Semi-martingales and rough paths theory

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Abstract

We prove that the theory of rough paths, which is used to define path-wise integrals and path-wise differential equations, can be used with continuous semi-martingales. We provide then an almost sure theorem of type Wong-Zakai. Moreover, we show that the conditions UT and UCV, used to prove that one can interchange limits and Itô or Stratonovich integrals, provide the same result when one uses the rough paths theory.

Keywords: Semi-martingales, p-variation, iterated integrals, rough paths, Wong-Zakai theorem, conditions UT and UCV

1 Introduction

The theory of rough paths allows to give a meaning to integrals like

$$z_t = z_0 + \int_0^t f(x_s)\,\mathrm{d}x_s$$

and controlled differential equations like

$$y_t = y_0 + \int_0^t g(y_s)\,\mathrm{d}x_s$$

when x is a continuous, irregular path in a Banach space, and f and g are differential forms and vector field smooth enough (See [Lyo98, LQ02, Lej03]). But for that, one needs to know the equivalent of the iterated integrals of x, that is $\int_{0\leq s_1\leq\cdots\leq s_k\leq T}\mathrm{d}x_{s_1}\otimes\cdots\otimes\mathrm{d}x_{s_k}$, and to use the topology of p-variation, which is defined using the semi-norm

$$\mathrm{Var}_{p,[0,T]}(x) = \left(\sup_{\text{partition }\Pi\text{ of }[0,T]}\sum_{i=1}^{k-1}|x(t_{i+1})-x(t_i)|^p\right)^{1/p}, \tag{1}$$

where we use the convention that the points of the partitions Π are $t_1 \leq t_2 \leq \cdots \leq t_k$. The real p define how irregular the path x is, and there is no canonical way to define the iterated integrals of x up to the order $\lfloor p \rfloor$.

However, for a large class of stochastic processes, it is possible to define the equivalent of the iterated integrals for the trajectories. Although the rough path theory is completely deterministic, it has been used for a large class of processes, including Brownian motion, some Gaussian processes in a Banach space [LLQ02], fractional Brownian motion [CQ02], free Brownian motion [CDM01], symmetric Markov process [BHL02],...

In this article, we study the use of rough paths theory for general continuous semi-martingales. Let us remark first that any one-dimensional, continuous local martingale M may be written as $M_t = B_{\langle M\rangle_t}$, where B is a Brownian motion and $\langle M\rangle$ the quadratic variation of M. Since almost every trajectory of the Brownian motion is $1/p$-Hölder continuous as soon as $p > 2$, then it is also of finite p-variation for any $p > 2$. Hence, as a continuous time-change does no change the p-variation, there exists a modification of M whose trajectories are of finite p-variation for any $p > 2$ (See Lemma 3). Moreover, if X is a semi-martingale, the following Itô and Stratonovich integrals exist

$$\Theta_{s,t}(X) = \int_s^t (X_r - X_s)\otimes\,\mathrm{d}X_r \text{ and } \overline{\Theta}_{s,t}(X) = \int_s^t (X_r - X_s)\otimes\circ\mathrm{d}X_r.$$

Hence, the results from the rough paths theory may be used if one can show that $\Theta(X)$ and $\overline{\Theta}(X)$ are of finite $p/2$-variation (See Proposition 1). The main idea of the proof is to use a random time-change φ such that $X_\varphi = X_{\varphi(0)} + M_\varphi + V_\varphi$ remains a semi-martingale and such that M_φ and V_φ are Hölder continuous trajectories. But, X_φ is then defined on a random time interval $[0, \widehat{T}]$. Yet, the value of $\widehat{T}$ depends on $\langle M \rangle_T$ and $\mathrm{Var}_{1,[0,T]}(V)$, which are assumed to be controlled.

Basically, we prove two results:

- If $(X^n)_{n\in\mathbb{N}}$ is a family of semi-martingales converging in distribution to X and satisfying the conditions UT or UCV, then $(X^n, \Theta(X^n))_{n\in\mathbb{N}}$ and $(X^n, \overline{\Theta}(X^n))_{n\in\mathbb{N}}$ converge respectively in distribution to $(X, \Theta(X))$ and $(X, \overline{\Theta}(X))$ in the topology used in the rough paths theory. Hence, this is fully coherent with the results concerning interchanging limits and stochastic integrals in the semi-martingales theory.

- For almost every trajectory of the continuous semi-martingale X, there exists a family $(X(n))_{n\in\mathbb{N}}$ of piecewise linear approximations of X such that $(X(n), \overline{\Theta}(X(n)))_{n\in\mathbb{N}}$ converges almost surely in the appropriate topology to $(X, \overline{\Theta}(X))$. In other words, one obtains the almost sure convergence of the ordinary integral (resp. the ordinary differential equation)

$$Z(n)_t = Z_0 + \int_0^t f(X(n)_s)\,\mathrm{d}X(n)_s$$
$$(\text{resp. } Y(n)_t = Y_0 + \int_0^t g(Y(n)_s)\,\mathrm{d}X(n)_s)$$

  to the Stratonovich integral (resp. the stochastic differential equation) $Z_t = Z_0 + \int_0^t f(X_s) \circ \mathrm{d}X_s$ (resp. $Y_t = Y_0 + \int_0^t g(Y_s) \circ \mathrm{d}X_s$).

A similar convergence result is also given for Itô's integrals and Itô's SDEs.

The particular case of the Brownian motion was already known (See [LQZ02, BHL02],...) and was developed initially in [Sip93]. In this Ph.D. thesis, this method can be used for martingales with α-Hölder continuous trajectories for $\alpha < 1/2$. Thus, the Wong-Zakai result will not be surprising. Here, we extend this result to a general continuous semi-martingales, and our proof includes recent techniques in the theory of rough path, that leads to some simplification of results.

Besides, these proofs show the role played by Hölder continuous paths among paths of finite p-variation, and that it is not a big deal to use Hölder continuous path instead of path of finite p-variation using a time-change.

Moreover, this time-change gives us information on the way to get some uniform control for a family of paths, since we are reduced in controlling the length of the image of the time interval on which the path is initially defined. Then, the conditions UT and UCV appear naturally in this context.

Recent results show that the *enhanced Brownian motion*, that is a multiplicative functional lying above a Brownian motion, could be manipulated as the Brownian motion regarding many of its properties: support theorem, large deviation result, ... See [LQZ02, Fri03, FV03].

Hence, one could think that this technique of using a time-change could be applied to generalized almost immediately to continuous semi-martingales that type of results given for the Brownian motion, as long as only the martingale property of the Brownian motion is involved in their proofs.

2 A few notations

We refer the reader to [Lyo98, LQ02] or [Lej03] for a detailed insight on the theory of rough paths and the objects we introduce now.

For a continuous function x from $[0,T]$ to a Banach space V with a norm $|\cdot|$, denote by $\mathrm{Var}_{p,[0,T]}(x)$ its p-variation defined by (1).

Denote by $\mathbb{V}^p([0,T];\mathrm{V})$ the Banach space of continuous functions of finite p-variation with the norm $\mathrm{Var}_{p,[0,T]}(\cdot)+\|\cdot\|_\infty$. Note that $\mathbb{V}^p([0,T];\mathrm{V})$ is not separable.

Set $\Delta^+ = \{\, 0 \le s \le t \le T \,\}$.

For a continuous function x from Δ^+ to V, denote also by $\mathrm{Var}_{p,[0,T]}(x)$ its p-variation, that is

$$\mathrm{Var}_{p,[0,T]}(x) = \left(\sup_{\text{partition } \Pi \text{ of } [0,T]} \sum_{i=1}^{k-1} |x(t_i,t_{i+1})|^p \right)^{1/p}.$$

If $x(s,t) = y(t) - y(s)$ for some continuous function y, then $\mathrm{Var}_{p,[0,T]}(x) = \mathrm{Var}_{p,[0,T]}(y)$.

Let N be a fixed integer. From now, we use the Banach space $\mathrm{V} = \mathbb{R}^N$, and throughout all this article, N denotes the dimension of the space. Besides, we equip $\mathrm{V}\otimes\mathrm{V}$ with a norm $\|\cdot\|_{\mathrm{V}\otimes\mathrm{V}}$ such that $\|x\otimes y\|_{\mathrm{V}\otimes\mathrm{V}} \le |x|\times|y|$ for any $x, y \in \mathrm{V}$.

We denote by $\mathcal{M}\mathbb{V}^p([0,T];\mathrm{V})$ the space of functions $(x_0, x^1_{s,t}, x^2_{s,t})_{(s,t)\in\Delta^+}$

such that

$$x^1_{s,t} = x_t - x_s \text{ for a continuous function } x \text{ from } [0,T] \text{ to } \mathbb{R}^N, \tag{2a}$$
$$x^2 \text{ is continuous from } \Delta^+ \text{ to } \mathbb{R}^N \otimes \mathbb{R}^N, \tag{2b}$$
$$\mathrm{Var}_{p,[0,T]}(x^1) < +\infty, \tag{2c}$$
$$\mathrm{Var}_{p/2,[0,T]}(x^2) < +\infty, \tag{2d}$$
$$x^{2,i,j}_{s,t} = x^{2,i,j}_{s,u} + x^{2,i,j}_{u,t} + x^{1,i}_{s,u} \cdot x^{1,j}_{u,t} \text{ for } i,j \in \{1,\dots,N\} \tag{2e}$$

for all $0 \le s \le u \le t \le T$. Sometimes, it could be useful to use tensor product notations instead of indices. This means that $\mathbf{x} = (x^1_{s,t}, x^2_{s,t})$ is seen as $\mathbf{x}_{s,t} = 1 + x^1_{s,t} + x^2_{s,t} \in \mathbb{R} \oplus \mathrm{V} \oplus \mathrm{V}^{\otimes 2}$ (Here, the starting point x_0 of x is not taken into account). Accordingly, (2e) could be rewritten as $x^2_{s,t} = x^2_{s,u} + x^2_{u,t} + x^1_{s,u} \otimes x^1_{u,t}$.

When there is no ambiguity, we identify (x_0, x^1) with x, that is a function on Δ^+ and a starting point with a function on $[0,T]$. Thus, (x_0, x^1, x^2) is also denoted by (x, x^2). The elements of $\mathcal{MV}^p([0,T];\mathrm{V})$ are called *multiplicative functionals*.

A sequence $(x^n_0, x^{1,n}, x^{2,n})_{n\in\mathbb{N}}$ is said to converge to (x_0, x^1, x^2) in $\mathcal{MV}^p([0,T];\mathrm{V})$ if

$$x^n_0 \xrightarrow[n\to\infty]{} x_0,$$
$$\mathrm{Var}_{p,[0,T]}(x^{1,n} - x^1) + \sup_{t\in[0,T]} |x^{1,n}_t - x^1_t| \xrightarrow[n\to\infty]{} 0,$$
$$\mathrm{Var}_{p/2,[0,T]}(x^{2,n} - x^2) + \sup_{(s,t)\in\Delta^+} |x^{2,n}_{s,t} - x^2_{s,t}| \xrightarrow[n\to\infty]{} 0.$$

To study the convergence in $\mathcal{MV}^p([0,T];\mathrm{V})$, let us recall two useful lemmas.

Lemma 1 (See [Lej03]). *A sequence $(X^n_0, X^{1,n}, X^{2,n})_{n\in\mathbb{N}}$ is tight in the space $\mathcal{MV}^p([0,T];\mathrm{V})$ if $(X^n_0)_{n\in\mathbb{N}}$ is tight, if the sequences $(X^{1,n}_{0,\cdot})_{n\in\mathbb{N}}$ and $(X^{2,n}_{0,\cdot})_{n\in\mathbb{N}}$ are tight in the space of continuous functions with the uniform norm, and if there exists some real $q < p$ such that for any $\varepsilon > 0$, there exists C large enough such that*

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left[\, \mathrm{Var}_{q,[0,T]}(X^{1,n}) > C \,\right] + \sup_{n\in\mathbb{N}} \mathbb{P}\left[\, \mathrm{Var}_{q/2,[0,T]}(X^{2,n}) > C \,\right] < \varepsilon.$$

Lemma 2 (See for example [HL98, BHL02, LLQ02]). *Let (X^1, X^2) be a multiplicative functional of finite p-variation with $2 < p < 3$. Then for any*

$\gamma > p/2 - 1$, *there exists a constant* C *such that*

$$\mathrm{Var}_{p/2,[0,T]}(X^2)^{p/2} + \mathrm{Var}_{p,[0,T]}(X^1)^p \leq C \sum_{n\geq 1} n^\gamma \sum_{k=0}^{2^n-1} \left(|X^1_{i/2^n,(i+1)/2^n}|^p + |X^2_{i/2^n,(i+1)/2^n}|^{p/2} \right).$$

3 Rough path theory

The main result in the rough paths theory is to extend continuously to paths with irregular variations the maps

$$\mathfrak{K} : x \mapsto z_0 + \sum_{i=1}^N \int_0^\cdot g_i(x_s)\,\mathrm{d}x^i_s, \tag{3}$$

$$\mathfrak{I} : x \mapsto \left(y_t = y_0 + \sum_{i=1}^N \int_0^t f_i(y_s)\,\mathrm{d}x^i_s \right)_{t\in[0,T]}, \tag{4}$$

which are well defined for smooth paths x, where g is from V (here assumed to be a space of finite dimension N) to a separable Banach space W and f is from W to W.

If one want to deal with irregular paths of finite p-variation with $p \in (2,3)$, such as the trajectories of the Brownian motion, f and g need to be regular enough, and $\mathfrak{K}$ and $\mathfrak{I}$ are continuous from $\mathcal{M}\mathbb{V}^p([0,T];\mathrm{V})$ to $\mathcal{M}\mathbb{V}^p([0,T];\mathrm{W})$. This means that one need to know not only a path x, but some element of $\mathcal{M}\mathbb{V}^p([0,T];\mathrm{V})$ lying above x.

The term $x^2_{s,t}$ could be interpreted by the following way: Let us remark that

$$x^{2,i,j}_{s,t} = \frac{1}{2} x^{1,i}_{s,t} \times x^{1,j}_{s,t} + \frac{1}{2}(x^{2,i,j}_{s,t} - x^{2,j,i}_{s,t}),$$

and $\frac{1}{2}(x^{2,i,j}_{s,t} - x^{2,j,i}_{s,t})$ is the area enclosed between the path (x^i, x^j) and the chord between (x^i_s, x^j_s) and (x^i_t, x^j_t). Thus, it could be defined as x integrated against itself. Besides, if x is a piecewise smooth path, then $x^2_{s,t} = \int_s^t (x_r - x_s) \otimes \mathrm{d}x_r$, that is $x^{2,i,j}_{s,t} = \int_s^t (x^i_r - x^i_s)\,\mathrm{d}x^j_r$.

If x is of finite p-variation with $2 < p < 3$, there are different way to choose an element x^2 such that, if $x^1_{s,t} = x_t - x_s$, then $(x_0, x^1_{s,t}, x^2_{s,t})$ belongs to $\mathcal{M}\mathbb{V}^p([0,T];\mathrm{V})$.

The main theorem of the rough paths theory we use in this article is the following.

Theorem 1. *Let $p \in (2,3)$. Let* V *be a Banach space. To simplify, we assume that the dimension of* V *is* N.

(i) *If $g_1, \dots, g_N$ are bounded functions from* V *to a Banach space* W *with bounded derivatives which are also α-Hölder continuous with $\alpha > p-2$, then the map $\mathfrak{K}$ defined by* (6), (7) *and* (8) *below (and corresponding to* (3)*) is continuous from $\mathcal{MV}^p([0,T];\mathrm{V})$ to $\mathcal{MV}^p([0,T];\mathrm{W})$.*

(ii) *Let $\mathbf{x} = (x_0, x^1, x^2)$ be an element of $\mathcal{MV}^p([0,T];\mathrm{V})$. If $f = (f_1, \dots, f_N)$ is a bounded functions from* W *to* W^N *with bounded derivatives which are also α-Hölder continuous with $\alpha > p-2$, then there exists a solution in $\mathcal{MV}^p([0,T];\mathrm{W})$ to*

$$\mathbf{y}_t = \mathbf{y}_0 + \int_0^t f(\mathbf{y}_s)\,\mathrm{d}\mathbf{x}_s. \tag{5}$$

Moreover, if the f_i's are twice differential, with bounded, α-Hölder continuous second derivatives with $\alpha > p-2$, then $\mathbf{y}$ *is unique. Besides, the application $\mathfrak{I} : \mathbf{x} \mapsto \mathbf{y}$ defined by* (5) *is continuous from $\mathcal{MV}^p([0,T];\mathrm{V})$ to $\mathcal{MV}^p([0,T];\mathrm{W})$.*

If $\mathbf{x} = (x_0, x^1, x^2)$ belongs to $\mathcal{MV}^p([0,T];\mathrm{V})$ with $x^1_{s,t} = x_t - x_s$, then

$$\mathfrak{K}(\mathbf{x})_{s,t} = \lim_{n\to\infty} \mathbf{y}_{t_0^n,t_1^n} \otimes \mathbf{y}_{t_1^n,t_2^n} \otimes \cdots \otimes \mathbf{y}_{t^n_{k^n-2},t^n_{k^n-1}} \otimes \mathbf{y}_{t^n_{k^n-1},t^n_{k^n}} \tag{6}$$

$$y^1_{s,t} = \sum_{i=1}^N g_i(x_s) x^{1,i}_{s,t} + \sum_{i,k=1}^N \frac{\partial g_i}{\partial x_k}(x_s) x^{2,k,i}_{s,t}, \tag{7}$$

$$y^2_{s,t} = \sum_{i,j=1}^N g_i(x_{t_j^n}) g_j(x_{t_j^n}) x^{2,i,j}_{s,t}, \tag{8}$$

where $\mathbf{y} = 1 + y^1 + y^2 \in \mathbb{R} \oplus \mathrm{W} \oplus \mathrm{W}^{\otimes 2}$ and $t_0^n \le \ldots \le t_{k^n}^n$ is a partition of $[s,t]$ whose mesh decreases to 0 as $n \to \infty$.

There is a sub-family of $\mathcal{MV}^p([0,T];\mathrm{V})$ which contains the multiplicative functionals (x_0, x^1, x^2) that are limits in $\mathcal{MV}^p([0,T];\mathrm{V})$ of piecewise smooth paths $(x_0, x(n)^1, x(n)^2)$ with $x(n)^1_{s,t} = x(n)_t - x(n)_s$ for some sequence $(x(n))_{n\in\mathbb{N}}$ of piecewise, smooth paths approximating $x = x^1_{0,\cdot}$ and $x(n)^2_{s,t} = \int_s^t (x(n)_r - x(n)_s) \otimes \mathrm{d}x(n)_r$.

Such elements are called *geometric multiplicative functionals*. Not all the elements of $\mathcal{MV}^p([0,T];\mathrm{V})$ are geometric multiplicative functionals. Moreover, with the appropriate changes in the hypotheses, Theorem 1 is valid for $p > 3$. But in this case, one can only define integration against geometric multiplicative functionals.

4 The conditions UT and UCV for semi-martingales

Let X be a continuous semi-martingale with respect to a filtration $\mathcal{F}_\cdot = (\mathcal{F}_t)_{t\geq 0}$ with values in $\mathbb{R}^N$, which is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration $\mathcal{F}_\cdot$ may be assumed to satisfy the usual conditions. There exists a unique decomposition of X as the sum of a local martingale M and a process V of locally of finite variation, both $\mathcal{F}_\cdot$-adapted. The decomposition $X = X_0 + M + V$ is called the *canonical decomposition* of X.

Consider a sequence $(X^n)_{n\in\mathbb{N}}$ of continuous $\mathcal{F}^n_\cdot$-semi-martingales on probability spaces $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$. To simplify the notation, denote $\mathbb{P}^n$ by $\mathbb{P}$ when there is no ambiguity.

In all this section, we consider the convergence of semi-martingales in the space of continuous functions with the uniform norm. The convergence in p-variation is considered in the next sections.

The following definitions and results are taken from the review article [KP96].

Definition 1 (Condition UT). Let $\mathfrak{H}^n$ be the class of $\mathcal{F}^n_\cdot$-predictable, simple process bounded by 1 on $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$. The sequence $(X^n)_{n\in\mathbb{N}}$ is said to satisfy the *condition UT* if for each $t > 0$ and for any $\varepsilon > 0$, there exists C large enough such that

$$\sup_{n\in\mathbb{N}} \sup_{H\in\mathfrak{H}^n} \mathbb{P}[\, |H \cdot X^n_t| > C \,] < \varepsilon. \tag{UT}$$

Theorem 2. *Assume that $(X^n)_{n\in\mathbb{N}}$ satisfies the condition UT.*

(i) *The sequence $(X^n)_{n\in\mathbb{N}}$ is tight and any cluster point X of the sequence $(X^n)_{n\in\mathbb{N}}$ is a semi-martingale with respect to the smallest filtration $\mathcal{F}^X_\cdot$ satisfying the usual hypotheses it generates.*

(ii) *Let $(H^n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}^n_\cdot$-progressively measurable càdlàg processes such that $(H^n, X^n)_{n\in\mathbb{N}}$ converges to (H, X) in the Skorohod topology, then X is a semi-martingale with respect to the filtration generated by (H, X), and, when all the stochastic integrals are defined, $(H^n, X^n, H^n \cdot X^n)_{n\in\mathbb{N}}$ converges to $(H, X, H \cdot X)$.*

(iii) *The sequence $(X^n, \langle X^n\rangle)_{n\in\mathbb{N}}$ converges to $(X, \langle X\rangle)$.*

From now, consider a time $T > 0$ and restrict only to processes on $[0, T]$.

Definition 2 (Condition UCV). The sequence $(X^n)_{n\in\mathbb{N}}$ of semi-martingales with canonical decompositions $X^n = X^n_0 + M^n + V^n$ is said to satisfy the *condition UCV* on $[0, T]$ if

$$(\langle M^n\rangle_T)_{n\in\mathbb{N}} \text{ is tight,} \tag{9}$$

$$(\mathrm{Var}_{1,[0,T]}(V^n))_{n\in\mathbb{N}} \text{ is tight.} \tag{10}$$

Remark 1. This definition is slightly different from the one in [KP96], but it is easily seen that both definitions are equivalent.

Theorem 3. (i) *Assume that the sequence $(X^n)_{n\in\mathbb{N}}$ converges weakly to X. Then $(X^n)_{n\in\mathbb{N}}$ satisfies the condition UT if and only if it satisfies the condition UCV.*

(ii) *Let $(M^n)_{n\in\mathbb{N}}$ be a sequence of local martingales converging in distribution to M. Then M is a local martingale and $(M^n)_{n\in\mathbb{N}}$ satisfies the conditions UT and UCV. In particular, $(\langle M^n\rangle_T)_{n\in\mathbb{N}}$ is tight.*

(iii) *Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of continuous semi-martingales converging in distribution to X and satisfying the conditions UT or UCV. Assume that $X^n = X_0^n + M^n + V^n$ is the canonical decomposition of X^n. Then $(X^n, M^n, V^n)_{n\in\mathbb{N}}$ converges in distribution to (X, M, V), where $X = X_0 + M + V$ is the canonical decomposition of the $\mathcal{F}_\cdot^X$-semi-martingale X (In particular, M and V are $\mathcal{F}_\cdot^X$-adapted).*

Proof. Point (i) corresponds to Theorem 7.6 in [KP96, p. 24].

For (ii), consider an arbitrary H^n in $\mathfrak{H}^n$ for any integer n. Then for any $C > 0$ and any $K > 0$,

$$\begin{aligned}&\mathbb{P}\left[\,|H^n \cdot M_t^n| > C\,\right]\\ &\qquad\leq \mathbb{P}\left[\,|H^n \cdot M_t^n| > C;\, \sup_{s\in[0,t]} |M_s^n| \leq K\,\right] + \mathbb{P}\left[\,\sup_{s\in[0,t]} |M_s^n| > K\,\right].\end{aligned}$$

Denote by τ^n the stopping time $\tau^n = \inf\{\, t \geq 0 \,|\, |M_t^n| \geq K\,\}$. Since $H^n \in \mathfrak{H}^n$, $|H_s^n| \leq 1$, and by the Burkholder-Davis-Gundy inequality (See for example Theorem IV.4.1 in [RY90, p. 153]), there exists a universal constant C_{bdg} such that

$$\begin{aligned}&\mathbb{P}\left[\,|H^n \cdot M_t^n| > C;\, \sup_{s\in[0,t]} |M_s^n| \leq K\,\right]\\ &\qquad\leq \mathbb{P}\left[\,|H^n \cdot M_{t\wedge\tau^n}^n| > C\,\right] \leq \frac{1}{C^2}\mathbb{E}\left[\,\langle M^n\rangle_{t\wedge\tau^n}\,\right]\\ &\qquad\leq \frac{C_{\text{bdg}}}{C^2}\mathbb{E}\left[\,\sup_{s\in[0,t]} |M_{t\wedge\tau^n}^n|^2\,\right] \leq \frac{C_{\text{bdg}}}{C^2}K^2.\end{aligned}$$

On the other hand, as $(M^n)_{n\in\mathbb{N}}$ is tight, $(\sup_{s\in[0,t]} |M_s^n|)_{n\in\mathbb{N}}$ is also tight, and for any $\varepsilon > 0$, for a constant K large enough,

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left[\,\sup_{s\in[0,t]} |M_s^n| > K\,\right] < \varepsilon/2.$$

Thus, choosing C large enough so that $C_{\text{bdg}}K^2/C^2 < \varepsilon/2$, it is easily proved that $(M^n)_{n\in\mathbb{N}}$ satisfies the condition UT.

That the limit of $(M^n)_{n\in\mathbb{N}}$ is a local martingale follows from Theorem IX.1.19 in [JS87, p. 486].

The proof of (iii) is more complicated. From Theorem 2(iii), since $\langle X^n\rangle = \langle M^n\rangle$, it is clear that $(X^n, \langle M^n\rangle)_{n\in\mathbb{N}}$ is tight, and then that $(X^n, M^n)_{n\in\mathbb{N}}$ is also tight. Denote by (X, M, V) a cluster point of this sequence. From (ii), M is a local martingale, but it is *a priori* only a local martingale with respect to the filtration generated by X and M (See Proposition IX.1.17 in [JS87, p. 485]).

However, a $\mathcal{F}_\cdot$-semi-martingale is also a $\mathcal{F}_\cdot$-Dirichlet process in the sense of Föllmer [Föl81], that is the sum of a local $\mathcal{F}_\cdot$-martingale and a term V which is $\mathcal{F}_\cdot$-adapted and locally with zero quadratic variation. This means that if $(\Pi_m)_{m\in\mathbb{N}}$ be a family of partitions of $[0,T]$ whose mesh decreases to 0, then

$$\lim_{m\to\infty} \mathbb{P}[\, Q_m(V) > C\,] = 0$$

for any $C > 0$, where

$$Q_m(V) = \sum_{i=1}^{k-1} |V_{t_{i+1}} - V_{t_i}|^2$$

if Π_m is the partition $t_1 \le t_2 \le \cdots \le t_k$. Of course, when it exists, this decomposition is unique.

Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}^n_\cdot$-adapted Dirichlet processes with canonical decomposition $X^n = x + M^n + N^n$. Assume that $(X^n)_{n\in\mathbb{N}}$ converges weakly to X, which is $\mathcal{F}_\cdot$-adapted. In [RS98], it is proved that if $(\langle M^n\rangle_T)_{n\in\mathbb{N}}$ and $(\sup_{t\in[0,T]} |V^n_t|)_{n\in\mathbb{N}}$ are tight, and

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} \mathbb{P}[\, Q_m(V^n) > C\,] = 0$$

for any $C > 0$, then X is a $\mathcal{F}_\cdot$-Dirichlet process along $(\Pi^m)_{m\in\mathbb{N}}$ with decomposition $X = x + M + V$ and (X^n, M^n, V^n) converges in distribution to (X, M, V). In particular, both M and V are $\mathcal{F}_\cdot$-adapted.

Now, if $(X^n)_{n\in\mathbb{N}}$ is a family of semi-martingale satisfying the condition UT, since

$$Q^m(V^n) \le 2\,\mathrm{osc}(V^n, \mathrm{mesh}\,\Pi^m)\,\mathrm{Var}_{1,[0,T]}(V^n),$$

where $\mathrm{osc}(x,\delta)$ is $\sup_{|t-s|<\delta} |x_t - x_s|$ for a continuous function x, and $\mathrm{mesh}\,\Pi^m$ is the mesh of the partition Π^m.

Thus, using the tightness of V, the condition (10) and the previous result on convergence of Dirichlet processes, $X = X_0 + M + V$ and $(X^n, M^n, V^n)_{n\in\mathbb{N}}$

converges weakly to (X, M, V), where M is a $\mathcal{F}_\cdot^X$-local martingale, and V a $\mathcal{F}_\cdot^X$-adapted process locally of zero quadratic variation.

One the other hand, one knows that X is a $\mathcal{F}_\cdot^X$-semi-martingale. Thus, V is almost surely of finite variation. □

Counter-example 1. Of course, any sequence of semi-martingale does not satisfy the conditions UT and UCV. Let V be a smooth function from $\mathbb{R}$ to $\mathbb{R}$ which is one-periodic, and set

$$X_t^\varepsilon = B_t + \frac{1}{\varepsilon}\int_0^t b(X_s^\varepsilon/\varepsilon)\,\mathrm{d}s, \tag{11}$$

where B is a Brownian motion and $b = \nabla V$. Then, it is well known that X^ε is equal in distribution to $\varepsilon X_{\cdot/\varepsilon}$, where $X_t = B_t + \int_0^t b(X_s)\,\mathrm{d}s$, and that X^ε converges in distribution to $\sigma\beta$, where σ is a constant matrix which is not in general the identity matrix, and β is a Brownian motion (This is the homogenization problem: See for example [BLP78]). Hence $(X^\varepsilon)_{\varepsilon>0}$ cannot satisfy the conditions UT and UCV, since it contradicts Theorem 2-(iii) (See also [Lej02] and [LL03]).

In facts, it is easy to construct family of semi-martingales that does not satisfy the conditions UT and UCV. For that, one has consider semi-martingales where the term of finite-variation becomes a martingale when one passes to the limit. The example above comes from an homogenization problem and appears in a completely natural situation.

5 Semi-martingales and p-variation

The norm $|\cdot|$ on $\mathbb{R}^N$ we use is $|x| = \sup_{i=1,\dots,N}|x^i|$ for all $x = (x^1, \dots, x^N)$.

For any one-dimensional, continuous local martingale M on $[0,T]$, there exists a one-dimensional Brownian motion B such that $M_t = B_{\langle M\rangle_t}$ for all $t \in [0,T]$. Let $p > 2$ be fixed. Using the scaling property of the Brownian motion, for any $T > 0$,

$$\mathbb{E}\left[\,\mathrm{Var}_{p,[0,T]}(B)\,\right] \le C_p T^{1/p} \tag{12}$$

with $C_p = \mathbb{E}[\,\mathrm{Var}_{p,[0,1]}(B)\,]$. It follows that for any continuous local martingale

M with values in $\mathbb{R}^N$ and any $C > 0$, if B^i is such that $B^i_{\langle M^i \rangle_t} = M^i_t$,

$$\begin{aligned}
\mathbb{P}\left[\, \mathrm{Var}_{p,[0,T]}(M) > C \,\right] =& \mathbb{P}\left[\sup_{i=1,\dots,N} \mathrm{Var}_{p,[0,\langle M^i \rangle_T]}(B^i) > C \right] \\
&\leq \sum_{i=1}^{N} \mathbb{P}\left[\, \mathrm{Var}_{p,[0,K]}(B^i) > C; \langle M^i \rangle_T \leq K \,\right] \\
&\quad + \sum_{i=1}^{N} \mathbb{P}\left[\, \mathrm{Var}_{p,[0,\langle M^i \rangle_T]}(B^i) > C; \langle M^i \rangle_T > K \,\right] \\
&\leq \frac{C_{K,p}}{C} K^{1/2} + \sum_{i=1}^{N} \mathbb{P}\left[\, \langle M^i \rangle_T > K \,\right].
\end{aligned} \tag{13}$$

Remark that if V is of finite variations, then for $N = 1$ and for all $s \leq t \leq u$,

$$|V_t - V_s|^p + |V_u - V_t|^p \leq \left(\int_s^t |\mathrm{d}V_r| \right)^p + \left(\int_t^u |\mathrm{d}V_r| \right)^p \leq \left(\int_s^u |\mathrm{d}V_r| \right)^p.$$

Clearly,

$$\mathrm{Var}_{p,[0,T]} V \leq \mathrm{Var}_{1,[0,T]}(V). \tag{14}$$

The next lemma follows immediately from (13) and (14). That a local martingale, even discontinuous, is of finite p-variation as soon as $p > 2$ is well known: See [Lép76, Bru79] for example.

Lemma 3. *Let X be a $\mathcal{F}_\cdot$-semi-martingale with decomposition $X = X_0 + M + V$. Then X is almost surely of finite p-variation as soon as $p > 2$. Moreover, if $(X^n)_{n\in\mathbb{N}}$ is a sequence of semi-martingales satisfying the conditions UCV or UT, then for any $\varepsilon > 0$ there exists C large enough such that*

$$\mathbb{P}\left[\, \mathrm{Var}_{p,[0,T]}(X^n) > C \,\right] < \varepsilon. \tag{15}$$

This implies that if X^n converges in distribution in $\mathbb{C}([0,T];\mathbb{R}^N)$ to X (which is then a semi-martingale), then X^n converges in $\mathbb{V}^p([0,T];\mathbb{R}^N)$ to X for any $p > 2$.

For a semi-martingale X, define $\Theta^{i,j}_{s,t}(X)$ by

$$\Theta^{i,j}_{s,t}(X) = \int_s^t (X^i_r - X^i_s)\, \mathrm{d}X^j_r$$

and $\Theta(X) = (\Theta^{i,j}_{s,t}(X); 0 \leq s \leq t \leq T, i, j \in \{\, 1, \dots, N \,\})$.

Define also

$$\begin{aligned}\overline{\Theta}^{i,j}_{s,t}(X) &= \int_s^t (X^i_r - X^i_s) \circ \mathrm{d}X^j_r \\ &= \Theta^{i,j}_{s,t}(X) + \frac{1}{2}\left(\langle X^i, X^j\rangle_t - \langle X^i, X^j\rangle_s\right).\end{aligned}$$

Clearly, $(X, \Theta(X))$ and $(X, \overline{\Theta}(X))$ satisfy (2a), (2b) and (2e). We have seen in Lemma 3 that they also satisfy (2c). To prove that they belong to $\mathcal{MW}^p([0,T];\mathbb{R}^N)$, it remains to prove that $\Theta(X)$ and $\overline{\Theta}(X)$ satisfy (2d).

Proposition 1. *Let X be a $\mathcal{F}_\cdot$-semi-martingale. Then $\Theta(X)$ and $\overline{\Theta}(X)$ are almost surely of finite $p/2$-variation for all $p > 2$. Consequently, both $(X, \Theta(X))$ and $(X, \overline{\Theta}(X))$ belong to $\mathcal{MW}^p([0,T];\mathbb{R}^N)$ for all $p > 2$.*

Furthermore, if $(X^n)_{n\in\mathbb{N}}$ converges in distribution to X in $\mathbb{C}([0,T];\mathbb{R}^N)$ and satisfies the condition UCV, then $(X^n, \Theta(X^n))$ and $(X^n, \overline{\Theta}(X^n))$ converge respectively to $(X, \Theta(X))$ and $(X, \overline{\Theta}(X))$ in $\mathcal{MW}^p([0,T];\mathbb{R}^N)$.

Proof. Let us decompose the semi-martingale X as $X = X_0 + M + V$. Besides, let V^+ and V^- be the increasing, continuous functions such that $V = V^+ - V^-$. We consider now the process (M, V^+, V^-) in $\mathbb{R}^{3N}$, and denote by $\mathrm{Var}_{p,[s,t]}(M, V^+, V^-)$ its p-variation. Associate to the semi-martingale X the function φ defined by

$$\varphi(t) = \inf\left\{\, s > 0 \,\middle|\, \mathrm{Var}_{p,[0,s]}(M, V^+, V^-)^p > t \,\right\}.$$

The process $t \mapsto \mathrm{Var}_{p,[0,t]}(X)$ is continuous and $\mathcal{F}_\cdot$-adapted. Assume that $\varphi(t) = \varphi(s)$ for some $s < t$. Then for $y = M$ or $y = V$, $\mathrm{Var}_{p,[s,t]}(y) = 0$ since $\mathrm{Var}_{p,[0,s]}(y)^p + \mathrm{Var}_{p,[s,t]}(y)^p \le \mathrm{Var}_{p,[0,t]}(y)^p$. Consequently, y is constant on $[s,t]$, and thus that X, M and V are constant on the intervals $[\varphi(t-), \varphi(t)]$ for all $t \in [0,T]$.

Let $\mathcal{F}^\varphi_\cdot = (\mathcal{F}^\varphi_t)_{t\ge 0}$ be the filtration defined by $\mathcal{F}^\varphi_t = \mathcal{F}_{\varphi(t)}$. Because M and V are constant on the intervals $[\varphi(t-), \varphi(t)]$, $M_\varphi = (M_{\varphi(t)})_{t\in[0,T]}$ is a local $\mathcal{F}^\varphi_\cdot$-martingale. Moreover, $(V_{\varphi(t)})_{t\in[0,T]}$ is locally of finite variation (See Propositions V.1.4 and V.1.5 in [RY90]).

Let us set $\widehat{T} = \mathrm{Var}_{p,[0,T]}(M, V^+, V^-)^p$. Then

$$\mathrm{Var}_{p,[0,\varphi(\widehat{T})]}(M, V^+, V^-) = \mathrm{Var}_{p,[0,T]}(M, V^+, V^-) \tag{16}$$

and

$$\begin{aligned}\mathrm{Var}_{p/2,[0,T]}\,\Theta(X) &= \left(\sup_{t_0\le\cdots\le t_n \text{ partition of } [0,T]} \sum_{i=0}^{n-1} \Theta_{t_i,t_{i+1}}(X)^{p/2}\right)^{2/p} \\ &\le \left(\sup_{t_0\le\cdots\le t_n \text{ partition of } [0,\widehat{T}]} \sum_{i=0}^{n-1} \Theta_{\varphi(t_i),\varphi(t_{i+1})}(X)^{p/2}\right)^{2/p}.\end{aligned}$$

According to Propositions V.1.4 and V.1.5 in [RY90],

$$\int_{\varphi(s)}^{\varphi(t)} (X_r^i - X_{\varphi(s)}^i)\,\mathrm{d}X_r^j = \int_s^t (X_{\varphi(r)}^i - X_{\varphi(s)}^i)\,\mathrm{d}X_{\varphi(r)}^j$$

for $i, j \in \{\, 1, \ldots, N \,\}$. Thus, $\Theta_{s,t}(X_\varphi) = \Theta_{\varphi(s),\varphi(t)}(X)$ and

$$\mathrm{Var}_{p/2,[0,T]}\,\Theta(X) = \mathrm{Var}_{p/2,[0,\widehat{T}]}\,\Theta(X_\varphi). \tag{17}$$

Assume in a first time that the semi-martingale X is such that

$$\mathbb{E}[\,\widehat{T}^{q/p}\,] = \mathbb{E}\left[\,\mathrm{Var}_{p,[0,T]}(M, V^+, V^-)^{q/p}\,\right] < +\infty \tag{18}$$

for some $q > p > 2$.

The Burkholder-Davis-Gundy inequality may be applied: There exists some constant C_1 such that

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_{\varphi(s)}^{\varphi(t)} (X_r^i - X_{\varphi(s)}^i)\,\mathrm{d}M_r^j\right|^{q/2}\right] \\
&\qquad\qquad \le C_1\mathbb{E}\left[\left|\int_{\varphi(s)}^{\varphi(t)} |X_r^i - X_{\varphi(s)}^i|^2\,\mathrm{d}\langle M^j\rangle_r\right|^{q/4}\right] \\
&\quad \le \frac{C_1}{2}\mathbb{E}\left[\sup_{r\in[\varphi(s),\varphi(t)]} |X_r^i - X_{\varphi(s)}^i|^q\right] + \frac{C_1}{2}\mathbb{E}\left[\,|\langle M^j\rangle_{\varphi(t)} - \langle M^j\rangle_{\varphi(s)}|^{q/2}\,\right].
\end{aligned}$$

Applying the other side of the Burkholder-Davis-Gundy inequality, there exists some constant C_2 such that

$$\mathbb{E}\left[\,|\langle M^j\rangle_{\varphi(t)} - \langle M^j\rangle_{\varphi(s)}|^{q/2}\,\right] \qquad \le \qquad C_2\mathbb{E}\left[\sup_{r\in[\varphi(s),\varphi(t)]} |M_r^j - M_{\varphi(s)}^j|^q\right].$$

Still using the inequality $ab \le (a^2 + b^2)/2$ for any $a, b > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_{\varphi(s)}^{\varphi(t)} (X_r^i - X_{\varphi(s)}^i)\,\mathrm{d}V_r^{\pm,j}\right|^{q/2}\right] \\
&\qquad\qquad \le \frac{1}{2}\mathbb{E}\left[\sup_{r\in[\varphi(s),\varphi(t)]} |X_r^i - X_{\varphi(s)}^i|^q\right] + \frac{1}{2}\mathbb{E}\left[\left|\int_{\varphi(s)}^{\varphi(t)} |\mathrm{d}V_r^{\pm,j}|\right|^q\right].
\end{aligned}$$

For any continuous function y and all $0 \leq s \leq t$,

$$\sup_{r \in [s,t]} |y_r - y_s|^q \leq \mathrm{Var}_{q,[s,t]}(y)^q. \tag{19}$$

Moreover, let us remark that $\mathrm{Var}_{q,[s,t]}(X) \leq \mathrm{Var}_{q,[s,t]}(M, V^+, V^-)$.

Besides, for any continuous, increasing function y of finite variation,

$$\left(\int_s^t \mathrm{d}y_s \right)^q \leq \mathrm{Var}_{q,[s,t]}(y)^q. \tag{20}$$

Applying (20) to V^+ and V^- and using (19) to M and X, one gets that for some constant C depending only on q,

$$\mathbb{E}\left[\, |\Theta^{i,j}_{\varphi(s),\varphi(t)}(X)|^{q/2} \, \right] \leq C \mathbb{E}\left[\, \mathrm{Var}_{q,[\varphi(s),\varphi(t)]}(M, V^+, V^-)^q \, \right].$$

Since $\mathrm{Var}_{p,[0,\varphi(s)]}(M, V^+, V^-)^p = s$ for all $s \geq 0$ and

$$\mathrm{Var}_{p,[0,s]}(y)^p + \mathrm{Var}_{p,[s,t]}(y)^p \leq \mathrm{Var}_{p,[0,t]}(y)^p$$

for all $0 \leq s \leq t$ and any continuous function y of finite p-variation,

$$\mathrm{Var}_{p,[\varphi(s),\varphi(t)]}(M, V^+, V^-)^p \leq |t - s|. \tag{21}$$

As $\mathrm{Var}_{q,[s,t]}(y) \leq \mathrm{Var}_{p,[s,t]}(y)$ and $a^q + b^q \leq (a+b)^q$ for any $q \geq 1$ and all $a, b > 0$, it follows from (21) that

$$\mathrm{Var}_{q,[\varphi(s),\varphi(t)]}(M, V^+, V^-)^q \leq |t - s|^{q/p}.$$

It follows that

$$\mathbb{E}\left[\left| \int_{\varphi(s)}^{\varphi(t)} (X^i_r - X^i_{\varphi(s)}) \, \mathrm{d}X^j_r \right|^{q/2} \right] \leq K |t - s|^{q/p}$$

for some constant K that depends only on q. It follows that

$$\sum_{k=0}^{2^n - 1} \mathbb{E}\left[\left| \int_{\varphi(\widehat{T} k/2^n)}^{\varphi(\widehat{T}(k+1)/2^n)} (X^i_r - X^i_{\varphi(s)}) \, \mathrm{d}X^j_r \right|^{q/2} \right] \leq 2^{n(1-q/p)} \mathbb{E}[\, \widehat{T}^{q/p} \,]$$

which is finite from (18). From (17) and from Lemma 2, there exists some constant $c_{p,q,T}$ such that

$$\begin{aligned} \mathbb{E}\left[\, \mathrm{Var}_{q/2,[0,T]} \, \Theta(X)^{q/2} \, \right] &\leq \mathbb{E}\left[\, \mathrm{Var}_{q/2,[0,\widehat{T}]} \, \Theta(X_\varphi)^{q/2} \, \right] \\ &\leq c_{p,q,T} \mathbb{E}[\, \widehat{T}^{q/p} \,]. \end{aligned}$$

Now, to consider the general case, fix K and let $\tau = \varphi(K)$. Then τ is a stopping time and for a $\mathcal{F}_\cdot$-adapted process Y, set $Y^\tau = Y_{\cdot\wedge\tau}$. The continuous process X^τ is a $\mathcal{F}_\cdot$-semi-martingales with canonical decomposition $X^\tau = X_0 + M^\tau + V^\tau$. Remark that $\tau > T$ is equivalent to $\widehat{T} \leq K$. Hence, it follows from (13) that for any $L > 0$,

$$\begin{aligned}
&\mathbb{P}\left[\, \mathrm{Var}_{q/2,[0,T]}\, \Theta(X) > C \,\right] \\
&\qquad \leq \mathbb{P}\left[\, \mathrm{Var}_{q/2,[0,T]}\, \Theta(X^\tau) > C \,\right] + \mathbb{P}\left[\, \tau < T \,\right] \\
&\qquad \leq \frac{1}{C^{q/2}} \mathbb{E}\left[\, \mathrm{Var}_{q/2,[0,T]}\, \Theta(X^\tau)^{q/2} \,\right] + \mathbb{P}\left[\, \widehat{T} > K \,\right] \\
&\leq \frac{c_{q,p,T}}{C^{q/2}} K^{q/p} + \frac{C_{K,p}}{K}\sqrt{L} + \sum_{i=1}^{N} \mathbb{P}\left[\, \langle M^i \rangle_T > L^{1/p} \,\right] + \mathbb{P}\left[\, \mathrm{Var}_{1,[0,T]}(V^+, V^-) > L^{1/p} \,\right].
\end{aligned}$$

Now, remark that $V_t^\pm = \frac{1}{2}(V_t \pm \mathrm{Var}_{1,[0,t]}(V))$. Hence, $\mathrm{Var}_{1,[0,t]}(V^\pm) \leq \mathrm{Var}_{1,[0,t]}(V)$. Thus, $\mathbb{P}\left[\, \mathrm{Var}_{1,[0,T]}(V^+, V^-) > L^{1/p} \,\right] \leq \mathbb{P}\left[\, \mathrm{Var}_{1,[0,T]}(V) > L^{1/p} \,\right]$.

For any $\varepsilon > 0$, choosing first L, then K and C, it follows that

$$\mathbb{P}\left[\, \mathrm{Var}_{q/2,[0,T]}\, \Theta(X) > C \,\right] < \varepsilon. \tag{22}$$

Hence, $\Theta(X)$ is of finite $q/2$-variation almost surely.

Now, let $(X^n)_{n\in\mathbb{N}}$ be a sequence of semi-martingales satisfying the condition UCV. In (22), when $\varepsilon > 0$ is fixed, the choice of C is uniform in n, since with (22), if L is large enough such that

$$\sup_{n\in\mathbb{N}} \sum_{i=1}^{N} \mathbb{P}\left[\, \langle M^{i,n} \rangle_T > L^{1/p} \,\right] + \sup_{n\in\mathbb{N}} \mathbb{P}\left[\, \mathrm{Var}_{1,[0,T]}(V^n) > L^{1/p} \,\right] < 2\varepsilon, \tag{23}$$

then there exists C large enough such that

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left[\, \mathrm{Var}_{q/2,[0,T]}\, \Theta(X^n) > C \,\right] < 3\varepsilon.$$

Thus, if $(X^n)_{n\in\mathbb{N}}$ converges in distribution to X, it follows from the condition UT that $\Theta_{0,\cdot}(X^n)$ converges in distribution to $\Theta_{0,\cdot}(X)$ in $\mathbb{C}([0,T];\mathbb{R}^N)$. This convergence, together with (15) and (23) implies that the convergence of $(X^n, \Theta(X^n))$ holds in $\mathcal{MV}^p([0,T];\mathbb{R}^N)$.

Concerning $\overline{\Theta}(M)$, note that $\langle M^i_\varphi, M^j_\varphi \rangle_t = \langle M^i, M^j \rangle_{\varphi(t)}$ for all $t \geq 0$. Thus, if $\Xi^{i,j}_{s,t}(M) = \frac{1}{2}(\langle M^i, M^j \rangle_t - \langle M^i, M^j \rangle_s)$, then

$$\mathrm{Var}_{p/2,[0,T]}\, \Xi^{i,j}(M) = \mathrm{Var}_{p/2,[0,\widehat{T}]}\, \Xi^{i,j}(M_\varphi).$$

Besides, one knows that

$$\mathbb{E}\left[\, |\Xi^{i,j}_{s,t}(M_\varphi)|^{q/2} \,\right] \leq \mathbb{E}\left[\, |\Xi^{i,i}_{s,t}(M_\varphi)|^{q} \,\right]^{1/2} + \mathbb{E}\left[\, |\Xi^{j,j}_{s,t}(M_\varphi)|^{q} \,\right]^{1/2}.$$

Thus, one has only to act as previously. □

Counter-example 2 (Counter-example 1 (continuation)). In the case of $(X^\varepsilon)_{\varepsilon>0}$ defined by (11), $(X^\varepsilon, \overline{\Theta}(X^\varepsilon))_{\varepsilon>0}$ converges in p-variation to $(\sigma\beta_t, \overline{\Theta}_{s,t}(\sigma\beta) + c(t-s))_{0\leq s\leq t\leq T}$, where c is an antisymmetric matrix (See [Lej02, LL03]). This implies that the limit of the stochastic integral $\int_0^\cdot f(X_s^\varepsilon)\circ \mathrm{d}X_s^\varepsilon$ is not $\int_0^\cdot f(\sigma\beta_s)\circ \mathrm{d}\beta_s$, but

$$\int_0^\cdot f(\sigma\beta_s)\sigma\circ \mathrm{d}\beta_s + \frac{1}{2}\int_0^\cdot c_{i,j}\left(\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}\right)(\sigma\beta_s)\,\mathrm{d}s.$$

6 An almost sure Wong-Zakai theorem

Let X be a $\mathcal{F}_\cdot$-semi-martingale with values in $\mathbb{R}^N$. We prove a result of type Wong-Zakai in the topology in p-variation.

We have seen in the previous section that $(X, \overline{\Theta}(X))$ is a multiplicative functional in $\mathcal{MV}^p([0,T];\mathbb{R}^N)$ for any $p>2$. From now, we set

$$X^1_{s,t} = X_t - X_s \text{ and } X^2_{s,t} = \overline{\Theta}_{s,t}(X).$$

Notation and hypothesis. Let $p>2$ and K be a positive random variable. Let φ be a compatible random time-change (that is φ is the right-continuous inverse of a non-decreasing, continuous, $\mathcal{F}_\cdot$-adapted function and X is constant on the intervals $[\varphi(t-), \varphi(t)]$) such that

$$|M_{\varphi(t)} - M_{\varphi(s)}|^p + |V^+_{\varphi(t)} - V^+_{\varphi(s)}|^p + |V^-_{\varphi(t)} - V^-_{\varphi(s)}|^p \leq K|t-s|. \tag{24}$$

Let us denote by φ^{-1} the right-continuous inverse of φ.

Remark 2. We have seen in the proof of Proposition 1 how to construct a random time change such that (24) is satisfied with $K=1$.

The properties of φ and the inequality (24) imply that

$$\mathrm{Var}_{p,[0,T]}(X) = \mathrm{Var}_{p,[0,\varphi^{-1}(T)]}\, X_{\varphi(\cdot)} \leq K^{1/p}\varphi^{-1}(T)^{1/p}$$

and

$$X^2_{\varphi(s),\varphi(t)} = \int_s^t (X_{\varphi(r)} - X_{\varphi(s)})\otimes \circ \mathrm{d}X_{\varphi(r)}.$$

For any $n\in\mathbb{N}$, let $0\leq s_1^n \leq s_2^n \leq \ldots \leq s_{2^n}^n \leq T$ be a partition of $[0,T]$. For any $n\in\mathbb{N}$ and any $k=1,2,\ldots,2^n$, we set $\Delta_k^n X = X_{s_k^n} - X_{s_{k-1}^n}$ and

$$X(n)_t = X_{s_{k-1}^n} + \frac{t-s_{k-1}^n}{s_k^n - s_{k-1}^n}\Delta_k^n X \text{ for } t\in[s_{k-1}^n, s_k^n]. \tag{25}$$

6.1 A Wong-Zakai theorem for the Stratonovich SDEs

The path $X(n)$ is the piecewise linear approximation of X that coincide with X at the points s_k^n. We could then construct a geometric rough path $(X(n)^1, X(n)^2)$ from $X(n)$ by setting

$$X(n)^1_{s,t} = X(n)_t - X(n)_s$$
$$\text{and } X(n)^2_{s,t} = \int_s^t (X(n)_r - X(n)_s) \otimes \mathrm{d}X(n)_r.$$

Let $0 \leq s \leq t \leq T$, and k and ℓ such that $s^n_{k-1} < s \leq s^n_k \leq s^n_\ell \leq t < s^n_{\ell+1}$,

$$X(n)^2_{s,t} = \int_s^{s^n_j} (X(n)_r - X(n)_s) \otimes \mathrm{d}X(n)_r + \int_{s^n_\ell}^t (X(n)_r - X(n)_{s^n_\ell}) \otimes \mathrm{d}X(n)_r$$
$$+ \sum_{j=k}^{\ell-1} \left(\frac{X(n)_{s^n_{j+1}} + X(n)_{s^n_j}}{2} - X(n)_s \right) \otimes (X(n)_{s^n_{j+1}} - X(n)_{s^n_j}).$$

With this expression, it is easily seen that $X(n)^2_{s,t}$ converges in probability to the Stratonovich integral $X^2_{s,t} = \int_s^t (X_r - X_s) \otimes \circ \mathrm{d}X_r$ for any $s \leq t$ (See for example [RY90, exercice 2.18, p. 136]).

The following proposition was already known for semi-martingales, but for the convergence in probability with respect to the uniform norm: See [KP91]. Yet, combining the next proposition and the usual Wong-Zakai result for semi-martingales implies that the integral defined by the rough-path theory agrees with the Stratonovich integral.

Besides, it is important here that the approximation is piecewise linear: See for example [IW89, KP91, LL03, CE00] for various discussions and results when one uses other types of interpolation.

Proposition 2. *Let p, K and φ as above. Let $X(n)$ be the piecewise linear approximation of X along the partition $(\varphi^{-1}(\varphi(T)k/2^n))_{k=0,\ldots,2^n}$. Then, the geometric multiplicative functional $(X(n)^1, X(n)^2)$ converges almost surely in $\mathcal{M}\mathbb{V}^p([0,T];\mathbb{R}^N)$ to (X^1, X^2) as $n \to \infty$, and (X^1, X^2) is a* geometric multiplicative functional.

Remark 3. If X is a semi-martingale satisfying the condition

$$\mathbb{E}\left[\, |X_t - X_s|^\beta \,\right] \leq C|t-s|^{\beta/2}, \text{ for any } 0 \leq s, t \leq T,$$

for all $\beta > 2$ and some constant $C > 0$, then one knows from the Kolmogorov lemma, that there exists a α-Hölder continuous version of X for any $\alpha < 1/2$.

Moreover, there exists a random variable K such that $|X_t(\omega) - X_s(\omega)|^{1/\alpha} \leq K(\omega)|t-s|$. Thus, Proposition 2 may be applied with the time change $\varphi(t) = t$.

In particular, this proposition is true for the Brownian motion with $\varphi(t) = t$. Hence, we recover a result from [Sip93].

Remark 4. The partition $(\varphi^{-1}(\varphi(T)k/2^n))_{k=0,\dots,2^n}$ is a random partition unless the semi-martingale X has Hölder continuous paths. We have to note that φ^{-1} is not necessarily continuous, but the discontinuities of φ corresponds to the intervals on which X is constant, and are not taken into account in the integrals.

Remark 5. If the partition is deterministic, but no longer related to dyadics, then computations similar to the ones used in the proof of Proposition 2 prove that the convergence in probability of $(X(n)^1, X(n)^2)$ to (X^1, X^2) holds in p-variation.

Corollary 1. *Under the hypotheses of Proposition 2 on X and Theorem 1 on f and g, then the ordinary integral $Z(n)$ and the solution of the ordinary differential equation $Y(n)$ defined by*

$$Z(n)_t = z + \int_0^t g(X(n)_s)\,\mathrm{d}X(n)_s \text{ and } Y(n)_t = y + \int_0^t f(Y(n)_s)\,\mathrm{d}X(n)_s$$

converge respectively almost surely in p-variation to

$$Z_t = z + \int_0^t g(X_s) \circ \mathrm{d}X_s \text{ and } Y_t = y + \int_0^t f(Y_s) \circ \mathrm{d}X_s,$$

where the integrals are understood as Stratonovich integrals.

Proof. Let us set $\mathbf{Z}(n) = \mathfrak{K}(X(n))$ and $\mathbf{Y}(n) = \mathfrak{I}(X(n))$. By Theorem 1, $\mathbf{Z}(n)$ and $\mathbf{Y}(n)$ converges almost surely respectively to the geometric multiplicative functionals $\mathbf{Y} = \mathfrak{K}(\mathbf{X})$ and $\mathbf{Y} = \mathfrak{I}(\mathbf{X})$. But $Y(n)_t - y = \mathbf{Y}(n)^1_{0,t}$ and $Z(n)_t - y = \mathbf{Z}(n)^1_{0,t}$. Besides, by the Wong-Zakai theorem (See for example [KP91]), one knows that $Y(n)_t$ and $Z(n)_t$ converges in probability to Y_t and Z_t. Using the continuity of the trajectories, it is now clear that $Y_t = y + \mathbf{Y}^1_{0,t}$ and $Z_t = z + \mathbf{Z}^1_{0,t}$ for any $t \in [0,T]$. □

Remark 6. It could be important to use a piecewise linear approximation of the trajectories of X, since it is known that other approximations could lead to different integrals: See for example [Kun90, Chapter 5.7], Section VI-7 of [IW89], [CE00] and more specifically, [LL03] concerning the rough paths theory.

First part of the proof of Proposition 2. Since X is continuous, it is clear that $X(n)$ converges to X almost surely in the space of continuous functions with the uniform norm.

Let $2 < q < p$. Let $\Pi = \{ u_i \,|\, 0 \leq u_1 \leq \cdots \leq u_n \leq T \}$ be a partition of $[0,T]$. We introduce two sets of indices (with the convention that $s_0^n = u_0 = 0$ and $u_{n+1} = s_{2^n+1}^n = T$): For $j = 0, \ldots, 2^n$, we set

$$\Pi_j = \left\{ i \,\middle|\, u_i \in [s_j^n, s_{j+1}^n] \right\},$$
$$\Pi_{\text{left}} = \left\{ i \,\middle|\, \exists j, j' \text{ s.t. } u_i \leq s_j^n \leq s_{j'}^n < u_{i+1} \right\}.$$

For each j, we remark that (with the convention that a sum over the empty set is 0),

$$\sum_{i \in \Pi_j,\ i \neq \max \Pi_j} |X(n)_{u_{i+1}} - X(n)_{u_i}|^q \leq \sum_{i \in \Pi_j,\ i \neq \max \Pi_j} \frac{|u_{i+1} - u_i|^q}{|s_{j+1}^n - s_j^n|^q} |X_{s_{j+1}^n} - X_{s_j^n}|^q.$$

But

$$\sum_{i \in \Pi_j,\ i \neq \max \Pi_j} |u_{i+1} - u_i|^q \leq |u_{\max \Pi_j} - u_{\min \Pi_j}|^q \leq |s_{j+1}^n - s_j^n|^q.$$

Thus,

$$\sum_{j=0}^{2^n} \sum_{i \in \Pi_j,\ i \neq \max \Pi_j} |X(n)_{u_{i+1}} - X(n)_{u_i}|^q \leq \sum_{j=0}^{2^n} |X_{s_{j+1}^n} - X_{s_j^n}|^q \leq \operatorname{Var}_q(X)^q.$$

For i in Π_{left}, since $|X(n)_s - X_{s_j^n}| \leq |X_{s_{j+1}^n} - X_{s_j^n}|$ and $|X(n)_s - X_{s_{j+1}^n}| \leq |X_{s_{j+1}^n} - X_{s_j^n}|$ for any $s \in [s_j^n, s_{j+1}^n]$,

$$\begin{aligned} |X(n)_{u_i} - X(n)_{u_{i+1}}|^q \leq & 3^{q-1} |X(n)_{u_i} - X_{s_j^n}|^q + 3^{q-1} |X_{s_j^n} - X_{s_{j'}^n}|^q \\ & + 3^{q-1} |X_{s_{j'}^n} - X(n)_{u_{i+1}}|^q \\ \leq & 3^{q-1} |X_{s_{j-1}^n} - X_{s_j^n}|^q + 3^{q-1} |X_{s_j^n} - X_{s_{j'}^n}|^q \\ & + 3^{q-1} |X_{s_{j'}^n} - X_{s_{j'+1}^n}|^q. \end{aligned}$$

Hence,

$$\sum_{i \in \Pi_{\text{left}}} |X(n)_{u_{i+1}} - X(n)_{u_i}|^q \leq 3^{q-1} \operatorname{Var}_{q,[0,T]}(X)^q.$$

Let us remark that

$$\begin{aligned} \sum_{i=0}^{n-1} |X(n)_{u_{i+1}} - X(n)_{u_i}|^q = & \sum_{i \in \Pi_{\text{left}}} |X(n)_{u_{i+1}} - X(n)_{u_i}|^q \\ & + \sum_{j=0}^{2^n} \sum_{i \in \Pi_j,\ \text{s.t. } i \neq \max \Pi_j} |X_{u_{i+1}} - X(n)_{u_i}|^q \leq (3^{q-1} + 1) \operatorname{Var}_{q,[0,T]}(X)^q. \end{aligned}$$

So, one deduce that

$$\mathrm{Var}_{q,[0,T]}(X(n)) \leq (3^{q-1}+1)^{1/q}\,\mathrm{Var}_{q,[0,T]}(X). \tag{26}$$

Finally, let us remark that for any $p > q$,

$$\begin{aligned}&\mathrm{Var}_{p,[0,T]}(X - X(n))^p \\ &\qquad \leq 2^{q-1} \sup_{t\in[0,T]} |X_t - X(n)_t|^{p-q} \left(\mathrm{Var}_{q,[0,T]}(X)^q + \mathrm{Var}_{q,[0,T]}(X(n))^q\right).\end{aligned}$$

Thus, one obtains the almost sure convergence of $X(n)$ to X in p-variation. □

Let K_0 be fixed, and let T_0 be the $\mathcal{F}^\varphi$-stopping time such that

$$T_0 = \inf\left\{ r > 0 \,\middle|\, \sup_{0\leq s<t\leq r} \frac{|(M,V^+,V^-)_{\varphi(t)} - (M,V^+,V^-)_{\varphi(s)}|^p}{t-s} \geq K_0 \right\}.$$

Let also $\widehat{T}$ be a fixed real number. Let us set $Y_t = X_{\varphi(\cdot)\wedge T_0 \wedge \widehat{T}}$, which is a $\mathcal{F}^\varphi$-semi-martingale.

Let $Y(n)$ denotes the piecewise linear approximation of Y along the dyadics at level n, *i.e.*, X and $X(n)$ are replaced by Y and $Y(n)$ in (25), but with the partition $(\widehat{T}k/2^n)_{k=0,\ldots,2^n}$ Similarly, one can define $Y(n)^2_{s,t}$. From now, we use the notation $t^n_k = \widehat{T}k/2^n$ for $k = 0, 1, \ldots, 2^n$.

The proof relies on two lemmas.

Lemma 4. *For all $n \in \mathbb{N}$ and $j, \ell \in \{0, \ldots, 2^n\}$ such that $\ell > j$, we have*

$$Y(n)^2_{t^n_j, t^n_\ell} = Y^2_{t^n_j, t^n_\ell} + [\theta^n_{t^n_j, t^n_\ell}]^{\mathrm{a}},$$

where $[\theta^n_{s,t}]^{\mathrm{a}}$ denotes the anti-symmetric part (i.e., $[A]^{\mathrm{a}} = \frac{1}{2}(A - A^{\mathrm{t}})$ for a $d \times d$-matrix A) of

$$\theta^n_{s,t} = \int_s^t \delta^n_u Y \otimes \mathrm{d}Y_u \text{ and } \delta^n_u Y = \sum_{k=1}^{2^n} (Y_u - Y_{t^n_{k-1}}) \mathbf{1}_{[t^n_{k-1}, t^n_k]}(u).$$

Proof. We set $s = t^n_j$ and $t = t^n_k$. Then, a direct computation shows that

$$Y(n)^2_{s,t} = \sum_{\ell=j+1}^{k} \left((Y_{t^n_{\ell-1}} - Y_s) \otimes \Delta^n_\ell Y + \frac{1}{2} \Delta^n_\ell Y \otimes \Delta^n_\ell Y \right).$$

As the random variable $Y_{t^n_{\ell-1}} - Y_s$ is $\mathcal{F}_{\varphi(t^n_{\ell-1})}$-measurable,

$$\begin{aligned}(Y_{t^n_{\ell-1}} - Y_s) \otimes \Delta^n_\ell Y &= \int_{t^n_{\ell-1}}^{t^n_\ell} (Y_{t^n_{\ell-1}} - Y_s) \otimes \mathrm{d}Y_u \\ &= \int_{t^n_{\ell-1}}^{t^n_\ell} (Y_u - Y_s) \otimes \mathrm{d}Y_u - \int_{t^n_{\ell-1}}^{t^n_\ell} \delta^n_u Y \otimes \mathrm{d}Y_u.\end{aligned}$$

On the other hand, the Itô formula implies that

$$\frac{1}{2}\Delta^n_\ell Y \otimes \Delta^n_\ell Y = \left[\int_{t^n_{\ell-1}}^{t^n_\ell} \delta^n_u Y \otimes \mathrm{d}Y_u\right]^{\mathrm{s}} + \frac{1}{2}\big(\langle Y\rangle_{t^n_\ell} - \langle Y\rangle_{t^n_{\ell-1}}\big),$$

where $[A]^{\mathrm{s}} = \frac{1}{2}(A + A^{\mathrm{t}})$ denotes by symmetric part of a matrix A. Since

$$Y^2_{s,t} = \int_s^t (Y_u - Y_s) \circ \mathrm{d}Y_u + \frac{1}{2}\big(\langle Y\rangle_t - \langle Y\rangle_s\big),$$

the result is now clear. □

The next lemma is a variation of Lemma 2.

Lemma 5 (See for example [HL98, BHL02, LLQ02]). *Let $p > 2$ and $\gamma > p/2 - 1$. Let (X^1, X^2) and (Y^1, Y^2) be multiplicative functionals with finite p-variation. Then for any partition $0 \le s_1 \le \ldots \le s_k \le T$ of $[0,T]$, there exists a constant C depending only on p and γ such that*

$$\begin{aligned}\sum_{i=1}^{k-1} |X^2_{s_i,s_{i+1}} - Y^2_{s_i,s_{i+1}}|^{p/2} \le C\Big(\sum_{n\ge1} n^\gamma \sum_{j=0}^{2^n-1} |X^1_{t^n_j,t^n_{j+1}} - Y^1_{t^n_j,t^n_{j+1}}|^p\Big)^{1/2} \\ \times \sum_{n\ge1} n^\gamma \sum_{j=0}^{2^n-1} \Big(|X^1_{t^n_j,t^n_{j+1}}|^p + |Y^1_{t^n_j,t^n_{j+1}}|^p\Big)^{1/2} \\ + \sum_{n\ge1} n^\gamma \sum_{j=1}^{2^n} |X^2_{t^n_j,t^n_{j+1}} - Y^2_{t^n_j,t^n_{j+1}}|^p.\end{aligned}$$

Final part of the proof of Proposition 2. Using the Hölder continuity of Y and (26), one get easily that for any $q \ge p$ and any $m \in \mathbb{N}$,

$$|Y^1_{s,t}|^q + |Y(m)^1_{s,t}|^q \le c_q K_0^q |t-s|^{q/p},$$

where c_q is a constant depending only on q. Besides, as $Y(m)_{s,t} = Y_{s,t}$ if $s = \widehat{T}i/2^m$, $t = \widehat{T}j/2^m$ for any $i, j \in \{0, \ldots, 2^m\}$,

$$|Y^1_{t^n_i,t^n_{i+1}} - Y(m)^1_{t^n_i,t^n_{i+1}}|^q \le \begin{cases} c_q \widehat{T}^{q/p} K_0^{q/p} 2^{-mq/p} & \text{if } m \ge n, \\ c_q \widehat{T}^{q/p} K_0^{q/p} 2^{-nq/p} & \text{if } m < n. \end{cases}$$

We decompose Y as $Y = Y_0 + N + V$, where N is a martingale and V a term of finite variation.

Lemma 5 asserts that one can evaluate the $p/2$-variation of $Y^2 - Y(m)^2$ if one knows $Y^2_{t_j^n,t_{j+1}^n} - Y(m)^2_{t_j^n,t_{j+1}^n}$, that is the difference between Y^2 and $Y(m)^2$ only at the dyadic points of $[0,\widehat{T}]$.

If $m \geq n$, for any $j = 0,\ldots,2^n$, there exists j' and j'' in $\{0,\ldots,2^m\}$ such such $t_j^n = t_{j'}^m$ and $t_{j+1}^n = t_{j''}^m$. With Lemma 4, $Y^2_{t_j^n,t_{j+1}^n} - Y(m)^2_{t_j^n,t_{j+1}^n} = -[\theta^m_{t_{j'}^m,t_{j''}^m}]^{\mathrm{a}}$. We focus on $\theta^m_{s,t}$. For $m \geq n$,

$$\mathbb{E}\left[\,|\theta^m_{t_j^n,t_{j+1}^n}|^{q/2}\,\right] \leq 2^{q-1}\mathbb{E}\left[\left|\int_{t_j^n}^{t_{j+1}^n} \delta^m_u Y \otimes \mathrm{d}N_u\right|^{q/2}\right] + 2^{q-1}\mathbb{E}\left[\left|\int_{t_j^n}^{t_{j+1}^n} \delta^m_u Y \otimes \mathrm{d}V_u\right|^{q/2}\right].$$

We follow the arguments of the proof of Proposition 1. Using the both side of the Burkholder-Davies-Gundy inequality and the Cauchy-Schwarz inequality, there exists a constant C_q depending only on q such that

$$\mathbb{E}\left[\left|\int_{t_j^n}^{t_{j+1}^n} \delta^m_u Y \otimes \mathrm{d}N_u\right|^{q/2}\right] \leq C_q\mathbb{E}\left[\sup_{t\in[t_j^n,t_{j+1}^n]}|\delta^m_t Y|^q\right]^{1/2}\mathbb{E}\left[\sup_{t\in[t_j^n,t_{j+1}^n]}|N_t - N_{t_j^n}|^q\right]^{1/2}.$$

Let us remark that

$$\sup_{t\in[t_j^n,t_{j+1}^n]}|\delta^m_t Y|^p \leq \frac{K_0\widehat{T}}{2^m}$$

and that

$$\mathbb{E}\left[\sup_{t\in[t_j^n,t_{j+1}^n]}|N_t - N_{t_j^n}|^q\right] \leq C\left(\frac{K_0\widehat{T}}{2^n}\right)^{q/p}.$$

From (24), $\mathrm{Var}_{p,[s,t]}(V) \leq 2^{p-1}\,\mathrm{Var}_{p,[s,t]}(V^+,V^-)^p \leq 2^{p-1}K_0|t-s|$ and then

$$\mathbb{E}\left[\left|\int_{t_j^n}^{t_{j+1}^n} \delta^m_u Y \otimes \mathrm{d}V_u\right|^{q/2}\right] \leq \left(2^{p-1}\frac{K_0\widehat{T}}{2^{m/2}2^{n/2}}\right)^{q/p}.$$

If $m < n$, a direct computation shows that if $[t_j^n, t_{j+1}^n] \subset [t_i^m, t_{i+1}^m]$, then

$$|Y(m)^2_{t_j^n, t_{j+1}^n}| = \frac{(t_{j+1}^n - t_j^n)^2}{(t_{i+1}^m - t_i^m)^2} |Y_{t_{i+1}^m} - Y_{t_i^m}|^2 \leq \frac{2^{2m}}{2^{2n}} \frac{(K_0 \widehat{T})^{2/p}}{2^{2m/p}}.$$

Choose $q > q' > p$. It follows that there exists a constant C depending only on p, q and q' such that

$$\sum_{n>m} n^\gamma \sum_{j=0}^{2^n-1} |Y(m)^2_{t_j^n, t_{j+1}^n}|^{q/2} \leq C \frac{(K_0 \widehat{T})^{q/p}}{1 - 2^{1-q'}} 2^{m(1-q'+q-q/p)} \sup_{n>m} n^\gamma 2^{n(q-q')}.$$

As $q/p > 1$, if we choose q' so that $q - q' < q/p - 1$, then there exists a constant C depending only on q, q', p and γ such that

$$\sum_{n>m} n^\gamma \sum_{j=0}^{2^n-1} |Y(m)^2_{t_j^n, t_{j+1}^n}|^{q/2} \leq C K_0^{q/p} \widehat{T}^{q/p} 2^{m\delta}$$

with $\delta = 1 - q' + q - q/p < 0$.

On the other hand, we have seen in the proof of Proposition 1 that

$$\sum_{j=0}^{2^n-1} \mathbb{E}\left[|Y^2_{t_j^n, t_{j+1}^n}|^{q/2} \right] \leq C 2^{n(1-q/p)} \widehat{T}^{q/p}.$$

Hence, using all these estimates and Lemma 5, one gets for m large enough,

$$\mathbb{E}\left[\mathrm{Var}_{q/2}(Y^2 - Y(m)^2)^{q/2} \right] \leq C K_0^{q/p} \widehat{T}^{q/p} \left(2^{m\delta} + \sum_{n=1}^{m} \frac{n^\gamma}{2^{(m+n)q/2p}} + \sum_{n>m} \frac{n^\gamma}{2^{nq/p}} \right)^{1/2},$$

where the constant C depends only on p, q and γ.

The Bienaymé-Tchebitchev inequality implies that for all $\alpha_m > 0$ and any $\delta > 0$

$$\mathbb{P}\left[\mathrm{Var}_{q/2,[0,\widehat{T}]}(Y^2 - Y(m)^2) \geq \alpha_m \right] \leq \frac{C'}{\alpha_m} \left(2^{m\delta} + \frac{m^{\gamma+1}}{2^{mq/2p}} + S_m \right)$$

with $C' = C K_0^{q/p} \widehat{T}^{q/p}$ and $S_m = \sum_{n>m} n^\gamma 2^{-nq/p}$. Clearly, $S_m \leq \int_m^{+\infty} t^\gamma \exp(-\zeta t)\, \mathrm{d}t$ for $\zeta = p(1-\beta)\ln 2$. Hence, we deduce easily that $S_m \leq 1/3m^3$ for m large enough. Setting

$$\alpha_m = m^2(2^{m\delta} + m^{\gamma+1} 2^{-mq/2p} + 1/3m^3),$$

we remark that $\alpha_m \xrightarrow[m\to\infty]{} 0$ and that

$$\sum_{m\geq 1} \mathbb{P}\left[\mathrm{Var}_{q/2,[0,\widehat{T}]}(Y^2 - Y(m)^2) \geq \alpha_m \right] < +\infty.$$

The Borel-Cantelli Lemma implies that

$$\mathrm{Var}_{q/2,[0,\widehat{T}]}(Y^2 - Y(m)^2) \xrightarrow[m\to\infty]{} 0$$

$\mathbb{P}$-almost surely. Now, for a given trajectory, let us choose K_0 and T_0 large enough such that $\varphi(T) \leq \widehat{T}$ and $T_0 \geq \widehat{T}$. Since

$$Y(m)^2_{s,t} = X(m)^2_{\varphi(s),\varphi(t)} \text{ and } Y^2_{s,t} = X^2_{\varphi(s),\varphi(t)},$$

we deduce that

$$\mathrm{Var}_{q/2,[0,\widehat{T}]}(Y^2 - Y(m)^2) = \mathrm{Var}_{q/2,[0,T]}(X^2 - X(m)^2).$$

Besides, since

$$\sup_{0\leq s\leq t\leq T} |X(m)^2_{s,t} - X^2_{s,t}| \leq \mathrm{Var}_{q/2,[0,T]}(X(m)^2 - X^2)^{q/2},$$

one obtains the uniform convergence of $X(m)^2_{s,t}$ to $X^2_{s,t}$ with respect to (s,t) such that $0 \leq s \leq t \leq T$. □

6.2 A Wong-Zakai theorem for the Itô SDEs

The difference between an Itô integral and a Stratonovich integral depends of the cross-variation of the process that is integrated.

Let us recall that we have denoted the family of cross-variations of X by Ξ. For $t \in [0,T]$, we set $j^n(t) = \sup\{\, i \in \{\, 0,\ldots,2^n \,\} \mid s^n_i \leq t \,\}$ and

$$\Xi(n)_t = \frac{1}{2}\frac{t - s_{j^n(t)}}{s_{j^n(t)+1} - s_{j^n(t)}}(X_{s_{j^n(t)+1}} - X_{s_{j^n(t)}}) \otimes (X_{s_{j^n(t)+1}} - X_{s_{j^n(t)}}) + \frac{1}{2}\sum_{i=1}^{j^n(t)} (X_{s^n_i} - X_{s^n_{i-1}}) \otimes (X_{s^n_i} - X_{s^n_{i-1}}).$$

We set $\Xi(n)_{s,t} = \Xi(n)_t - \Xi(n)_s$. We consider now the (non-geometric) multiplicative functional $(X(n)^1, X(n)^2 + \Xi(n))$.

Proposition 3. *Let p, K and φ as above. Let $X(n)$ be the piecewise linear approximation of X along the partition $(\varphi^{-1}(\varphi(T)k/2^n))_{k=0,\ldots,2^n}$. Then, the multiplicative functional $(X(n)^1, X(n)^2 + \Xi(n))$ converges almost surely in $\mathcal{MV}^p([0,T];\mathbb{R}^N)$ to $(X^1, X^2 + \Xi)$ as $n \to \infty$.*

Proof. Clearly, we have only to prove that $\Xi(n)$ converges in $q/2$-variation to Ξ. The proof is similar to the one of Proposition 2. One has to work with Y instead of X. Let us define $\Xi(n)^Y$ and Ξ^Y by a substitution of X to Y in the definition of $\Xi(n)$ and Ξ. We use the same notations as in the proof of Proposition 2 for K_0, $\widehat{T}$, ... We assume in a first time that $m \geq n$. Then, for $i \in \{\, 0, \ldots, 2^n \,\}$ and for $j < k$ in $\{\, 0, \ldots, 2^m \,\}$ such that $t_j^m = t_i^n$ and $t_j^m - t_k^m = \widehat{T} 2^{-n}$,

$$\begin{aligned}\Xi(m)^Y_{t_i^n, t_i^n} - \Xi^Y_{t_i^n, t_{i+1}^n} &= \Xi(m)^Y_{t_j^m, t_k^n} - \Xi^Y_{t_j^m, t_k^n} \\ &= \sum_{i=j+1}^{k} \frac{1}{2} (Y_{t_{i-1}^m} - Y_{t_i^n})^{\otimes 2} - \Xi^Y_{t_{i-1}^m, t_i^n} = \sum_{i=j+1}^{k} \frac{1}{2} \int_{t_{i-1}^m}^{t_i^n} (Y_r - Y_{t_i^m}) \otimes \mathrm{d}Y_r.\end{aligned}$$

And if $n > m$, for $i \in \{\, 0, \ldots, 2^n \,\}$,

$$|\Xi(m)^Y_{t_i^n, t_{i+1}^n} - \Xi^Y_{t_i^n, t_{i+1}^n}| = |(Y_{t_{i-1}^n} - Y_{t_i^n})^{\otimes 2}| \leq \frac{2^{2m}}{2^{2n}} \frac{(K_0 \widehat{T})^{2/p}}{2^{2m/p}}.$$

Besides, we have already seen that

$$\mathbb{E}\left[\, |\Xi^Y_{t_{i-1}^n, t_i^n}|^{q/2} \,\right] \leq \frac{K_0^{q/p} \widehat{T}^{q/p}}{2^{nq/p}}.$$

We have already used these estimates in the proof of Proposition 2, and the proof of Proposition 3 is similar to the one of Proposition 2. □

Finally, let us remark that (7), (8) and Corollary 1 implies that $\mathfrak{K}((X^1, X^2 + \Xi))$ and $\mathfrak{I}((X^1, X^2 + \Xi))$ are multiplicative functionals lying above the Itô integral $z + \int_0^{\cdot} g(X_s)\, \mathrm{d}X_s$ and the solution Y to the Itô's SDE $Y_t = y + \int_0^t f(X_s)\, \mathrm{d}X_s$. Again with (6)–(8), the proof of the following corollary is now clear.

Corollary 2. *Under the hypotheses of Proposition 2 and Theorem 1 on g and $f = (f_j^i)_{i=1,\ldots,m, j=1,\ldots,N}$ (Y takes its values in a space of dimension m), the ordinary integral*

$$Z(n)_t = z + \int_0^t g(X(n)_s)\, \mathrm{d}X(n)_s + \sum_{i,j=1}^{N} \int_0^t \frac{\partial g_i}{\partial x_j}(X(n)_s)\, \mathrm{d}\Xi^{i,j}(n)_s$$

and the ordinary differential equation defined by

$$Y(n)_t = y + \int_0^t f(Y(n)_s)\, \mathrm{d}X(n)_s + \sum_{j=1}^{m} \sum_{i,k=1}^{N} \int_0^t f_k^j \frac{\partial f_i}{\partial x_j}(Y(n)_s)\, \mathrm{d}\Xi^{i,k}(n)_s$$

converge almost surely to Z and Y defined above.

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