Volterra Bridges and Applications

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Abstract

We introduce and study the bridges over a given Volterra process. As an application, we show that if a Gaussian process admits in law a Volterra type representation, then this representation is never unique.

1 Introduction

Fractional Brownian motion, the Wiener Spiral of [11], was introduced the first time by Kolmogorov in 1940 . It was then set in a wider framework by Yaglom (1958), [18], and Molchan et al. (1969), [15]. In 1969 Mandelbrot et al. [13] emphasized its relevance to the modelling of natural phenomena.

Although, this is not a semimartingale nor a Markov process, Decreusefond and Üstünel, [6] have defined a stochastic integral with respect to it. Their approach is based on the fact that fractional Brownian motion may be represented by the integral of a Volterra deterministic kernel with respect to a Brownian motion. Then, Alos-Mazet and Nualart, in [2], extend this stochastic integral with respect to more general Volterra Gaussian processes.
The main subject of this paper is to define and to study the bridges over a given Gaussian Volterra process. As for the Brownian bridge, it is simpler to define it (in law) in a filtration which is greater than the filtration of the starting Volterra Gaussian process. Using these bridges, we can obtain, in the spirit of [5], [9], and [12], initial enlargement formulas of the natural filtration of a Volterra process.

A by product is to give some non canonical representations of a Gaussian Volterra processes. The study of canonical representation for a Gaussian process has been started by

Lévy, [10]. More recently, Hibino et al., in [7], present a systematic method for the construction of non canonical representations of a Brownian motion and Gaussian Volterra processes. Using these bridges, we recover very quickly a non canonical representation of a Volterra process given in [7].

The first part of this paper is devoted to the definition of a bridge over a given Volterra process. Then in the second part, we obtain the decomposition of the bridges in their own filtration, and in the third one, the representation of the Volterra process in an initially enlarged filtration. In the last part, using ergodic theory, the bridges allow to show that there always exist non canonical representation of Volterra processes.

2 Construction and elementary properties of bridges over a given Volterra process

Let $T > 0$ be a fixed horizon time. Let K denote a Volterra kernel, that is K is an application

$$\begin{array}{ccc} [0,T] \times [0,T] & \rightarrow & \mathbb{R} \\ (t,s) & \rightarrow & K(t,s) \end{array}$$

such that:

1. $\int\int_{[0,T]^2} K(t,s)^2\, dtds < +\infty$;
2. For every $t \in [0,T]$, $\int_0^t K(t,s)ds < +\infty$;
3. $K(t,s) = 0$ for $s > t$.

This kernel induces a Hilbert-Schmidt operator $\mathcal{K}$ on $L^2([0,T], dx)$, in the following way:

$$\begin{aligned} \mathcal{K} : L^2([0,T], dx) &\rightarrow L^2([0,T]^2, dx), \\ \phi &\mapsto \left\{ t \mapsto \int_0^T K(t,s)\phi(s)ds \right\}. \end{aligned}$$

We shall assume in the sequel that the kernel K is non degenerate in the sense that the family

$$\{K(t,\cdot), t \in (0,T]\}$$

is free and generates a vector space which is dense in $L^2([0,T], dx)$. With K, we can associate the Volterra process

$$X_t = \int_0^t K(t,s)\, dB_s,\ 0 \leq t \leq T,$$

where $(B_t)_{0\leq t\leq T}$ is a standard one-dimensional Brownian motion. The covariance function of X shall be denoted R. Observe that the non-degeneracy assumption on K implies that

R is positive definite on $(0,T]$.

Indeed, if for some $(a_i)_{1\le i\le n} \in \mathbf{R}^n$ and $0 < t_1 < ... < t_n \le T$, we have $\sum_{i,j} a_i a_j R(t_i,t_j) = 0$, then it follows from the definition of R and X that the function $\sum_i a_i K(t_i,.)$ coincides with the null function in $L^2([0,T],\mathbf{R})$, wich implies that $a_i = 0$. An important another consequence of our assumptions the non-degeneracy of the kernels is the following equality

$$\mathcal{F}^B = \mathcal{F}^X,$$

where $\mathcal{F}^Z$ denotes the natural filtration of a process Z. Indeed, the non-degeneracy of the kernel implies the injectivity of the Hilbert-Schmidt operator $\mathcal{K}$, and the fact that $\mathcal{K}(L^2([0,T],dx)$ is dense in $L^2([0,T],dx)$.
As it is pointed out in the pioneering paper of Decreusefond-Üstünel, [6], a now celebrated example of Volterra process which satisfies the previous assumptions is the fractional Brownian motion with Hurst parameter $H \in (0,1]$. This is the Volterra process associated with the kernel (see [17]):

$$\begin{aligned} K(t,s) &= c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \; t > s, \quad \text{for} \quad H > \frac{1}{2}, \\ &= c_H s^{\frac{1}{2}-H} \left[\frac{(t-s)^{H-\frac{1}{2}} t^{H-\frac{1}{2}}}{H-\frac{1}{2}} - \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} du \right] \; t > s, \quad \text{for} \quad H < \frac{1}{2} \end{aligned}$$

where $c_H > 0$ is the constant such that the covariance function is

$$R(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

Proposition 2.1 *Let $x \in \mathbb{R}$. The bridge over X from 0 to x with length T is the process*

$$P_t^x = X_t + \frac{R(t,T)}{R(T,T)}(x - X_T), \; 0 \le t \le T.$$

More precisely, the law of the process $(P_t^x)_{0\le t\le T}$ defined above is the same as the law of X conditionally to $X_T = x$.

Proof. It is enough to check two things. The first thing is that, almost surely, $P_T^x = x$ and the second thing is that the whole process $(P_t^x)_{0\le t\le T}$ is independent of X_T. The first thing stems directly from the definition of $(P_t^x)_{0\le t\le T}$ and the second thing stems from the fact that for any $t \in [0,T]$,

$$\mathbb{E}\left(P_t^x X_T\right) = R(t,T) + \frac{R(t,T)}{R(T,T)}\left(x\mathbb{E}\left(X_T\right) - R(T,T)\right) = 0.$$

□

Remark 2.2 *We have*

$$\mathbb{E}(P_t^x) = x\frac{R(t,T)}{R(T,T)},$$

and

$$\mathbb{E}\left(\left(P_t^x - x\frac{R(t,T)}{R(T,T)}\right)\left(P_s^x - x\frac{R(s,T)}{R(T,T)}\right)\right) = R(s,t) - \frac{R(s,T)R(t,T)}{R(T,T)}.$$

Moreover, the process P^x is not a Volterra process with respect to B since they do not have the same filtration.

Example 2.3 (Fractional Brownian bridge) *The fractional Brownian bridge from* 0 *to x with length T is therefore the process*

$$P_t^x = X_t + \frac{1}{2}\left(\left(\frac{t}{T}\right)^{2H} + 1 - \left(1 - \frac{t}{T}\right)^{2H}\right)(x - X_T),\ 0 \le t \le T,$$

where X is a fractional Brownian motion.

3 Decomposition of the bridges in their own filtration

Proposition 2 gives the construction of the bridges over any Volterra process (actually it also holds for any Gaussian process). Nevertheless this construction is not adapted to the natural filtration of the starting process X. So, it is a natural question to study the bridges over a Volterra process in their own filtrations.
Following [2], let us introduce some additional hypothesis on the kernel K. For some $\alpha > 0$,

(i) K is differentiable with respect to his first variable t in $\{0 < s < t < T\}$ and both K and $\frac{\partial K}{\partial t}$ are continuous in $\{0 < s < t < T\}$,

(ii) $|\frac{\partial K}{\partial t}(t,s)| \le c(t-s)^{-\alpha-1}$ in $\{0 < s < t < T\}$,

(iii) $\int_s^t K(t,u)^2 du \le c(t-s)^{1-2\alpha}$ in $\{0 < s < t < T\}$.

Conditions (ii) and (iii) imply that B has Hölder continuous path of order strictly less than $\frac{1}{2} - \alpha$.

Theorem 3.1 *Assume that all the above hypothesis on K are fulfilled.*
Let $x \in \mathbb{R}$ and let $(P_t^x)_{0 \le t \le T}$ denote the bridge from 0 to x with length T over X. The processes $(P_t^x)_{0 \le t < T}$ and $(X_t)_{0 \le t < T}$ are locally equivalent and there exists moreover a standard Brownian motion $(W_t)_{0 \le t < T}$ such that for any $t < T$:

1. $\sigma(W_s, s \le t) = \sigma(P_s^x, s \le t)$;

2.

$$P_t^x = \int_0^t K(t,s)dY_s,$$

where $(Y_t)_{0\le t<T}$ is the solution of the following stochastic differential equation

$$dY_t = K(T,t)\frac{x - \int_0^t K(T,s)dY_s}{\int_t^T K(T,s)^2 ds}dt + dW_t \tag{3.1}$$

associated with the initial condition $Y_0 = 0$.

Proof. Since

$$X_t = \int_0^t K(t,s)\, dB_s, \; 0 \le t \le T,$$

we have

$$(P_t^x)_{0\le t<T} =^{law} \left(\int_0^t K(t,s)d\beta_s\right)_{0\le t<T},$$

where $(\beta_t)_{0\le t<T}$ is any standard Brownian motion conditioned by

$$\int_0^T K(T,s)d\beta_s = x.$$

Now, from [3], we know that such a process can be realized in the natural filtration of B by solving the stochastic differential equation

$$Y_t = \int_0^t K(T,s)\frac{x - \int_0^s K(T,u)dY_u}{\int_s^T K(T,u)^2 du}ds + W_t, \; 0 \le t < T,$$

where $(W_t)_{0\le t<T}$ is a standard Brownian motion which has the same filtration as $(B_t)_{0\le t<T}$. Indeed, let us denote by $\tilde{T} = \int_0^T K(T,s)^2 ds$. Then, according to Theorem 25 of [3], a Brownian motion $(\tilde{B})_{0\le t\le T}$ conditioned by $\tilde{B}_{\tilde{T}} = x$ has the same law as the solution of

$$d\tilde{Z}_t = \frac{x - \tilde{Z}_t}{\tilde{T} - t}dt + d\tilde{\beta}_t,$$

starting from 0, where $\tilde{\beta}$ is a Brownian motion. Therefore, the process $(\int_0^t K(T,s)dB_s)_{0\le t\le T}$ conditioned by $\int_0^T K(T,s)dB_s = x$ has the same law as the solution of

$$dZ_t = \frac{x - Z_t}{\int_t^T K(T,s)^2 ds}K^2(T,t)dt + K(T,t)d\beta_t,$$

starting from 0, where β is a Brownian motion. Then using the first step of the proof with the kernel $(K(T,s)\mathbf{1}_{[0,t]})_{(s,t)\in[0,T]^2}$ instead of K, the process B conditioned by $\int_0^T K(T,s)dB_s = x$ has the same law as the solution of

$$dY_t = \frac{x - \int_0^t K(T,s)dY_s}{\int_t^T K(T,s)^2 ds}K(T,t)dt + d\beta_t,$$

starting from 0. $\square$

Remark 3.2 *By solving the stochastic differential equation (3.1) we find:*

$$Y_t = \int_0^t \frac{dZ_s}{K(T,s)},$$

where $(Z_t)_{0\leq t<T}$ *is the Gaussian semimartingale*

$$Z_t = x\frac{\int_0^t K(T,s)^2 ds}{\int_0^T K(T,s)^2 ds} + \int_t^T K(T,s)^2 ds \int_0^t \frac{K(T,s)}{\int_s^T K(T,u)^2 du} dW_s.$$

Remark 3.3 *Observe that the bridges over* X *are therefore solutions of a stochastic integro-differential equation that can be written*

$$P_t^x = \int_0^t \frac{K(t,s)K(T,s)}{\int_s^T K(T,u)^2 du}\left(x - \int_0^s \tilde{K}(s,u)dP_u^x\right) ds + \tilde{X}_t,$$

where $\tilde{X}$ *is a process which has the same law as* X *and where* $\tilde{K}$ *is a Volterra kernel (which does not depend on* x*). Moreover, under the assumption that the covariance function* $R(t,s)$ *is differentiable for* $t<s$*,* $\tilde{K}$ *solves the following (complicated!) equation*

$$\frac{\partial_t R(t,T)}{R(T,T)} = \int_0^t \frac{K(t,s)K(T,s)}{\int_s^T K(T,u)^2 du}\left(1 - \int_0^s \tilde{K}(s,u)\frac{\partial_u R(u,T)}{R(T,T)} du\right) ds.$$

Indeed, the kernel $\tilde{K}$ *is such that* $\mathbf{E}(X_T/\mathcal{F}_t) = \int_0^T \tilde{K}(T,t,s)dX_s$*. When* X *is a fractional Brownian motion, an explicit formula for* $\tilde{K}$ *is given in [16].*

4 Initial enlargement of the natural filtration of a Volterra process

After the study of the bridges over the Volterra process X, it is interesting to understand what becomes X in the initially enlarged filtration $\mathcal{F}^X \vee \sigma(X_T)$. For the classical results concerning the initial enlargement of filtration of semimartingales, we refer to [9].

Proposition 4.1 *In the initially enlarged filtration* $\mathcal{F}^X \vee \sigma(X_T)$*, we have*

$$X_t = \int_0^t K(t,s)\left(K(T,s)\frac{\int_s^T K(T,u)dB_u}{\int_s^T K(T,u)^2 du} ds + d\tilde{B}_s\right),\ 0 \leq t < T,$$

where $(\tilde{B}_t)_{0\leq t<T}$ *is a standard Brownian motion in the initially enlarged filtration.*

Proof. Indeed, we know from [5], that in the filtration

$$\mathcal{F}^X \vee \sigma(X_T) = \mathcal{F}^B \vee \sigma\left(\int_0^T K(T,s)dB_s\right),$$

the process $(B_t)_{0\le t<T}$ is a semimartingale whose decomposition reads

$$B_t = \int_0^t K(T,s)\frac{\int_s^T K(T,u)dB_u}{\int_s^T K(T,u)^2du}ds + d\tilde{B}_s,\ 0\le t<T,$$

where $(\tilde{B}_t)_{0\le t<T}$ is a standard Brownian motion in the initially enlarged filtration. □

Remark 4.2 *In the case $K=1$, that is the case of the Brownian motion, we recover the classical enlargement formula*

$$dX_t = \frac{X_T - X_t}{T-t}dt + d\tilde{B}_t.$$

5 Application: Non canonical representations of Volterra processes

We recover the non canonical representation of Volterra processes given in Proposition 2.1 in the case $T=1$, $N=1$ and $g_1(t)=K(1,t)$.

Proposition 5.1 *Let $T>0$, and K be a Volterra kernel which satisfies the assumptions of section 2 and 3. If $(B_t)_{0\le t\le T}$ is a standard Brownian motion, then*

$$\left(\int_0^t K(t,s)dB_s\right)_{0\le t\le T} =^{law} \left(\int_0^t \tilde{K}(t,s)dB_s\right)_{0\le t\le T}$$

where,

$$\tilde{K}(t,s) = K(t,s) - K(T,s)\int_s^t \frac{K(t,v)K(T,v)}{\int_0^v K(T,u)^2du}dv.$$

Proof. We could, of course, do a direct computation by showing that the two processes have the same covariance function. Nevertheless, we would like to give a more elegant proof of this fact which uses the bridges introduced in this work. Moreover, this proof shall give us much more than the equality in law of the two processes. As before, we denote $(X_t)_{0\le t\le T}$ the Volterra process

$$X_t = \int_0^t K(t,s)\,dB_s,\ 0\le t\le T.$$

There is a natural action on the bridges over $(X_t)_{0\le t\le T}$ of a one-parameter group of transformations $(\mathbf{T}_\mu)_{\mu\in\mathbb{R}}$ defined on the space $C([0,T],\mathbb{R})$. Indeed, let us consider for $\mu\in\mathbb{R}$, the transformed process

$$\mathbf{T}_\mu(X)_t = X_t + \mu R(t,T),\ 0\le t\le T.$$

Then, the laws of the processes $(X_t)_{0\le t\le T}$ and $(\mathbf{T}_\mu(X)_t)_{0\le t\le T}$ are equivalent. Indeed, we have

$$\mathbf{T}_\mu(X)_t = \int_0^t K(t,s)\left(dB_s + \mu K(T,s)ds\right).$$

So, if $\mathbb{P}$ denotes the probability measure under which the process $\left(B_t + \mu \int_0^t K(T,s)ds\right)_{0\le t\le T}$ is a standard Brownian motion, then under $\mathbb{P}$ the process $(\mathbf{T}_\mu(X)_t)_{0\le t\le T}$ has the same law as X under the probability for which $(B_t)_{0\le t\le T}$ is a standard Brownian motion.
It is moreover easily seen that for any $x \in \mathbb{R}$,

$$(\mathbf{T}_\mu(P^x)_t)_{0\le t\le T} =^{law} \left(P_t^{x+\mu R(T,T)}\right)_{0\le t\le T}.$$

Let us now consider the process

$$S(X)_t = X_t - \int_0^t \frac{K(t,s)K(T,s)}{\int_0^s K(T,v)^2 dv} \int_0^s K(T,v)dB_v ds = \int_0^t \tilde{K}(t,s)dB_s,\ 0 \le t \le T.$$

Since, for every $\mu \in \mathbb{R}$,

$$S \circ \mathbf{T}_\mu = S,$$

and since $\mathbf{T}_\mu$ is ergodic, the process $S(X)$ is independent of X_T and has the same law as the process X (see [4]). $\square$

Example 5.2 *In the case $K = 1$, that is the case of the Brownian motion, then we have*

$$\tilde{K}(t,s) = 1 + \ln \frac{s}{t}.$$

We recognize the well-known non canonical representation of the Brownian motion which is due to Jeulin-Yor (see e.g. [14], [19]).

The previous proof has moreover shown that the natural filtration of

$$\left(\int_0^t \tilde{K}(t,s)dB_s\right)_{0\le t\le T}$$

is strictly included in the natural filtration of $(B_t)_{0\le t\le T}$. Actually, it is easily checked that the kernel $\tilde{K}$ is degenerate in the sense that the family

$$\{\tilde{K}(t,\cdot), t \in (0,T]\}$$

does not generates a vector space which is dense in $L^2([0,T], dx)$. It is indeed orthogonal to the function

$$s \to K(T,s).$$

Observe that if we have the equality in law

$$\left(\int_0^t K(t,s)dB_s\right)_{0\le t\le T} =^{law} \left(\int_0^t \tilde{K}(t,s)dB_s\right)_{0\le t\le T}$$

where the two kernels K and $\tilde{K}$ are non-degenerate, then we necessarily have $K(t,s) = \varepsilon_s \tilde{K}(t,s)$ with $\varepsilon \in \{-1,1\}$. Indeed, the previous equality, implies, due to the non degeneracy of the kernels the equality

$$\left(\int_0^t K(t,s)d\beta_s\right)_{0\leq t\leq T} = \left(\int_0^t \tilde{K}(t,s)dB_s\right)_{0\leq t\leq T}$$

where β is a standard Brownian motion which has the same filtration as B.

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References

[1] L. Alili, Canonical decomposition of certain generalized Brownian bridges, Electronic Communications in Probability, Volume 7, pp. 27-35, (2002).

[2] E. Alòs, O. Mazet and D. Nualart, Stochastic calculus with respect to Gaussian process, The Annals of Probability 29 (2), 2001, 766-801.

[3] F. Baudoin, Conditioned stochastic differential equations: Theory, Examples and Applications to finance. Stochastic Processes and their Applications, Vol. 100, pp. 109-145, (2002).

[4] F. Baudoin, M. Thieullen: Pinning Class of the Wiener Measure by a Functional: Related Martingales and Invariance Properties, Probability Theory and Related Fields, Vol. 127, 1-36 (2003).

[5] M. Chaleyat-Maurel, T. Jeulin, Grossissement gaussien de la filtration brownienne. LNM 1118, Grossissements de filtration: exemples et applications, ed. T. Jeulin and M. Yor, 59-109, Springer-Verlag, (1985).

[6] L. Decreusefond and A. S. Üstünel, Stochastic analysis of the fractional Brownian motion, Potential Analysis. 10 (1998) 177-214.

[7] Y. Hibino, M. Hitsuda and H. Muraoka, Construction of noncanonical representations of a Brownian motion, Hiroshima Math. J. 27 (1997), no. 3, 439-448.

[8] T. Hida, Canonical representations of Gaussian processes and their applications. Mem. Coll. Sci. Univ Kyoto Ser. A. 33., (1960), 109-155.

[9] J. Jacod, Grossissement initial, hypothèse (H') et théorème de Girsanov, in Grossissement de filtrations: exemples et applications, LNM, 1118, 15-35, Springer, Berlin, Heidelberg, New-York 1985.

[10] P; Lévy, Sur une classe de courbes de l'espace de Hilbert et sur une équation intégrale non linéaire. Ann. Sci. Norm. Superiare. 73 (1956), 121-156.

[11] A. N. Kolmogorov, Wienersche Spiralen und einige andere interessante Kurven in Hilbertschen Raum, Comptes Rendus (Doklady) Acad. Sci. USSR (N.S.) 26 (1940), 115-118.

[12] T. Jeulin and M. Yor, Filtration des ponts browniens et équations différentielles stochastiques linéaires, Séminaire de Probabilités XXIV, LNM 1426, 227-265, 1990.

[13] B.B. Mandelbrot and Van Ness, Fractional Brownian motions, fractional noises and applications, SIAM Review 10, 4, 422-437, 1968.

[14] P.A. Meyer, Sur une transformation du mouvement brownien due à Jeulin et Yor. Sém. Prob. XXVIII, 98-101, LNM 1583, Springer 1994.

[15] G. Molchan and J. Golosov; Gaussian stationary processes with asymptotic power spectrum. Soviet Math. Dokl., (1969), 134-137.

[16] I. Norros, E. Valkeila, J. Virtamo, An elementary approach to a Girsanov formula and other analytic results on fractional Brownian motions, Bernoulli 5 (199) 571-588.

[17] V. Pipiras and M.S. Taqqu, Are classes of deterministic integrands for fractional Brownian motion on interval complete? Bernoulli 7 (6), 2001, 873-897.

[18] A. M. Yaglom; correlation theory of processes with random stationary nth increments. AMS Transl. (2) 8 (1958), 87-141.

[19] M. Yor, Some Aspects of Brownian Motion, Part two : Some recent martingales problems, Lectures in Math., ETH Zurich, Birkhauser, Basel, 1997.