GOOD ROUGH PATH SEQUENCES AND APPLICATIONS TO ANTICIPATING & FRACTIONAL STOCHASTIC CALCULUS

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ABSTRACT. We consider anticipative Stratonovich stochastic differential equations driven by some stochastic process (not necessarily a semi-martingale). No adaptedness of initial point or vector fields is assumed. Under a simple condition on the stochastic process, we show that the unique solution of the above SDE understood in the rough path sense is actually a Stratonovich solution. This condition is satisfied by the Brownian motion and the fractional Brownian motion with Hurst parameter greater than 1/4. As application, we obtain rather flexible results such as support theorems, large deviation principles and Wong-Zakai approximations for SDEs driven by fractional Brownian Motion along anticipating vectorfields. In particular, this unifies many results on anticipative SDEs.

1. INTRODUCTION

We fix a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\leq t\leq 1}\right)$ satisfying the usual conditions. Itô's theory tells us that there exists a unique solution to the Stratonovich stochastic differential equation

$$\left\{ \begin{array}{l} dY_t = V_0(Y_t)dt + \sum_{i=1}^d V_i(Y_t) \circ dB_t^i \\ Y_0 = y_0, \end{array} \right. \tag{1.1}$$

where $B = \left(B_t^1, \cdots, B_t^d\right)_{0\leq t\leq 1}$ is a standard d-dimensional $(\mathcal{F}_t)$-Brownian motion, V_0 a C^1 vector field on $\mathbb{R}^d$, $V_1, \cdots, V_d$ some C^2 vector fields on $\mathbb{R}^d$, and y_0 a $\mathcal{F}_0$-measurable random variable.

We remind the reader that, by definition (see [25]), a process z is Stratonovich integrable with respect to a process x if for all t there exists a random variable denoted $\int_0^t z_u \circ dx_u$ (the Stratonovich integral of z with respect to x between time 0 and t) such that for all sequences $\left(D^n = (t_i^n)_i\right)_{n\geq 0}$of subdivisions of $[0,t]$ such that $|D^n| \to_{n\to\infty} 0$, the following convergence holds in probability

$$\sum_i \left(\frac{1}{t_{i+1}^n - t_i^n} \int_{t_i^n}^{t_{i+1}^n} z_u du \right) \left(x_{t_{i+1}^n} - x_{t_i^n} \right) \to_{n\to\infty} \int_0^t z_u \circ dx_u.$$

Another way to express this is by introducing x^D the D-linear approximation of x, where $D = (t_i)$ is a subdivision of $[0,1]$:

$$x^D(t) = x_{t_i} + \frac{t - t_i}{t_{i+1} - t_i} \left(x_{t_{i+1}} - x_{t_i} \right) \text{ if } t_i \leq t \leq t_{i+1}. \tag{1.2}$$

Then, z is Stratonovich integrable with respect to x if and only if for all sequences of subdivisions D^n which mesh size tends to 0 and for all $t \in [0,1]$, $\int_0^t z_u dx^{D^n}(u)$ converges in probability to $\int_0^t z_u \circ dx_u$.

Ocone and Pardoux [26] showed that there exists a unique solution to equation (1.1) even if the vector field V_0 and the initial condition y_0 were allowed to be $(\mathcal{F}_1)$-random variables. They did so relating the Skorokhod integral and the Stratonovich one, and using Malliavin Calculus techniques. This solution has been studied in various directions: existence and study of the density of Y [2, 28, 30], a Freidlin-Wentzell's type theorem [23], results on the support of the law of $\{Y_t, 0 \leq t \leq 1\}$ [3, 22], approximation of Y_t by some Euler's type schemes [1]... The case where the vector fields $V_1, \cdots, V_d$ are allowed to be $(\mathcal{F}_1)$-measurable was dealt in [12, 13], under the strong condition that the $(V_i)_{1 \leq i \leq d}$ commute. This is an application of the Doss-Sussman theorem. The latter says that if $V = (V_1, \cdots, V_d)$ are d vector fields smooth enough such that $[V_i, V_j] = 0$, for all $i, j \geq 1$, and if V_0 is another vector fields, then the map $\varphi^{y_0}_{(V_0,V)}$ which at a smooth path $x : [0,1] \to \mathbb{R}^d$ associates the path y which is the solution of the differential equation

$$\begin{cases} dy_t = V_0(y_t)dt + \sum_{i=1}^d V_i(y_t)dx_t^i \\ y_0 = y_0, \end{cases} \tag{1.3}$$

is continuous when one equips the space of continuous functions with the uniform topology. One can then define $\varphi^{y_0}_{(V_0,V)}$ on the whole space of continuous function, in particular $\varphi^{y_0}_{(V_0,V)}(B)$ is then well defined, and is almost-surely the solution of the Stratonovich differential equation (1.1). This remains true even if the vector fields and the initial condition are allowed to be random.

Rough path theory can be seen as a major extension of the Doss-Sussman result. One of the main thing to remember from this theory is that it is not x which controls the differential equation (1.3), but the lift of x to a path in a Lie group lying over $\mathbb{R}^d$. The choice of the Lie group depends on the roughness of x.

If x is a $\mathbb{R}^d$-valued path of finite p-variation, $p \geq 1$, one needs to lift x to a path $\mathbf{x}$ with values in $G^{[p]}\left(\mathbb{R}^d\right)$, the free nilpotent Lie group of step $[p]$ over $\mathbb{R}^d$. When x is smooth, there exists a canonical lift of x to a path denoted $S(x)$ with values in $G^{[p]}\left(\mathbb{R}^d\right)$ ($S(x)$ is obtain from x by computing the "first $[p]$" iterated integrals of x). If x is a smooth path, then there exists a solution y to the differential equation (1.3). We denote by $I^{y_0}_{(V_0,V)}$ the map which at $S(x)$ associates $S(x \oplus y)$. Denoting $C\left(G^{[p]}\left(\mathbb{R}^d\right)\right)$ the set of continuous paths from $[0,1]$ into $G^{[p]}\left(\mathbb{R}^d\right)$, we see that $I^{y_0}_{(V_0,V)}$ is a map from a subset of $C\left(G^{[p]}\left(\mathbb{R}^d\right)\right)$ to $C\left(G^{[p]}\left(\mathbb{R}^d \oplus \mathbb{R}^n\right)\right)$. Lyons showed that this map is (locally uniformly) continuous when one equips $C\left(G^{[p]}\left(\mathbb{R}^d\right)\right)$ and $C\left(G^{[p]}\left(\mathbb{R}^d \oplus \mathbb{R}^n\right)\right)$ with a "p-variation distance". Hence one can define $I^{y_0}_{(V_0,V)}$ on the closure (in this p-variation topology) of the canonical lift of smooth paths. This latter set is the set of geometric p-rough paths.

In the case $x = B$, the Brownian motion being almost surely $1/p$-Hölder, $2 < p < 3$, one needs to lift B to a process with values in $G^2(\mathbb{R}^d)$ to obtain the solution in the rough path sense of equation (1.1). This is equivalent to define its area process. The standard choice for the area process of the Brownian motion is the Lévy area [18, 20], although one could choose very different area processes [16]. Choosing this area, we lift B to a geometric p-rough path $\mathbf{B}$, $I^{y_0}_{(V_0,V)}(\mathbf{B})$ is then the Stratonovich solution of the stochastic differential equation (1.1), together with its lift (i.e. here its area process).

Just as before, when the vector fields V_i are almost surely "smooth enough" and y_0 is almost surely finite, the Itô map $I^{y_0}_{(V_0,V)}$ is still well defined and continuous almost surely, and there is no problem at all of definition of $I^{y_0}_{(V_0,V)}(\mathbf{B})$. Therefore, the theory of rough path provides a meaning and a unique solution to the stochastic differential equation (1.1), even when the vector fields and the initial condition depend on the whole Brownian path. Moreover, the continuity of the Itô map provides for free a Wong-Zakai theorem, and is very well adapted to obtaining large deviation principles and support theorems.

The only work not completely given for free by the theory of rough path is to prove that the solution $\mathbf{y}$ of equation (1.1) using the rough path approach is actually solution of the Stratonovich differential equation, i.e. that for all t,

$$y_t = y_0 + \int_0^t V_0(y_u)du + \sum_{i=1}^d \int_0^t V_i(y_u) \circ dB^i_u.$$

When the vector fields and the initial condition are deterministic, this is usually proved using the standard Wong-Zakai theorem.

We provide in this general case here a solution typically in the spirit of rough path, by separating neatly probability theory and differential equation theory. We will show via a deterministic argument that to obtain our result we only need to check that, if D^n is a sequence of subdivisions which steps tends to 0, $\int_0^\cdot B_u \otimes dB^{D^n}_u$ and $\int_0^\cdot B^{D^n}_u \otimes dB^{D^n}_u$ converges in an appropriate topology to $\int_0^\cdot B_u \otimes \circ dB_u$.

The paper is organized as follow: in the first section, we present quickly the theory of rough path (for a more complete presentation, see [19, 20] or [15]). The second section introduce the notion of good rough path sequence and its properties. We will then show that B^n defines a good rough path sequence associated to $\mathbf{B}$, and this will imply that the solution via rough path of equation (1.3) with signal $\mathbf{B}$ is indeed solution of the Stratonovich stochastic differential equation (1.1).

We conclude with a few applications: an approximation/Wong-Zakai result, a large deviation principle, and some remarks on the support theorem.

2. Rough Paths

By path we will always mean a continuous function from $[0,1]$ into a (Lie) group. If x is such a path, $x_{s,t}$ is a notation for $x_s^{-1}.x_t$.

2.1. **Algebraic preliminaries.**

We present the theory of rough paths, in the finite dimensional case purely for simplicity. All arguments are valid in infinite dimension. We equip $\mathbb{R}^d$ with the Euclidean scalar product $\langle .,. \rangle$ and the Euclidean norm $|x| = \langle x,x\rangle^{1/2}$. We denote by $\left(G^n(\mathbb{R}^d), \otimes\right)$ the free nilpotent of step n over $\mathbb{R}^d$, which is imbedded in the tensor algebra $T^n(\mathbb{R}^d) = \bigoplus_{k=0}^n \left(\mathbb{R}^d\right)^{\otimes k}$. We define a family of dilations on the group $G^n(\mathbb{R}^d)$ by the formula

$$\delta_\lambda(1, x_1, \cdots, x_n) = (1, \lambda x_1, \cdots, \lambda^n x_n),$$

where $x_i \in \left(\mathbb{R}^d\right)^{\otimes i}$, $(1, x_1, \cdots, x_n) \in G^n(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$. Inverse, exponential and logarithm functions on $T^n(\mathbb{R}^d)$ are defined by mean of their power series [19, 27].

We define on $\left(\mathbb{R}^d\right)^{\otimes k}$ the Hilbert tensor scalar product and its norm

$$\begin{aligned}\left\langle \sum_{i=1}^n x_i^1 \otimes \cdots \otimes x_i^k, \sum_{j=1}^n y_j^1 \otimes \cdots \otimes y_i^k \right\rangle &= \sum_{i,j} \left\langle x_i^1, y_j^1 \right\rangle \cdots \left\langle x_i^k, y_j^k \right\rangle, \\ |x| &= \langle x, x \rangle^{1/2}.\end{aligned}$$

This yields a family of comptible tensor norms on $\mathbb{R}^d$ and its tensor product spaces. Since all finite dimensional norms are equivalent the Hilbert structure of $\mathbb{R}^d$ was only used for convenience. In fact, one can replace $\mathbb{R}^d$ by a Banach-space and deal with suited tensor norms but this can be rather subtle, see [20] and the references therein.

For $(1, x_1, \cdots, x_n) \in G^n(\mathbb{R}^d)$, with $x_i \in \left(\mathbb{R}^d\right)^{\otimes i}$, we define

$$\|(1, x_1, \cdots, x_n)\| = \max_{i=1,\cdots,n} \left\{ \left(i! \left|x_i\right|\right)^{1/i} \right\}.$$

$\|.\|$ is then a symmetric sub-additive homogeneous norm on $G^n(\mathbb{R}^d)$ ($\|g\| = 0$ iff $g = 1$, and $\|\delta_\lambda g\| = |\lambda| \cdot \|g\|$ for all $(\lambda, g) \in \mathbb{R} \times G^n(\mathbb{R}^d)$, for all $g, h \in G^n(\mathbb{R}^d)$, $\|g \otimes h\| \leq \|g\| + \|h\|$, and $\left\|g^{-1}\right\| = \|g\|$). This homogeneous norm allows us to define on $G^n(\mathbb{R}^d)$ a left-invariant distance with the formula

$$d(g, h) = \left\|g^{-1} \otimes h\right\|.$$

From this distance we can define some distances on the space of continuous paths from $[0, 1]$ into $G^{[p]}(\mathbb{R}^d)$ (p is a fixed real greater than or equal to 1):

(1) the p-variation distance

$$d_{p-var}(x, y) = d\left(x_0, y_0\right) + \sup \left(\sum_i d\left(x_{t_i, t_{i+1}}, y_{t_i, t_{i+1}}\right)^p \right)^{1/p}$$

where the supremum is over all subdivision $(t_i)_i$ of $[0, 1]$. We also define $\|x\|_{p-var} = d_{p-var}(1, x)$, and $\|x\|_{p-var} < \infty$ means that x has finite p-variation.

(2) modulus distances: We say that $\omega : \{(s, t), 0 \leq s \leq t \leq 1\} \to \mathbb{R}^+$ is a control if

$$\begin{cases} \omega \text{ is continuous.} \\ \omega \text{ is super-additive, i.e. } \forall s < t < u,\ \omega(s, t) + \omega(t, u) \leq \omega(t, u). \\ \omega \text{ is zero on the diagonal, i.e. } \omega(t, t) = 0 \text{ for all } t \in [0, 1] \end{cases}$$

When ω is non-zero off the diagonal we introduce the distance $d_{\omega,p}$ by

$$d_{\omega,p}(x, y) = d\left(x_0, y_0\right) + \sup_{0 \leq s < t \leq 1} \frac{d\left(x_{s,t}, y_{s,t}\right)}{\omega(s, t)^{1/p}}.$$

The simplest example of such a control is given by $\omega(s, t) = t - s$. In this case, $d_{\omega,p}$ defines then a notion of $1/p$-Hölder distance on $G^n(\mathbb{R}^d)$-valued paths. In general, if $\|x\|_{\omega,p} = d_{\omega,p}(1, x) < \infty$, we say that x has finite p-variation controlled by ω.

2.2. $G^n(\mathbb{R}^d)$**-Valued Paths.** Let $x : [0,1] \to \mathbb{R}^d$ be a path of bounded variation and define $S(x)$ to be the solution of the ordinary differential equation

$$\begin{aligned} dS(x)_t &= S(x)_t \otimes dx_t \\ S(x)_0 &= \exp(x_0), \end{aligned}$$

where $\otimes$ is the multiplication of $T^n(\mathbb{R}^d)$ and exp is the exponential on $T^n(\mathbb{R}^d)$. $S(x)$ actually takes its values in $G^n(\mathbb{R}^d)$. Moreover, if the 1-variation of x is controlled by ω, then so is the 1-variation of $S(x)$ [19, Theorem 1].

Example 1. *If $n = 2$, $S(x)_t = \exp\left(x(t) + \frac{1}{2}\int_0^t x_u \otimes dx_u - \frac{1}{2}\int_0^t dx_u \otimes x_u\right)$. The term $\frac{1}{2}\int_0^t x_u \otimes dx_u - \frac{1}{2}\int_0^t dx_u \otimes x_u$ is the area between the line joining x_0 and x_t and the path $(x_u)_{0\le u\le t}$.*

Definition 1. *A path $\mathbf{x} : [0,1] \to G^{[p]}(\mathbb{R}^d)$ is a geometric p-rough path if there exists a sequence of paths of bounded variation $(x_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} d_{p-var}\left(S\left(x_n\right), \mathbf{x}\right) = 0.$$

Notation 1. *Let $C^{0,p-var}\left(G^{[p]}(\mathbb{R}^d)\right)$ denote the space of geometric p-rough paths, and $C^{0,\omega,p}\left(G^{[p]}(\mathbb{R}^d)\right)$ the set of paths $\mathbf{x} : [0,1] \to G^{[p]}(\mathbb{R}^d)$ for which there exists a sequence of paths $(x_n)_{n\in\mathbb{N}}$ of bounded variation such that*

$$\lim_{n\to\infty} d_{\omega,p}\left(S\left(x_n\right), \mathbf{x}\right) = 0.$$

In particular, any $G^{[p]}(\mathbb{R}^d)$-valued path with finite q-variation, $q < p$, is a geometric p-rough path, and a geometric p-rough path has finite p-variation. We refer to [11] for a "Polish" study of the space of geometric rough paths.

If $\mathbf{x}$ is a geometric p-rough path, we denote by $\mathbf{x}^i_{s,t}$ the projection of $\mathbf{x}$ onto $\left(\mathbb{R}^d\right)^{\otimes i}$. We also say that $\mathbf{x}$ lies above $\mathbf{x}^1$.

2.3. **The Itô map.** We fix $p \ge 1$, and a control ω.

When A is vector field on $\mathbb{R}^d$, we denote by d^kA its k^{th} derivative (with the convention $d^0A = A$), and its γ-Lipschitz norm by

$$\|A\|_{Lip(\gamma)} = \max\left\{\max_{k=0,\cdots,[\gamma]} \left\|d^kA\right\|_\infty, \left\|d^{[\gamma]}A\right\|_{\gamma-[\gamma]}\right\},$$

where $\|.\|_\infty$ is the sup norm and $\|.\|_\beta$ the β-Hölder norm, $0 \le \beta < 1$. If $\|A\|_{Lip(\gamma)} < \infty$, we say that A is a $Lip\left(\gamma\right)$-vector fields on $\mathbb{R}^d$.

We consider $V = (V_1, \cdots, V_d)$, where the V_i are $Lip\left(p+\varepsilon\right)$-vector fields on $\mathbb{R}^n$, $\varepsilon > 0$. V can be identified with a linear map from $\mathbb{R}^d$ into $Lip\left(p+\varepsilon\right)$-vector fields on $\mathbb{R}^n$,

$$V(y)(dx^1, \cdots, dx^d) = \sum_{i=1}^d V_i(y)dx^i.$$

For a $\mathbb{R}^d$-valued path x of bounded variation, we define y to be the solution of the ordinary differential equation

$$\left\{\begin{array}{l} dy_t = V(y_t)dx_t \\ y_0 = y_0. \end{array}\right. \tag{2.1}$$

Lifting x and $(x \oplus y)$ to $S(x)$ and $S(x \oplus y)$ (their canonical lift to paths with values in the free nilpotent group of step p), we consider the map which at $S(x)$ associates $S(x \oplus y)$. We denote it $I_{y_0,V}$. We refer to [19, 20] for the following theorem:

Theorem 1 (Universal Limit Theorem (Lyons))**.** *The map $I_{y_0,V}$ is continuous from $\left(C^{0,\omega,p}\left(G^{[p]}(\mathbb{R}^d)\right), d_{\omega,p}\right)$ into $\left(C^{0,\omega,p}\left(G^{[p]}(\mathbb{R}^d \oplus \mathbb{R}^n)\right), d_{\omega,p}\right)$.*

Let x_n be a sequence of path of bounded variation such that $S(x_n)$ converges in the $d_{\omega,p}$-topology to a geometric p-rough path $\mathbf{x}$, and define y_n to be the solution of the differential equation (2.1), where x is replaced by x_n. Then, the Universal Limit Theorem says that $S(x_n \oplus y_n)$ converges in the $d_{\omega,p}$-topology to a geometric p-rough path $\mathbf{z}$. We say that $\mathbf{y}$**,** the projection of $\mathbf{z}$ onto $G^{[p]}(\mathbb{R}^n)$ is the solution of the rough differential equation

$$d\mathbf{y} = V(\mathbf{y})d\mathbf{x}$$

with initial condition y_0. It is interesting to observe that Lyons' estimates actually give that for all $R > 0$ and sequence $\mathbf{x}_n \in C^{0,\omega,p}\left(G^{[p]}(\mathbb{R}^d \oplus \mathbb{R}^n)\right)$ converging to $\mathbf{x} \in C^{0,\omega,p}\left(G^{[p]}(\mathbb{R}^d \oplus \mathbb{R}^n)\right)$ in the $d_{\omega,p}$-topology,

$$\sup_{\substack{|y_0| \leq R \\ \|V\|_{Lip(p+\varepsilon)} \leq R}} d_{\omega,p}\left(I_{y_0,V}(\mathbf{x}_n), I_{y_0,V}(\mathbf{x})\right) \to_{n\to\infty} 0. \tag{2.2}$$

To see this, one first observes that the continuity of integration along of a one-form is uniform over the set of one-forms with Lipschitz norm bounded by a given R. Then, the path $I_{y_0,V}(\mathbf{x})$ over small time is the fixed point of a map (integrating along a one-form) which is a contraction. Reading the estimate in [20], one sees that this map is uniformly, over the set $\left\{|y_0| \leq R; \|V\|_{Lip(p+\varepsilon)} \leq R\right\}$, a contraction with parameter strictly less than 1.

The next theorem was also proved in [20], and deals with the continuity of the flow.

Theorem 2. *If $(y_0^n)_n$ is a $\mathbb{R}^d$-valued sequence converging to y_0, then for all $R > 0$,*

$$\sup_{\substack{\|\mathbf{x}\|_{\omega,p} \leq R \\ \|V\|_{Lip(p+\varepsilon)} \leq R}} d_{\omega,p}\left(I_{y_0^n,V}(\mathbf{x}), I_{y_0,V}(\mathbf{x})\right) \to_{n\to\infty} 0.$$

The next theorem shows that the Itô map it continuous when one varies the vector fields defining the differential equation. It does not seem to have appeared anywhere, despite that its proof does not involve any new ideas.

Theorem 3. *Let $V^n = (V_1^n, \cdots, V_d^n)$ be a sequence of d $Lip(p+\varepsilon)$-vector fields on $\mathbb{R}^n$ such that*

$$\lim_{n\to\infty} \max_i \|V_i^n - V_i\|_{Lip(p+\varepsilon)} = 0.$$

Then, for all $R > 0$,

$$\lim_{n\to\infty} \sup_{\substack{\|\mathbf{x}\|_{\omega,p} \leq R \\ |y_o| \leq R}} d_{\omega,p}\left(I_{y_0,V^n}(\mathbf{x}), I_{y_0,V}(\mathbf{x})\right) = 0.$$

Proof. We use the notations of [19]. First consider the $Lip(p+\varepsilon-1)$-one-forms $\theta_i : \mathbb{R}^d \to Hom\left(\mathbb{R}^d, \mathbb{R}^n\right)$, $i \in \mathbb{N} \cup \{\infty\}$. We assume that θ_n converges to θ_∞ in the

$(p+\varepsilon-1)$-Lipschitz topology when $n\to\infty$. For $n\in\mathbb{N}\cup\{\infty\}$, $\int\theta_n(\mathbf{x})\,d\mathbf{x}$ is the unique rough path associated to the almost multiplicative functional

$$\left(Z(\mathbf{x})_{s,t}^n\right)^i=\sum_{l_1,\cdots,l_i=0}^{[p]-1}d^{l_1}\theta_n\left(\mathbf{x}_s^1\right)\otimes\cdots\otimes d^{l_i}\theta_n\left(\mathbf{x}_s^1\right)\left(\sum_{\pi\in\Pi_{(l_1,\cdots,l_i)}}\pi\left(\mathbf{x}_{s,t}^{i+l_1+\cdots+l_i}\right)\right)$$

(see [19] for the definition of Π). It is obvious that

$$\sup_{\|\mathbf{x}\|_{\omega,p}\leq R}\max_i\frac{\left|\left(Z(\mathbf{x})_{s,t}^n\right)^i-\left(Z(\mathbf{x})_{s,t}^\infty\right)^i\right|}{\omega(s,t)^{i/p}}\to_{n\to\infty}0,$$

which implies by theorem 3.1.2 in [19] that

$$\sup_{\|\mathbf{x}\|_{\omega,p}\leq R}d_{\omega,p}\left(\int\theta_n(\mathbf{x})\,d\mathbf{x},\int\theta_\infty(\mathbf{x})\,d\mathbf{x}\right)\to_{n\to\infty}0.$$

Now consider the Picard iteration sequence $(\mathbf{z}_m^n)_{m\geq0}$ introduced in [19, formula (4.10)] to construct to $I_{y_0,V^n}(\mathbf{x})$:

$$\begin{aligned}\mathbf{z}_0^n&=(0,y_0)\\ \mathbf{z}_{m+1}^n&=\int h_n(\mathbf{z}_m^n)\,d\mathbf{z}_m^n,\end{aligned}$$

where h_n is the one-form defined by the formula $h_n(x,y)(dX,dY)=(dX,V(y)\,dX)$ (by $V(y)dX$ we mean $\sum V_i(y)dX^i$). Lyons proved that

$$\lim_{m\to\infty}\sup_{\substack{\|\mathbf{x}\|_{\omega,p}\leq R\\|y_o|\leq R}}\sup_n d_{\omega,p}(\mathbf{z}_m^n,\mathbf{z}_\infty^n)=0;$$

we have the supremum over all n here because the $p+\varepsilon$-Lipschitz norm of the V^n are uniformly bounded in n. Moreover, we have just seen that for all fixed m, $\lim_{n\to\infty}\sup_{|y_0|\leq R}d_{\omega,p}(\mathbf{z}_m^n,\mathbf{z}_m^\infty)=0$. Therefore, with a 3ε-type argument, we obtain our theorem. □

The three previous theorems actually gives that the map

$$(y_0,V,\mathbf{x})\to I_{y_0,V}(\mathbf{x})$$

is continuous in the product topology $\mathbb{R}^n\times(Lip(p+\epsilon)\text{ on }\mathbb{R}^n)^d\times C^{0,\omega,p}\left(G^{[p]}(\mathbb{R}^d)\right)$.

In the reminder of this section, p is a real in $[2,3)$. Now consider a $Lip(1+\varepsilon)$-vector field on $\mathbb{R}^n$ denoted V_0, and for a path x of bounded variation, we consider y to be the solution of

$$\left\{\begin{array}{l}dy_t=V_0(y_t)dt+V(y_t)dx_t\\ y_t=y_0,\end{array}\right.\tag{2.3}$$

Lifting x and $(x\oplus y)$ to $S(x)$ and $S(x\oplus y)$ we consider the map which at $S(x)$ associates $S(x,y)$. We denote it $I_{y_0,(V_0,V)}$. The following extension of the Universal Limit Theorem was obtained in [17].

Theorem 4. *The map $I_{y_0,(V_0,V)}$ is continuous from $\left(C^{0,\omega,p}\left(G^{[p]}(\mathbb{R}^d)\right),d_{\omega,p}\right)$ into $\left(C^{0,\omega,p}\left(G^{[p]}(\mathbb{R}^d\oplus\mathbb{R}^n)\right),d_{\omega,p}\right)$. More precisely, for all $R>0$,*

$$\sup_{\substack{|y_0|\leq R\\\|V\|_{Lip(p+\varepsilon)}\leq R}}d_{\omega,p}\left(I_{y_0,(V_0,V)}(\mathbf{x}_n),I_{y_0,(V_0,V)}(\mathbf{x})\right)\to_{n\to\infty}0.\tag{2.4}$$

Let x_n be a sequence of paths of bounded variation such that $S(x_n)$ converges in the $d_{\omega,p}$-topology to a geometric p-rough path $\mathbf{x}$, and define y_n to be the solution of equation (2.3) replacing x by x_n. Then, the previous theorem says that $S(x_n \oplus y_n)$ converges in the $d_{\omega,p}$-topology to a geometric p-rough path $\mathbf{z}$. We say that $\mathbf{y}$, the projection of $\mathbf{z}$ onto $G^{[p]}(\mathbb{R}^n)$ is the solution of the rough differential equation

$$d\mathbf{y}_t = V_0(\mathbf{y}_t)dt + V(\mathbf{y}_t)d\mathbf{x}_t$$

with initial condition y_0.

We obtain, as before, the following two theorems:

Theorem 5. *If $(y_0^n)_n$ is a $\mathbb{R}^d$-valued sequence converging to y_0, then*

$$\sup_{\substack{\|\mathbf{x}\|_{\omega,p}\leq R \\ \|V\|_{Lip(p+\varepsilon)}\leq R \\ \|V_0\|_{Lip(1+\varepsilon)}\leq R}} d_{\omega,p}\left(I_{y_0^n,(V_0,V)}(\mathbf{x}), I_{y_0,(V_0,V)}(\mathbf{x})\right) \to_{n\to\infty} 0.$$

Theorem 6. *Let $(V^n = (V_1^n, \cdots, V_d^n))_{n\geq 0}$ be a sequence of d $Lip(p+\varepsilon)$-vector fields on $\mathbb{R}^n$ and $(V_0^n)_{n\geq 0}$ a sequence of $Lip(1+\varepsilon)$-vector fields on $\mathbb{R}^n$, such that*

$$\lim_{n\to\infty} \max\left\{\|V_0^n - V_0\|_{Lip(1+\varepsilon)}, \max_{1\leq i\leq d}\|V_i^n - V_i\|_{Lip(p+\varepsilon)}\right\} = 0.$$

Then, if $\mathbf{x} \in C^{0,\omega,p}\left(G^{[p]}(\mathbb{R}^d)\right)$,

$$\lim_{n\to\infty} \sup_{\substack{\|\mathbf{x}\|_{\omega,p}\leq R \\ |y_o|\leq R}} d_{\omega,p}\left(I_{y_0,\left(V_0^n,V^n\right)}(\mathbf{x}), I_{y_0,(V_0,V)}(\mathbf{x})\right) = 0.$$

2.4. **Solving Anticipative Stochastic Differential Equations Via Rough Paths.** We fix a $p \in (2,3)$ and, for simplicity, the control $\omega(s,t) = t - s$ (i.e. we deal with Hölder topologies), although we could have been more general and have considered a wide class of controls as in [10] (i.e. we could have consider modulus type topologies). We define $\mathbf{B}$ the Stratonovich lift to a geometric p-rough path of the Brownian motion B with the formula

$$\mathbf{B}_t = \left(1, B_t, \int_0^t B_u \otimes \circ dB_u\right).$$

$\mathbf{B}$ is a $G^2\left(\mathbb{R}^d\right)$-valued path, and almost surely, $\|\mathbf{B}\|_{\omega,p} < \infty$.

Consider V_0 a random vector field on $\mathbb{R}^n$ almost surely in $Lip(1+\varepsilon)$, i.e. a measurable map from

$$V_0 : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$$

such that $V_0(\omega, .) \in Lip(1+\varepsilon)$ for ω in a set a full measure, and $V = (V_1, \cdots, V_d)$, where $V_1, \cdots, V_d$ are random vector fields on $\mathbb{R}^n$ almost surely in $Lip(2+\varepsilon)$, and a random variable $y_0 \in \mathbb{R}^n$ finite almost surely.

$I_{y_0,(V_0,V)}(\mathbf{B})$ is then almost surely well defined, and its projection onto $G^2(\mathbb{R}^n)$ is the solution, in the rough path sense, of the anticipative stochastic differential equation

$$d\mathbf{y}_t = V_0(\mathbf{y}_t)dt + V(\mathbf{y}_t)d\mathbf{B}_t \tag{2.5}$$

with initial condition y_0.

The next section introduces the notion of good rough paths sequence, and its properties. Showing that linear approximations of Brownian motion form good rough path sequences (in some sense that will be precise later on) will prove that

$\mathbf{y}^1$ is solution of the anticipative Stratonovich stochastic differential equation (1.1). In particular, the solution that we construct coincides with the one constructed in [26].

3. Good Rough Path Sequence

3.1. **Definitions.** We fix a parameter $p > 2$, and a control ω. $\mathbb{R}^d$ and $\widetilde{\mathbb{R}^d}$ will denote two identical copies of $\mathbb{R}^d$.

Let $p > 2$, and q such that $1/p + 1/q > 1$. We consider x and y two $\mathbb{R}^d$-valued paths of bounded variation. We let $\mathbf{y} = S(y)$ to be the canonical lift of y to a $G^{[p]}\left(\mathbb{R}^d\right)$-valued path. We let

$$S'(x, S(y)) := S(x \oplus y) \tag{3.1}$$

be the canonical lift of $x \oplus y$ to a $G^{[p]}\left(\mathbb{R}^d \oplus \mathbb{R}^d\right)$-valued path and

$$S''(S(x)) := S'(x, S(x)) = S(x \oplus x) \tag{3.2}$$

be the canonical lift of $y \oplus y$ to A $G^{[p]}\left(\mathbb{R}^d \oplus \mathbb{R}^d\right)$-valued path.

Proposition 1. *Let x be a R^d-valued path of finite q-variation, and $\mathbf{y}$ a $G^{[p]}\left(\mathbb{R}^d\right)$-valued path of finite p-variation. Let (x_n, y_n) be a sequence of $\mathbb{R}^d \oplus \mathbb{R}^d$-valued path such that $d_{\omega,p}(x_n, x) \to_{n\to\infty} 0$ and $d_{\omega,p}(S(y_n), \mathbf{y}) \to_{n\to\infty} 0$. Then,*
(i) $S'(x_n, S(y_n))$ converges in the $d_{\omega,p}$-topology, and the limit is independent of the choice of the sequence (x_n, y_n). We denote this limit element $S'(x, \mathbf{y})$.
(ii) $S''(S(y_n), S(y_n))$ converges in $d_{\omega,p}$-topology, and the limit is independent of the choice of the sequence y_n. We denote this elememt $S''(\mathbf{y}, \mathbf{y})$.

Proof. This is simply obtained using theorem 3.1.2 in [19], which says that the procedure which at an almost multiplicative functional associates a rough path is continuous, and we leave the details to the reader. □

Example 2. *If $2 \le p < 3$,*

$$\begin{aligned} & S'(x, \mathbf{y})_t \\ = \; & \left(1, x_t \oplus \mathbf{y}^1_t, \int_0^t x_u \otimes dx_u \oplus \int_0^t x_u \otimes d\mathbf{y}^1_u \oplus \int_0^t \mathbf{y}^1_u \otimes dx_u \oplus \mathbf{y}^2_t\right). \end{aligned}$$

The three integrals are well defined Young's integrals [33].

We introduce the notion of a good p-rough path sequence.

Definition 2. *Let $(x_n)_n$ be a sequence of $\mathbb{R}^d$-valued paths of bounded variation, and $\mathbf{x}$ a geometric p-rough path. We say that $(x_n)_{n\in\mathbb{N}}$ is a good p-rough path sequence (associated to $\mathbf{x}$) (for the control ω) if and only if*

$$\lim_{n\to\infty} d_{\omega,p}(S'(x_n, \mathbf{x}), S''(\mathbf{x})) = 0. \tag{3.3}$$

In particular, if $(x_n)_n$ is a good p-rough path sequence associated to $\mathbf{x}$, for the control ω, x_n converges to $\mathbf{x}$ in the topology induced by $d_{\omega,p}$.

Proposition 2. *Assume $2 \le p < 3$. The sequence $(x(n))_n$ of paths of bounded variation is good rough path sequence associated to $\mathbf{x}$, for the control ω, if and only*

if

$$\begin{aligned}
\lim_{n\to\infty} \sup_{0\leq s<t\leq 1} \frac{\left|x(n)_{s,t} - \mathbf{x}^1_{s,t}\right|}{\omega(s,t)^{1/p}} &= 0,\\
\lim_{n\to\infty} \sup_{0\leq s<t\leq 1} \frac{\left|\int_s^t x(n)_{s,u}\otimes dx(n)_u - \mathbf{x}^2_{s,t}\right|}{\omega(s,t)^{2/p}} &= 0,\\
\lim_{n\to\infty} \sup_{0\leq s<t\leq 1} \frac{\left|\int_s^t \mathbf{x}^1_{s,u}\otimes dx(n)_u - \mathbf{x}^2_{s,t}\right|}{\omega(s,t)^{2/p}} &= 0.
\end{aligned}$$

Proof. $d_{\omega,p}\left(S'\left(x(n),\mathbf{x}\right), S''\left(\mathbf{x},\mathbf{x}\right)\right)$ is bounded by a constant times

$$\max\left\{A_1^n, \sqrt{A_2^n}, \sqrt{A_3^n}, \sqrt{A_4^n}\right\},$$

where

$$\begin{aligned}
A_1^n &= \sup_{0\leq s<t\leq 1} \frac{\left|\mathbf{x}^1_{s,t} - x(n)_{s,t}\right|}{\omega(s,t)^{1/p}},\\
A_2^n &= \sup_{0\leq s<t\leq 1} \frac{\left|\int_s^t x(n)_{s,u}\otimes dx(n)_u - \mathbf{x}^2_{s,t}\right|}{\omega(s,t)^{2/p}},\\
A_3^n &= \sup_{0\leq s<t\leq 1} \frac{\left|\int_s^t x(n)_{s,u}\otimes d\mathbf{x}^1_u - \mathbf{x}^2_{s,t}\right|}{\omega(s,t)^{2/p}},\\
A_4^n &= \sup_{0\leq s<t\leq 1} \frac{\left|\int_s^t \mathbf{x}^1_{s,u}\otimes dx(n)_u - \mathbf{x}^2_{s,t}\right|}{\omega(s,t)^{2/p}}.
\end{aligned}$$

Let π be the linear operator from $\mathbb{R}^d\otimes\mathbb{R}^d$ onto itself defined by $\pi\left(x\otimes y\right) = \pi\left(y\otimes x\right)$. Observe that $\pi\left(\mathbf{x}^2_{s,t}\right) = \left(\mathbf{x}^1_{s,t}\right)^{\otimes 2} - \mathbf{x}^2_{s,t}$. Also, we have for all $z\in\mathbb{R}^d\otimes\mathbb{R}^d$, $|z| = |\pi(z)|$. Using this property and an integration by part, we see that

$$\begin{aligned}
&\left|\int_s^t \mathbf{x}^1_{s,u}\otimes dx(n)_u - \mathbf{x}^2_{s,t}\right|\\
=\ &\left|\mathbf{x}^1_{s,t}\otimes x(n)_{s,t} - \int_s^t d\mathbf{x}^1_{s,u}\otimes x(n)_u - \mathbf{x}^2_{s,t}\right|\\
=\ &\left|\mathbf{x}^1_{s,t}\otimes\left(x(n)_{s,t} - \mathbf{x}^1_{s,t}\right) - \int_s^t d\mathbf{x}^1_{s,u}\otimes x(n)_u + \left(\left(\mathbf{x}^1_{s,t}\right)^{\otimes 2} - \mathbf{x}^2_{s,t}\right)\right|\\
\leq\ &\left|\mathbf{x}^1_{s,t}\right|.\left|x(n)_{s,t} - \mathbf{x}^1_{s,t}\right| + \left|\mathbf{x}^2_{s,t} - \int_s^t x(n)_{s,u}\otimes d\mathbf{x}^1_u\right|.
\end{aligned}$$

Hence, $A_4^n \leq \|\mathbf{x}\|_{\omega,p}.A_1^n + A_3^n$, which proves the proposition. □

By definition of a geometric p-rough path, there always exists a sequence of smooth x_n such that $S(x_n)$ converges to $\mathbf{x}$ in the topology induced by $d_{\omega,p}$. However, this does not imply that equation (3.3) holds (or equivalently for $\lceil p\rceil = 2$: that the conditions in proposition 2 hold). We now give an example of a geometric rough path $\mathbf{x}$ for which there exists no good sequence associated to it.

Example 3. *Consider*

$$\begin{aligned}\mathbf{x}_t &= \exp(t[e_1,e_2]) \\ &= \exp(t(e_1\otimes e_2 - e_2\otimes e_1))\end{aligned}$$

where e_1, e_2 is a basis of $\mathbb{R}^2$. Note that $\mathbf{x}$ is a geometric p-rough path and

$$\mathbf{x}_t^1 \equiv 0 \in \mathbb{R}^2, \qquad x_t^2 \equiv t[e_1,e_2] \in \mathbb{R}^2\otimes\mathbb{R}^2.$$

If x_n is a sequence of smooth paths in $\mathbb{R}^2$, then $\int_0^t \mathbf{x}^1 \otimes dx_n \equiv 0$ trivially converges to $0 \in \mathbb{R}^2\otimes\mathbb{R}^2$. On the other hand, if x_n was a good p-rough path sequence associated to $\mathbf{x}$ it should converge to $t[e_1,e_2]$

3.2. **Stability of Good Rough Path Sequences.** Nonetheless, there are some good news. Good rough path sequences are stable under integration, as shows the following theorem. We fix a control ω.

Theorem 7. *Let $(x_n)_{n\in\mathbb{N}}$ denote a good p-rough path sequence associated to $\mathbf{x}$ for the control ω, and $\theta : \mathbb{R}^d \to Hom(\mathbb{R}^d, \mathbb{R}^n)$ be a $Lip\,(p-1+\varepsilon)$ one-form, with $\varepsilon > 0$. Then $\left(\int \theta(x_n)dx_n\right)_n$ is a good rough path sequence associated to $\int \theta(\mathbf{x})d\mathbf{x}$ for the control ω. Moreover,*

$$\lim_{n\to\infty} S\left(\int \theta(\mathbf{x}^1)dx_n\right) = \int \theta(\mathbf{x})d\mathbf{x}. \tag{3.4}$$

Proof. Consider the $Lip\,(p-1+\varepsilon)$ one-form

$$\begin{aligned}\widetilde{\theta} \quad &: \quad \mathbb{R}^d\oplus\mathbb{R}^d \to Hom(\mathbb{R}^d\oplus\mathbb{R}^d, \mathbb{R}^n\oplus\mathbb{R}^n) \\ ((x,y),(dX,dY)) \quad &\to \quad \theta(x)dX \oplus \theta(y)dY.\end{aligned}$$

All the limits considered in this proof are to be understood to be in the topology induced by $d_{\omega,p}$. By the continuity of the integral [19, 20] and of the operator S' (by proposition 1),

$$\begin{aligned}S'\left(\int\theta(x_n)dx_n, \int\theta(\mathbf{x})d\mathbf{x}\right) &= \lim_{m\to\infty} S'\left(\int\theta(x_n)dx_n, S\left(\int\theta(x_m)dx_m\right)\right) \\ &= \lim_{m\to\infty} S\left(\int\theta(x_n)dx_n \oplus \int\theta(x_m)dx_m\right) \\ &= \lim_{m\to\infty} S\left(\int\widetilde{\theta}(x_n,x_m)d(x_n,x_m)\right) \\ &= \int\tilde{\theta}\left(S\left(x_n\oplus x_m\right)\right)dS\left(x_n\oplus x_m\right) \\ &= \int\widetilde{\theta}(S'(x_n,\mathbf{x}))dS'(x_n,\mathbf{x}).\end{aligned}$$

In in the last line we have used once again the continuity of the integral and the assumption that (x_n) is a good p-rough path sequence associated to $\mathbf{x}$.

Therefore, $S'\left(\int \theta(x_n)dx_n, \int \theta(\mathbf{x})d\mathbf{x}\right)$ converges when $n \to \infty$ to

$$\begin{aligned}\int \widetilde{\theta}(S''(\mathbf{x}))\, dS''(\mathbf{x}) &= \lim_n \int \widetilde{\theta}(S''(S(x_n)))dS''(S(x_n)) \\ &= \lim_n \int \widetilde{\theta}(S(x_n \oplus x_n))dS(x_n \oplus x_n) \\ &= \lim_n S\left(\int \theta(x_n)dx_n \oplus \int \theta(x_n)dx_n\right) \\ &= S''\left(\int \theta(\mathbf{x})d\mathbf{x}\right),\end{aligned}$$

which proves the first assertion.

For the second one, consider the map

$$\begin{array}{c}\widehat{\theta} : \mathbb{R}^d \oplus \mathbb{R}^d \to Hom(\mathbb{R}^d \oplus \mathbb{R}^d, \mathbb{R}^n) \\ ((x,y),(dX,dY)) \to \theta(y)dX.\end{array}$$

By the continuity of the integral, we obtain that $\int \widehat{\theta}(S'(x_n,\mathbf{x}))dS'(x_n,\mathbf{x})$ converges in the topology induced by $d_{\omega,p}$ to $\int \widehat{\theta}(S''(\mathbf{x}))dS''(\mathbf{x})$. This is our result as $\int \widehat{\theta}(S'(x_n,\mathbf{x}))dS'(x_n,\mathbf{x}) = S\left(\int \theta(\mathbf{x})dx_n\right)$. □

We show now that good rough path sequences are stable under the Itô map.

Theorem 8. *Let $V_1, \cdots, V_d$ be d elements of $Lip(p+\varepsilon)$-vector fields on $\mathbb{R}^n$, and $V = (V_1, \cdots, V_d)$ thought as a linear map from $\mathbb{R}^d$ into $Lip(p+\varepsilon)$ vector fields on $\mathbb{R}^n$. Assume that $(x_n)_n$ is a good p-rough path sequence associated to $\mathbf{x}$ for the control ω. Denote by y_n the solution of the ordinary differential equation*

$$\left\{\begin{array}{l} dy_n(t) = V(y_n(t))dx_n(t) \\ y_n(0) = y_0.\end{array}\right.$$

Then $(x_n \oplus y_n)_n$ is a good p-rough path sequence associated to $\mathbf{z} = I_{y_0,V}(\mathbf{x})$ for the control ω.

Proof. The proof is similar to the proof of the theorem 7. Denote by $\widetilde{V}$ the linear map from $\mathbb{R}^d \oplus \mathbb{R}^d$ into the $Lip(p+\varepsilon)$ vector fields on $\mathbb{R}^n$ by the formula $\widetilde{V}(y_1,y_2)(dx_1,dx_2) = (V(y_1)dx_1, V(y_2)dx_2)$. First notice that

$$I_{(y_0,y_0),\widetilde{V}}(S''(\mathbf{x})) = S''(I_{y_0,V}(\mathbf{x}))$$

and that

$$S'(z_n,\mathbf{z}) = I_{(y_0,y_0),\widetilde{V}}(S'(x_n,\mathbf{x})).$$

Hence, since x_n is a good sequence and by continuity of the Itô map, $S'(z_n,\mathbf{z})$ converges as $n \to \infty$ to

$$I_{(y_0,y_0),\widetilde{V}}(S''(\mathbf{x})) = S''(I_{y_0,V}(\mathbf{x})) = S''(\mathbf{z}).$$

□

From our two previous theorems, we immediately obtain the following corollary:

Corollary 1. *We keep the notation of theorem 8. Then, in the topology induced by $d_{\omega,p}$,*

$$\mathbf{y} = \lim_{n\to\infty} S\left(y_0 + \int V\left(\mathbf{y}^1_u\right) dx_n(u)\right).$$

In particular, looking at the first level of this equation, we obtain that

$$y_0 + \int_0^{\cdot} V(y_u) dx_n(u) \to y.$$

Remark 1. *In the previous theorem and its corollary, with no modification in the proof, for $2 \leq p < 3$, one can obtain the same results replacing the map $I_{y_0,V}$ by the map $I_{y_0,(V_0,V)}$, where V_0 is a $Lip(1+\varepsilon)$-vector fields on $\mathbb{R}^n$. In other words, one can consider differential equations with a time drift and (almost) minimal smoothness condition on V_0.*

3.3. **Piecewise-linear Approximation of Brownian motion as a Good Rough Path Sequence.** We fix a $p \in (2,3)$ and $\omega(s,t) = t-s$. We recall from the introduction that B is a d-dimensional Brownian motion, and that $\mathbf{B}$ is the Stratonovich lift of B to a geometric p-rough path. Let B^D be the D-linear approximation of B (equation (1.2)). Let D^n a sequence of subdivisions of $[0,1]$ which steps tends to 0 when n tends to infinity. If $D^n = \left(\frac{k}{2^n}, 0 \leq k \leq 2^n\right)$, we know from [20] that, almost surely, $S\left(B^{D^n}\right)$ converges in p-variation to $\mathbf{B}$. If D^n is an increasing sequence of subdivision, i.e. if $D^{n+1} \subset D^n$ for all n, a martingale argument proved in [10] that almost surely and in L^q, $q \geq 1$, $S\left(B^{D^n}\right)$ converges in $1/p$-Hölder distance (and even some modulus distances) to $\mathbf{B}$. The following theorem goes a bit deeper in the convergence of piecewise linear approximations of the Brownian motion.

Theorem 9. *Let D^n be a sequence of subdivision which steps size tends to 0. Then $d_{\omega,p}\left(S'\left(B^{D^n}, \mathbf{B}\right), S''(\mathbf{B},\mathbf{B})\right)$ converges when n tends to infinity to 0 in L^q, $q \geq 1$ and in probability.*
If $D^n = \left(\frac{k}{2^n}, 0 \leq k \leq 2^n\right)$, the convergence also holds almost surely, i.e. $B^n = B^{D^n}$ is almost surely a good p-rough path sequence associated to $\mathbf{B}$.

We decompose the proof in four lemmas.

Lemma 1. *For all $q > p > 1$, the norm L^q and L^p norm on the k^{th} Wiener chaos.*

Proof. This is a simple consequence of the hypercontractivity of the Ornstein-Uhlenbeck semigroup, see [25, p57] for example. □

Lemma 2. *Let D be a subdivision of $[0,1]$. Then, for all $q \geq 1$ and $p' > 1/H$, there exists $\mu > 0$ and $C_{p',q,\mu} < \infty$ such that for all $s < t \in D$*

$$\left\| \int_s^t B_{s,u} \otimes dB_u^D - \mathbf{B}_{s,t}^2 \right\|_{L^q} \leq C_{p',q,\mu} |D|^{\mu} |t-s|^{2/p'}$$

$$\left\| \int_s^t B_{s,u}^D \otimes B_{s,u}^D - \mathbf{B}_{s,t}^2 \right\|_{L^q} \leq C_{p',q,\mu} |D|^{\mu} |t-s|^{2/p'}$$

Proof. From the previous lemma, we can take $q = 2$. We write $(s,t) = (t_m, t_n)$, with $0 \leq m < n \leq |D|$, where $D = (t_i)_{0 \leq i \leq D}$. It is easy to see that

$$\begin{aligned} \int_s^t B_{s,u} \otimes dB_u^D - \mathbf{B}_{s,t}^2 &= \sum_{k=m}^{n-1} \left(\int_{t_k}^{t_{k+1}} B_{t_k,u} \otimes dB_u^D - \mathbf{B}_{t_k,t_{k+1}}^2 \right) \\ \int_s^t B_{s,u}^D \otimes B_{s,u}^D - \mathbf{B}_{s,t}^2 &= \sum_{k=m}^{n-1} \left(\left(B_{t_k,t_{k+1}}\right)^{\otimes 2} - \mathbf{B}_{t_k,t_{k+1}}^2 \right). \end{aligned}$$

Therefore, by independence of incrememement,

$$\begin{aligned}\left\|\int_s^t B_{s,u}\otimes dB_u^D-\mathbf{B}_{s,t}^2\right\|_{L^2}^2 &= \sum_{k=m}^{n-1}E\left(\left(\int_{t_k}^{t_{k+1}}B_{t_k,u}\otimes dB_u^D-\mathbf{B}_{t_k,t_{k+1}}^2\right)^2\right)\\ &= C\sum_{k=m}^{n-1}\left(t_{k+1}-t_k\right)^2\\ &\leq C\left|D\right|^{2-4/p'}\sum_{k=m}^{n-1}\left(t_{k+1}-t_k\right)^{4/p'}\\ &\leq C\left|D\right|^{2-4/p'}\left(t-s\right)^{4/p'}.\end{aligned}$$

We also obtain the same estimate for $\left\|\int_s^t B_{s,u}^D\otimes B_{s,u}^D-\mathbf{B}_{s,t}^2\right\|_{L^q}$, which concludes the proof. □

Lemma 3. *Let D be a subdivision of $[0,1]$. Then, for all $q\geq 1$, there exists $\mu>0$ and $C_{q,\mu}<\infty$ such that*

$$\left\|\sup_{0\leq s<t\leq 1}\frac{\left|\int_s^t B_{s,u}\otimes dB_u^D-\mathbf{B}_{s,t}^2\right|}{\left|t-s\right|^{2/p}}\right\|_{L^q}\leq C_{q,\mu}\left|D\right|^{\mu} \tag{3.5}$$

$$\left\|\sup_{0\leq s<t\leq 1}\frac{\left|\int_s^t B_{s,u}^D\otimes B_{s,u}^D-\mathbf{B}_{s,t}^2\right|}{\left|t-s\right|^{2/p}}\right\|_{L^q}\leq C_{q,\mu}\left|D\right|^{\mu} \tag{3.6}$$

Proof. We only prove equation (3.5) as the proof for the other estimates is similar. We define $X_{s,t}^D=\int_s^t B_{s,u}\otimes dB_u^D-\mathbf{B}_{s,t}^2$, where $D=\left(0=t_0\leq t_1<\cdots<t_{|D|}=1\right)$ is fixed subdivision of $[0,1]$. First assume that $t_i\leq s<t\leq t_{i+1}$. We let $p'\in(2,p)$; it is easy to check that there exists $C<\infty$ independent of D such that $\left\|B^D\right\|_{\omega,p'}\leq C\left\|B\right\|_{\omega,p'}$. Therefore,

$$\begin{aligned}\frac{\left|X_{s,t}^D\right|}{\left|t-s\right|^{2/p}} &= \frac{\left|\mathbf{B}_{s,t}^2\right|+\left|\left(\int_s^t B_{s,u}du\right)\otimes\frac{B_{t_i,t_{i+1}}}{t_{i+1}-t_i}\right|}{\left|t-s\right|^{2/p}}\\ &\leq \frac{C\left\|B\right\|_{\omega,p'}^2\left(\left|t-s\right|^{2/p'}+\left(\int_s^t(u-s)^{1/p'}du\right)\left(t_{i+1}-t_i\right)^{1/p'-1}\right)}{\left|t-s\right|^{2/p}}\\ &\leq C\left\|B\right\|_{\omega,p'}^2\left(\left|t-s\right|^{2/p'-2/p}+\left(t-s\right)^{1+1/p'-2/p}\left(t_{i+1}-t_i\right)^{1/p'-1}\right).\end{aligned}$$

Bounding $t-s$ and $t_{i+1}-t_i$ by $|D|$, the mesh size of D, we obtain that

$$\max_{k\in\{0,\cdots,|D|\}}\sup_{t_k\leq s<t\leq t_{k+1}}\frac{\left|X_{s,t}^D\right|}{\left|t-s\right|^{2/p}}\leq C\left\|B\right\|_{\omega,p'}^2\left|D\right|^{2\left(\frac{1}{p'}-\frac{1}{p}\right)}. \tag{3.7}$$

Defining t_D to be the biggest real in D less than or equal to t, and s^D the smallest real in D greater than to s. The above estimate rewrites

$$\sup_{\substack{0\leq s<t\leq 1\\ t_D<s^D}}\frac{\left|X_{s,t}^D\right|}{\left|t-s\right|^{2/p}}\leq C\left\|B\right\|_{\omega,p'}^2\left|D\right|^{2\left(\frac{1}{p'}-\frac{1}{p}\right)} \tag{3.8}$$

and an L^q-estimate is immediate. Hence we are left to prove that

$$\left\| \sup_{\substack{0\le s<t\le 1 \\ s^D\le t_D}} \frac{\left|X^D_{s,t}\right|}{|t-s|^{2/p}} \right\|_{L^q} \le C_\mu |D|^\mu .$$

Now observe that for all $s<t<u$,

$$X^D_{s,u} = X^D_{s,t} + X^D_{t,u} + B_{s,t} \otimes \left(B^D_{t,u} - B_{t,u}\right). \tag{3.9}$$

Hence, for all $s<t$ such that $s^D \le t_D$, so that

$$X^D_{s,t} = X^D_{s,s^D} + X^D_{s^D,t_D} + X^D_{t_D,t} + B_{s,t_D} \otimes \left(B^D_{t_D,t} - B_{t_D,t}\right).$$

From (3.8))

$$\left\| \sup_{\substack{0\le s<t\le 1 \\ s^D\le t_D}} \frac{\left|X^D_{s,s^D}\right|}{|s^D-s|^{2/p}} + \frac{\left|X^D_{t_D,t}\right|}{|t-t_D|^{2/p}} \right\|_{L^q} \le C_\mu |D|^\mu .$$

Compatibility of tensor norms shows - similar as above but easier -

$$\frac{\left|B_{s,t_D} \otimes \left(B^D_{t_D,t} - B_{t_D,t}\right)\right|}{|t-s|^{2/p}} \le C \|B\|^2_{\omega,p'} |D|^{1/p'-1/p};$$

therefore, we just need to check that for some $\mu>0$,

$$\left\| \sup_{0\le s<t\le 1} \frac{\left|X^D_{s^D,t_D}\right|}{|t-s|^{2/p}} \right\|_{L^q} \overset{\text{trivial}}{\le} \left\| \max_{s<t\in D} \frac{\left|X^D_{s,t}\right|}{|t-s|^{2/p}} \right\|_{L^q} \le C_\mu |D|^\mu . \tag{3.10}$$

Now consider $\widetilde{X}^D : [0,1] \to \mathbb{R}^d \otimes \mathbb{R}^d$ the linear path on the intervals $[t_i, t_{i+1}]$, such that for all i, $\widetilde{X}^D_{t_i,t_{i+1}} = X^D_{t_i,t_{i+1}}$, i.e. for all t,

$$\widetilde{X}^D_{t_D,t} = X^D_{t_i,t} + \frac{t-t_D}{t^D-t_D} X^D_{t_D,t^D}.$$

The previous lemma showed that for all $s,t\in D$, $\left\|\widetilde{X}^D_{s,t}\right\|_{L^q} \le C_{p',q,\mu} |D|^\mu |t-s|^{2/p'}$. From this, it is easy to check that (changing the constant), this equality remains true for all $s,t\in[0,1]$. Define $B_\alpha = \int_0^1\int_0^1 \left|\frac{\widetilde{X}^D_{s,t}}{(t-s)^\alpha}\right|^{2m} dsdt$. If $\alpha < \frac{2}{p'} + \frac{1}{2m}$, then $\mathbb{E}(B_\alpha) \le C_m |D|^{\mu m}$. By Garsia, Rodemich and Rumsey's theorem [31], we obtain that

$$\sup_{0\le s<t\le 1} \frac{\left|\widetilde{X}^D_{s,t}\right|}{(t-s)^{\alpha-1/m}} \le C B_\alpha^{1/2m}.$$

Therefore, there exists two constants $\vartheta>0$ and $C_\vartheta<\infty$ (independent of D) such that

$$\left\| \max_{s<t\in D} \frac{\left|X^D_{s,t}\right|}{|t-s|^{2/p}} \right\|_{L^q} \le C_\vartheta |D|^\vartheta .$$

□

Lemma 4. *Let D be a subdivision of $[0,1]$. Then, for all $q \geq 1$, there exists $\nu > 0$ and $C_{q,\nu} < \infty$ such that*

$$\left\| d_{\omega,p}\left(S'\left(B^D, \mathbf{B}\right), S''\left(\mathbf{B}, \mathbf{B}\right)\right)\right\|_{L^q} \leq C_{q,\nu} |D|^{\nu}.$$

Proof. It is easy to check that $\sup_{0\leq s<t\leq 1} \frac{\left|\mathbf{B}^1_{s,t} - B^D_{s,t}\right|}{|t-s|^{1/p}} \leq C \|\mathbf{B}\|_{\omega,q} |D|^{1/q-1/p}$,where $2 < q < p$. The previous lemmae together with proposition 2 give the result. □

We can now turn to the proof of theorem 9.

Proof. The first part is obvious from the previous lemma. From the second part,

$$\begin{aligned}\mathbb{P}\left(d_{\omega,p}\left(S'\left(B^n, \mathbf{B}\right), S''\left(\mathbf{B}, \mathbf{B}\right)\right) \geq \frac{1}{n}\right) &\leq n^2 \left\| d_{\omega,p}\left(S'\left(B^n, \mathbf{B}\right), S''\left(\mathbf{B}, \mathbf{B}\right)\right)\right\|^2_{L^2} \\ &\leq C_{q,\nu} n^2 2^{-n\nu}.\end{aligned}$$

Hence, by Borel-Cantelli's lemma, we obtain that, almost surely, $S\prime\left(B^n, \mathbf{B}\right)$ converges to $S''\left(\mathbf{B}, \mathbf{B}\right)$ in the topology induced by $d_{\omega,p}$. □

3.4. Piecewise-linear Approximation of Fractional Brownian motion as a

Good Rough Path Sequence. Let W_H be fractional Browian motion of Hurst paramter $H \in (0,1)$, defined as Gaussian process on $[0,1]$ with zero mean and covariance

$$\mathbb{E}\left(W_H(t) W_H(s)\right) = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right].$$

From Kolmogorov's criterion, W_H has "$H-\epsilon$"-Hölder regularity for any $\epsilon \in (0,H)$. For $H > 1/2$ this is enough regularity to determine all iterated integrals as Young-integrals. Then a pathwise ODE-theory is possible, see [20]. When $H = 1/2$ we are dealing with usual Brownian motion. For $H \in \left(\frac{1}{4}, \frac{1}{2}\right)$ the canonical lift to a geometric rough path $\mathbf{W}_H$ was constructed in [5]. According to rough path theory $H \in (\frac{1}{3}, \frac{1}{2})$ requires $\mathbf{W}_H = \left(1, \mathbf{W}^1_H, \mathbf{W}^2_H\right)$ while for $H \in (\frac{1}{4}, \frac{1}{3}]$ we need additionally the third iterated integrals,

$$\mathbf{W}_H = \left(1, \mathbf{W}^1_H, \mathbf{W}^2_H, \mathbf{W}^3_H\right).$$

As in the Brownian case, we work with the Hölder control $\omega(s,t) = t - s$ and take

$$1/H < p < \lfloor 1/H \rfloor + 1.$$

The proofs of the following two theorems are found in the appendix.

Theorem 10. *In $d_{\omega,p}$-topology*

$$S\left(W^{D^n}\right) \to \mathbf{W}_H$$

in L^q for any $q \in [1,\infty)$ and in probability.

Theorem 11. *In $d_{\omega,p}$-topology*

$$S'\left(W_H^{D^n}, \mathbf{W}\right) \to S''\left(\mathbf{W}_H, \mathbf{W}_H\right)$$

in L^q for any $q \in [1,\infty)$ and in probability. If $D^n = \left(\frac{k}{2^n}, 0 \leq k \leq 2^n\right)$, the convergence also holds almost surely. In other words, $\left(W_H^{D^n}\right)_{n\in\mathbb{N}}$ is a.s. a good p-rough path sequence associated to $\mathbf{W}_H$.

4. Anticipative and Fractional Stochastic Analysis

Combing the results of the preceding sections yields a powerful theory which unifies anticipation with non-semimartingale driving noise such as fractional Brownian motion. For the reader's convenience, we always discuss the case of driving Brownian motion separately (although the arguments we developed allow for rather similar proofs). As concrete applications, we discuss Wong-Zakai results, support theorems and large deviations in the simultaneous context of anticipation and driving fractional Brownian motion.

4.1. **Rough Paths Solution Equals Stratonovich Solution.**

4.1.1. *The case of driving Brownian motion.* Fix $p \in (2,3)$ and $\omega(s,t) = t-s$. Consider random vector-fields V_0 and $V = (V_1, \cdots, V_d)$, where V_0 (resp. $V_1, \cdots, V_d$) is almost surely a $Lip(1+\varepsilon)$ (resp. $Lip(2+\varepsilon)$) vector field on $\mathbb{R}^n$, and an a.s. finite random variable $y_0 \in \mathbb{R}^n$. As earlier, $\mathbf{B}$ denotes the canonical lift of Brownian motion to a geometric p-rough path. Then, there exists a unique rough path solution of the (anticipative) stochastic differential equation

$$d\mathbf{y}_t = V_0(\mathbf{y}_t)\, dt + V(\mathbf{y}_t)\, d\mathbf{B}_t$$

with (random) initial condition y_0.

Theorem 12. *Let $\mathbf{y}^1$ denote the projection to path-level of the rough-path $\mathbf{y}$. Then $\mathbf{y}^1$ solves the anticipative Stratonovich stochastic differential equation,*

$$\mathbf{y}^1_t = y_0 + \int_0^t V_0(\mathbf{y}^1_u)du + \int_0^t V(\mathbf{y}^1_u) \circ dB_u. \tag{4.1}$$

Moreover, almost surely, $\mathbf{y}^1$ is the limit in $1/p$-Hölder topology of

$$t \to y_0 + \int_0^t V_0(\mathbf{y}^1_u)du + \int_0^t V(\mathbf{y}^1_u)dB^n_u,$$

where B^n is the dyadic linear approximation of B of level n.

Proof. For any sequence of subdivisions $(D^n)_n$ which mesh size tends to 0 when $n \to \infty$, $d_{\omega,p}\left(S'\left(B^{D^n}, \mathbf{B}\right), S''(\mathbf{B}, \mathbf{B})\right)$ converges in probability to 0 (theorem 9) hence by corollary 1,

$$d_{\omega,p}\left(S\left(y_0 + \int_0^t V_0(\mathbf{y}^1_u)du + \int_0^t V(\mathbf{y}^1_u)dB^{D^n}_u\right), \mathbf{y}\right)$$

converges in probability to 0. The first level of this equation says precisely that $\mathbf{y}^1$ is solution of the anticipative Stratonovich stochastic differential equation (4.2).

The same argument and the fact that $d_{\omega,p}(S'(B^n, \mathbf{B}), S''(\mathbf{B}, \mathbf{B}))$ converges almost surely to 0 gives the second part of the theorem. □

4.1.2. *The case of driving fBM with $H \in (1/4, 1/2]$.* Of course, this section covers the one above with $H = 1/2$.We fix $H \in (1/4, 1/2]$ and $p \in (1/H, \lfloor 1/H \rfloor + 1)$. As earlier, $\mathbf{W}_H$ denotes the $\lfloor p \rfloor$-level rough path associated to fractional Brownian Motion and we work with Hölder-control $\omega(s,t) = t-s$.

Consider random vectorfields V_0 and $V = (V_1, \cdots, V_d)$, where V_0 (resp. $V_1, \cdots, V_d$) is almost surely a $Lip(1+\varepsilon)$ (resp. $Lip(1/H+\varepsilon)$) vector field on $\mathbb{R}^n$, and an a.s.

finite random variable $y_0 \in \mathbb{R}^n$. Then, there exists a unique rough path solution of the (anticipative) stochastic differential equation

$$d\mathbf{y}_t = V_0(\mathbf{y}_t)\, dt + V(\mathbf{y}_t)\, d\mathbf{W}_H(t)$$

with (random) initial condition y_0.

Theorem 13. $\mathbf{y}^1$ *solves the anticipative Stratonovich stochastic differential equation*

$$\mathbf{y}_t^1 = y_0 + \int_0^t V_0(\mathbf{y}_u^1)du + \int_0^t V(\mathbf{y}_u^1) \circ dW_H(u). \tag{4.2}$$

Moreover, almost surely, $\mathbf{y}^1$ *is the limit in* $1/p$*-Hölder topology of*

$$t \to y_0 + \int_0^t V_0(\mathbf{y}_u^1)du + \int_0^t V(\mathbf{y}_u^1) dW_H^n(u),$$

where W_H^n *is the* n^{th} *dyadic linear approximation of* W_H*.*

Proof. For any sequence of subdivisions $(D^n)_n$ which mesh size tends to 0 when $n \to \infty$, $d_{\omega,p}\left(S'\left(W_H^{D^n}, \mathbf{W}\right), S''(\mathbf{W}_H, \mathbf{W}_H)\right)$ converges in probability to 0 by theorem 11 hence by corollary 1,

$$d_{\omega,p}\left(S\left(y_0 + \int_0^t V_0(\mathbf{y}_u^1)du + \int_0^t V(\mathbf{y}_u^1) dB_u^{D^n}\right), \mathbf{y}\right)$$

converges in probability to 0. The first level of this equation says precisely that $\mathbf{y}^1$ is solution of the anticipative Stratonovich stochastic differential equation (4.2).

The same argument and the fact that $d_{\omega,p}\left(S'\left(W_H^n, \mathbf{W}\right), S''(\mathbf{W}_H, \mathbf{W}_H)\right)$ converges almost surely to 0 gives the second part of the theorem. □

4.2. **A Wong-Zakai Theorem.** The Universal Limit Theorem gives us for free a Wong-Zakai theorem for our solution of the Stratonovich differential equation. The situation is so simple that we feel no need to discuss $H = 1/2$ seperately.

Theorem 14. *Under the same assumptions and notation than the previous theorem, let* y^n *be the solution of the differential equation*

$$\begin{cases} dy_t^n = V_0(y_t^n)dt + V(y_t^n)\, dW_H^n(t), \\ y_0^n = y_0. \end{cases}$$

Then almost surely, y^n *converges to* y *in the topology induced by the* $1/p$*-Hölder distance.*

Proof. Let $\mathbf{z}^n = I_{V,y_0}(S(W_H^n)))$. Then by the continuity of the Itô map, $d_{\omega,p}(\mathbf{z}^n, \mathbf{z}) \to 0$. As $\mathbf{z}^n$ projects down onto y^n, this is a stronger result that the stated theorem. □

4.3. **A Large Deviation Principle.**

4.3.1. *The case of driving Brownian motion.* Due to the universal limit theorem, Freidlin-Wentzell's type theorems have easy proofs via rough paths [21, 11]. We give here an extension of the Freidlin-Wentzell's theorem, and of the main theorem in [23].

Theorem 15. *Let $(y_0^\alpha)_{\alpha\geq 0}$ be a family of random elements of $\mathbb{R}^d$, $(V_0^\alpha)_{\alpha\geq 0}$ be a family of random $Lip\,(1+\varepsilon)$-vector fields, $(V_1^\alpha,\cdots,V_d^\alpha)_{\alpha\geq 0}$, $i=1,\cdots,d$ d families of random $Lip\,(2+\varepsilon)$-vector fields, such that for all $\beta>0$*

$$\begin{aligned}\lim_{\alpha\to 0}\alpha\log\mathbb{P}\left(\left\|V_0^\alpha-V_0^0\right\|_{Lip(1+\varepsilon)}>\beta\right) &= -\infty,\\ \lim_{\alpha\to 0}\alpha\log\mathbb{P}\left(\max_{1\leq i\leq d}\left\|V_i^\alpha-V_i^0\right\|_{Lip(2+\varepsilon)}>\beta\right) &= -\infty,\\ \lim_{\alpha\to 0}\alpha\log\mathbb{P}\left(\left|y_0^\alpha-y_0^0\right|>\beta\right) &= -\infty.\end{aligned}$$

Define $\mathbf{y}^\alpha$ to be the rough path solution of the differential equation

$$\begin{cases} d\mathbf{y}_t^\alpha = V_0^\alpha\left(\mathbf{y}_t^\alpha\right)dt+\sqrt{\alpha}V^\alpha\left(\mathbf{y}_t^\alpha\right)d\mathbf{B}_t\\ \mathbf{y}_0^\alpha = y_0^\alpha.\end{cases}$$

Then $(\mathbf{y}^\alpha)_{\alpha>0}$ satisfies a large deviation principle in the topology induced by $d_{\omega,p}$ with good rate function

$$J(\mathbf{x})=\inf_{I_{y_0^0,V^0}(\mathbf{y})=\mathbf{x}}I(\mathbf{y}),$$

where

$$I(\mathbf{x})=\begin{cases}\frac{1}{2}\int_0^1|x_u'|^2\,du, \text{ if } S(x)=\mathbf{x} \text{ for some } x\in W^{1,2}\\ +\infty \text{ otherwise.}\end{cases}$$

Proof. In [10], it was proved that $\left(\delta_{\sqrt{\alpha}}\mathbf{B}\right)_{\alpha>0}$ satisfies a large deviation principle in the topology induced by $d_{\omega,p}$ with good rate function I. The assumptions on the vector fields and the initial conditions give that $\left((V_i^\alpha)_{0\leq i\leq d},y_0^\alpha,\delta_{\sqrt{\alpha}}\mathbf{B}\right)$ satisfies a large deviation (in the topology product $Lip\,(1+\varepsilon),Lip\,(2+\varepsilon)^d,|.|$, and $d_{\omega,p}$) with good rate function

$$\begin{cases}\frac{1}{2}\int_0^1|x_u'|^2\,du, \text{ if } S(x)=\mathbf{x} \text{ for some } x\in W^{1,2},\ V_i=V_i^0,\ 1\leq i\leq d \text{ and } y_0=y_0^0\\ +\infty \text{ otherwise.}\end{cases}$$

By the continuity of the Itô map (theorem 4,5 and 6), we obtain our large deviation principle. □

4.3.2. *The case of driving fBM with $H\in(1/4,1/2]$.* Recent work [24] establishes

a large deviation principle for fractional Brownian motion lifted to a rough path w.r.t. p-variation topology with good rate function

$$I_H(\mathbf{x})=\begin{cases}\frac{1}{2}\left\|i^{-1}(x)\right\|_{\mathcal{H}_H}^2 & \text{if } S(x)=\mathbf{x} \text{ for some } x\in i(\mathcal{H}_H)\\ +\infty \text{ otherwise}\end{cases}$$

This shows that the above LD principle is true in p-variation topology with rate function I_H when (enhanced) Brownian motion $\mathbf{B}$ is replaced by (enhanced) fractional Brownian motion $\mathbf{W}_H$.

Remark 2. *We are certain the LD principle for SDEs driven by (enhanced) fractional Brownian motion holds in Hölder topology. We shall discuss this in forthcoming work.*

4.4. **Support theorem.** It turns out that the notion of good rough path sequence leads to neat support description of fractional Brownian motion lifted to a rough path. In fact, given that piecewise linear approximations form a good rough path sequence to the fractional Brownian rough path, our proof is no more complicated than the well-known proof on the characterization of the support of the law of the Brownian motion as seen in Revuz and Yor [29]. The method here works for any $H \in (1/4, 1)$ but one should keep in mind that for $H > 1/2$ no lift is necessary and the support description is trivial. The case of Brownian motion, $H = 1/2$, has been dealt with several times, including by the two last named authors of this article ([8], [11]). Hence, the real interest of the result below is for $H < 1/2$. It was also discussed, with different methods, in recent work [10] and [6].

Lemma 5. *The map*

$$\begin{array}{rcl} Minus & : & \mathbb{R}^d \oplus \mathbb{R}^d \to \mathbb{R}^d \\ x, y & \to & y - x \end{array}$$

induces a homeomorphism $Minus$ from $G^{[p]}\left(\mathbb{R}^d \oplus \mathbb{R}^d\right)$ onto $G^{[p]}\left(\mathbb{R}^d\right)$.

Example 4. *Let $[p] = 2$. Given $Z \in G^2\left(\mathbb{R}^d \oplus \mathbb{R}^d\right)$*

$$Z = \left(1,\ Z^{1;1} \oplus Z^{1;2},\ Z^{2;1,1} \oplus Z^{2;1,2} \oplus Z^{2;2,1} \oplus Z^{2;2,2}\right).$$

with $Z^{1;i} \in \mathbb{R}^d$, $Z^{2;i,j} \in \mathbb{R}^d \otimes \mathbb{R}^d$. Then

$$Minus\left(Z\right) = \left(1,\ Z^{1;2} - Z^{1;1},\ Z^{2;2,2} - Z^{2;1,2} - Z^{2;2,1} + Z^{2;1,1}\right) \in G^{[p]}\left(\mathbb{R}^d\right).$$

The map $Minus$ induces a continuous surjection, still denoted $Minus$, from $C^{0,p-var}\left(G^{[p]}(\mathbb{R}^d \oplus \mathbb{R}^d)\right)$ onto $C^{0,p-var}\left(G^{[p]}(\mathbb{R}^d)\right)$. We define for $(h, \mathbf{x})$ in the space $C^{0,q-var}\left(\mathbb{R}^d\right) \times C^{0,p-var}\left(G^{[p]}(\mathbb{R}^d)\right)$, with $q^{-1} + p^{-1} > 1$ the translation of $\mathbf{x}$ by $-h$

$$T_{-h}\left(\mathbf{x}\right) = Minus\left(S^{'}\left(h, \mathbf{x}\right)\right).$$

The following lemma was obtained in [10] for $\mathbf{W}_{1/2} = \mathbf{B}$. The generalization is easy.

Lemma 6. *Let h be a bounded variation path in the Cameron-Martin space associated to fBM with $H > 1/4$. Then the law of $T_h\left(\mathbf{W}_H\right)$ is equivalent to the law of $\mathbf{W}_H$.*

Theorem 16. *Let $H \in (1/4, 1)$ and $p \in (1/H, \lfloor 1/H \rfloor + 1)$. The support of the law of $\mathbf{W}_H$ in the $d_{\omega,p}$-topology is the set of path starting at 0 in $C^{0,\omega,p}\left(G^{[p]}\left(\mathbb{R}^d\right)\right)$ where $\omega(s,t) = t - s$.*

Proof. Almost surely, $W_H \in C^{0,\omega,p}\left(G^{[p]}\left(\mathbb{R}^d\right)\right)$ and $W_H\left(0\right) = 0$ therefore the support of its law is included in the set of path starting at 0 in $C^{0,\omega,p}\left(G^{[p]}\left(\mathbb{R}^d\right)\right)$. Reciprocally, the support of the law of $\mathbf{W}_H$ contains at least one point $\mathbf{x}$ such that its sequence of dyadic linear approximation of level n $\left(x^n\right)_n$ is a good p rough path sequence associated to $\mathbf{x}$ (due to theorem 11). By lemma 6, the support of the law of $\mathbf{W}_H$ contains the $d_{\omega,p}$-closure of $\{T_h\left(\mathbf{x}\right),\ h$ in the Cameron-Martin space$\}$. As (x^n) is a good p-rough path sequence associated to $\mathbf{x}$, $T_{-x^n}\left(\mathbf{x}\right) = Minus\left(S'\left(x^n, \mathbf{x}\right)\right)$ converges to 0 (in the $d_{\omega,p}$-topology). From section 5.2, piecewise linear approximation are Cameron Martin paths and since the support of a measure is always closed (by definition) it follows that 0 belongs to the support of the law of $\mathbf{W}_H$. Clearly, by lemma 6, the support contains the closure of the translation of all smooth paths. Therefore, the support of the law of $\mathbf{W}_H$ contains the closure in the $d_{\omega,p}$-topology

of the set of smooth paths starting at 0. This concludes the proof with the results in [10]. □

Denote by I the map which maps $(y_0, (V_0, V), \mathbf{x})$ to $I_{y_0,(V_0,V)}(\mathbf{x})$. The following proposition is an obvious corollary of the the continuity of the map I (theorems 4,5,6) and of theorem 16.

Proposition 3. *Let* $\mathbf{y}$ *be the solution of the rough differential equation*

$$\begin{cases} d\mathbf{y}_t = V_0(\mathbf{y}_t)\,dt + V(\mathbf{y}_t)\,d\mathbf{W}_H(t) \\ \mathbf{y}_0^1 = y_0, \end{cases}$$

where V_0 *is almost surely a* $Lip(1+\varepsilon)$ *vector field and* V_i, $i \in \{1, \cdots, d\}$ *are almost surely* $Lip(1/H+\varepsilon)$ *vector fields,* $y_0 \in \mathbb{R}^d$ *is almost surely finite. The support of the law of* $\mathbf{y}$ *in the* $d_{\omega,p}$*-topology is equal to the image by the map* I *of the support of the law of* $(y_0, V, \mathbf{W}_H)$*, in the product of the Euclidean, Lipschitz and* $d_{\omega,p}$ *topology.*

In particular, if y_0 and the vector fields V_i are deterministic, in the $d_{\omega,p}$-topology, the support of the law of $\mathbf{y}$ is equal to the set $I_{y_0,(V_0,V)}\left(C^{0,\omega,p}\left(G^{[p]}\left(\mathbb{R}^d\right)\right)\right)$, which, at the first level and specialized to $H = 1/2$ is the classical support theorem of Stroock-Varadhan [32].

If y_0 and the V_is are the image by a continuous function of $\mathbf{B}$, then the support is still trivially characterized. One could then ask for more specific conditions on y_0 and the V_is, in the spirit of [22], and obtain a detailed support theorem. Thanks to the Universal Limit, one would obtain stronger results than in [22] (stronger topology and without the assumption of deterministic vector fields) but we shall not pursue this here.

5. Appendix 1

5.1. Linear Approximation of the Fractional Brownian motion as Good Rough Path Sequences.

We fix $\omega(s,t) = t - s$.

5.1.1. *Fractional Brownian motion framework.* We use the framework of [4] or [5]. The starting point of the approach devellopped in [4] is the following representation of fractional Brownian motion given by Decreusefond-Üstünel, [7]: $\mathbf{P}$ almost every where

$$W_H(t) = \int_0^t K_H(t,s)dB_s,\ \forall t \in [0,1] \tag{5.1}$$

where $K_H(t,s) = \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} F(H-\frac{1}{2}, \frac{1}{2}-H, H+\frac{1}{2}, 1-\frac{t}{s}),\ s<t.$
F denotes Gauss hypergeometric function, [14] and $(B_t)_{t\in[0,1]}$ is a Brownian motion. According to Lemma 2.7 of [4], the function $t \mapsto K_H(t,s)$ is differentiable on $]s, +\infty[$, with derivative

$$\partial_t K_H(t,s) = \frac{(s/t)^{H-\frac{1}{2}}}{\Gamma(\frac{1}{2}-H)}(s-t)^{H-\frac{1}{2}},\ 0<s<t.$$

The parameter H will be fixed in $\left(\frac{1}{4}, \frac{1}{2}\right)$ (if $H > 1/2$, things are trivial, $H = \frac{1}{2}$ is the Brownian motion case, while our techniques do not allow us to deal with the case $H \leq 1/4$).

5.1.2. *"Canonical" lift of fractional Brownian motion to a geometric p-rough path.* Let be $d \in \mathbf{N}^*$ and $B = (B^1, \cdots, B^d)$ be a d dimensional Brownian motion, $H \in]\frac{1}{4}, 1[$, and $W_H = (W_H^1, \cdots, W_H^d)$ given by

$$W_H^i(t) = \int_0^t K_H(t,s) dB_s^i, \ \forall t \in [0,1].$$

Indeed, W_H is a d dimensional fractional Brownian motion with Hurst parameter H. We fix $p \in (\frac{1}{H}, \left[\frac{1}{H}\right] + 1)$.

Definition 3. *For $H > \frac{1}{4}$, we define, according to* [5]*, the lift of W_H to a p-geometric rough path $\mathbf{W}_H$ is formally given by*

$$\begin{aligned}
\mathbf{W}_H^1(0,t) &= W_H(t), \\
\mathbf{W}_H^2(0,t) &= \int_0^t W_H(s) \otimes \circ dW_H(s), \\
\mathbf{W}_H^3(0,t) &= \int_0^t \mathbf{W}_H^2(0,s) \otimes \circ dW_H(s).
\end{aligned}$$

The 2nd level is rigourosly given as

$$\begin{aligned}
\mathbf{W}_H^2(0,t)^{i,j} &= \int_0^1 I_{0,t}^{K_H}(W_H^i)(u) dB^j(u) \text{ for } i \neq j \\
\mathbf{W}_H^2(0,t)^{i,i} &= \frac{1}{2} W_H^i(t)^2
\end{aligned}$$

with

$$I_{t,-}^{K_H}(f)(s) = K_H(t,s) f(s) + \int_s^t (f(u) - f(s)) \partial_u K_H(u,s) du.$$

From [5]

$$\begin{aligned}
&\mathbf{W}_H^3(0,t)^{i,j,k} \\
= &\int_0^1 \left[\int_0^1 \left[\int_0^1 I_{0,t}^{K_H} \left(I_{0,\cdot}^{K_H} \left(K_H(.,u) \right)(v) \right)(r) \, dB^i(u) \right] dB^j(v) \right] dB^k(r) \\
&+ \frac{\delta_{i,j}}{2} \int_0^1 I_{0,t}^{K_H} \left(\mathbb{E} \left(\mathbf{W}_H^1(0,.)^k \right)^2 \right)(z) \, dB^k(z) \\
&+ \frac{\delta_{k,j}}{2} \int_0^1 I_{0,t}^{K_H} \left(\mathbb{E} \left(\mathbf{W}_H^1(.,t)^i \right)^2 \right)(u) \, dB^i(u) \\
&+ \delta_{i,k} \int_0^1 I_{0,t}^{K_H} \left(\mathbb{E} \left(\mathbf{W}_H^1(0,.)^j \mathbf{W}_H^1(.,t)^j \right) \right)(v) \, dB^j(v)
\end{aligned}$$

The stochastic integrals with respect to the Brownian motion B are Skorokhod integrals. We then define for $s < t$

$$\mathbf{W}_H(s,t) = \mathbf{W}_H(0,s)^{-1} \otimes \mathbf{W}_H(0,t).$$

Remark 3. *Using scaling property of fractional Brownian motion and the fact that it has stationnary increment, we have for a small enough*

$$\sup_{s<t} \mathbb{E} \left(\exp a \frac{\|\mathbf{W}_H(s,t)\|^2}{(t-s)^{2H}} \right) < \infty$$

Then using Garsia-Rudomich-Rumsey lemma as in [11]*, it is easy to see that*

$$\mathbb{E}\exp\left(a\left\|\mathbf{W}_H\right\|_{\omega,p}^2\right)<\infty$$

for some positive a small enough. In particular $\left\|\mathbf{W}_H\right\|_{\omega,p}$ *belongs to* L^q *for all* $q\geq 1$.

5.1.3. *Good Sequence for Fractional Brownian motion.* We fix a sequence D^n of subdivision which steps size tends to 0. We first start by slightly extending the result of [5]: we prove that piecewise linear approximation of the fractional Brownian motion, lifted to a (smooth) rough path converges to the lift of the fractional Brownian motion constructed in [5]. Our technique is quite different, and our convergence stronger. Recall the notation of 1.2.

Theorem 17. *In the* $d_{\omega,p}$*-topology,* $S\left(W_H^{D^n}\right)$ *converge to* $\mathbf{W}_H$ *in* L^q *and in probability.*

Proof. For flexibel notation, we agree that

$$\mathbf{W}_H(t)\equiv\mathbf{W}_t,\ \ \mathbf{W}_H(s,t)\equiv\mathbf{W}_{s,t}\,.$$

We fix a subdivision $D=(0\leq t_1<\cdots<t_{|D|}\leq 1)$ of $[0,1]$. Similarly as for the Brownian motion, it is enough to prove that, for $p'>1/H$, there exists $\mu>0$ such that for all s and t in D, we have

$$\left\|\sup_{0\leq s<t\leq 1}\frac{d\left(S\left(W^D\right)_{s,t},\mathbf{W}_{s,t}\right)}{|t-s|^{1/p}}\right\|_{L^q}\leq C_{q,\mu}\,|D|^{\mu}\,.$$

Working just as in the case of the Brownian motion, we see that it is enough to prove that for all $s=t_m,t=t_n\in D$,

$$\left\|\int_s^t W_{s,u}^D\otimes dW_u^D-\mathbf{W}_{s,t}^2\right\|_{L^2} \tag{5.2}$$

$$\leq\ C_{q,\mu}\,|D|^{\mu}\,|t-s|^{2/p'}\,, \tag{5.3}$$

$$\left\|\int_{s<u_1<u_2<u_3<t} dW_{u_1}^D\otimes dW_{u_2}^D\otimes dW_{u_3}^D-\mathbf{W}_{s,t}^3\right\|_{L^2} \tag{5.4}$$

$$\leq\ C_{q,\mu}\,|D|^{\mu}\,|t-s|^{3/p'}\,. \tag{5.5}$$

We will divide the proof into two parts, one corresponding to the second level, and the other one the third level.

1st part: A simple computation leads to

$$\int_s^t W_{s,u}^D\otimes dW_u^D-\mathbf{W}_{s,t}^2=-\sum_{l=m}^{n-1}\left(\mathbf{W}^2(t_l,t_{l+1})-\frac{1}{2}W(t_l,t_{l+1})^{\otimes 2}\right).$$

Therefore, to prove (5.2), we just need to prove that

$$\mathbb{E}\left(\left|\sum_{l=m}^{n-1}\mathbf{A}_{t_l,t_{l+1}}\right|^2\right)\leq C_{q,\mu}\,|D|^{\mu}\,|t_n-t_m|^{4/p\prime}$$

where $\mathbf{A}_{s,t} = \mathbf{W}^2(s,t) - \frac{1}{2}W(s,t)^{\otimes 2}$, which is the Levy area of the fractional Brownian motion. We are going to decompose furthermore

$$\left(\mathbf{A}_{s,t}\right)^{i,j} = \mathbf{W}^2(s,t)^{i,j} - \mathbf{W}^2(s,t)^{j,i}$$

By definition of $\mathbf{W}_H^2$ (or see [5]), that for $i \neq j$,

$$\begin{aligned}\mathbf{W}^2(s,t)^{i,j} &= \int_0^1 K_H(t,u)W^i(s,u)1_{]s,t[}(u)\,dB_u^j \\ &+ \int_0^1 \left(\int_u^t W^i(u,r)\partial K_H(r,u)dr\right)1_{]s,t[}(u)\,dB_u^j \\ &+ \int_0^1 \left(\int_s^t W^i(s,r)\partial K_H(r,u)dr\right)1_{]0,s[}(u)\,dB_u^j.\end{aligned}$$

Writing $\left(\mathbf{A}_{s,t}\right)^{i,j} = \left(\mathbf{A}^1_{s,t}\right)^{i,j} + \left(\mathbf{A}^2_{s,t}\right)^{i,j} - \left(\mathbf{A}^1_{s,t}\right)^{j,i} - \left(\mathbf{A}^2_{s,t}\right)^{j,i}$, where

$$\begin{aligned}\left(\mathbf{A}^1_{s,t}\right)^{i,j} &= 1_{i\neq j}\int_0^1 1_{]s,t[}(u)\left(K_H(t,u)W^i(s,u) + \int_u^t \partial K_H(r,u)\,W^i(u,r)\,dr\right)dB_u^j \\ \left(\mathbf{A}^2_{s,t}\right)^{i,j} &= 1_{i\neq j}\int_0^1 1_{[0,s[}(u)\left(\int_s^t \partial K_H(r,u)W^i(s,r)dr\right)dB_u^j.\end{aligned}$$

First observe that

$$\begin{aligned}\left|\sum_{l=m}^{n-1}\mathbf{A}_{t_l,t_{l+1}}\right|^2 &= \sum_{i,j=1}^d\left|\sum_{l=m}^{n-1}\left(\mathbf{A}_{t_l,t_{l+1}}\right)^{i,j}\right|^2 \\ &\leq C\sum_{i,j}\left|\sum_{l=m}^{n-1}\left(\mathbf{A}^1_{t_l,t_{l+1}}\right)^{i,j}\right|^2 + C\sum_{i,j}\left|\sum_{l=m}^{n-1}\left(\mathbf{A}^2_{t_l,t_{l+1}}\right)^{i,j}\right|^2.\end{aligned}$$

The lemmae in the next appendix (lemmae 10,11) show that for all $s' < t' \leq s < t$

$$\left|\mathbb{E}\left(\left(\mathbf{A}^1_{s',t'}\right)^{i,j}\left(\mathbf{A}^1_{s,t}\right)^{i,j}\right)\right| + \left|\mathbb{E}\left(\left(\mathbf{A}^2_{s',t'}\right)^{i,j}\left(\mathbf{A}^2_{s,t}\right)^{i,j}\right)\right| \leq \mathbb{E}(W^i(s,t)W^i(s',t'))^2.$$

From the scaling property of the Fractional Brownian Motion, we have

$$\mathbb{E}\left(\left|\mathbf{A}_{s,t}\right|^2\right) \leq C\left|t-s\right|^{4H} = C\mathbb{E}\left(\left|W^i(s,t)\right|^2\right)^2$$

With lemma 12, we therefore obtain for $s < t$,

$$\mathbb{E}\left(\left|\left(\mathbf{A}^1_{s,t}\right)^{i,j}\right|^2\right) + \mathbb{E}\left(\left|\left(\mathbf{A}^2_{s,t}\right)^{i,j}\right|^2\right) \leq C\mathbb{E}\left(\left|W^i_H(s,t)\right|^2\right)^2.$$

Hence, we obtain that

$$\mathbb{E}\left(\left|\sum_{l=m}^{n-1}\mathbf{A}_{t_l,t_{l+1}}\right|^2\right) \leq C\sum_{k,l=m}^{n-1}\mathbb{E}\left(W^1(t_k,t_{k+1})W^1(t_l,t_{l+1})\right)^2.$$

Therefore, by lemma 17, we obtain

$$\mathbb{E}\left(\left|\sum_{l=m}^{n-1}\mathbf{A}_{t_l,t_{l+1}}\right|^2\right) \leq C\left|D\right|^{4H-4/p'}\left|t_n - t_m\right|^{4/p'},$$

i.e. we have proved equation (5.2).

2nd part: First observe that

$$\left(\mathbf{W}^3_{s,t}\right)^{i,k,j} = \left(\mathbf{W}^2_{s,t}\right)^{i,k}\left(\mathbf{W}^1_{s,t}\right)^{j} - \left(\mathbf{W}^{3,i,j,k}_{s,t} + \mathbf{W}^{3,j,i,k}_{s,t}\right). \tag{5.6}$$

Therefore, using twice (5.6), for (i,j,j) and $(i,j,k) = (j,i,j)$, $\left(\mathbf{W}^3_{s,t}\right)^{i,j,i} = \left(\mathbf{W}^2_{s,t}\right)^{i,j}\left(\mathbf{W}^1_{s,t}\right)^{i} - 2\mathbf{W}^{i,i,j}_{s,t}$ and

$$\begin{aligned} 2\left(\mathbf{W}^3_{s,t}\right)^{i,j,j} &= \left(\mathbf{W}^2_{s,t}\right)^{i,j}\left(\mathbf{W}^1_{s,t}\right)^{j} - \mathbf{W}^{j,i,j}_{s,t} \\ &= \left\{\left(\mathbf{W}^2_{s,t}\right)^{i,j} - \left(\mathbf{W}^2_{s,t}\right)^{j,i}\right\}\left(\mathbf{W}^1_{s,t}\right)^{j} + 2\mathbf{W}^{j,j,i}_{s,t}. \end{aligned}$$

In particular, as, for $s < t \in D$,

$$\begin{aligned} \left\|\left\{\mathbf{W}^2_{s,t} - S\left(W^D\right)^2_{s,t}\right\}\otimes \mathbf{W}^1_{s,t}\right\|_{L^2} &\leq \left\|\left\{\mathbf{W}^2_{s,t} - S\left(W^D\right)^2_{s,t}\right\}\right\|_{L^4}\left\|\mathbf{W}^1_{s,t}\right\|_{L^4} \\ &\leq C\left|D\right|^{4H-4/p'}\left|t-s\right|^{2/p\prime+H}, \end{aligned}$$

we see that we just need to prove that for all i,j,k with $k \notin \{i,j\}$,

$$\left\|\mathbf{W}^{3,i,j,k}_{s,t} + \mathbf{W}^{3,j,i,k}_{s,t} - \left(S\left(W^D\right)^3_{s,t}\right)^{i,j,k} - \left(S\left(W^D\right)^3_{s,t}\right)^{j,i,k}\right\|_{L^2} \leq C_{q,\mu}\left|D\right|^{\mu}\left|t-s\right|^{3/p'}. \tag{5.7}$$

The multiplicative (or Chen) property of iterated integrals quickly gives that $\mathbf{W}^{3,i,j,k}_{t_m,t_n} + \mathbf{W}^{3,j,i,k}_{t_m,t_n} - \left(S\left(W^D\right)^3_{t_m,t_n}\right)^{i,j,k} - \left(S\left(W^D\right)^3_{t_m,t_n}\right)^{j,i,k}$ is equal to

$$\begin{aligned} &\sum_{l=m}^{n-1}\mathbf{W}^{3,i,j,k}_{t_l,t_{l+1}} + \mathbf{W}^{3,j,i,k}_{t_l,t_{l+1}} - \left(S\left(W^D\right)^3_{t_l,t_{l+1}}\right)^{i,j,k} - \left(S\left(W^D\right)^3_{t_l,t_{l+1}}\right)^{j,i,k} \\ &-\sum_{l=m}^{n-1} W^i_{t_m,t_l}\left(\frac{1}{2}W^j_{t_l,t_{l+1}}W^k_{t_l,t_{l+1}} - \mathbf{W}^{2,j,k}_{t_l,t_{l+1}}\right) \\ &-\sum_{l=m}^{n-1} W^j_{t_m,t_l}\left(\frac{1}{2}W^i_{t_l,t_{l+1}}W^k_{t_l,t_{l+1}} - \mathbf{W}^{2,i,k}_{t_l,t_{l+1}}\right). \end{aligned}$$

We first start by bounding the L^2 norm of the second line (we bound the third line in just the same way):

$$\begin{aligned} \left\|\sum_{l=m}^{n-1} W^i_{t_m,t_l}\left(\frac{1}{2}W^j_{t_l,t_{l+1}}W^k_{t_l,t_{l+1}} - \mathbf{W}^{2,j,k}_{t_l,t_{l+1}}\right)\right\|_{L^2} &= \frac{1}{2}\left\|\sum_{l=m}^{n-1} W^i_{t_m,t_l}W^j_{t_l,t_{l+1}}W^k_{t_l,t_{l+1}}\right\|_{L^2} \\ &+ \sum_{\alpha=1}^{2}\left\|\sum_{l=m}^{n-1} W^i_{t_m,t_l}\left(\mathbf{A}^{\alpha,1}_{t_l,t_{l+1}}\right)^{j,k}\right\|_{L^2}, \end{aligned}$$

where $A^\alpha,\ \alpha = 1,2$ are defined earlier in this proof. Because $k \neq j$, using the Gaussian formula (6.1)

$$\begin{aligned}
&\mathbb{E}\left(\left|\sum_{l=m}^{n-1} W^i_{t_m,t_l} W^{,j}_{t_l,t_{l+1}} W^k_{t_l,t_{l+1}}\right|^2\right) \\
= &\sum_{l,l'=m}^{n-1} \mathbb{E}\left(W^i_{t_m,t_l} W^j_{t_l,t_{l+1}} W^i_{t_m,t_{l'}} W^{,j}_{t_{l'},t_{l'+1}}\right) \mathbb{E}\left(W^k_{t_l,t_{l+1}} W^k_{t_{l'},t_{l'+1}}\right) \\
\leq &\sum_{l,l'=m}^{n-1} \left\|W^i_{t_m,t_l}\right\|_{L^4} \left\|W^j_{t_l,t_{l+1}}\right\|_{L^4} \left\|W^i_{t_m,t_{l'}}\right\|_{L^4} \left\|W^{,j}_{t_{l'},t_{l'+1}}\right\|_{L^4} \mathbb{E}\left(W^k_{t_l,t_{l+1}} W^k_{t_{l'},t_{l'+1}}\right) \\
\leq &\ C\left|t_n - t_m\right|^{2H} \sum_{l,l'=m}^{n-1} \left\|W^j_{t_l,t_{l+1}}\right\|_{L^2} \left\|W^j_{t_{l'},t_{l'+1}}\right\|_{L^2} \mathbb{E}\left(W^k_{t_l,t_{l+1}} W^k_{t_{l'},t_{l'+1}}\right) \\
\leq &\ C\left|t_n - t_m\right|^{6/p'} |D|^\mu \quad \text{by lemma 17.}
\end{aligned}$$

In lemma 16, we proved that

$$\begin{aligned}
&\left\|\sum_{l=m}^{n-1} W^i_{t_m,t_l} \left(\mathbf{A}^2_{t_l,t_{l+1}}\right)^{j,k}\right\|^2_{L^2} \\
&\leq \sum_{l,l'=m}^{n-1} \mathbb{E}\left(W^i_{t_m,t_l} W^j_{t_l,t_{l+1}} W^i_{t_m,t_{l'}} W^j_{t_{l'},t_{l'+1}}\right) \mathbb{E}\left(W^k_{t_l,t_{l+1}} W^k_{t_{l'},t_{l'+1}}\right) \\
&\leq \left|t_n - t_m\right|^{6/p'} |D|^\mu \quad \text{as above.}
\end{aligned}$$

Using the fact that $k \notin \{i,j\}$ and lemma 10,

$$\begin{aligned}
&\left\|\sum_{l=m}^{n-1} W^i_{t_m,t_l} \left(\mathbf{A}^1_{t_l,t_{l+1}}\right)^{j,k}\right\|_{L^2} \\
= &\left(\sum_{l=m}^{n-1} \mathbb{E}\left(\left|W^i_{t_m,t_l} \left(\mathbf{A}^1_{t_l,t_{l+1}}\right)^{j,k}\right|^2\right)\right)^{1/2} \\
\leq &\left(\sum_{l=m}^{n-1} \mathbb{E}\left(\left|\left(\mathbf{A}^1_{t_l,t_{l+1}}\right)^{j,k}\right|^4\right)^{1/2} \mathbb{E}\left(\left|W^i_{t_m,t_l}\right|^4\right)^{1/2}\right)^{1/2} \\
\leq &\left(\left|t_n - t_m\right|^{2H} \sum_{l=m}^{n-1} \left|t_{l+1} - t_l\right|^{4H}\right)^{1/2} \\
\leq &\left(C\left|t_n - t_m\right|^{2H} \left|t_n - t_m\right|^{4/p'} |D|^{4h-4/p'}\right)^{1/2} \\
\leq &\ C_{q,\mu} |D|^\mu \left|t - s\right|^{3/p'}.
\end{aligned}$$

Therefore, to prove equation (5.7), it remains to prove that for $k \notin \{i,j\}$

$$\left\|\sum_{l=m}^{n-1} \mathbf{W}^{3,i,j,k}_{t_l,t_{l+1}} - \left(S\left(W^D\right)^3_{t_l,t_{l+1}}\right)^{i,j,k}\right\|_{L^2} \leq C_{q,\mu} |D|^\mu \left|t - s\right|^{3/p'}.$$

First observe that $\left(S\left(W^D\right)^3_{t_l,t_{l+1}}\right)^{i,j,k} = \frac{1}{6}W^i_{t_l,t_{l+1}}W^j_{t_l,t_{l+1}}W^k_{t_l,t_{l+1}}$ and that (using the Gaussian formula (6.1))

$$\begin{aligned}
&\left\|\sum_{l=m}^{n-1} W^i_{t_l,t_{l+1}}W^j_{t_l,t_{l+1}}W^k_{t_l,t_{l+1}}\right\|^2_{L^2}\\
\leq\ & \sum_{l,l'=m}^{n-1}\mathbb{E}\left(W^i_{t_l,t_{l+1}}W^j_{t_l,t_{l+1}}W^i_{t_{l'},t_{l'+1}}W^j_{t_{l'},t_{l'+1}}\right)\mathbb{E}\left(W^k_{t_l,t_{l+1}}W^k_{t_{l'},t_{l'+1}}\right)\\
\leq\ & C\left|t_n-t_m\right|^{6/p'}|D|^{\mu} \quad \text{by lemma 17.}
\end{aligned}$$

Hence, we need to prove that

$$\left\|\sum_{l=m}^{n-1}\mathbf{W}^{3,i,j,k}_{t_l,t_{l+1}}\right\|_{L^2} \leq C_{q,\mu}|D|^{\mu}|t-s|^{3/p'}.$$

Recall that for i,j,k as above, we have

$$\begin{aligned}
\mathbf{W}^3(s,t)^{i,j,k} =& \int_0^1 K_H(t,u)\mathbf{W}^2(s,u)^{i,j}1_{]s,t[}(u)\,dB^k_u\\
&+\int_0^1\left(\int_u^t \mathbf{W}^2(u,r)^{i,j}\partial K_H(r,u)dr\right)1_{]s,t[}(u)\,dB^k_u\\
&+\int_0^1\left(\mathbf{W}^1(s,u)^i\int_u^t \mathbf{W}^1(u,r)^j\partial K_H(r,u)dr\right)1_{]s,t[}(u)\,dB^k_u\\
&+\int_0^1\left(\int_s^t \mathbf{W}^2(s,r)^{i,j}\partial K_H(r,u)dr\right)1_{]0,s[}(u)\,dB^k_u.
\end{aligned}$$

Since $\mathbf{W}^2(s,t)=\mathbf{A}^1_{s,t}+\mathbf{A}^2_{s,t}$, we let

$$\begin{aligned}
\mathbf{G}^1_H(s,t)^{i,j,k} =& \int_0^1 K_H(t,u)\mathbf{W}^2(s,u)^{i,j}1_{]s,t[}(u)\,dB^k_u\\
&+\int_0^1\left(\int_u^t \mathbf{W}^2(u,r)^{i,j}\partial K_H(r,u)dr\right)1_{]s,t[}(u)\,dB^k_u\\
&+\int_0^1\left(\mathbf{W}^1(s,u)^i\int_u^t \mathbf{W}^1(u,r)^j\partial K_H(r,u)dr\right)1_{]s,t[}(u)\,dB^k_u
\end{aligned}$$

and for $\alpha\in\{1,2\}$

$$\mathbf{G}^{2,\alpha}_H(s,t)^{i,j,k} = \int_0^1\left(\int_s^t\left(\mathbf{A}^{\alpha}_{s,r}\right)^{i,j}\partial K_H(r,u)dr\right)1_{]0,s[}(u)\,dB^k_u,$$

so that $\mathbf{W}^3_H(s,t)^{i,j,k}=\mathbf{G}^1_H(s,t)^{i,j,k}+\mathbf{G}^{2,1}_H(s,t)^{i,j,k}+\mathbf{G}^{2,2}_H(s,t)^{i,j,k}$. Arguments of lemmae 10 and 11 leads to, for $\alpha\in\{1,2\}$ and $s'<t'\leq s<t$ or $(s'<t')=(s<t)$,

$$\mathbb{E}\left(\mathbf{G}^{2,\alpha}_H(s,t)^{i,j,k}\mathbf{G}^{2,\alpha}_H(s',t')^{i,j,k}\right)\leq -C\mathbb{E}(W^i_H(s,t)W^i_H(s',t'))^3.$$

Therefore, working as above, we get

$$\left\|\sum_{l=m}^{n-1}\mathbf{G}^{2,\alpha}_H(t_l,t_{l+1})^{i,j,k}\right\|_{L^2} \leq C_{q,\mu}|D|^{\mu}|t-s|^{3/p'}.$$

Then, lemma 10 gives that

$$\mathbb{E}\left(\mathbf{G}_H^1(s,t)^{i,j,k}\mathbf{G}_H^1(s',t')^{i,j,k}\right)=0$$

for $s'<t'\leq s<t$ and scaling property of the fractional Brownian motion gives

$$\left\|\mathbf{G}_H^1(s,t)^{i,j,k}\right\|_{L^2}=C\left|t-s\right|^{3H},$$

so that we once again easily find

$$\left\|\sum_{l=m}^{n-1}\mathbf{G}_H^{2,\alpha}(t_l,t_{l+1})^{i,j,k}\right\|_{L^2}\leq C_{q,\mu}\left|D\right|^{\mu}\left|t-s\right|^{3/p'}.$$

That concludes the proof. □

We go one step further than in the paper [5], by proving the following theorem:

Theorem 18. *When n tends to ∞, $d_{\omega,p}\left(S'\left(W_H^{D^n},\mathbf{W}\right),S''\left(\mathbf{W},\mathbf{W}\right)\right)$ converges to 0 in L^q, $q\geq 1$ and in probability.*
If $D^n=\left(\frac{k}{2^n},0\leq k\leq 2^n\right)$, the convergence also holds almost surely, i.e. $W_H^n=W_H^{D^n}$ is almost surely a good p-rough path sequence associated to $\mathbf{W}_H$.

We first start by two lemmae.

Lemma 7. *Let D be a subdivision of $[0,1]$. Then, there exists $\mu>0$ and $C_\mu<\infty$ such that for all $s,t\in D$,*

$$\left\|\int_s^t W\left(s,u\right)\otimes dW^D\left(u\right)-\mathbf{W}_{s,t}^2\right\|_{L^2}\leq C_\mu\left|D\right|^{\mu}\left|t-s\right|^{2/p'}. \tag{5.8}$$

Proof. The subdivision $D=(0\leq t_1<\cdots<t_{|D|}\leq 1)$ is fixed. As

$$\left\|\int_s^t W^D\left(s,u\right)\otimes dW^D\left(u\right)-\mathbf{W}_{s,t}^2\right\|_{L^2}\leq C_\mu\left|D\right|^{\mu}\left|t-s\right|^{4/p'},$$

we just need to prove that for all $0\leq m<n\leq|D|$,

$$\mathbb{E}\left(\left|\int_{t_m}^{t_n}\left(W^D(t_m,u)-W\left(t_m,u\right)\right)\otimes dW^D\left(u\right)\right|^2\right)\leq C\left|D\right|^{\mu}\left|t_n-t_m\right|^{4/p\prime},$$

i.e. that

$$\mathbb{E}\left(\left|\sum_{j=m}^{n-1}\int_{t_j}^{t_{j+1}}\left(W^D(t_j,u)-W\left(t_j,u\right)\right)\otimes dW^D\left(u\right)\right|^2\right)\leq C\left|D\right|^{\mu}\left|t_n-t_m\right|^{4/p\prime}$$

For $s<t$, we let

$$\mathbf{Y}_{s,t}=\frac{1}{|t-s|^2}\int_s^t[(u-s)W(u,t)-(t-u)W(s,u)]du\otimes\mathbf{W}^1(s,t).$$

As for s,t are two consecutive points in D, $\mathbf{Y}_{s,t}=\int_s^t\left(W^D(s,u)-W\left(s,u\right)\right)\otimes dW^D\left(u\right)$, to prove (5.8), we just need to prove that

$$\mathbb{E}\left(\left|\sum_{l=m}^{n-1}\mathbf{Y}_{t_l,t_{l+1}}\right|^2\right)\leq C\left|D\right|^{\theta}\left|t_n-t_m\right|^{2/p\prime}. \tag{5.9}$$

Lemma 13 that for all $s' < t' \leq s < t$

$$\mathbb{E}\left(\left(\mathbf{Y}_{s',t'}\right)^{i,j}\left(\mathbf{Y}_{s,t}\right)^{i,j}\right) \leq C\mathbb{E}(W^i(s,t)W^i(s',t'))^2.$$

From the scaling property of the fractional Brownian Motion, we have

$$\mathbb{E}\left(\left|\mathbf{Y}_{s,t}\right|^2\right) \leq C\left|t-s\right|^{4H} = C\mathbb{E}\left(\left|W^i(s,t)\right|^2\right)^2$$

Hence, we obtain

$$\mathbb{E}\left(\left|\sum_{l=m}^{n-1}\mathbf{Y}_{t_l,t_{l+1}}\right|^2\right) \leq C\sum_{k,l=m}^{n-1} E\left(W^1(t_k,t_{k+1})W^1(t_l,t_{l+1})\right)^2.$$

which gives by lemma 17

$$\mathbb{E}\left(\left|\sum_{l=m}^{n-1}\mathbf{Y}_{t_l,t_{l+1}}\right|^2\right) \leq C\left|D\right|^{4H-4/p'}\left|t_n-t_m\right|^{4/p\prime}.$$

□

Lemma 8. *Let D be a subdivision of $[0,1]$. Then, for all $q \geq 1$, there exists $\mu > 0$ and $C_{q,\mu} < \infty$ such that for all $s < t \in D$,*

$$\begin{aligned}&\left\|\int_{s<u_1<u_2<u_3<t}\left\{dW\left(u_1\right)-dW^D\left(u_1\right)\right\}\otimes dW^D\left(u_2\right)\otimes dW^D\left(u_3\right)\right\|_{L^q}\\&\leq C_{q,\mu}\left|D\right|^{\mu}\left|t-s\right|^{3/p'},\end{aligned}$$

and

$$\left\|\int_{s<u<t}\left\{\mathbf{W}^2\left(s,u\right)-S\left(W^D\right)^2_{s,u}\right\}\otimes dW^D\left(u\right)\right\|_{L^q} \leq C_{q,\mu}\left|D\right|^{\mu}\left|t-s\right|^{3/p'}$$

Proof. The subdivision $D = (0 \leq t_1 < \cdots < t_{|D|} \leq 1)$ is fixed. With similar techniques as in theorem 17, we see that to obtain the first inequality, we just need to prove that

$$\begin{aligned}&\mathbb{E}\left(\left|\sum_{l=m}^{n-1}\int_{t_l<u_1<u_2<u_3<t_{l+1}}\left\{dW\left(u_1\right)-dW^D\left(u_1\right)\right\}\otimes dW^D\left(u_2\right)\otimes dW^D\left(u_3\right)\right|^2\right)\\&\leq C\left|D\right|^{\mu}\left|t_n-t_m\right|^{6/p\prime},\end{aligned} \tag{5.10}$$

while to obtain the second one, it is enough to prove that

$$\mathbb{E}\left(\left|\sum_{l=m}^{n-1}\int_{t_l}^{t_{l+1}}\mathbf{W}^{2,i,j}_{t_l,u}dW^{D,k}\left(u\right)\right|^2\right) \leq C\left|D\right|^{\mu}\left|t_n-t_m\right|^{6/p\prime}. \tag{5.11}$$

We first start by proving equation (5.10). We define

$$\tilde{\mathbf{Y}}_{s,t} = \frac{1}{|t-s|^3}\int_{s<u_1<u_2<t}[(u_1-s)W(u_1,t)-(t-u_1)W(s,u_1)]du_1du_2\otimes\left(\mathbf{W}^1(s,t)\right)^{\otimes 2}$$

so that $\tilde{\mathbf{Y}}_{s,t} = \int_{s<u_1<u_2<u_3<t} \left\{dW(u_1) - dW^D(u_1)\right\} \otimes dW^D(u_2) \otimes dW^D(u_3)$. From lemma 14, we have that

$$\begin{aligned}
&\mathbb{E}\left(\left|\sum_{l=m}^{n-1} \tilde{\mathbf{Y}}_{t_l,t_{l+1}}\right|^2\right) \\
\leq\ & \sum_{k,l} \mathbb{E}\left(\left|W^1_{t_k,t_{k+1}}\right|^2\right) \mathbb{E}\left(\left|W^1_{t_l,t_{l+1}}\right|^2\right) \mathbb{E}\left(W^1_{t_k,t_{k+1}} W^1_{t_l,t_{l+1}}\right) \\
\leq\ & |t_n - t_m|^{2h'} \sum_{k,l} \mathbb{E}\left(\left|W^1_{t_k,t_{k+1}}\right|^2\right)^{1/2} \mathbb{E}\left(\left|W^1_{t_l,t_{l+1}}\right|^2\right)^{1/2} \mathbb{E}\left(W^1_{t_k,t_{k+1}} W^1_{t_l,t_{l+1}}\right) \\
\leq\ & C\,|t_n - t_m|^{6/p'} |D|^{\mu} \quad \text{by lemma 17.}
\end{aligned}$$

We can prove (5.11) when $i = j$, with a simple use of the Gaussian formula (6.1), in a similar way as the proof or equation (5.10). The case $k \notin \{i, j\}$ is not too difficult, using the independence between $W_H^{D,k}$ and $\mathbf{W}^{2,i,j}_{t_l,u}$. So we take $k \in \{i, j\}$, with $i \neq j$. An integration by part shows that the cases $k = i$ and $k = j$ are equivalent. Hence, we just need to prove equation (5.11) for $k = i$ different than j. We let for $\alpha \in \{1, 2\}$,

$$\mathbf{Z}^{\alpha}_{s,t} = \left(\frac{1}{t-s}\int_s^t \left(\mathbf{A}^{\alpha}_{s,u}\right)^{i,j} du\right) W^i(s,t)$$

so that for s and t in D, we have $\int_{t_l<u<t_{l+1}} \mathbf{W}^{2,i,j}_{t_l,u} dW_H^{D,k}(u) = \mathbf{Z}^1_{s,t} + \mathbf{Z}^2_{s,t}$. Lemma 10 gives $\mathbb{E}\left(\mathbf{Z}^1_{s,t}\mathbf{Z}^1_{s',t'}\right) = 0$ whenever $s < t \leq s' < t'$, while lemma 15 gives $\mathbb{E}\left(\mathbf{Z}^2_{s,t}\mathbf{Z}^2_{s',t'}\right) \leq C\mathbb{E}\left(W^j(s,t)W^j(s',t')\right) \mathbb{E}\left(\left|W^j(s,t)\right|^2\right) \mathbb{E}\left(\left|W^j(s',t')\right|^2\right)$ for all $s < t \leq s' < t'$ or $(s < t) = (s' < t')$. Then, the scaling property of the fractional Brownian motion gives $\mathbb{E}\left(\left|\mathbf{Z}^1_{s,t}\right|^2\right) = C\mathbb{E}\left(\left|W^j(s,t)\right|^2\right)^3$, so that

$$\begin{aligned}
&\mathbb{E}\left(\left|\sum_{l=m}^{n-1} \int_{t_l<u<t_{l+1}} \mathbf{W}^{2,i,j}_{t_l,u} dW^{D,k}(u)\right|^2\right) \\
\leq\ & C\sum_{k,l} \mathbb{E}\left(\left|W^1_{t_k,t_{k+1}}\right|^2\right) \mathbb{E}\left(\left|W^1_{t_l,t_{l+1}}\right|^2\right) \mathbb{E}\left(W^1_{t_k,t_{k+1}} W^1_{t_l,t_{l+1}}\right) \\
\leq\ & C\,|t_n - t_m|^{6/p'} |D|^{\mu} \quad \text{by lemma 17.}
\end{aligned}$$

□

These previous lemmae, a few more integrations by part, and techniques as in theorem 9 easily lead to theorem 18.

5.2. $W^{1,\infty}$ **paths are in the Cameron-Martin Space.** We check that the piecewise linear combinations are in the Cameron-Martin space associated to Fractional Brownian motion, $\mathcal{H}_H$. Since the case of Brownian motion, that is $H = 1/2$, is trivial we focus on $H < 1/2$.Recall [7] that the integral kernel $K_H(t,s)$ gives rise to a well-defined integral transform for $L^2([0,1])$-functions and

$$\mathcal{H}_H = K_H\left[L^2([0,1])\right]$$

Lemma 9. *Let $H \in (0,1/2)$. Every function f with $f(0)=0$ and $|f'| \in L^\infty$ is in the Cameron-Martin space $\mathcal{H}_H$.*

Proof. Theorem 2.1 of [7] states

$$\mathcal{H}_H = I_{0+}^{H+1/2}\left[L^2([0,1])\right]$$

where fractional integral operator

$$I_{0+}^{\alpha}[f](x) = \frac{1}{\Gamma(\alpha)}\int_0^x f(t)(x-t)^{\alpha-1}\,dt, \qquad x \in [0,1].$$

Restricting to $\alpha \in (0,1)$, its inverse is given by the fractional derivative

$$D_{0+}^{\alpha}[f](t) \equiv \frac{d}{dt} I_{0+}^{1-\alpha}[f](t).$$

So it will be enough to prove that for f smooth and q large enough,

$$\left\| D_{0+}^{h+1/2}[f] \right\|_{L^2} \leq C_{H,q} \|f'\|_{L^q}.$$

This is easy as

$$\begin{aligned}
\left| D_{0+}^{H+1/2}[f](t) \right| &= \left| \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t f(r)(t-r)^{-H-1/2}\,dr \right| \\
&= \frac{H+1/2}{\Gamma(\alpha)} \left| \frac{d}{dt} \left[\int_0^t f'(r)(t-r)^{-H+1/2}\,dr \right] \right| \text{ by IBP} \\
&= \frac{1}{\Gamma(\alpha)} \left| \left[\int_0^t f'(r)(t-r)^{-H-1/2}\,dr \right] \right| \\
&\leq \frac{c_{H,q}\|f'\|_{L^q}}{\Gamma(\alpha)} < \infty \text{ by Hölder inequality and for } q > \frac{1}{1/2-H}.
\end{aligned}$$

In particular, $\left\| D_{0+}^{h+1/2}[f] \right\|_{L^2} \leq \left\| D_{0+}^{h+1/2}[f] \right\|_{L^\infty} \leq C_H \|f'\|_{L^q}$. □

6. Appendix 2

We use a couple times the following formula: for X_i centered Gaussian random variable

$$\mathbb{E}\left(\prod_{i=1}^{2n} X_i\right) = \sum_{\substack{(i_1^1,i_2^1),\ldots,(i_1^n,i_2^n) \\ \text{distinct pairs}}} \prod_{j=1}^{n} \mathbb{E}\left(X_{i_1^j} X_{i_2^j}\right) \tag{6.1}$$

Proposition 4. *For all $u<v$ in $[s,t]$, $u'<v'$ in $[s',t']$, where $s'<t' \leq s<t$*

$$0 \leq -\mathbb{E}(W^i(u',v')W^i(u,v)) \leq -\mathbb{E}(W^i(s',t')W^i(s,t))$$

and for

$$0 \leq \mathbb{E}(W^i(s,u)W^i(s,v)) \leq \mathbb{E}(W^i(s,u)W^i(s,t)) \leq \mathbb{E}\left(W^i(s,t)^2\right).$$

Proof. The first inequality is obvious from the formula

$$\begin{aligned}
\mathbb{E}(W(u',v')W(u,v)) &= \frac{1}{2}[|v-u'|^{2H} + |u-v'|^{2H} - |u-u'|^{2H} - |v-v'|^{2H}] \\
&= H(2H-1)\int_u^v \int_{u'}^{v'} |y-x|^{2H-2} dy dx.
\end{aligned}$$

We prove the second inequality with basic inequalities:

$$\begin{aligned}
& \mathbb{E}(W^i(s,u)W^i(s,v)) \\
= & \frac{1}{2}\left[|u-s|^{2H}+|v-s|^{2H}-|v-u|^{2H}\right] \\
\leq & \frac{1}{2}\left[|u-s|^{2H}+|t-s|^{2H}-|t-u|^{2H}\right] = \mathbb{E}(W^i(s,u)W^i(s,t)) \\
\leq & \frac{1}{2}\left[|t-s|^{2H}+|t-s|^{2H}-0\right] = \mathbb{E}\left(W^i(s,t)^2\right).
\end{aligned}$$

□

In the lemmae below, we will use the notation of the proof of lemma 7.

Lemma 10. *Let f_u and g_u two measurable processes independent of B, and Skorokhod integrable. Then, for all $s' < t' \leq s < t$,*

$$\mathbb{E}\left(\left(\int_s^t f(u)\,dB_u\right)\left(\int_{s'}^{t'} g(u)\,dB_u\right)\right) = 0.$$

Proof. Obvious from the property of Skorokhod integrals. □

Lemma 11. *For all $s' < t' \leq s < t$,*

$$\mathbb{E}\left(\left(\mathbf{A}^2_{s',t'}\right)^{i,j}\left(\mathbf{A}^2_{s,t}\right)^{i,j}\right) \leq \mathbb{E}(W^i(s,t)W^i(s',t'))^2.$$

Proof. Once again, we can take $i \neq j$. Hence, denoting $X = \left|\mathbb{E}\left(\left(\mathbf{A}^2_{s',t'}\right)^{i,j}\left(\mathbf{A}^2_{s,t}\right)^{i,j}\right)\right|$, we have that X equals

$$\begin{aligned}
& \left|\mathbb{E}\left(\int_0^s\left(\int_s^t \partial K_H(r,u)W^i_H(s,r)dr\right)dB^j_u \int_0^{s'}\left(\int_{s'}^{t'}\partial K_H(r',u)W^i_H(s',r')dr'\right)dB^j_u\right)\right| \\
& \quad = \int_{u=0}^{s'}\int_{r=s}^{t}\int_{r'=s'}^{t'} \partial K_H(r,u)\partial K_H(r',u)\mathbb{E}\left(W^i(s,r)W^i(s',r')\right)drdr'du.
\end{aligned}$$

Now we bound $\mathbb{E}\left(W^i_H(s,r)W^i_H(s',r')\right)$ by $|\mathbb{E}\left(W^i(s,t)W^i(s',t')\right)|$, thanks to proposition 4. Hence, we obtain that

$$\begin{aligned}
X &\leq \left|\mathbb{E}\left(W^i(s,t)W^i(s',t')\right)\right| \int_{u=0}^{s'}\int_{r=s}^{t}\int_{r'=s'}^{t'} \partial K_H(r,u)\partial K_H(r',u)drdr'du \\
&= \left|\mathbb{E}\left(W^i(s,t)W^i(s',t')\right)\right| \int_{u=0}^{s'} \left[K_H(t,u)-K(s,u)\right]\left[K_H(t',u)-K(s',u)\right]du.
\end{aligned}$$

Then,

$$\begin{aligned}
\int_{u=0}^{s'}\left[K_H(t,u)-K(s,u)\right]K_H(t',u)du &\leq \int_{u=0}^{t'}\left[K_H(t,u)-K(s,u)\right]K_H(t',u)du \\
&= \mathbb{E}\left(W^i(s,t)W^i(t')\right)
\end{aligned}$$

and

$$\int_{u=0}^{s'}\left[K_H(t,u)-K(s,u)\right]K(s',u)du = \mathbb{E}\left(W^i(s,t)W^i(s')\right).$$

That gives our result. □

The same argument gives the following lemma:

Lemma 12. *For all $s < t$,*

$$\mathbb{E}\left(\left|\left(\mathbf{A}^2_{s,t}\right)^{i,j}\right|^2\right) \leq \mathbb{E}(\left|W^i(s,t)\right|^2)^2,$$

Lemma 13. *For all $s' < t' \leq s < t$,*

$$\mathbb{E}\left(\left(\mathbf{Y}_{s',t'}\right)^{i,j}\left(\mathbf{Y}_{s,t}\right)^{i,j}\right) \leq \mathbb{E}(W^i(s,t)W^i(s',t'))^2$$

Proof. Recall that

$$\mathbf{Y}_{s,t} = \frac{1}{|t-s|^2}\int_s^t [(u-s)W(u,t) - (t-u)W(s,u)]du \otimes W(s,t)$$

Moreover, since

$$\mathbb{E}(W(s)W(t)) = \frac{1}{2}[s^{2H} + t^{2H} - |t-s|^{2H}]$$

we obtain $\mathbb{E}(\mathbf{Y}_{s,t}) = 0$. Therefore, $2|t-s|^2|t'-s'|^2\mathbb{E}\left(\left(\mathbf{Y}_{s,t}\right)^{i,j}\left(\mathbf{Y}_{s',t'}\right)^{i,j}\right)$ is bounded by $M_1M_2 + M_3M_4$, where

$$\begin{aligned}
M_1 &= \mathbb{E}\left(\int_s^t [(u-s)W^i(u,t) - (t-u)W^i(s,u)]duW^j_H(s',t')\right),\\
M_2 &= \mathbb{E}\left(\int_{s'}^{t'} [(u-s')W^i(u,t') - (t'-u)W^i(s',u)]duW^j_H(s,t)\right),\\
M_3 &= \mathbb{E}\left(\int_s^t [(u-s)W^i(u,t) - (t-u)W^i(s',u)]du\right.\\
&\qquad \left.\times \int_{s'}^{t'} [(v-s')W^j(u,t') - (t'-v)W^j(s',v)]dv\right),\\
M_4 &= \mathbb{E}\left(W^j(s,t)W^j(s',t')\right),
\end{aligned}$$

Using proposition 4, we obtain that

$$\begin{aligned}
M_1 &\leq \int_s^t ((u-s)+(t-u))\,du\left|\mathbb{E}\left(W^i(s,t)W^j(s',t')\right)\right|\\
&\leq (t-s)^2\left|\mathbb{E}\left(W^i(s,t)W^j(s',t')\right)\right|.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
M_2 &\leq (t'-s')^2\left|\mathbb{E}\left(W^i(s,t)W^j(s',t')\right)\right|,\\
M_3 &\leq (t-s)^2(t'-s')^2\left|\mathbb{E}\left(W^i(s,t)W^i(s',t')\right)\right|.
\end{aligned}$$

Hence, we have shown that

$$\left|\mathbb{E}\left(\left(\mathbf{Y}_{s,t}\right)^{i,j}\left(\mathbf{Y}_{s',t'}\right)^{i,j}\right)\right| \leq \mathbb{E}\left(W^i(s,t)W^i(s',t')\right)^2$$

□

Lemma 14. *For all $s' < t' \leq s < t$ or $(s',t') = (s,t)$,*

$$\begin{aligned}
&\mathbb{E}\left(\left(\tilde{\mathbf{Y}}_{s',t'}\right)^{i,j}\left(\tilde{\mathbf{Y}}_{s,t}\right)^{i,j}\right)\\
\leq\ & C\mathbb{E}\left(W^j(s,t)W^j(s',t')\right)\mathbb{E}\left(\left|W^j(s,t)\right|^2\right)\mathbb{E}\left(\left|W^j(s',t')\right|^2\right)
\end{aligned}$$

Proof. Recall that

$$\tilde{\mathbf{Y}}_{s,t} = \frac{1}{|t-s|^3} \int_{s<u_1<u_2<t} [(u_1-s)W(u_1,t) - (t-u_1)W(s,u_1)] du_1 du_2 \otimes W(s,t)^{\otimes 2}.$$

The proof is quite similar to the previous one. By equality (6.1),

$$\begin{aligned}(t-s)^3 (t'-s')^3 E\left(\tilde{\mathbf{Y}}_{s,t}\tilde{\mathbf{Y}}_{s',t'}\right) &= \tilde{M}_3\left(\tilde{M}_5\tilde{M}_6 + 2\tilde{M}_4^2\right) \\ &\quad +2\tilde{M}_7\left(2\tilde{M}_8\tilde{M}_4 + \tilde{M}_2\tilde{M}_6\right) \\ &\quad +2\tilde{M}_1\left(2\tilde{M}_2\tilde{M}_4 + \tilde{M}_8\tilde{M}_5\right)\end{aligned}$$

where

$$\begin{aligned}
\tilde{M}_1 &= \mathbb{E}\left(\left(\int_{s<u_1<u_2<t} [(u_1-s)W(u_1,t) - (t-u_1)W(s,u_1)] du_1 du_2\right) W^j(s',t')\right), \\
\tilde{M}_2 &= \mathbb{E}\left(\left(\int_{s'<u_1<u_2<t'} [(u_1-s')W^i(u_1,t') - (t'-u_1)W^i(s',u_1)] du_1 du_2\right) W^j(s,t)\right), \\
\tilde{M}_3 &= \mathbb{E}\left(\int_{s<u_1<u_2<t} [(u_1-s)W^i(u_1,t) - (t-u_1)W^i(u_1,t)] du_1 du_2\right. \\
&\quad \left.\int_{s'<u_1<u_2<t'} [(u_1-s')W^i(u_1,t') - (t'-u_1)W^i(u_1,t')] du_1 du_2\right), \\
\tilde{M}_4 &= \mathbb{E}\left(W^j(s,t)W^j(s',t')\right), \\
\tilde{M}_5 &= \mathbb{E}\left(\left|W^j(s,t)\right|^2\right), \\
\tilde{M}_6 &= \mathbb{E}\left(\left|W^j(s',t')\right|^2\right), \\
\tilde{M}_7 &= \mathbb{E}\left(\left(\int_{s<u_1<u_2<t} [(u_1-s)W(u_1,t) - (t-u_1)W(s,u_1)] du_1 du_2\right) W^j(s,t)\right), \\
\tilde{M}_8 &= \mathbb{E}\left(\left(\int_{s'<u_1<u_2<t'} [(u_1-s')W^i(u_1,t') - (t'-u_1)W^i(s',u_1)] du_1 du_2\right) W^j(s',t')\right).
\end{aligned}$$

Working just as in the previous lemma , we see that

$$\frac{\left|\tilde{M}_1\right|}{(t-s)^3} + \frac{\left|\tilde{M}_2\right|}{(t'-s')^3} + \frac{\left|\tilde{M}_3\right|}{(t-s)^3(t'-s')^3} \leq C\tilde{M}_4,$$

and $\left|\tilde{M}_7\right| \leq C(t-s)^3 \tilde{M}_5$ and $\left|\tilde{M}_8\right| \leq C(t-s)^3 \tilde{M}_6$. Therefore, we obtain that

$$\mathbb{E}\left(\left(\tilde{\mathbf{Y}}_{s',t'}\right)^{i,j}\left(\tilde{\mathbf{Y}}_{s,t}\right)^{i,j}\right) \leq C\left(\tilde{M}_4^3 + \tilde{M}_4\tilde{M}_5\tilde{M}_6\right).$$

Finally, by Cauchy-Schwarz, $\tilde{M}_4^2 \leq \tilde{M}_5\tilde{M}_6$, which implies that

$$\mathbb{E}\left(\left(\tilde{\mathbf{Y}}_{s',t'}\right)^{i,j}\left(\tilde{\mathbf{Y}}_{s,t}\right)^{i,j}\right) \leq C\tilde{M}_4\tilde{M}_5\tilde{M}_6.$$

□

Lemma 15. *For $i \neq j$, all $s' < t' \leq s < t$ or $s = s' < t = t'$, with $(s', t', s, t) \in D$,*

$$\mathbb{E}\left(\left(\frac{1}{t-s}\int_s^t \left(\mathbf{A}^2_{s,u}\right)^{i,j} du\right) W^i(s,t) \left(\frac{1}{t'-s'}\int_{s'}^{t'} \left(\mathbf{A}^2_{s',u}\right)^{i,j} du\right) W^i(s',t')\right)$$
$$\leq C\mathbb{E}\left(W^1(s,t)W^1(s',t')\right)\mathbb{E}\left(\left|W^1(s,t)\right|^2\right)\mathbb{E}\left(\left|W^1(s',t')\right|^2\right).$$

Proof. First observe that

$$\int_s^t \left(\mathbf{A}^2_{s,u}\right)^{i,j} dW^{D,i}(u) = \frac{W^i_{s,t}}{t-s}\int_s^t \left(\int_0^s \left(\int_s^u \partial K_H(r,v) W^i(s,r) dr\right) dB^j_v\right) du$$
$$= \frac{1}{t-s}\int_s^t \left(\int_0^s \left(W^i_{s,t}\int_s^u \partial K_H(r,v) W^i(s,r) dr\right) dB^j_v\right) du.$$

Therefore,

$$(t-s)(t'-s')\,\mathbb{E}\left(\int_s^t \left(\mathbf{A}^2_{s,u}\right)^{i,j} dW^{D,i}(u) \int_{s'}^{t'} \left(\mathbf{A}^2_{s,u}\right)^{i,j} dW^{D,i}(u)\right)$$
$$= \int_s^t\int_{s'}^{t'} \mathbb{E}\left(\left(\int_0^s \left(W^i_{s,t}\int_s^u \partial K_H(r,v) W^i_{s,r} dr\right) dB^j_v\right)\right.$$
$$\left.\left(\int_0^{s'} \left(W^i_{s',t'}\int_{s'}^{u'} \partial K_H(r',v') W^i_{s',r'} dr'\right) dB^j_{v'}\right)\right) du du'$$
$$= \int_s^t\int_{s'}^{t'} \mathbb{E}\left(\int_0^{s'} W^i_{s',t'} W^i_{s,t} \left(\int_s^u \partial K_H(r,v) W^i_{s,r} dr\right)\right.$$
$$\left.\left(\int_{s'}^{u'} \partial K_H(r',v) W^i_{s',r'} dr'\right) dv\right) du du'$$
$$= \int_s^t\int_{s'}^{t'}\int_0^{s'}\int_s^u\int_{s'}^{u'} \partial K_H(r',v)\partial K_H(r,v)\mathbb{E}\left(W^i_{s',t'} W^i_{s,t} W^i_{s,r} W^i_{s',r'}\right) dr dr' dv du du'.$$

Now, using equality (6.1) and proposition 4, we see that

$$\begin{aligned}\mathbb{E}\left(W^i_{s',t'} W^i_{s,t} W^i_{s,r} W^i_{s',r'}\right) &\leq 2\mathbb{E}\left(W^i_{s',t'} W^i_{s,t}\right)^2 + \mathbb{E}\left(\left|W^i_{s,t}\right|^2\right)\mathbb{E}\left(\left|W^i_{s',t'}\right|^2\right) \\ &\leq 3\mathbb{E}\left(\left|W^i_{s,t}\right|^2\right)\mathbb{E}\left(\left|W^i_{s',t'}\right|^2\right) \text{ by Cauchy-Schwartz.}\end{aligned}$$

Hence

$$\begin{aligned}&(t-s)(t'-s')\,\mathbb{E}\left(\int_s^t \left(\mathbf{A}^2_{s,u}\right)^{i,j} dW^{D,i}(u) \int_{s'}^{t'} \left(\mathbf{A}^2_{s,u}\right)^{i,j} dW^{D,i}(u)\right) \\ \leq\ & 3\mathbb{E}\left(\left|W^i_{s,t}\right|^2\right)\mathbb{E}\left(\left|W^i_{s',t'}\right|^2\right) \\ &\times \int_{u=s}^t \int_{u'=s'}^{t'} \int_{v=0}^{s'} \int_{r=s}^u \int_{r'=s'}^{u'} \partial K_H(r',v)\partial K_H(r,v) dr dr' dv du du'.\end{aligned}$$

By positivity of the derivative of the kernel K_H, we therefore obtain

$$\begin{aligned}
&(t-s)(t'-s')\,\mathbb{E}\left(\int_s^t \left(\mathbf{A}^2_{s,u}\right)^{i,j} dW^{D,i}(u) \int_{s'}^{t'} \left(\mathbf{A}^2_{s,u}\right)^{i,j} dW^{D,i}(u)\right)\\
\leq\ & 3\mathbb{E}\left(\left|W^i_{s,t}\right|^2\right)\mathbb{E}\left(\left|W^i_{s',t'}\right|^2\right)(t-s)(t'-s')\\
&\times \int_{v=0}^{s'}\int_{r=s}^{t}\int_{r'=s'}^{t'} \partial K_H(r',v)\partial K_H(r,v)drdr'dv.
\end{aligned}$$

We proved in lemma 11 that

$$\int_{v=0}^{s'}\int_{r=s}^{t}\int_{r'=s'}^{t'} \partial K_H(r',v)\partial K_H(r,v)drdr'dv \leq \mathbb{E}\left(W^i(s,t)W^i(s',t')\right),$$

which conclude the case $s' < t' \leq s < t$. We leave the easier case $s = s' < t = t'$ to the reader. □

Lemma 16. *For all $x \leq s' < t' \leq s < t$, and $k \notin \{i,j\}$*

$$\begin{aligned}
&\mathbb{E}\left(W^i_{x,s'}\left(\mathbf{A}^2_{s',t'}\right)^{j,k} W^i_{x,s}\left(\mathbf{A}^2_{s,t}\right)^{j,k}\right)\\
\leq\ & \mathbb{E}(W^1_{x,s}W^1_{x,s'}W^1(s,t)W^1(s',t'))\mathbb{E}\left(W^1(s,t)W^1(s',t')\right).
\end{aligned}$$

and

$$\mathbb{E}\left(\left|W^i_{x,s}\left(\mathbf{A}^2_{s,t}\right)^{j,k}\right|^2\right) \leq \mathbb{E}(\left|W^i_{x,y}\right|^2\left|W^i(s,t)\right|^2)\mathbb{E}\left(\left|W^i(s,t)\right|^2\right)$$

Proof. Using the notation $X = \left|\mathbb{E}\left(W^i_{x,s'}\left(\mathbf{A}^2_{s',t'}\right)^{j,k} W^i_{x,s}\left(\mathbf{A}^2_{s,t}\right)^{j,k}\right)\right|$, we have that X equals

$$\begin{aligned}
&\left|\mathbb{E}\left(\int_0^s \left(\int_s^t \partial K_H(r,u)W^i_{x,s}W^j(s,r)dr\right)dB^j_u \right.\\
&\qquad\left. \int_0^{s'}\left(\int_{s'}^{t'}\partial K_H(r',u)W^i_{x,s'}W^j(s',r')dr'\right)dB^j_u\right)\right|\\
=\ & \int_{u=0}^{s'}\int_{r=s}^{t}\int_{r'=s'}^{t'} \partial K_H(r,u)\partial K_H(r',u)\mathbb{E}\left(W^i_{x,s}W^i_{x,s'}W^j(s,r)W^j(s',r')\right)drdr'du.
\end{aligned}$$

Now we bound $\mathbb{E}\left(W^i_{x,s}W^i_{x,s'}W^j(s,r)W^j(s',r')\right)$ by $\mathbb{E}\left(W^1_{x,s}W^1_{x,s'}W^1(s,t)W^1(s',t')\right)$, thanks to equation (6.1) and proposition 4. Hence, we obtain that

$$X \leq \mathbb{E}\left(W^1_{x,s}W^1_{x,s'}W^1(s,t)W^1(s',t')\right)\int_{u=0}^{s'}\int_{r=s}^{t}\int_{r'=s'}^{t'} \partial K_H(r,u)\partial K_H(r',u)drdr'du.$$

We conclude just as in lemma 11. The second part of the lemma follow in a similar way. □

Lemma 17. *Let $D = (t_k)_{k\in\{m,\cdots,n\}}$ be a subdivision. Then, for $p' > 1/H$,, there exists a constant $C_{p'}$ such that*

$$\begin{aligned}
&\sum_{k,l=m}^{n} \left\|W^j(t_k,t_{k+1})\right\|_{L^2}\left\|W^j(t_l,t_{l+1})\right\|_{L^2}\mathbb{E}\left(W^i(t_k,t_{k+1})W^i(t_l,t_{l+1})\right)\\
\leq\ & C_{p'}\left(t_n - t_m\right)^{4/p'}|D|^{4H-4/p'}.
\end{aligned}$$

In particular,

$$\sum_{k,l=m}^{n} \mathbb{E}\left(W^i(t_k,t_{k+1})W^i(t_l,t_{l+1})\right)^2 \leq C_{p'}\left(t_n-t_m\right)^{4/p'}|D|^{4h-4/p'}.$$

Proof. We observed in the proof of Proposition 4 that, for $s'<t'\leq s<t$,

$$\mathbb{E}(W(s',t')W(s,t)) = C_H\int_{u=s}^{t}\int_{v=s'}^{t'}(u-v)^{2H-2}dudv$$

with $C_H = H(2H-1)$.We bound the term $(u-v)^{2H-2}$ in the integral by $(u-t')^{-1+\varepsilon}(t'-v)^{2H-\varepsilon-1}$ (for a fixed $\varepsilon\in(0,2H)$) to obtain

$$|\mathbb{E}(W(s',t')W(s,t))| \leq C_H\int_{u=s}^{t}\int_{v=s'}^{t'}(u-t')^{-1+\varepsilon}du(t'-v)^{2H-\varepsilon-1}dv.$$

Integrating with respect to $v\in[s',t']$, we obtain

$$\mathbb{E}(W(s',t')W(s,t)) \leq C_{H,\varepsilon}\int_{s}^{t}(u-t')^{-1+\varepsilon}du|t'-s'|^{2H-\varepsilon}. \tag{6.2}$$

Multiplying inequality (6.2) by $(t-s)^h(t'-s')^h$, we get

$$\begin{aligned}(t-s)^H(t'-s')^H\,&\mathbb{E}(W(s',t')W(s,t))\\ &\leq C_{H,\varepsilon}\left(\int_s^t(u-t')^{-1+\varepsilon}du\right)(t'-s')^{3H-\varepsilon}|t-s|^H.\end{aligned}$$

Fix $l\in\{m,\cdots,n\}$, and apply the above inequality to $(s',t',s,t)=(t_l,t_{l+1},t_k,t_{k+1})$:

$$\begin{aligned}&\sum_{k=l+1}^{n}\mathbb{E}\left(W(t_l,t_{l+1})W(t_k,t_{k+1})\right)^2\\ &\leq C_{H,\varepsilon}\sum_{k=l+1}^{n}\left(\int_{t_k}^{t_{k+1}}(u-t_{l+1})^{-1+\varepsilon}du\right)(t_{l+1}-t_l)^{3H-\varepsilon}|t_{k+1}-t_k|^H.\end{aligned}$$

We then bound $|t_{k+1}-t_k|^H$ by $|D|^H$:

$$\begin{aligned}&\sum_{k=l+1}^{n}(t_{l+1}-t_l)^H(t_{k+1}-t_k)^H\,\mathbb{E}\left(W(t_l,t_{l+1})W(t_k,t_{k+1})\right)^2\\ \leq\;& C_{H,\varepsilon}\sum_{k=l+1}^{n}\left(\int_{t_k}^{t_{k+1}}(u-t_{l+1})^{-1+\varepsilon}du\right)(t_{l+1}-t_l)^{3H-\varepsilon}|D|^H\\ =\;& C_{H,\varepsilon}\left(\int_{t_{l+1}}^{t_n}(u-t_{l+1})^{-1+\varepsilon}du\right)(t_{l+1}-t_l)^{3H-\varepsilon}|D|^H\\ \leq\;& C_{H,\varepsilon}\left(t_n-t_{l+1}\right)^{\varepsilon}(t_{l+1}-t_l)|D|^{4H-1-\varepsilon}\end{aligned}$$

And therefore, we obtain

$$\begin{aligned}&\sum_{l=m}^{n-2}\sum_{k=l+1}^{n-1}(t_{l+1}-t_l)^H(t_{k+1}-t_k)^H\,\mathbb{E}\left(W^i(t_k,t_{k+1})W^i(t_l,t_{l+1})\right)\\ \leq\;& C_{H,\varepsilon}\sum_{l=m}^{n-2}\left(t_n-t_{l+1}\right)^{\varepsilon}(t_{l+1}-t_l)|D|^{4H-1-\varepsilon}.\end{aligned}$$

Comparing $\sum_{l=m}^{n-2}(t_n - t_{l+1})^{\varepsilon}(t_{l+1}-t_l)$ to $\int_{t_m}^{t_n}(t_n-u)^{\varepsilon}du$, we see that

$$\sum_{l=m}^{n-2}(t_n - t_{l+1})^{\varepsilon}(t_{l+1}-t_l) \leq C(t_n-t_m)^{1+\varepsilon}.$$

Hence,

$$\begin{aligned}&\sum_{l=m}^{n-2}\sum_{k=l+1}^{n-1}(t_{l+1}-t_l)^H(t_{k+1}-t_k)^H\mathbb{E}\left(W^i(t_k,t_{k+1})W^i(t_l,t_{l+1})\right)\\ \leq\ & C_{H,\varepsilon}(t_n-t_m)^{1+\varepsilon}|D|^{4H-1-\varepsilon}\end{aligned}$$

Also, we easily obtain

$$\begin{aligned}\sum_{l=m}^{n-1}(t_{l+1}-t_l)^{2H}\mathbb{E}\left(W(t_l,t_{l+1})^2\right) &= C_H\sum_{l=m}^{n-1}(t_{l+1}-t_l)^{4H}\\ &\leq C_H|D|^{4H-1-\varepsilon}\sum_{l=m}^{n-1}(t_{l+1}-t_l)^{1+\varepsilon}\\ &\leq C_{H,\varepsilon}(t_n-t_m)^{1+\varepsilon}|D|^{4H-1-\varepsilon}\end{aligned}$$

Adding the two previous inequalities give our results:

$$\begin{aligned}&\sum_{k,l=m}^{n}\left\|W^j(t_k,t_{k+1})\right\|_{L^2}\left\|W^j(t_l,t_{l+1})\right\|_{L^2}\mathbb{E}\left(W^i(t_k,t_{k+1})W^i(t_l,t_{l+1})\right)\\ &\qquad\leq C_{H,\varepsilon}(t_n-t_m)^{1+\varepsilon}|D|^{4H-1-\varepsilon}\end{aligned}$$

which is our result with $1+\varepsilon = 4/p'$ (which is coherent with $\varepsilon \in (0,2H)$, as $H \in (1/4,1/2)$). □

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