Yoneda map $h\colon \underline{A} \longrightarrow \mathbb{H}\mathrm{o}([A^{op}, sSet])$ induces an equivalence of derivators

$$\mathbb{H}\mathrm{om}_!(\mathbb{H}\mathrm{o}([A^{op}, sSet]), \mathbb{D}) \simeq \mathbb{H}\mathrm{om}(\underline{A}, \mathbb{D}) = \mathbb{D}^A\,.$$

Similarly, Theorem 3·6 implies that the Yoneda map $h\colon \underline{A} \longrightarrow \mathbb{H}\mathrm{o}(L_A[A^{op}, sSet])$ induces an equivalence of derivators

$$\mathbb{H}\mathrm{om}_!(\mathbb{H}\mathrm{o}(L_A[A^{op}, sSet]), \mathbb{D}) \simeq \mathbb{L}\mathbb{C}(A, \mathbb{D})\,.$$

Thanks to Lemma 3·9 and to the fact that $\mathbb{H}\mathrm{om}_!(-, \mathbb{D})$ is a 2-functor, the adjunction

$$\gamma\colon \mathbb{H}\mathrm{o}([A^{op}, sSet]) \rightleftarrows \mathbb{H}\mathrm{o}(L_A[A^{op}, sSet])\colon i$$

thus induces an adjunction

$$i^*\colon \mathbb{H}\mathrm{om}_!(\mathbb{H}\mathrm{o}([A^{op}, sSet]), \mathbb{D}) \rightleftarrows \mathbb{H}\mathrm{om}_!(\mathbb{H}\mathrm{o}(L_A[A^{op}, sSet]), \mathbb{D})\colon \gamma^*\,.$$

In particular, we see that $\mathbb{L}\mathbb{C}(A, \mathbb{D})$ is a derivator (as it is equivalent to the derivator $\mathbb{H}\mathrm{om}_!(\mathbb{H}\mathrm{o}(L_A[A^{op}, sSet]), \mathbb{D})$), and we get an adjunction of derivators

$$\mathbb{D}^A \rightleftarrows \mathbb{L}\mathbb{C}(A, \mathbb{D})\,.$$

Applying this to the opposite derivator $\mathbb{D}^{op}$ gives the other adjoint.

Remark 3·14. The preceding result can be interpreted as follows in terms of model categories. Consider a small category A and a complete and cocomplete model category $\mathcal{V}$. Then $\mathbb{H}\mathrm{o}(\mathcal{V})$ is a derivator, so that $\mathbb{L}\mathbb{C}(A, \mathbb{H}\mathrm{o}(\mathcal{V}))$ is a derivator as well. Denote by $LC(A, \mathcal{V})$ the full subcategory of $[A, \mathcal{V}]$ whose objects are the locally constant functors. One can then verify that the prederivator associated to the category $LC(A, \mathcal{V})$ (by inverting the termwise weak equivalences) is canonically equivalent to $\mathbb{L}\mathbb{C}(A, \mathbb{H}\mathrm{o}(\mathcal{V}))$; this can be expressed by the formula

$$\mathbb{H}\mathrm{o}(LC(A, \mathcal{V})) \simeq \mathbb{L}\mathbb{C}(A, \mathbb{H}\mathrm{o}(\mathcal{V})).$$

This means that the left Bousfield localizations discussed in Subsection 2·1 for combinatorial model categories always exist in the setting of derivators. Theorem 3·13 implies that such Bousfield localizations actually exist in the setting of ABC cofibration categories developped in [**RB06**].

4. *Galois correspondence and homotopy distributors*

4·1. Let A and B be small categories. We get from Theorem 3·6 the following canonical equivalence of categories

$$\begin{aligned}\mathcal{H}om_!(\mathbb{H}\mathrm{o}(L_B[B^{op}, sSet]), \mathbb{H}\mathrm{o}(L_A[A^{op}, sSet])) &\simeq \mathcal{H}om_B(\underline{B}, \mathbb{H}\mathrm{o}(L_A[A^{op}, sSet])) \\ &\simeq \mathbf{Ho}(L_{A\times B^{op}}[A^{op}\times B, sSet])\,.\end{aligned} \tag{4·1·1}$$

Moreover, we have an equivalence of categories

$$\mathbf{Ho}(L_{A\times B^{op}}[A^{op}\times B, sSet]) \simeq \mathbf{Ho}(Cat/A\times B) \qquad (4{\cdot}1{\cdot}2)$$

where $\mathbf{Ho}(Cat/A\times B)$ denotes the localization of the category of small categories over $A\times B$ by the class of functors (over $A\times B$) whose nerve are simplicial weak equivalences; this follows for example from [**Cis06**, corollaries 4·4·20 and 6·4·27] and from the fact B and B^{op} have the same homotopy type.

The induced equivalence of categories

$$S\colon \mathbf{Ho}(Cat/A\times B) \longrightarrow \mathcal{H}om_!(\mathbb{H}\mathrm{o}(L_A[A^{op}, sSet]), \mathbb{H}\mathrm{o}(L_B[B^{op}, sSet])) \qquad (4{\cdot}1{\cdot}3)$$

can be described very explicitly: its composition with the localization functor from $Cat/A\times B$ to $\mathbf{Ho}(Cat/A\times B)$ is the functor

$$s\colon Cat/A\times B \longrightarrow \mathcal{H}om_!(\mathbb{H}\mathrm{o}(L_A[A^{op}, sSet]), \mathbb{H}\mathrm{o}(L_B[B^{op}, sSet])) \qquad (4{\cdot}1{\cdot}4)$$

which can be described as follows. Consider a functor $C \longrightarrow A\times B$. It is determined by a pair of functors $p\colon C\longrightarrow A$ and $q\colon C\longrightarrow B$. The functor q induces an inverse image morphism

$$q^*\colon L_B[B^{op}, sSet] \longrightarrow L_C[C^{op}, sSet] \qquad (4{\cdot}1{\cdot}5)$$

which happens to be a right Quillen functor for the projective local model structures; see Proposition 2·6. It thus defines a continuous morphism of derivators (see [**Cis03b**, proposition 6·12])

$$\mathbf{R}q^*\colon \mathbb{H}\mathrm{o}(L_B[B^{op}, sSet]) \longrightarrow \mathbb{H}\mathrm{o}(L_C[C^{op}, sSet])\ . \qquad (4{\cdot}1{\cdot}6)$$

Using the equivalences of type $\mathbb{H}\mathrm{o}(L_B[B^{op}, sSet]) \simeq \mathbb{L}\mathbb{C}(B^{op}, \mathbb{H}\mathrm{o}(sSet))$, we see that $\mathbf{R}q^*$ corresponds to the restriction to $\mathbb{L}\mathbb{C}(B^{op}, \mathbb{H}\mathrm{o}(sSet))$ and $\mathbb{L}\mathbb{C}(C^{op}, \mathbb{H}\mathrm{o}(sSet))$ of the inverse image map $q^*\colon \mathbb{D}^{B^{op}} \longrightarrow \mathbb{D}^{C^{op}}$ for $\mathbb{D} = \mathbb{H}\mathrm{o}(sSet)$ (which is cocontinuous, by virtue of [**Cis08b**, proposition 2·8]). We thus conclude from Lemma 3·9 that (4·1·6) is also cocontinuous. The functor p induces a left Quillen functor for the projective local model structures (by Proposition 2·6 again)

$$p_!\colon L_C[C^{op}, sSet] \longrightarrow L_A[A^{op}, sSet]\ . \qquad (4{\cdot}1{\cdot}7)$$

This defines a cocontinuous morphism of derivators (by the dual of [**Cis03b**, proposition 6·12])

$$\mathbf{L}p_!\colon \mathbb{H}\mathrm{o}(L_C[C^{op}, sSet]) \longrightarrow \mathbb{H}\mathrm{o}(L_A[A^{op}, sSet])\ . \qquad (4{\cdot}1{\cdot}8)$$

The functor (4·1·4) is simply defined by sending the pair (p, q) to the composition of (4·1·6) and (4·1·8).

$$s(p, q) = \mathbf{L}p_!\mathbf{R}q^*\colon \mathbb{H}\mathrm{o}(L_B[B^{op}, sSet]) \longrightarrow \mathbb{H}\mathrm{o}(L_A[A^{op}, sSet])\ . \qquad (4{\cdot}1{\cdot}9)$$

PROPOSITION 4·2. *Given a functor* $(p, q)\colon C \longrightarrow A\times B$*, the following conditions are equivalent*;

(a) *the morphism* (4·1·9) *is continuous (i.e. preserves homotopy limits)*;

(b) *the morphism* (4·1·8) *is continuous*;

(c) *the functor* $\mathbf{L}p_!\mathbf{R}q^*\colon \mathbf{Ho}(L_B[B^{op}, sSet]) \longrightarrow \mathbf{Ho}(L_A[A^{op}, sSet])$ *preserves terminal objects*;

(d) *the functor* $\mathbf{L}p_!\colon \mathbf{Ho}(L_C[C^{op}, sSet]) \longrightarrow \mathbf{Ho}(L_A[A^{op}, sSet])$ *preserves terminal objects*;

(e) *the morphism* (4·1·8) *is an equivalence of derivators*;
(f) *the functor* $\mathbf{L}p_!\colon\ \mathbf{Ho}(L_C[C^{op}, sSet]) \longrightarrow \mathbf{Ho}(L_A[A^{op}, sSet])$ *is an equivalence of categories*;
(g) *the nerve of p is a simplicial weak equivalence.*

Proof. The functor (4·1·7) is a left Quillen equivalence (for the projective local model structures) if and only if for any small category I, the induced functor

$$p_!\colon [I, L_C[C^{op}, sSet]] \longrightarrow [I, L_A[A^{op}, sSet]]$$

is a left Quillen equivalence. This proves that the conditions (e) and (f) are equivalent. It is obvious that condition (e) implies condition (b). The fact that condition (g) implies condition (f) can be obtained, for example, using Theorem 1·3. It is clear that condition (b) implies conditions (a) and (d), and that conditions (a) or (d) implies condition (c). To finish the proof, we will show that the condition (c) implies (g).

Under the equivalences of type $\mathbf{Ho}(L_X[X^{op}, sSet]) \simeq \mathbf{Ho}(Cat/X)$, the functor $\mathbf{L}p_!$ corresponds to the functor from $\mathbf{Ho}(Cat/A)$ to $\mathbf{Ho}(Cat/A)$ which is induced by composition with p. Similarly, the functor $\mathbf{R}q^*$ corresponds to the functor from $\mathbf{Ho}(Cat/B)$ to $\mathbf{Ho}(Cat/C)$ which sends a functor $X \longrightarrow B$ to the projection $X \times^h_B C \longrightarrow C$ (where $X \times^h_B C$ denotes the homotopy fiber product of X and C over B). These descriptions show immediately that the condition (c) implies (g). This completes the proof.

4·3. We refer to [**Mal05a**, **Mal05b**, **Cis06**] for the notion of smooth functor and of proper functor (with respect to the minimal basic localizer). The first reason we are interested by this notion is that these functors have very good properties with respect to homotopy Kan extensions; see [**Mal05a**, section 3·2]. The second reason of our interest for this class of functors is the following statement.

PROPOSITION 4·4. *The category of small categories is endowed with a structure of category of fibrant objects in the sense of Brown* [**Bro73**], *for which the weak equivalences are the functors whose nerve is a simplicial weak equivalence, and the fibrations are the smooth and proper functors. Moreover, the factorizations into a weak equivalence followed by a fibration can be made functorially.*

Proof. Any functor to the terminal category is smooth and proper (so that any small category will be fibrant). Functors which are smooth and proper are stable under base change and composition (see [**Mal05a**, corollary 3·2·4 and proposition 3·2·10]). It follows from [**Cis06**, corollaries 6·4·8 and 6·4·18] and from [**Mal05a**, proposition 3·2·6] that the class of trivial fibrations (i.e. of smooth and proper functors which are weak equivalences) is stable by pullbacks. By virtue of [**Cis06**, proposition 6·4·14], the pullback of a weak equivalence by a smooth and proper functor is a weak equivalence. To finish the proof, it is sufficient to prove that any functor can factor (functorially) through a weak equivalence followed by a smooth and proper functor, which is a consequence of [**Cis06**, theorem 5·3·14].

COROLLARY 4·5. *The localization of the full subcategory of $Cat/A \times B$ whose objects are the functors $(p, q)\colon C \longrightarrow A \times B$ such that p and q are smooth and proper by the class of weak equivalences is canonicaly equivalent to* $\mathbf{Ho}(Cat/A \times B)$.

4·6. The simplicial localization $L(Cat)$ of Cat by the class of weak equivalences can be described using the structure of category of fibrant objects given by Proposition 4·4. In particular, the simplicial set $\mathrm{Hom}_{L(Cat)}(A, B)$ can be described as the nerve of the category

$Map(A, B)$, which is defined as the full subcategory of $Cat/A \times B$ whose objects are the functors $(p, q)\colon C \longrightarrow A \times B$ such that p is a trivial fibration (i.e. a functor which is smooth, proper, and a weak equivalence); see [**DK80**]. It is easy to see from Proposition 4·4 that the fundamental groupoid of $Map(A, B)$ is equivalent to the full subcategory of $\mathbf{Ho}(Cat/A \times B)$ whose objects are the functors $(p, q)\colon C \longrightarrow A \times B$ such that p is a weak equivalence. In other words, Proposition 4·2 can now be reformulated as follows.

COROLLARY 4·7 (Galois reconstruction theorem). *The groupoid $\pi_1(Map(A, B))$ is canonically equivalent to the category of cocontinuous morphisms of derivators which preserve finite homotopy limits from $\mathbb{Ho}(L_B[B^{op}, sSet])$ to $\mathbb{Ho}(L_A[A^{op}, sSet])$.*

4·8. Let us explain why the preceding corollary can be interpreted as a Galois reconstruction theorem. Given a small category A, if we think of $\mathbb{Ho}(L_A[A^{op}, sSet])$ as the "topos of representations of the ∞-groupoid associated to A", it is natural to define the functor of points of $\mathbb{Ho}(L_A[A^{op}, sSet])$ by

$$B \longmapsto \mathcal{H}om_!^{ex}(\mathbb{Ho}(L_B[B^{op}, sSet]), \mathbb{Ho}(L_A[A^{op}, sSet]))$$

(where $\mathcal{H}om_!^{ex}(\mathbb{Ho}(L_B[B^{op}, sSet]), \mathbb{Ho}(L_A[A^{op}, sSet]))$ denotes the category of cocontinuous morphisms of derivators which preserve finite homotopy limits). This is a 2-functor from $\tau^{\leqslant 2}L(Cat)$ to the category of groupoids which is corepresentable precisely by A. This can be reformulated by saying that we can reconstruct the homotopy type of A from the "topos" $\mathbb{Ho}(L_A[A^{op}, sSet])$. This is the derivator version of Toën's homotopy Galois theory [**Toë02**].

4·9. Corollary 4·5 can also be used to understand the compatibilities of the equivalences of categories of type (4·1·3) with composition of morphisms of derivators. More precisely, we have a bicategory $\mathbf{Ho}(\mathcal{D}ist)$, whose objects are the small categories, and whose category of morphisms from A to B is the homotopy category $\mathbf{Ho}(Cat/A \times B)$ (composition is defined by homotopy fiber products). We will finish this section by explaining how Corollary 4·5 implies that the functors (4·1·3) define a bifunctor from $\mathbf{Ho}(\mathcal{D}ist)$ to the 2-category of derivators. Define a bicategory SP as follows. The obects of SP are the small categories. Given two small categories A and B, the category of morphisms $SP(A, B)$ is the full subcategory of $Cat/A \times B$ whose objects are the functors $(p, q)\colon C \longrightarrow A \times B$ such that p and q are smooth and proper. The composition law of SP is defined by fiber products (which is meaningful, as the smooth and proper functors are stable by pullbacks and compositions).

We denote by $\mathcal{D}er_!$ the 2-category whose objects are the derivators, and whose morphisms are the cocontinuous morphims (2-cells are just 2-cells in the 2-category of prederivators).

LEMMA 4·10. *The functors* (4·1·4) *define a bifunctor*

$$s\colon SP^{op} \longrightarrow \mathcal{D}er_!.$$

Proof. Consider a commutative diagram

$$\begin{tikzcd}
& & G \arrow[dl, "t"'] \arrow[dr, "u"] & & \\
& E \arrow[dl, "p"'] \arrow[dr, "q"] & & F \arrow[dl, "r"'] \arrow[dr, "s"] & \\
A & & B & & C
\end{tikzcd}$$

in which the square is a pullback, and all the maps are smooth and proper. Note that for any smooth and proper map φ, both φ and φ^{op} are locally constant; see [**Cis06**, corollary 6·4·8]. By virtue of Propositions 1·24 and 2·8, we can apply [**Mal05a**, proposition 3·2·28] to get that the base change map

$$\mathbf{L}u_!\mathbf{R}t^* \longrightarrow \mathbf{R}r^*\mathbf{L}q_!$$

is an isomorphism in $\mathcal{H}om_!(\mathbb{H}\mathbb{o}(L_E[E^{op}, sSet]), \mathbb{H}\mathbb{o}(L_F[F^{op}, sSet]))$. In particular, we get a canonical isomorphism

$$\mathbf{L}s_!\mathbf{L}u_!\mathbf{R}t^*\mathbf{R}p^* \simeq \mathbf{L}s_!\mathbf{R}r^*\mathbf{L}q_!\mathbf{R}p^*\,.$$

These isomorphisms, together with the functors (4·1·4) define a bifunctor: to check the coherences, we are reduce to check that commutative squares with the Beck–Chevalley property are stable by compositions, which is well known to hold.

4·11. We define now a bicategory $\mathbf{Ho}(SP)$ as follows. The objects are the small categories, and given two objects A and B, the category of morphisms from A to B is $\mathbf{Ho}(SP(A, B))$, that is the localisation of $SP(A, B)$ by the class of weak equivalences. We have a localization bifunctor

$$\gamma\colon SP \longrightarrow \mathbf{Ho}(SP)\,. \qquad (4·11·1)$$

Corollary 4·5 can now be reformulated: the canonical bifunctor

$$j\colon \mathbf{Ho}(SP) \longrightarrow \mathbf{Ho}(\mathcal{D}ist) \qquad (4·11·2)$$

is a biequivalence.

PROPOSITION 4·12. *The equivalences of categories* (4·1·3) *define a bifunctor*

$$S\colon \mathbf{Ho}(\mathcal{D}ist)^{op} \longrightarrow \mathcal{D}er_!\,.$$

Proof. For any small categories A, B and C, we have

$$\mathbf{Ho}(SP(A, B) \times SP(A, B)) = \mathbf{Ho}(SP(A, B)) \times \mathbf{Ho}(SP(B, C))\,.$$

The universal property of localizations and Lemma 4·10 imply that we get a bifunctor

$$S'\colon \mathbf{Ho}(SP)^{op} \longrightarrow \mathcal{D}er_!\,.$$

We deduce from this and from the biequivalence (4·11·2) that there is a unique way to define a bifunctor S from $\mathbf{Ho}(\mathcal{D}ist)^{op}$ to $\mathcal{D}er_!$ from the equivalences of categories (4·1·3) such that $S'j = S$.

Scholium 4·13. Theorem 3·6 asserts that for any derivator $\mathbb{D}$, we have an equivalence of derivators

$$\mathbb{H}\mathbb{o}\mathbb{m}_!(\mathbb{H}\mathbb{o}(L_A[A^{op}, sSet]), \mathbb{D}) \simeq \mathbb{L}\mathbb{C}(A, \mathbb{D})\,.$$

As $\mathbb{H}\mathbb{o}\mathbb{m}_!(-, \mathbb{D})$ is a 2-functor, we deduce from the preceding proposition that we get a bifunctor

$$\mathbb{L}\mathbb{C}(-, \mathbb{D})\colon \mathbf{Ho}(\mathcal{D}ist) \longrightarrow \mathcal{D}er_!,$$

which sends a small category A to $\mathbb{L}\mathbb{C}(A, \mathbb{D})$. This defines a bifunctor

$$\mathbb{L}\mathbb{C}(-, -)\colon \mathbf{Ho}(\mathcal{D}ist) \times \mathcal{D}er_! \longrightarrow \mathcal{D}er_!,$$

which defines an enrichment of $\mathcal{D}er_!$ in homotopy distributors.

REFERENCES

[**Bar07**] C. BARWICK. On (enriched) left Bousfield localization of model categories, arXiv:0708.2067, (2007).

[**Bek00**] T. BEKE. Sheafifiable homotopy model categories. *Math. Proc. Camb. Phil. Soc.* **129** (2000), 447–475.

[**Bro73**] K. S. BROWN. Abstract homotopy and generalized sheaf cohomology. *Trans. Amer. Math. Soc.* **186** (1973), 419–458.

[**Cis03a**] D.-C. CISINSKI. Faisceaux localement asphériques, preprint. (2003). Available at the url address http://www.math.univ-paris13.fr/~cisinski/

[**Cis03b**] D.-C. CISINSKI. Images directes cohomologiques dans les catégories de modèles. *Ann. Math. Blaise Pascal* **10** (2003), 195–244.

[**Cis06**] D.-C. CISINSKI. Les préfaisceaux comme modèles des types d'homotopie. *Astérisque* vol. 308, (2006).

[**Cis08b**] D.-C. CISINSKI. Propriétés universelles et extensions de Kan dérivées. *Theory and Appl. Categ.* **20** (2008), no. 17, 605–649.

[**DHI04**] D. DUGGER, S. HOLLANDER and D. ISAKSEN. Hypercovers and simplicial presheaves. *Math. Proc. Camb. Phil. Soc.* **136** (2004), no. 1, 9–51.

[**DK80**] W. G. DWYER and D. M. KAN. Calculating simplicial localizations. *J. Pure Appl. Algebra* **18** (1980), no. 1, 17–35.

[**Dug01**] D. DUGGER. Combinatorial model categories have presentations. *Adv. Math.* **164** (2001), no. 1, 177–201.

[**GZ67**] P. GABRIEL and M. ZISMAN. Calculus of fractions and homotopy theory. *Ergeb. Math. Grenzgeb.* vol. 35 (Springer-Verlag, 1967).

[**Hir03**] P S. HIRSCHHORN. Model categories and their localizations. *Math. Surveys Monog.* vol. 99 (Amer. Math. Soc., 2003).

[**Hov99**] M. HOVEY. Model categories. *Math. Surveys Monog.* vol. 63 (Amer. Math. Soc., 1999).

[**JT07**] A. JOYAL and M. TIERNEY. Quasi-categories vs Segal spaces. Categories in Algebra, Geometry and Physics. *Contemp. Math.* vol. 431 (Amer. Math. Soc., 2007), p. 277–326. Preprint version available as arXiv:math/0607820.

[**Mal01**] G. MALTSINIOTIS. Introduction à la théorie des dérivateurs (d'après Grothendieck). preprint, (2001).

[**Mal05a**] G. MALTSINIOTIS. La théorie de l'homotopie de Grothendieck. *Astérisque* vol. 301 (2005).

[**Mal05b**] G. MALTSINIOTIS. Structures d'asphéricité. *Ann. Math. Blaise Pascal* **12** (2005), no. 1, 1–39.

[**Qui67**] D. QUILLEN. Homotopical algebra. Lecture Notes in Mathematics, vol. 43 (Springer-Verlag, 1967).

[**Qui73**] D. QUILLEN. Higher algebraic K-theory. in Higher K-theories I. Lecture Notes in Mathematics, vol. 341 (Springer-Verlag, 1973), p. 85–147.

[**RB06**] A. RĂDULESCU-BANU. Cofibrations in homotopy theory. arXiv:math/0610009, (2006).

[**Shu08**] M. SHULMAN. Parametrized spaces model locally constant homotopy sheaves. *Topology Appl.* **155** (2008), no. 5, 412–432.

[**Tab08**] G. TABUADA. Higher K-theory via universal invariants. *Duke Math. J.* **145** (2008), no. 1, 121–206.

[**Toë02**] B. TOËN. Vers une interprétation galoisienne de la théorie de l'homotopie. *Cahiers Topologie Geom. Différentielle Categ.* **XLIII-3** (2002), 257–312.