A general framework for simulation of fractional fields

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Abstract

Beside fractional Brownian motion most non-Gaussian fractional fields are obtained by integration of deterministic kernels with respect to a random infinitely divisible measure. In this paper, generalized shot noise series are used to obtain approximation of most of these fractional fields, including linear fractional stable fields and harmonizable fractional stable fields. Almost sure rate of convergence and in L^r-norm, that are related to asymptotic developments of the deterministic kernels, are given. They are consequences of an important Lemma concerning series of random symmetric series. Normal approximation of fractional fields are also used for some random measures. The general framework is illustrated by simulations of classical fractional fields.

1 Introduction

Irregular phenomena appear in various fields of scientific research: fluid mechanics, image processing and financial mathematics for example. Experts in those fields often ask mathematicians to develop models both easy to use and relevant for their applications. In this perspective, fractional fields are very often used to model irregular phenomena. Among the huge literature devoted to the topic, one can refer the reader to [6] for a recent overview of fractional fields for applications. To use fractional models, one needs algorithms, and simulations that are both flexible and easy to generate.

One of the simplest model is the fractional Brownian motion introduced in [9] and further developed in [14]. Simulation of fractional Brownian motion is now both theoretically and practically well understood (see [2] for a survey on this problem). However, it is a well known fact that in some fields of applications data do not fit Gaussian models, see for instance [13, 22, 25] for image modeling. Many other fields with heavy tailed marginals have been proposed for applications, see Chapter 7 in [21] for an introduction to fractional stable processes. More recently other processes that are neither Gaussian nor stable have been proposed to model Internet traffic (cf. [24, 5]). The common feature for many of these fields, see also [3, 4, 10], is the fact that they are obtained by a stochastic integration of a deterministic kernel with respect to some random measure. In term of models, we can think that the probabilistic structure of the irregular phenomena (light or heavy tails for instance) is implemented in the random measure and the correlation structure is built in the deterministic kernel. Engineers will have to try many kernels and random measures before finding the more appropriate one for their applications. Therefore, they need a common framework to simulate fractional fields to make many attempts.

In the literature, there exist articles for simulation of the fractional fields that are non Gaussian. In [7] a wavelet expansion is used to approximate harmonizable and well-balanced type of fractional

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stable processes. For the linear fractional stable processes the fast Fourier transform is the main tool for simulation in [23, 26]. One can also quote a recent work [15], where another integral representation of the linear fractional stable processes is used to obtain simulation of the sample paths. Even though, all these processes are stable, they have different distributions and for each one a specific method is used. Concerning non stable processes, generalized shot noise series introduced for simulation of Lévy processes in [18, 19, 20] were used for simulation of the sample paths of real harmonizable multifractional fields in [11]. One of the advantages of this method is the fact that it can be applied to fractional fields that are neither with stationary increments nor self-similar. Moreover, it is straightforward to apply this technique to the simulation of fields indexed by multidimensional spaces. In this article, our main goal is to show how this method can be applied to most of the fractional fields.

Let us describe how one can obtain an algorithm of simulation when an integral representation of the fractional field is known. We will be interested in the simulation of stochastic integrals of the form

$$X^f(x) = \int_{\mathbb{R}^d} f(x,\xi)\,\Lambda(d\xi), \quad x \in \mathbb{R}^d,$$

with Λ an infinitely divisible random measure.

Basically, you transform the random measure Λ by a sum of weighted Dirac masses at random point at the arrival times of a standard Poisson process. After the transformation, the integrals are series, that you can simulate by truncating the number of terms.

We would like also stress that we have obtained rate of convergence for the truncating series. More precisely, almost sure rates of convergence are given both for each marginal of the field, and uniformly if the field is simulated on a compact set. The almost sure convergence is related to asymptotic developments of the deterministic kernel in the integral representation of the field. Let us also emphasize Lemma 3.5 which is an important tool to reach rates of convergence for series of symmetric random variables under moment assumptions. This lemma, that we have not found in the literature, may have interest of its own and is needed in the heavy tail cases. Rates of convergence in L^r-norm with explicit constant are further obtained.

When the control measure of Λ has infinite mass, a technical complication arises. In the case of Lévy processes, [1] solves this problem giving a normal approximation for a part of the Lévy processes. Following [1], in [11], one of the authors approximates one part of some multifractional fields called RHMLMs (real harmonizable multifractional Lévy motions) by a multifractional Brownian motion. Here, we will also split X^f into two fields $X^f_{\varepsilon,1}$ and $X^f_{\varepsilon,2}$. As in the case of RHMLMs, $X^f_{\varepsilon,2}$ can be represented as a generalized shot noise series. The field $X^f_{\varepsilon,1}$ can be approximated by a Gaussian field and the error due to this approximation will be given in terms of Berry-Esseen bounds.

Section 2 is devoted to some basic facts concerning stochastic integrals with respect to random measures. Then, convergence and rates of convergence of the generalized shot noise series are given in Section 3. Section 4 gives an approximation of the stochastic integrals when the control measure has infinite mass and establishes Berry-Esseen bounds. Examples, that include most of the classical fractional fields, are given in Section 5, illustrated by simulations. Section 6 is devoted to the case of complex random measures, which are important for harmonizable fields. The proof of Lemmas 3.5 and 3.8 is postponed to the Appendix.

2 Stochastic integrals with respect to Poisson random measure

Let us recall some classical facts concerning stochastic integrals with respect to Poisson random measure (see [17] for more details). Let $N(d\xi, dv)$ be a Poisson random measure on $\mathbb{R}^d \times \mathbb{R}$ with intensity $n(d\xi, dv) = d\xi\nu(dv)$. Assume that the non-vanishing σ-finite measure $\nu(dv)$ is a symmetric measure such that

$$\int_{\mathbb{R}} \Big(|v|^2 \wedge 1\Big)\,\nu(dv) < +\infty,$$

where $a \wedge b = \min(a, b)$. In particular, the assumptions done in [4] may not be fulfilled. However, see for instance [8], the stochastic integral

$$\int_{\mathbb{R}^d \times \mathbb{R}} \varphi(\xi, v) \Big[N(d\xi, dv) - (1 \vee |\varphi(\xi, v)|)^{-1} \, n(d\xi, dv) \Big],$$

where $a \vee b = \max(a, b)$, is defined if and only if

$$\int_{\mathbb{R}^d \times \mathbb{R}} \Big(|\varphi(\xi, v)|^2 \wedge 1 \Big) \, n(d\xi, dv) < +\infty.$$

Then, we can consider a random measure $\Lambda(d\xi)$ on $\mathbb{R}^d$ defined by

$$\int_{\mathbb{R}^d} g(\xi) \, \Lambda(d\xi) = \int_{\mathbb{R}^d \times \mathbb{R}} g(\xi) v \Big(N(d\xi, dv) - (|g(\xi)v| \vee 1)^{-1} \, n(d\xi, dv) \Big), \tag{1}$$

for every $g : \mathbb{R}^d \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^d} \big(g^2(\xi) \wedge 1 \big) \, d\xi < +\infty.$$

We have that

$$\mathbb{E}\left[\exp\left(i \left(\int_{\mathbb{R}^d} g(\xi) \, \Lambda(d\xi) \right) \right) \right] = \exp\left[\int_{\mathbb{R}^d \times \mathbb{R}} \big[\exp(ig(\xi)v) - 1 - ig(\xi)v \mathbf{1}_{|g(\xi)v| \wedge 1} \big] \, d\xi \, \nu(dv) \right], \tag{2}$$

Then Λ is an infinitely divisible random measure.

As explained below (see Examples 2.1 and 2.2), Lévy random measures and stable random measures are examples of such infinitely divisible random measures represented by a Poisson random measure owing to (1). Here are some illustrations.

Example 2.1. If $\nu(dv)$ is a symmetric measure such that

$$\int_{\mathbb{R}^d} |v|^2 \, \nu(dv) < +\infty,$$

then, for every $g \in L^2\big(\mathbb{R}^d\big)$,

$$\begin{aligned} \int_{\mathbb{R}^d} g(\xi) \, \Lambda(d\xi) &= \int_{\mathbb{R}^d \times \mathbb{R}} g(\xi) v \, (N(d\xi, dv) - n(d\xi, dv)) \\ &= \int_{\mathbb{R}^d} g(\xi) \, M(d\xi), \end{aligned}$$

with $M(d\xi)$ a Lévy random measure, without Brownian component, represented by the Poisson random measure $N(d\xi, dv)$ in the sense of [4]. Therefore, if the symmetric measure $\nu(dv)$ satisfies the assumptions done in [4], i.e. if

$$\forall p \geq 2, \; \int_{\mathbb{R}} |v|^p \, \nu(dv) < +\infty,$$

the field $(X_H(x))_{x \in \mathbb{R}^d}$, defined by

$$X_H(x) = \int_{\mathbb{R}^d} \Big(\|x - \xi\|^{H - d/2} - \|\xi\|^{H - d/2} \Big) \, \Lambda(d\xi),$$

is a moving average fractional Lévy motion, in short MAFLM, with index H ($0 < H < 1$, $H \neq d/2$).

Example 2.2. In the case where

$$\nu(dv) = \frac{dv}{|v|^{1+\alpha}}$$

with $0 < \alpha < 2$, the random measure $\Lambda(d\xi)$, defined by (1), is a symmetric α-stable random measure in the sense of [21]. Then, for instance, set

$$X_H(x) = D(\alpha)^{-1/\alpha} \int_{\mathbb{R}^d} \Big(\|x - \xi\|^{H-d/\alpha} - \|\xi\|^{H-d/\alpha} \Big) \Lambda(d\xi), \quad x \in \mathbb{R}^d,$$

with

$$D(\alpha) = \int_{\mathbb{R}} \frac{1 - \cos(r)}{|r|^{1+\alpha}} \, dr \tag{3}$$

is a moving average fractional stable motion, in short MAFSM, with index H ($0 < H < 1$, $H \neq d/\alpha$).

Hence in the following, we will be interested in the simulation of stochastic integrals of the form

$$X^f(x) = \int_{\mathbb{R}^d} f(x, \xi) \, \Lambda(d\xi), \quad x \in \mathbb{R}^d, \tag{4}$$

where $\Lambda(d\xi)$ is defined by (1) and $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is such that for every $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \Big(|f(x, \xi)|^2 \wedge 1 \Big) \, d\xi < +\infty. \tag{5}$$

To analyze these stochastic integrals, we represent them as series (known as shot noise series) for which we carefully study the rates of convergence.

3 Generalized Shot Noise Series

As in the case of RHMLMs, the field X^f can be represented as a generalized shot noise series as soon as the control measure $\nu(dv)$ has finite mass. Hence, in this section,

$$\nu(\mathbb{R}) < +\infty. \tag{6}$$

Let us recall that $\nu(dv)$ is a non-vanishing measure, i.e. $\nu(\mathbb{R}) \neq 0$.

Remark that in the case of MAFSMs, assumption (6) is not fulfilled. Then, their simulation, which is proposed in section 5.2, is not only based on a series expansion but also on a normal approximation. Actually, we will split the MAFSM into two fields $X^f_{\varepsilon,1}$ and $X^f_{\varepsilon,2}$. Each field $X^f_{\varepsilon,i}$ ($i = 1, 2$) is associated by (4) with a infinitely divisible random measure $\Lambda_{\varepsilon,i}$. Theorem (4.1) allows to approximate $X^f_{\varepsilon,1}$ by a Gaussian field. Since the control measure of $\Lambda_{\varepsilon,2}$ will fulfill assumption (6), $X^f_{\varepsilon,2}$ will be represented by a generalized shot noise series as explained in this section.

Let us now introduce some notation that will be used throughout the paper.

Notation Let $(T_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(U_n)_{n\geq 1}$ be independent sequences of random variables.

- T_n is the nth arrival time of a Poisson process with intensity 1.
- $(V_n)_{n\geq 1}$ is a sequence of independent and identically distributed (i.i.d.) random variables with common law
$$\mathcal{L}(V_n) = \frac{\nu(dv)}{\nu(\mathbb{R})}.$$
- $(U_n)_{n\geq 1}$ is a sequence of i.i.d. random variables such that U_1 is uniformly distributed on the unit sphere S^{d-1} of the Euclidean space $\mathbb{R}^d$.

- c_d is the volume of the unit ball of $\mathbb{R}^d$.

The following statement is the main series representation we will be using in our investigation.

Proposition 3.1. *For every $x \in \mathbb{R}^d$, the series*

$$Y^f(x) = \sum_{n=1}^{+\infty} f\left(x, \left(\frac{T_n}{c_d\,\nu(\mathbb{R})}\right)^{1/d} U_n\right) V_n \tag{7}$$

converges almost surely. Furthermore,

$$\{X^f(x) : x \in \mathbb{R}^d\} \overset{(d)}{=} \{Y^f(x) : x \in \mathbb{R}^d\}.$$

Proof. Let us fix $x \in \mathbb{R}^d$ and consider the Borel measurable map

$$\begin{array}{rccl} H_x : &]0,+\infty[\times\mathcal{D} & \longrightarrow & \mathbb{R} \\ & (r,\widetilde{v}) & \longmapsto & f\left(x, \left(\frac{r}{c_d\nu(\mathbb{R})}\right)^{1/d} u\right) v, \end{array}$$

where $\widetilde{v} = (u, v) \in \mathcal{D} = S^{d-1} \times \mathbb{R}$. Then define a measure Q_x on the Borel σ-field $\mathcal{B}(\mathbb{R})$ by

$$\forall B \in \mathcal{B}(\mathbb{R}),\, Q_x(B) = \int_0^{+\infty}\!\!\int_{\mathcal{D}} \mathbf{1}_{B\setminus\{0\}}(H_x(r,\widetilde{v}))\,\lambda(d\widetilde{v})\,dr,$$

where λ is the law of $\widetilde{V_n} = (U_n, V_n)$. Hence,

$$\int_{\mathbb{R}} |y|^2 \wedge 1\, Q_x(dy) = \int_{]0,+\infty[\times\mathcal{D}} H_x^2(r,\widetilde{v}) \wedge 1\, dr\, \lambda(d\widetilde{v}).$$

Then, proceeding as in [11], one proves that

$$\int_{\mathbb{R}} |y|^2 \wedge 1\, Q_x(dy) = \int_{\mathbb{R}^d\times\mathbb{R}} \left[\left(f^2(x,\xi)\,v^2\right) \wedge 1\right] n(d\xi, dv).$$

Consequently,

$$\int_{\mathbb{R}} |y|^2 \wedge 1\, Q_x(dy) < +\infty$$

and Q_x is a Lévy measure on $\mathbb{R}$. Therefore, according to Theorem 2.4 in [19],

$$\sum_{n=1}^{N} H_x\left(T_n, \widetilde{V_n}\right) - A_x(T_N),$$

where for $s \geq 0$

$$A_x(s) = \int_0^s \int_{\mathcal{D}} H_x(r,\widetilde{v}) \mathbf{1}_{|H_x(r,\widetilde{v})|\leq 1}\,\lambda(d\widetilde{v})\,dr$$

converges almost surely as $N \to +\infty$. Moreover, since ν is a finite and symmetric measure, for every $R \geq 0$,

$$\int_{\mathbb{R}} v\mathbf{1}_{R|v|\leq 1}\,\nu(dv) = 0,$$

and thus, by definition of H_x and of the measure $\lambda(d\widetilde{v})$, $A_x(s) = 0$ for every $s \geq 0$. Therefore,

$$Y^f(x) = \sum_{n=1}^{+\infty} H_x\left(T_n, \widetilde{V_n}\right) = \sum_{n=1}^{+\infty} f\left(x, \left(\frac{T_n}{c_d\nu(\mathbb{R})}\right)^{1/d} U_n\right) V_n$$

converges almost surely. Furthermore, owing to Theorem 2.4 in [19] and symmetry of $\nu(dv)$, one easily sees that $Y^f(x) \overset{(d)}{=} X^f(x)$.

Furthermore, repeating the same arguments for

$$\begin{array}{rccl} H : &]0,+\infty[\times\mathcal{D} & \longrightarrow & \mathbb{R} \\ & (r,\widetilde{v}) & \longmapsto & \displaystyle\sum_{j=1}^{p} u_j H_{x_j}(r,\widetilde{v}), \end{array}$$

where $p \in \mathbb{N}\backslash\{0\}$, $(x_1,\ldots,x_p) \in \left(\mathbb{R}^d\right)^p$ and $(u_1,\ldots,u_p) \in \mathbb{R}^p$, we get that the finite dimensional marginals of Y^f are identical to those of X^f. The proof of Proposition 3.1 is then complete. □

On the basis of Proposition 3.1, an approximation of Y^f, which is equal in law to X^f, is Y_N^f given by

$$Y_N^f(x) = \sum_{n=1}^{N} f\left(x, \left(\frac{T_n}{c_d \nu(\mathbb{R})}\right)^{1/d} U_n\right) V_n, \quad x \in \mathbb{R}^d. \tag{8}$$

We now explain in a few words how the rate of convergence of Y_N^f to Y^f can be studied. Firstly, let us recall the following classical result for Poisson arrival times:

$$\lim_{N\to+\infty} \frac{T_n}{n} = 1 \text{ almost surely.} \tag{9}$$

Hence, the asymptotics of (8) is given by the asymptotics of $f(x,\xi)$ as $\|\xi\|$ tends to infinity. If we suppose that the expansion of $f(x,\xi)$ is, roughly speaking, uniform in x as $\|\xi\|$ tends to infinity, the rate of convergence of Y_N^f will be deduced from the rate of convergence of some series of the kind

$$S_N^\gamma = \sum_{n=1}^{N} T_n^{-1/\gamma} X_n, \tag{10}$$

where $0 < \gamma < 2$. As soon as X_n, $n \geq 1$, are i.i.d L^γ-symmetric random variables, (10) converges almost surely to a stable random variable with index γ, see for instance [12] and [21]. Under a stronger integrability assumption, a rate of almost sure convergence is established in Lemma 3.5.

Let us first study the almost sure and L^r errors for each fix x.

Proposition 3.2. *Let $x \in \mathbb{R}^d$. Assume that*

$$\forall \xi \neq 0,\ |f(x,\xi)| \leq \frac{C}{\|\xi\|^\beta}, \tag{11}$$

where $\beta > d/2$ and $C > 0$. Furthermore, assume there exists $r \in (d/\beta, 2]$ such that

$$\int_0^{+\infty} |v|^r \, \nu(dr) < +\infty.$$

1. *Then, for every $\varepsilon \in (0, \beta/d - 1/r)$, almost surely,*

$$\sup_{N\geq 1} N^\varepsilon \left| Y^f(x) - Y_N^f(x) \right| < +\infty.$$

2. *Moreover, for every integer $N > r\beta/d$,*

$$\mathbb{E}\Big(\left| Y_N^f(x) - Y^f(x) \right|^r\Big) \leq C(r,\beta)\, \frac{D(N,r,\beta)}{N^{r\beta/d-1}}, \tag{12}$$

where

$$D(N,r,\beta)=\frac{\Gamma(N+1-r\beta/d)\,(N+1)^{r\beta/d}}{\Gamma(N+1)} \tag{13}$$

and

$$C(r,\beta)=\frac{dC^r(c_d\nu(\mathbb{R}))^{r\beta/d}\mathbb{E}(|V_1|^r)}{r\beta-d}. \tag{14}$$

Remark 3.3. *Remark that*

$$\lim_{N\to+\infty} D(N,r,\beta)=1$$

by the Stirling formula. Hence, Proposition 3.2 gives a rate of convergence in L^r for the series Y_N^f. Furthermore, (12) *allows us to control the error in simulation.*

Remark 3.4. *Assume that* (11) *is only fulfilled for $\|\xi\|\geq A$, which is the case for MAFLMs. Then, let*

$$g(x,\xi)=f(x,\xi)\mathbf{1}_{\|\xi\|\geq A}$$

and remark that

$$Y^f=Y^g+Y^{f-g}, \tag{15}$$

where Y^h is associated with h by (7)*. Hence, since g satisfies the assumptions of Proposition 3.2, an almost sure or L^r error may be obtained. Furthermore, in view of* (9)*,*

$$Y^{f-g}(x)=\sum_{n=1}^{+\infty}(f-g)\left(x,\left(\frac{T_n}{c_d\nu(\mathbb{R})}\right)^{1/d}U_n\right)$$

is, almost surely, a finite sum since for n large enough,

$$\left(\frac{T_n}{c_d\nu(\mathbb{R})}\right)^{1/d}>A.$$

This remark is used for MAFSMs or MAFLMs in section 5.

Before we prove Proposition 3.2, we state a result about the rate of convergence of S_N^γ defined by (10) under integrability conditions. This Lemma is proved in the appendix.

Lemma 3.5. *Let $(X_n)_{n\geq1}$ be a sequence of i.i.d. symmetric random variables. Assume that $(X_n)_{n\geq1}$ is independent from $(T_n)_{n\geq1}$ and from a sequence $(Y_n)_{n\geq1}$ which satisfies*

$$|Y_n|\leq CT_n^{-1/\gamma} \tag{16}$$

for some $\gamma\in(0,2)$. Furthermore, assume $\mathbb{E}(|X_n|^r)<+\infty$ for some $r>\gamma$. Then, for every $\varepsilon\in(0,1/\gamma-1/(r\wedge2))$, almost surely,

$$\sup_{N\in\mathbb{N}\setminus\{0\}} N^\varepsilon\left|\sum_{n=N+1}^{+\infty}Y_nX_n\right|<+\infty.$$

Let us now prove Proposition 3.2.

Proof of Proposition 3.2.

In the following,

$$\xi_n=\left(\frac{T_n}{c_d\nu(\mathbb{R})}\right)^{1/d}U_n.$$

1. Proof of part 1: Rate of Almost Sure Convergence

In view of (11),

$$|f(x,\xi_n)| \leq \frac{C(c_d\nu(\mathbb{R}))^{\beta/d}}{T_n^{\beta/d}}. \tag{17}$$

Then, by applying Lemma 3.5 with $X_n = V_n$ and $Y_n = f(x,\xi_n)$,

$$\sup_{N\geq 1} N^{\varepsilon}\Big|Y^f(x) - Y_N^f(x)\Big| < +\infty \text{ almost surely}$$

for every $\varepsilon \in (0, \beta/d - 1/r)$.

2. Proof of part 2: L^r-error

Since V_n, $n \in \mathbb{N}\backslash\{0\}$, are i.i.d. symmetric random variables, by the Jensen inequality, applied for $r \in (0,2]$,

$$\mathbb{E}\left(\left|\sum_{n=N+1}^{P} f(x,\xi_n)V_n\right|^r\right) \leq \mathbb{E}\left[\left(\sum_{n=N+1}^{P} f^2(x,\xi_n)V_n^2\right)^{r/2}\right],$$

Furthermore, since $(a+b)^{r/2} \leq a^{r/2} + b^{r/2}$ $(r \in (0,2])$ for every $a, b \geq 0$,

$$\begin{aligned}
\mathbb{E}\left(\left|\sum_{n=N+1}^{P} f(x,\xi_n)V_n\right|^r\right) &\leq \mathbb{E}(|V_1|^r)\sum_{n=N+1}^{P}\mathbb{E}(|f(x,\xi_n)|^r) \\
&\leq C^r(c_d\nu(\mathbb{R}))^{r\beta/d}\mathbb{E}(|V_1|^r)\sum_{n=N+1}^{P}\mathbb{E}\left(T_n^{-r\beta/d}\right) \\
&\leq C^r(c_d\nu(\mathbb{R}))^{r\beta/d}\mathbb{E}(|V_1|^r)\sum_{n=N+1}^{P}\frac{\Gamma(n-r\beta/d)}{\Gamma(n)}.
\end{aligned}$$

Therefore,

$$\mathbb{E}\left(\left|\sum_{n=N+1}^{P} f(x,\xi_n)V_n\right|^r\right) \leq C^r(c_d\nu(\mathbb{R}))^{r\beta/d}\mathbb{E}(|V_1|^r)\sup_{n\geq N} D(n,r,\beta)\sum_{n=N+1}^{+\infty}\frac{1}{n^{r\beta/d}}$$

where $D(n,r,\beta)$ is defined by (13). Hence, as in proof of Proposition 3.2 in [11], one sees that

$$\sup_{n\geq N} D(n,r,\beta) = D(N,r,\beta)$$

and then that

$$\mathbb{E}\left(\left|\sum_{n=N+1}^{P} f(x,\xi_n)V_n\right|^r\right) \leq \frac{dC^r(c_d\nu(\mathbb{R}))^{r\beta/d}\mathbb{E}(|V_1|^r)D(N,r,\beta)}{(r\beta-d)N^{r\beta/d-1}}$$

since $r > d\beta$. Moreover, by Proposition 2.3 of [12],

$$\mathbb{E}\left(\sup_{N\leq M\leq P}\left|\sum_{n=N+1}^{M} f(x,\xi_n)V_n\right|^r\right) \leq 2\mathbb{E}\left(\left|\sum_{n=N+1}^{P} f(x,\xi_n)V_n\right|^r\right).$$

As a result,

$$\mathbb{E}\left(\left|\sum_{n=N+1}^{+\infty} f(x,\xi_n)V_n\right|^r\right) \leq \frac{dC^r(c_d\nu(\mathbb{R}))^{r\beta/d}\mathbb{E}(|V_1|^r)D(N,r,\beta)}{(r\beta-d)N^{r\beta/d-1}}.$$

The proof of Proposition 3.2 is complete. □

Actually, if f admits an expansion, roughly speaking uniform in x, as $\|\xi\|$ tends to infinity, the next proposition gives a rate of uniform convergence in x for Y_N^f.

Proposition 3.6. *Let $K \subset \mathbb{R}^d$ be a compact set, $p \geq 1$ and $(\beta_i)_{1\leq i\leq p}$ be a non-decreasing sequence such that $\beta_1 > d/2$ and $\beta_p > d$. Assume that for every $x \in K$ and $\xi \neq 0$,*

$$\left| f(x,\xi) - \sum_{j=1}^{p-1} \frac{a_j(x) b_j(\xi/\|\xi\|)}{\|\xi\|^{\beta_j}} \right| \leq \frac{b_p(\xi/\|\xi\|)}{\|\xi\|^{\beta_p}} \tag{18}$$

where a_j, $j = 1, \ldots, p-1$, are real-valued continuous functions. Furthermore, assume that there exists $r \in (d/\beta_1, 2]$ such that $\mathbb{E}(|V_n|^r) < +\infty$ and $\mathbb{E}(|b_j(U_n)|^r) < +\infty$ for $j = 1, \ldots, p$. Then for every $\varepsilon \in (0, \min(\beta_1/d - 1/r, \beta_p/d - 1/(1 \wedge r)))$,

$$\sup_{N\geq 1} N^{\varepsilon} \sup_{x\in K} \left| Y^f(x) - Y_N^f(x) \right| < +\infty \text{ almost surely.}$$

Remark 3.7. *As a consequence of Proposition 3.6, Y_N^f converges uniformly to Y^f at least at the rate N^{ε} for*

$$0 < \varepsilon < \min\left(\frac{\beta_1}{d} - \frac{1}{r}, \frac{\beta_p}{d} - \frac{1}{1 \wedge r} \right) = m.$$

Actually, the upper bound m of the rate of convergence comes from the comparison of the series S_N^{d/β_1} and S_N^{d/β_p} defined by (10) with some convenient X_n. More precisely, the sum

$$S_N^{d/\beta_1} = (c_d \nu(\mathbb{R}))^{\beta_1/d} \sum_{n=1}^{N} T_n^{-\beta_1/d} b_1(U_n) V_n$$

corresponds to the first term in the asymptotic expansion of the function f and

$$S_N^{d/\beta_p} = (c_d \nu(\mathbb{R}))^{\beta_p/d} \sum_{n=1}^{N} T_n^{-\beta_p/d} b_p(U_n) V_n$$

comes from the pth term in this expansion. Thus, we have to compare the rate of convergence of S_N^{d/β_1} to the rate of absolute convergence of S_N^{d/β_p}. However, if in the asymptotic expansion of f, β_p can be chosen sufficiently large, then the rate of uniform convergence of Y_N^f to Y^f is given by the series S_N^{d/β_1}.

Before we proceed to the proof Proposition 3.6, we state an other result, proved in the appendix, about the rate of absolute convergence of S_N^{γ}.

Lemma 3.8. *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d random variables and $\gamma \in (0,1)$. Assume that $(X_n)_{n\geq 1}$ is independent from $(T_n)_{n\geq 1}$ and that $\mathbb{E}(|X_n|^r) < +\infty$ for some $r > \gamma$. Then, for every $\varepsilon \in (0, 1/\gamma - 1/(r \wedge 1))$, almost surely,*

$$\sup_{N\geq 1} N^{\varepsilon} \sum_{n=N+1}^{+\infty} T_n^{-1/\gamma} |X_n| < +\infty.$$

It remains to prove Proposition 3.6.

Proof of Proposition 3.6. We have

$$\left|Y^f(x) - Y_N^f(x)\right| \leq \sum_{j=1}^{p-1} |a_j(x)| \left| \sum_{n=N+1}^{+\infty} \left(\frac{T_n}{c_d \nu(\mathbb{R})}\right)^{-\beta_j/d} b_j(U_n) V_n \right| + |a_p(x)| \sum_{n=N+1}^{+\infty} \left(\frac{T_n}{c_d \nu(\mathbb{R})}\right)^{-\beta_p/d} |b_p(U_n) V_n|.$$

Note that $(b_j(U_n)V_n)_{n\geq 1}$ are i.i.d. symmetric random variables such that $\mathbb{E}(|b_j(U_n)V_n|^r) < +\infty$. Hence, since $0 < d/\beta_j < r \leq 2$, by Lemma 3.5, for every $\varepsilon \in (0, \beta_j/d - 1/r)$,

$$\sup_{N\geq 1} N^{\varepsilon} \left| \sum_{n=N+1}^{+\infty} T_n^{-\beta_j/d} b_j(U_n) V_n \right| < +\infty \text{ almost surely.}$$

In addition, since $\mathbb{E}\Big(|b_p(U_n)V_n|^{1\wedge r}\Big) < +\infty$ and $d/\beta_p < 1$, for every $\varepsilon \in (0, \beta_p/d - 1/(1\wedge r))$,

$$\sup_{N\geq 1} N^{\varepsilon} \sum_{n=N+1}^{+\infty} T_n^{-\beta_p/d} |b_p(U_n)V_n| < +\infty \text{ almost surely,}$$

which ends the proof since a_j, $j = 1, \ldots, p-1$, are continuous and thus bounded on the compact set K. □

4 Normal Approximation

When the assumption (6) is not fulfilled, section 3 can not be directly applied. In this case, we will split the field X^f into two fields $X^f_{\varepsilon,1}$ and $X^f_{\varepsilon,2}$. It leads to a decomposition of Λ into two random measures $\Lambda_{\varepsilon,1}$ and $\Lambda_{\varepsilon,2}$ such that the control measure of $\Lambda_{\varepsilon,2}$ satisfies the assumption (6). As a consequence of section 3, $X^f_{\varepsilon,2}$ can be represented as a series. This section is thus devoted to the simulation of $X^f_{\varepsilon,1}$.

Suppose now that

$$\nu(\mathbb{R}) = +\infty, \tag{19}$$

which is the case for MAFSMs. Then let $\varepsilon > 0$ and let us split

$$X^f = X^f_{\varepsilon,1} + X^f_{\varepsilon,2}$$

into two random fields where

$$X^f_{\varepsilon,1}(x) = \int_{\mathbb{R}^d\times\mathbb{R}} f(x,\xi) v \mathbf{1}_{|v|<\varepsilon} \Big(N(d\xi, dv) - (|f(x,\xi)v| \vee 1)^{-1} n(d\xi, dv) \Big) \tag{20}$$

and

$$X^f_{\varepsilon,2}(x) = \int_{\mathbb{R}^d\times\mathbb{C}} f(x,\xi) v \mathbf{1}_{|v|\geq\varepsilon} \Big(N(d\xi, dv) - (|f(x,\xi)v| \vee 1)^{-1} n(d\xi, dv) \Big). \tag{21}$$

Let us consider the two independent Poisson random measures

$$N_{\varepsilon,1}(d\xi, dv) = \mathbf{1}_{|v|<\varepsilon} N(d\xi, dv)$$

and

$$N_{\varepsilon,2}(d\xi, dv) = \mathbf{1}_{|v|\geq\varepsilon} N(d\xi, dv).$$

Let $\Lambda_{\varepsilon,i}$ $(i = 1, 2)$ be the infinitely divisible random measure associated with $N_{\varepsilon,i}$ by (1). Remark that $X^f_{\varepsilon,1}$ and $X^f_{\varepsilon,2}$ are independent and that

$$X^f_{\varepsilon,i}(x) = \int_{\mathbb{R}^d} f(x,\xi)\,\Lambda_{\varepsilon,i}(d\xi), \quad i = 1, 2.$$

In addition, the control measure $\nu_{\varepsilon,2}(dv) = \mathbf{1}_{|v|\geq\varepsilon}\,\nu(dv)$ of $\Lambda_{\varepsilon,2}$ is finite and symmetric. Therefore, $X^f_{\varepsilon,2}$ can be simulated as a generalized shot noise series, (see section 3). It now remains to approximate $X^f_{\varepsilon,1}$. Let us notice that the control measure $\nu_{\varepsilon,1}(dv)$ of $\Lambda_{\varepsilon,1}$ has moments of every order greater than 2. Hence, $\Lambda_{\varepsilon,1}$ is a Lévy random measure in the sense of [4].

Set

$$\sigma(\varepsilon) = \left(\int_{-\varepsilon}^{\varepsilon} v^2\,\nu(dv)\right)^{1/2}. \tag{22}$$

Proposition 4.1. *Assume that for each $x \in \mathbb{R}^d$,*

$$f(x,\cdot) \in L^2\Big(\mathbb{R}^d\Big)$$

and

$$\lim_{\varepsilon\to 0_+} \frac{\sigma(\varepsilon)}{\varepsilon} = +\infty.$$

Then

$$\lim_{\varepsilon\to 0_+} \left(\frac{X^f_{\varepsilon,1}(x)}{\sigma(\varepsilon)}\right)_{x\in\mathbb{R}^d} \overset{(d)}{=} \Big(W^f(x)\Big)_{x\in\mathbb{R}^d}, \tag{23}$$

where, with $W(d\xi)$ a real Brownian random measure,

$$W^f(x) = \int_{\mathbb{R}^d} f(x,\xi)\,W(d\xi) \tag{24}$$

and where the limit is understood in the sense of finite dimensional distributions.

Proof. Let $r \in \mathbb{N}^*$, $u = (x_1, \ldots, x_r) \in \left(\mathbb{R}^d\right)^r$ and $y = (y_1, \ldots, y_r) \in \mathbb{R}^r$. Then

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^{r} y_k \frac{X^f_{\varepsilon,1}(x_k)}{\sigma(\varepsilon)}\right)\right] = \exp\left(\Psi_\varepsilon(x,y)\right)$$

with

$$\Psi_\varepsilon(x,y) = \int_{\mathbb{R}^d\times\mathbb{R}}\left(\exp\left(\frac{ig(\xi,x,y)v}{\sigma(\varepsilon)}\right) - 1 - \frac{ig(\xi,x,y)v}{\sigma(\varepsilon)}\mathbf{1}_{|g(\xi,x,y)v|\leq\sigma(\varepsilon)}\right) d\xi\,\nu_{\varepsilon,1}(dv)$$

and

$$g(\xi,x,y) = \sum_{k=1}^{r} y_k f(x_k,\xi).$$

Then, by the Fubini theorem,

$$\Psi_\varepsilon(x,y) = \int_{\mathbb{R}^d} I_\varepsilon(g(\xi,x,y))\,d\xi,$$

where for every $a \in \mathbb{R}$

$$I_\varepsilon(a) = \int_{\mathbb{R}}\left(\mathrm{e}^{i\frac{av}{\sigma(\varepsilon)}} - 1 - i\frac{av}{\sigma(\varepsilon)}\mathbf{1}_{|av|<\sigma(\varepsilon)}\right)\mathbf{1}_{|v|<\varepsilon}\,\nu(dv).$$

Since $\nu(dv)$ is a symmetric Lévy measure,

$$I_\varepsilon(a) = \int_{\mathbb{R}}\left(\mathrm{e}^{i\frac{av}{\sigma(\varepsilon)}} - 1 - i\frac{av}{\sigma(\varepsilon)}\right)\mathbf{1}_{|v|<\varepsilon}\,\nu(dv).$$

As $\lim_{\varepsilon\to 0_+} \sigma(\varepsilon)/\varepsilon = +\infty$, according to [1],

$$\lim_{\varepsilon\to 0_+} I_\varepsilon(a) = \frac{-a^2}{2}.$$

Since moreover $|I_\varepsilon(a)| \leq \frac{a^2}{2}$, for every $a \in \mathbb{R}$, a dominated convergence argument yields

$$\begin{aligned}\lim_{\varepsilon\to 0_+} \Psi_\varepsilon(x,y) &= -\tfrac{1}{2}\int_{\mathbb{R}^d}\left|\sum_{k=1}^r y_k f(x_k,\xi)\right|^2 d\xi \\ &= -\tfrac{1}{2}\mathrm{Var}\left(\sum_{k=1}^r y_k W^f(x_k)\right).\end{aligned}$$

The proof is thus complete. □

As in the case of RHMLMs, an estimation in terms of Berry-Esseen bounds of the rate of convergence stated in Proposition 4.1 can be given. The assumption of the following proposition insures the existence of the moment of order three for $X^f_{\varepsilon,1}(x)$. However, this assumption is only done for the sake of simplicity. Actually, owing to [16], Berry-Esseen bounds can be established under a weaker assumption. The following proposition is a consequence of Lemma 4.1 in [11].

Proposition 4.2. *Let $x \in \mathbb{R}^d$ and assume that*

$$f(x,\cdot) \in L^3\Big(\mathbb{R}^d\Big).$$

Then $\mathbb{E}\left(\left|X^f_{\varepsilon,1}(x)\right|^3\right) < +\infty$ *and*

$$\sup_{u\in\mathbb{R}}\left|\mathbb{P}\Big(X^f_{\varepsilon,1}(x)\leq u\Big) - \mathbb{P}\Big(\sigma(\varepsilon)W^f(x)\leq u\Big)\right| \leq A(x)\,\frac{m_3^3(\varepsilon)}{\sigma^3(\varepsilon)}$$

where W^f is defined by (24) *in Proposition 4.1, $m_3^3(\varepsilon) = \int_{-\varepsilon}^{\varepsilon}|v|^3\,\nu(dv)$ and*

$$A(x) = \frac{0.7975\int_{\mathbb{R}^d}|f(x,\xi)|^3\,d\xi}{3\left(\pi\int_{\mathbb{R}^d}|f(x,\xi)|^2\,d\xi\right)^{3/2}}.$$

As a consequence, we approximate $X^f_{\varepsilon,1}$ by the Gaussian field $\sigma(\varepsilon)W^f$. Moreover, according to section 3, an approximation of $X^f_{\varepsilon,2}$ is given by

$$Y^f_{\varepsilon,N,2}(x) = \sum_{n=1}^N f\left(x,\left(\frac{T_n}{c_d\,\nu(\mathbb{R})}\right)^{1/d}U_n\right)V_{\varepsilon,n},\quad x\in\mathbb{R}^d,$$

where $(V_{\varepsilon,n})_n$ is a sequence of i.i.d. random variables with common law $\nu_{\varepsilon,2}(dv)/\nu_{\varepsilon,2}(\mathbb{R})$. Note that T_n, U_n and $V_{\varepsilon,n}$ are independent. Since $X^f_{\varepsilon,1}$ and $X^f_{\varepsilon,2}$ are independent, W^f is assumed to be independent from $(T_n,U_n,V_{\varepsilon,n})$. As a result, in the case where $\nu(\mathbb{R}) = +\infty$, under assumptions of Proposition 4.1, an approximation of X^f is

$$Y^f_{\varepsilon,N}(x) = \sigma(\varepsilon)W^f(x) + \sum_{n=1}^N f\left(x,\left(\frac{T_n}{c_d\,\nu(\mathbb{R})}\right)^{1/d}U_n\right)V_{\varepsilon,n},\quad x\in\mathbb{R}^d.$$

5 Examples

This section illustrates with various examples the range of application of the preceding results. In all the following examples, $K \subset \mathbb{R}^d$ is a compact set and (11) is only fulfilled for $\|\xi\| \geq A$. Then, as noticed in Remark 3.4, we may split

$$Y_N^f = Y_N^g + Y_N^{f-g},$$

with $g(x,\xi) = f(x,\xi)\mathbf{1}_{\|\xi\|\geq A}$. Since, Y_N^g is in fact a finite sum (almost surely), the rate of convergence precised below is actually the rate of convergence of Y_N^{f-g}.

5.1 Moving Average Fractional Lévy Motions

Let $H \in (0,1)$ such that $H \neq d/2$. Suppose that

$$f_{H,2}(x,\xi) = \|x-\xi\|^{H-d/2} - \|\xi\|^{H-d/2}$$

and that for every $p \geq 2$, $\int_{\mathbb{R}} |v|^p \, \nu(dv) < +\infty$. Then, $X_{H,2} = X^{f_{H,2}}$ is a MAFLM in the sense of [4].

5.1.1 Case of finite control measures

An approximation, in law, of the MAFLM X_H is given by

$$Y_N^{f_{H,2}}(x) = \sum_{n=1}^{N} \left(\left\| x - \left(\frac{T_n}{c_d \nu(\mathbb{R})} \right)^{1/d} U_n \right\|^{H-d/2} - \left(\frac{T_n}{c_d \nu(\mathbb{R})} \right)^{H/d-1/2} \right) V_n.$$

Furthermore, for every $x \in K$, for $\|\xi\| > \max_K \|y\| = A$ and every $q \geq 1$,

$$\left| f_{H,2}(x,\xi) - \sum_{j=1}^{q-1} \|\xi\|^{H-j-d/2} a_j(x, \xi/\|\xi\|) \right| \leq A^q \|\xi\|^{H-d/2-q}, \tag{25}$$

where the a_j's are polynomial functions in x_i and u_j, $j = 1, \ldots, d$. Let $\beta_1 = 1 - H + d/2$ and

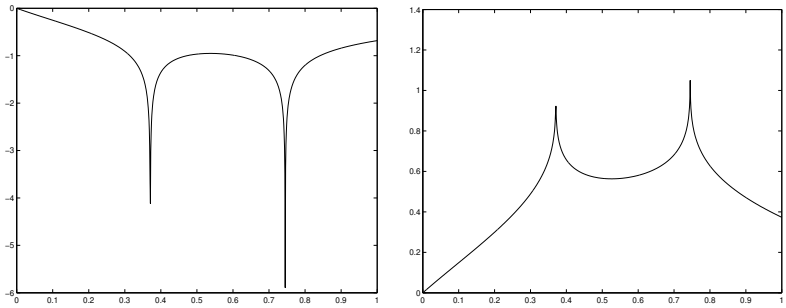
$$g_{H,2}(x,\xi) = f_{H,2}(x,\xi)\mathbf{1}_{\|\xi\|\geq \max_K \|y\|}.$$

Note that $\beta_1 > d/2$ since $1 > H$. Then, in view of Proposition 3.2, applied with $r = 2$,

$$\mathbb{E}\Big(\big| Y_N^{g_{H,2}}(x) - Y^{g_{H,2}}(x) \big|^2 \Big) \leq \frac{C(2,\beta) D(N,2,\beta)}{N^{2(1-H)/d}}$$

where $C(2,\beta)$ and $D(N,2,\beta)$ are defined by (14) and (13). Therefore, the mean square error converges at the rate $N^{(1-H)/d}$. Furthermore, since (25) holds for every integer $q \geq 1$, by Proposition 3.6, $Y_N^{g_{H,2}}$ converges uniformly at the rate N^{ε} for every $\varepsilon \in (0, (1-H)/d)$.

Let us now present one example (see figure 1) taking $\nu(dv) = (\delta_{-1} + \delta_1)/2$. In this example, we have first simulate a realization of the random variables (T_n, U_n, V_n). Then, for these values of $(T_n, U_n, V_n)_{1\leq n\leq N}$, we evaluate $Y_N^{f_{H,2}}$ for $H = 0.3$ and $H = 0.7$. We observe that the trajectories regularity does not depend on the value of H. Actually, one can see that the derivatives of $Y_N^{g_{H,2}}$ at each order converge uniformly on each compact set. Therefore, $Y^{g_{H,2}}$ has $\mathcal{C}^\infty$ sample paths almost surely. As a consequence, the sample paths of $Y^{f_{H,2}}$ are $\mathcal{C}^\infty$ except at points $\xi_n = (T_n/c_d\nu(\mathbb{R}))^{1/d} U_n$. At these points, the behaviour depends on H: whereas when $H < d/2$, $Y_N^{f_{H,2}}$ is not defined, when $H > d/2$ the pointwise Hölder exponent of $Y_N^{f_{H,2}}$ is given by $H - d/2$. In figure 1, we observe that the sample paths are smooth on $[0,1]$ except at two points.



$H = 0.3$ $H = 0.7$

Figure 1: Examples of MAFLMs

5.1.2 Case of infinite control measures

In this example,

$$\nu(dv) = \frac{\mathbf{1}_{|v|\leq 1}\, dv}{|v|^{1+\alpha}} \quad \text{with} \quad 0 < \alpha < 2.$$

Let $(V_{\varepsilon,n})_{n\geq 1}$ be a sequence of i.i.d variables with common law

$$\frac{\alpha\, \mathbf{1}_{\varepsilon<|v|<1}\, dv}{2(\varepsilon^{-\alpha}-1)|v|^{1+\alpha}}.$$

Moreover, let B_H be a standard FBM with index H and assume that B_H, $(U_n)_{n\geq 1}$, $(T_n)_{n\geq 1}$ and $(V_{\varepsilon,n})_{n\geq 1}$ are independent.

Hence, an approximation of the MAFLM X_H is given by

$$Y_{\varepsilon,N}^{f_{H,2}}(x) = \sum_{n=1}^{N} \left(\left\| x - \left(\frac{T_n}{c_d \nu_{\varepsilon,2}(\mathbb{R})} \right)^{1/d} U_n \right\|^{H-d/2} - \left(\frac{T_n}{c_d \nu_{\varepsilon,2}(\mathbb{R})} \right)^{H/d-1/2} \right) V_{\varepsilon,n} + \sigma(\varepsilon) W^{f_{H,2}}(x), \quad (26)$$

where

$$W^{f_{H,2}}(\cdot) = \int_{\mathbb{R}^d} f_{H,2}(\cdot,\xi)\, W(d\xi) \overset{(d)}{=} C_{H,d} B_H(\cdot)$$

with

$$C_{H,d} = \left(\int_{\mathbb{R}^d} |f_{H,2}(e_1,\xi)|^2\, d\xi \right)^{1/2}$$

and $e_1 = (1, 0, \ldots, 0)$. Actually, by a Fourier transform argument

$$C_{H,d} = \frac{2^{H-2}|d-2H|\Gamma(H/2+d/4)}{\Gamma(d/4+1-H/2)} \left(\int_{\mathbb{R}^d} \frac{\left|\mathrm{e}^{-ie_1\cdot\lambda}-1\right|^2}{\|\lambda\|^{2H+d}}\, d\lambda \right)^{1/2}.$$

As a result, owing to [21] for $d = 1$ and to [11] for $d \geq 2$,

$$C_{H,d} = \frac{2^{H-2}|d-2H|\Gamma(H/2+d/4)}{\Gamma(d/4+1-H/2)} \left(\frac{\pi^{(d+1)/2}\Gamma(H+1/2)}{H\Gamma(2H)\sin(\pi H)\Gamma(H+d/2)} \right)^{1/2}. \quad (27)$$

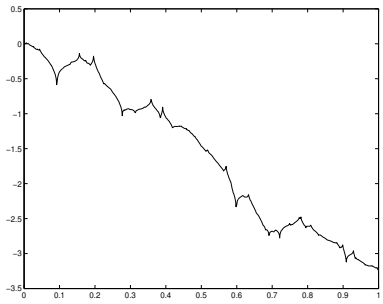
When $H > d/6$, $X_{\varepsilon,1}^f(x)$ has a moment of order three. Then, by Proposition 4.2, in terms of Berry-Esseen bounds, the rate of convergence of the error due to the approximation of $X_{\varepsilon,1}^f(x)$ is of the order

$$\delta(\varepsilon) = \frac{m_3^3(\varepsilon)}{\sigma^3(\varepsilon)} = \frac{(2-\alpha)^{3/2}\varepsilon^{\alpha/2}}{(3-\alpha)\sqrt{2}}.$$

However, as soon as $H \leq d/6$, there exists $\eta \in (0,1]$ such that $H > d/2 - d/(2+\eta)$. Then, $\mathbb{E}\Big(\Big|X^f_{\varepsilon,1}(x)\Big|^{2+\eta}\Big) < +\infty$. Therefore, a generalization of Proposition 4.2 leads to the rate of convergence

$$\delta(\varepsilon) = \frac{\int_{-\varepsilon}^{\varepsilon} |v|^{2+\eta} dv}{\sigma^{2+\eta}(\varepsilon)} = \frac{(2-\alpha)^{1+\eta/2}\varepsilon^{\alpha\eta/2}}{(2+\eta-\alpha)\sqrt{2}}.$$

Except at points $\xi_n = (T_n/c_d\nu_{\varepsilon,2}(\mathbb{R}))^{1/d}U_n$, the trajectories regularity of $Y^{f_{H,2}}_{\varepsilon,N}$ is given by the trajectory regularity of $W^{f_{H,2}}$ because of the smoothness of the part due to the sum. Hence, between ξ_n and ξ_{n+1}, the pointwise Hölder exponent of $Y^{f_{H,2}}_{\varepsilon,N}$ is equal to H. When $H > d/2$, the trajectories of $Y^{f_{H,2}}_{\varepsilon,N}$ are thus H'-holderian on each compact set for every $H' < H - d/2$. Following [4], that is exactly what we expect for the trajectories regularity of a MAFLM X_H associated to a infinite control measure. Figure 2 yields illustration of these facts in the case where $H = 0.8$, $\alpha = 1$ and for the preceding control measure.



$H = 0.8$

Figure 2: Examples of MAFLMs

5.2 Moving Average Fractional Stable Motions

In this example,

$$\nu(dv) = \frac{dv}{|v|^{1+\alpha}} \text{ with } 0 < \alpha < 2, \tag{28}$$

and

$$f_{H,\alpha}(x,\xi) = D(\alpha)^{-1/\alpha}\Big(\|x-\xi\|^{H-d/\alpha} - \|\xi\|^{H-d/\alpha}\Big),$$

with $0 < H < 1$ and $H \neq d/\alpha$ and where $D(\alpha)$ is defined by (3). Note that

$$D(\alpha) = \begin{cases} \dfrac{2\Gamma(2-\alpha)|\cos(\pi\alpha/2)|}{\alpha|\alpha-1|} & \text{if } \alpha \neq 1 \\ \dfrac{\pi}{2} & \text{if } \alpha = 1. \end{cases} \tag{29}$$

In this example,

$$\sigma^2(\varepsilon) = \frac{2\varepsilon^{2-\alpha}}{2-\alpha} \quad \text{and} \quad \nu_{\varepsilon,2}(\mathbb{R}) = \frac{2}{\alpha\varepsilon^{\alpha}}.$$

Then, the approximation of the MAFSM is given by formula (26), replacing $d/2$ by d/α in the summation and with

$$W^{f_{H,\alpha}}(\cdot) = \int_{\mathbb{R}^d} f_{H,\alpha}(\cdot,\xi)\, W(d\xi) \overset{(d)}{=} D(\alpha)^{-1/\alpha} C_{H+d/2-d/\alpha,d} B_{H+d/2-d/\alpha}.$$

More precisely, as previously, $B_{H+d/2-d/\alpha}$ is a standard FBM with index $H + d/2 - d/\alpha$ and $C_{H+d/2-d/\alpha,d}$ is defined by (27). Furthermore,

$$\nu_{\varepsilon,2}(dv) = \mathbf{1}_{|v|>\varepsilon}\,\nu(dv)$$

and $(V_{\varepsilon,n})_{n\geq 1}$, is a sequence of i.i.d variables with common law $\nu_{\varepsilon,2}(dv)/\nu_{\varepsilon,2}(\mathbb{R})$. Let us recall that the sequences B_H, $(U_n)_{n\geq 1}$, $(T_n)_{n\geq 1}$ and $(V_{\varepsilon,n})_{n\geq 1}$ are independent.
Thus, the approximation of the MAFSM $X_{H,\alpha} = X^{f_{H,\alpha}}$ is given by

$$Y_{\varepsilon,N}^{f_{H,\alpha}}(x) = D(\alpha)^{-1/\alpha}\sum_{n=1}^{N}\left(\left\|x - \left(\frac{T_n}{c_d\nu_{\varepsilon,2}(\mathbb{R})}\right)^{1/d}U_n\right\|^{H-d/\alpha} - \left(\frac{T_n}{c_d\nu_{\varepsilon,2}(\mathbb{R})}\right)^{H/d-1/\alpha}\right)V_{\varepsilon,n}+\sigma(\varepsilon)W^{f_{H,\alpha}}(x).$$

However, this approximation only holds if $f_{H,\alpha}(x,\cdot) \in L^2\big(\mathbb{R}^d\big)$, i.e. the FBM $B_{H+d/2-d/\alpha}$ is defined, that is if $1 > H > d/\alpha - d/2$.

Let us observe that the asymptotic expansion of $f_{H,\alpha}$ is given by (25), replacing $d/2$ by d/α. Then, let

$$g_{H,\alpha}(x,\xi) = f_{H,\alpha}(x,\xi)\mathbf{1}_{\|\xi\|\geq \max_K \|y\|}$$

and note that $Y_{\varepsilon,N}^{f_{H,\alpha}} = Y_{\varepsilon,N,2}^{g_{H,\alpha}} + Y_{\varepsilon,N,2}^{f_{H,\alpha}-g_{H,\alpha}} + \sigma(\varepsilon)W^{f_{H,\alpha}}(x)$ with

$$Y_{\varepsilon,N,2}^{h}(x) = D(\alpha)^{-1/\alpha}\sum_{n=1}^{N} h\left(x, \frac{T_n}{c_d\nu_{\varepsilon,2}(\mathbb{R})}\right)V_{\varepsilon,N}.$$

As noticed in Remark 3.4, $Y_{\varepsilon,N,2}^{f_{H,\alpha}-g_{H,\alpha}}$ is a finite sum. In addition, $g_{H,\alpha}$ satisfies the assumptions of Proposition 3.2 for every r such that $r < \alpha$. Hence, for every $r \in (0,\alpha)$,

$$\mathbb{E}\Big(\Big|Y_{\varepsilon,N,2}^{g_{H,\alpha}}(x) - Y_{\varepsilon,2}^{g_{H,\alpha}}(x)\Big|^r\Big) \leq \frac{C(r,\beta)D(N,r,\beta)}{N^{r(1/d+1/\alpha-H/d)-1}},$$

where $\beta = 1 + d/\alpha - H$. Furthermore, by Proposition 3.6, $Y_{\varepsilon,N,2}^{g_{H,\alpha}}$ converges uniformly at the rate N^{ε} for every $\varepsilon \in (0,(1-H)/d)$.

Finally, when $H > d/\alpha - d/2$, there exists $\eta \in (0,1]$ such that $H > d/\alpha - d/(2+\eta)$. Then, $\mathbb{E}\left(\left|X_{\varepsilon,1}^{f}(x)\right|^{2+\eta}\right) < +\infty$ and as in the case of MAFLMs, in terms of Berry-Esseen bounds, the rate of convergence of the error due to the approximation of $X_{\varepsilon,1}^{f}(x)$ is of the order

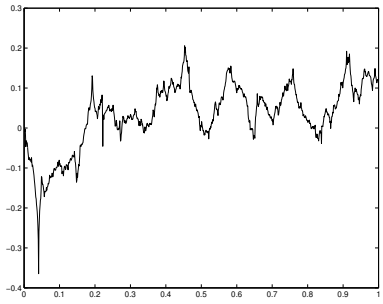
$$\delta(\varepsilon) = \frac{\int_{-\varepsilon}^{\varepsilon}|v|^{2+\eta}dv}{\sigma^{2+\eta}(\varepsilon)} = \frac{(2-\alpha)^{1+\eta/2}\varepsilon^{\alpha\eta/2}}{(2+\eta-\alpha)\sqrt{2}}.$$

Except at points $\xi_n = (T_n/c_d\nu_{\varepsilon,2}(\mathbb{R}))^{1/d}U_n$, the pointwise Hölder exponent of $Y_{\varepsilon,N,2}^{f_{H,\alpha}}$ is given by the $W^{f_{H,\alpha}}$'s one and thus is equal to $H - d/\alpha + d/2$. When $H > d/\alpha$, on each compact set, $Y_{\varepsilon,N}^{f_{H,\alpha}}$ has thus H'-Hölder sample paths for every $H' < H - d/\alpha$. Figure 5.2 presents a realization of a MAFSM when $\alpha = 1.5$ and $H = 0.7$.

5.3 Linear Fractional Stable Motions

Let $d = 1$. Here, we use the notation of section 5.2. Especially, $\nu(dv)$ is given by (28). In this example, the kernel function is

$$f(x,\xi) = D(\alpha)^{-1/\alpha}\Big((x-\xi)_+^{H-1/\alpha} - (-\xi)_+^{H-1/\alpha}\Big),$$



$H = 0.7$

Figure 3: Example of MAFSM

where $(a)_+ = a \vee 0$, $D(\alpha)$ is given by (29), $H \in (0,1)$, $H \neq 1/\alpha$ and with the convention $0^{H-1/\alpha} = 0$. Then, $L_{H,\alpha} = X^f$ is a linear fractional stable motion with index H. Furthermore, we approximate $L_{H,\alpha}$, in distribution by

$$Y^f_{\varepsilon,N}(x) = D(\alpha)^{-1/\alpha} \sum_{n=1}^{N} \left(\left(x - \frac{T_n U_n}{2\nu_{\varepsilon,2}(\mathbb{R})} \right)_+^{H-1/\alpha} - \left(\frac{-T_n U_n}{2\nu_{\varepsilon,2}(\mathbb{R})} \right)_+^{H-1/\alpha} \right) V_{\varepsilon,n} + \sigma(\varepsilon) W^f(x),$$

where W^f is defined by (24). As previously, W^f is independent of $((U_n, T_n, V_{\varepsilon,n}))_{n\geq 1}$. Moreover,

$$W^f(\cdot) = \int_{\mathbb{R}^d} f(\cdot, \xi)\, W(d\xi) \overset{(d)}{=} D(\alpha)^{-1/\alpha} \widetilde{C_H} B_{H+1/2-1/\alpha}$$

where $B_{H+1/2-1/\alpha}$ is a standard fractional Brownian motion (in short FBM) with index $H+1/2-1/\alpha$ and

$$\begin{aligned} \widetilde{C_H} &= \left(\int_{\mathbb{R}} \left((x-\xi)_+^{H-1/\alpha} - (-\xi)_+^{H-1/\alpha} \right)^2 d\xi \right)^{1/2} \\ &= \frac{\Gamma(H+1-1/\alpha)}{\sqrt{\Gamma(2H+2-2/\alpha)\sin\left((H+1/2-1/\alpha)\pi\right)}}. \end{aligned}$$

according to [15]. Obviously, this approximation only holds when $1 > H > 1/\alpha - 1/2$.

Furthermore, let us observe that

$$f(x,\xi) = \begin{cases} 0 & \text{if } \xi > \max_K |y| \\ f_{H,\alpha}(x,\xi) & \text{if } \xi < -\max_K |y|. \end{cases}$$

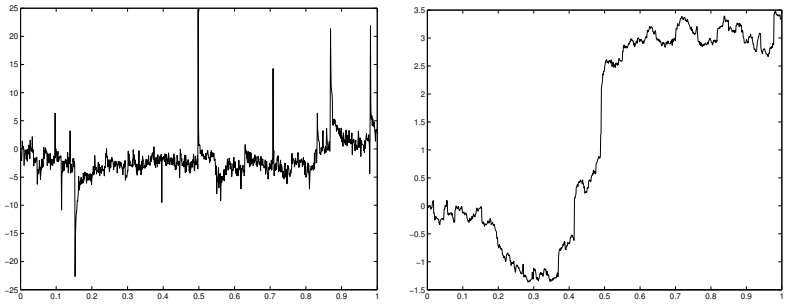
As a consequence, we obtain the same estimates for the almost sure, the L^r errors ($r < \alpha$) and the rate of convergence in terms of Berry-Esseen bounds as in the case of MAFSMs (see section 5.2).

Figure 4 presents two realizations of LFSMs for $\alpha = 1.5$. As noticed in [23], when $H = 0.2$, we observe spikes which take place at points ξ_n. Actually, since $H = 0.2 < 1/\alpha$, when x tends to a point ξ_n, $Y^f_{\varepsilon,N}(x)$ tends to infinity, which explains that spikes appear. When $H = 0.7 > 1/\alpha$, as in the case of MAFSMs, the sample paths of the approximation are H'-Hölder on each compact set for every $H' < H - 1/\alpha$.

5.4 Log-Fractional Stable Motion

Let $d = 1$ and $1 < \alpha < 2$ and assume that $\nu(dv)$ is given by (28). Furthermore, $(V_{\varepsilon,n})_{n\geq 1}$ and $\sigma(\varepsilon)$ are defined in section 5.2. Remark that here $(U_n)_{n\geq 1}$ is a sequence of i.i.d symmetric Bernoulli random variables. Then, let

$$f(x,\xi) = D(\alpha)^{-1/\alpha} (\ln|x-\xi| - \ln|\xi|).$$



$H = 0.3$ $H = 0.7$

Figure 4: Examples of LFSMs

Hence, X^f is a log-fractional stable motion and its approximation in law is given by

$$Y^f_{\varepsilon,N}(x) = D(\alpha)^{-1/\alpha} \sum_{n=1}^{N} \left(\ln \left| x - \frac{T_n U_n}{2\nu_{\varepsilon,2}(\mathbb{R})} \right| - \ln \left(\frac{T_n}{2\nu_{\varepsilon,2}(\mathbb{R})} \right) \right) V_{\varepsilon,n} + \sigma(\varepsilon) W^f(x),$$

where

$$W^f(x) = D(\alpha)^{-1/\alpha} \int_{\mathbb{R}} \left(\ln |x - \xi| - \ln |\xi| \right) W(d\xi)$$

is independent of $((U_n, T_n, V_{\varepsilon,n}))_{n\geq 1}$. Note that $W^f \overset{(d)}{=} D(\alpha)^{-1/\alpha} C B_{1/2}$ where $B_{1/2}$ is a standard Brownian motion and

$$C = \int_{\mathbb{R}} \left(\ln |x - \xi| - \ln |\xi| \right)^2 d\xi.$$

Furthermore, by a Fourier transform argument, one proves that

$$C = \left(\frac{\pi}{2} \int_{\mathbb{R}} \frac{\left| \mathrm{e}^{-i\lambda} - 1 \right|^2}{|\lambda|^2} \, d\lambda \right)^{1/2}.$$

Hence, by [21], $C = \pi$.

As previously, the rate of almost sure convergence can be studied. In particular, if

$$g(x, \xi) = f(x, \xi) \mathbf{1}_{\|\xi\| \geq \max_K \|y\|},$$

$Y^g_{\varepsilon,N,2}$ converges uniformly on K at least at the rate N^{ε} for every $\varepsilon \in (0, 1 - 1/\alpha)$. Furthermore, the L^r-error can be controlled and decreases in $N^{1-1/r}$ for every $r < \alpha$. Let us notice that X^f is a self-similar field with index $H = 1/\alpha$. Thus, we obtain the same rate of convergence for log-fractional stable motion and MAFSMs. Furthermore, since $f(x, \cdot) \in L^3(\mathbb{R})$, Proposition 4.2 gives the same rate of convergence in terms of Berry-Esseen bounds as in the cases of MAFSMs or MAFLMs.

Figure 5 presents a trajectory of a Log-fractional stable motion for $\alpha = 1.5$. Note that except at point $\xi_n = T_n U_n / (2\nu_{\varepsilon,2}(\mathbb{R}))$, the sample paths are locally H'-Hölder for every $H' < 1/2$: actually the regularity of the trajectories is given by the Brownian part. At point ξ_n, $Y^f_{\varepsilon,N}$ is not defined, which explains that spikes appear in figure 5.

6 Extension to complex random measure

Owing to arguments used in section 3 and 4, the results obtained in the case of RHMLMs in [11] can be extended to a larger class of infinitely divisible fields, in particular to the complex case. More

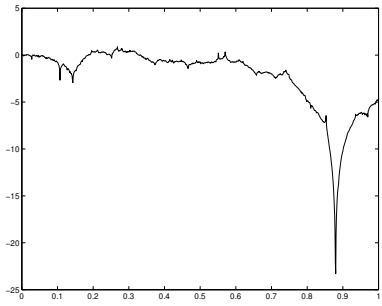


Figure 5: Example of Log-fractional stable motion

precisely, let $N(d\xi, dz)$ be a Poisson random measure on $\mathbb{R}^d \times \mathbb{C}$ with intensity $n(d\xi, dz) = d\xi\nu(dz)$. Assume that the σ-finite measure $\nu(dz)$ satisfies

$$\int_{\mathbb{R}} \Big(|z|^2 \wedge 1 \Big) \, \nu(dz) < +\infty.$$

Furthermore, the control measure $\nu(dz)$ is assumed to be rotationally invariant, i.e.

$$P(\nu(dz)) = d\theta \, \nu_\rho(d\rho), \tag{30}$$

where $d\theta$ is the uniform measure on $[0, 2\pi)$ and $P\big(\rho e^{i\theta}\big) = (\theta, \rho) \in [0, 2\pi) \times \mathbb{R}_*^+$.

Then, following the definition of complex Lévy random measure (see [3]), we can consider a complex random measure $\Lambda(d\xi)$ on $\mathbb{R}^d$ defined by

$$\int_{\mathbb{R}^d} g(\xi) \, \Lambda(d\xi) = \int_{\mathbb{R}^d \times \mathbb{R}} (g(\xi) z + g(-\xi)\overline{z}) \, \Big(N(d\xi, dv) - (|g(\xi) z + g(-\xi)\overline{z}| \vee 1)^{-1} \, n(d\xi, dv) \Big) \tag{31}$$

for every $g : \mathbb{R}^d \to \mathbb{C}$ such that

$$\int_{\mathbb{R}^d} \big(|g^2(\xi)| \wedge 1 \big) \, d\xi < +\infty.$$

Hence, owing to the arguments used in [11] in the case of RHFLMs, analogous results to those of sections 3 and 4 can be obtained and a way to simulate

$$X^f(x) = \int_{\mathbb{R}^d} f(x, \xi) \, \Lambda(d\xi).$$

can be proposed. However, in this part, we will just focus on the case where

$$\nu_\rho(d\rho) = \frac{\mathbf{1}_{\rho > 0} \, d\rho}{\rho^{1+\alpha}}, \quad \alpha \in (0, 2),$$

and the kernel function is

$$f_{H,\alpha}(x, \xi) = \frac{\big(2^{\alpha+1} \pi D(\alpha)\big)^{-1/\alpha} \big(e^{-ix\cdot\xi} - 1\big)}{\|\xi\|^{H + d/\alpha}}$$

with $D(\alpha)$ given by (29). In this case,

$$X_{H,\alpha}(x) = \int_{\mathbb{R}^d} f_{H,\alpha}(x, \xi) \, \Lambda(d\xi), \quad x \in \mathbb{R}^d,$$

is a real harmonizable fractional stable motion with index $H \in (0, 1)$, i.e.

$$\{X_{H,\alpha}(x), \, x \in \mathbb{R}^d\} \overset{(d)}{=} \{S_{H,\alpha}(x), \, x \in \mathbb{R}^d\}$$

where

$$S_H(x) = \Re\Big(\int_{\mathbb{R}^d} f_{H,\alpha}(x,\xi)\, M_\alpha(d\xi)\Big)$$

with $M_\alpha(d\xi)$ a complex isotropic α-stable random measure with control measure the Lebesgue measure in the sense of [21].

Furthermore, in the case we are interested in, $\nu(\mathbb{C}) = +\infty$. As we know, we have to split in this case the random field

$$X_{H,\alpha} = X_{\varepsilon,1}^{f_{H,\alpha}} + X_{\varepsilon,2}^{f_{H,\alpha}}$$

into two random fields where

$$X_{\varepsilon,1}^{f_{H,\alpha}}(x) = 2\int_{\mathbb{R}^d\times\mathbb{C}} \Re(f_{H,\alpha}(x,\xi)z)\mathbf{1}_{|z|<\varepsilon}\left(N(d\xi,dz) - (|2\Re(f_{H,\alpha}(x,\xi)z)|\vee 1)^{-1}\, n(d\xi,dz)\right) \tag{32}$$

and

$$X_{\varepsilon,2}^{f_{H,\alpha}}(x) = 2\int_{\mathbb{R}^d\times\mathbb{C}} \Re(f_{H,\alpha}(x,\xi)z)\mathbf{1}_{|z|\geq\varepsilon}\Big(N(d\xi,dz) - (|2\Re(f_{H,\alpha}(x,\xi)z)|\vee 1)^{-1}\, n(d\xi,dz)\Big). \tag{33}$$

As previously, $X_{\varepsilon,1}^{f_{H,\alpha}}$ and $X_{\varepsilon,2}^{f_{H,\alpha}}$ can be simulated independently. Furthermore,

$$X_{\varepsilon,2}^{f}(x) = \int_{\mathbb{R}^d} f_{H,\alpha}(x,\xi)\,\Lambda_{\varepsilon,2}(d\xi), \quad x\in\mathbb{R}^d,$$

where the complex random measure $\Lambda_{\varepsilon,2}$ is associated by (31) to a Poisson random measure $N_{\varepsilon,2}$ whose control measure $\nu_{\varepsilon,2}(dz) = \mathbf{1}_{|z|\geq\varepsilon}\,\nu(dz)$ is finite. Therefore, $X_{\varepsilon,2}^{f}$ can be simulated as a generalized shot noise series. More precisely, let $(Z_{\varepsilon,n})_{n\geq 1}$ be a sequence of i.i.d. random variables with common law

$$\mathcal{L}(Z_{\varepsilon,n}) = \frac{\nu_{\varepsilon,2}(dz)}{\nu_{\varepsilon,2}(\mathbb{C})}.$$

Moreover, $(Z_{\varepsilon,n})_{n\geq 1}$, $(T_n)_{n\geq 1}$ and $(U_n)_{n\geq 1}$ are independent. Then, as in the case of RHMLMs, a series expansion of $X_{\varepsilon,2}^{f}$ can be given and this series converges in the space of continuous function endowed with the topology of the uniform convergence on compact sets.

Proposition 6.1. *For every $x\in\mathbb{R}^d$, the series*

$$Y_{\varepsilon,N}^{f_{H,\alpha}}(x) = 2\sum_{n=1}^{N}\Re\left(f_{H,\alpha}\left(x,\left(\frac{T_n}{c_d\,\nu(\mathbb{C})}\right)^{1/d}U_n\right)Z_{\varepsilon,n}\right) \tag{34}$$

converges almost surely to $Y_{\varepsilon}^{f_{H,\alpha}}(x)$ as $N\to+\infty$. Furthermore, $Y_{\varepsilon,N}^{f_{H,\alpha}}$ converges uniformly on each compact set almost surely and

$$\{X_{\varepsilon,2}^{f_{H,\alpha}}(x) : x\in\mathbb{R}^d\} \overset{(d)}{=} \{Y_{\varepsilon}^{f_{H,\alpha}}(x) : x\in\mathbb{R}^d\}.$$

Proof. The arguments of proof of Proposition 3.2 lead to the almost sure convergence of $Y_{\varepsilon,N}^{f_{H,\alpha}}(x)$ for each fixed x. They also give the equality of the finite dimensional marginals of $X_{\varepsilon,2}^{f_{H,\alpha}}$ and $Y_{\varepsilon}^{f_{H,\alpha}}$. In order to obtain the uniform convergence, one may follow the proof of Proposition 3.1 in [11]. □

Due to the rotational invariance of $Z_{\varepsilon,n}$ and to Proposition 3.2, a rate of almost sure convergence for $Y_{\varepsilon,N}^{f_{H,\alpha}}(x)$ can be given and the L^r-error can be controlled.

Proposition 6.2. *Let $x\in\mathbb{R}^d$.*

1. *Then, for every* $\varepsilon \in (0, H/d)$, *almost surely,*

$$\sup_{N\geq 1} N^{\varepsilon}\left|Y_{\varepsilon}^{f_{H,\alpha}}(x) - Y_{\varepsilon,N}^{f_{H,\alpha}}(x)\right| < +\infty.$$

2. *Moreover, for every* $r < \alpha$ *and every integer* $N > r(1/\alpha + H/d)$,

$$\mathbb{E}\left(\left|Y_{\varepsilon,N}^{f_{H,\alpha}}(x) - Y_{\varepsilon}^{f_{H,\alpha}}(x)\right|^r\right) \leq C(r)\,\frac{D(N,r,H+d/\alpha)}{N^{r/\alpha+rH/d-1}}, \tag{35}$$

where $D(N,r,\beta)$ *is defined by* (13) *and*

$$C(r) = \frac{\left(2^{1-\alpha}\pi D(\alpha)\right)^{-r/\alpha} d(c_d\nu(\mathbb{R}))^{rH/d+r/\alpha}\mathbb{E}(|\Re(V_1)|^r)}{rH - d + rd/\alpha}.$$

Proof. Since $(Z_{\varepsilon,n})_{n\geq 1}$ is a sequence of i.i.d. with common law invariant by rotation,

$$Y_{\varepsilon,N}^{f_{H,\alpha}}(x) \overset{(d)}{=} 2\sum_{n=1}^{N}\left|f_{H,\alpha}\left(x, \left(\frac{T_n}{c_d\,\nu(\mathbb{C})}\right)^{1/d} U_n\right)\right| \Re(Z_{\varepsilon,n}).$$

Hence, taking $V_n = \Re(Z_{\varepsilon,n})$, $C = 2^{1-1/\alpha}(\pi D(\alpha))^{-1/\alpha}$ and $\beta = H + d/\alpha$, the proof of Proposition 3.2 leads to the conclusion. □

Finally, the next proposition gives the expected approximation of $X_{\varepsilon,1}$. Let

$$\sigma(\varepsilon) = \left(\int_0^{\varepsilon}\rho^2\,\nu_\rho(d\rho)\right)^{1/2} = \sqrt{\frac{2\varepsilon^{2-\alpha}}{2-\alpha}}. \tag{36}$$

Proposition 6.3. *Assume that* $0 < H + d/\alpha - d/2 < 1$ *then*

$$\lim_{\varepsilon\to 0_+}\left(\frac{X_{\varepsilon,1}(x)}{\sigma(\varepsilon)}\right)_{x\in\mathbb{R}^d} \overset{(d)}{=} \left(A_{H+d/\alpha-d/2}B_{H+d/\alpha-d/2}(x)\right)_{x\in\mathbb{R}^d},$$

where the convergence is in distribution on the space of continuous functions endowed with the topology of uniform convergence on compact sets, $B_{H+d/\alpha-d/2}$ *is a standard FBM with index* $H + d/\alpha - d/2$ *and for* $u \in (0,1)$

$$A_u = \left(2^{\alpha+1}\pi D(\alpha)\right)^{-1/\alpha}\left(\frac{4\,\pi^{(d+3)/2}\,\Gamma(u+1/2)}{u\,\Gamma(2u)\sin(\pi u)\,\Gamma(u+d/2)}\right)^{1/2}. \tag{37}$$

Remark 6.4. *In Proposition 6.3,* $0 < H + d/\alpha - d/2 < 1$ *means that*

$$f_{H,\alpha}(x,\cdot) \in L^2\left(\mathbb{R}^d\right)$$

for every $x \in \mathbb{R}^d$.

Proof. Actually,

$$X_{\varepsilon,1}^f(x) = \int_{\mathbb{R}^d} f_{H,\alpha}(x,\xi)\,\Lambda_{\varepsilon,1}(d\xi), \quad x \in \mathbb{R}^d,$$

where the complex random measure $\Lambda_{\varepsilon,1}$ is associated by (31) with a Poisson random measure $N_{\varepsilon,1}$ whose control measure $\nu_{\varepsilon,1}(dz) = \mathbf{1}_{|z|<\varepsilon}\,\nu(dz)$. Also, for every $p \geq 2$,

$$\int_{\mathbb{C}}|z|^p\,\nu_{\varepsilon,1}(dz) < +\infty$$

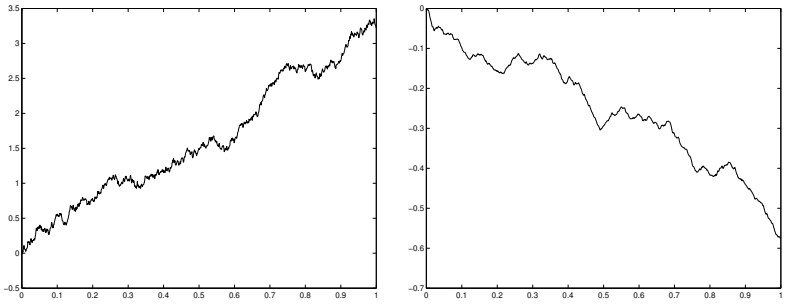
and then $\left(2^{\alpha+1}\pi D(\alpha)\right)^{1/\alpha}X_{\varepsilon,1}^f(x)$ is a RHFLM (real harmonizable fractional Lévy motion) since $f_{H,\alpha}(x,\cdot) \in L^2\left(\mathbb{R}^d\right)$ for every $x \in \mathbb{R}^d$. Then, Proposition 4.1 in [11] yields the conclusion. □

As a consequence, as soon as the assumptions of Proposition 6.3 are fulfilled, we may approximate the RHFSM $X_{H,\alpha}$ by

$$Y_{\varepsilon,N}(x) = 2\sum_{n=1}^{N} \Re\left(f_{H;\alpha}\left(x, \left(\frac{T_n}{c_d\,\nu(\mathbb{C})}\right)^{1/d} U_n \right) Z_{\varepsilon,n} \right) + \sigma(\varepsilon) A_{H+d/\alpha-d/2} B_{H+d/\alpha-d/2}(x), \quad x \in \mathbb{R}^d,$$

where $B_{H+d/\alpha-d/2}$, T_n, U_n and $Z_{\varepsilon,n}$ are independent.

Figure 6 exhibit some examples of trajectories of RHFSMs for $\alpha = 1.5$.



$H = 0.3$ $\qquad$ $H = 0.7$

Figure 6: Examples of RHFSMs

Appendix

A Proof of Lemma 3.5

Set $R_N = \sum_{n=N+1}^{+\infty} Y_n X_n$. Then, let us fix $M > 0$ and set

$$\Omega_M = \left\{ \sup_{n\geq 1} n^{-1/r} |X_n| \leq M \right\}.$$

Hence for any $\delta > 0$,

$$\mathbb{P}\Big(\Omega_M \cap \Big\{ |R_N| \geq N^{-\delta} \Big\}\Big) \leq \mathbb{P}\left(\left| \sum_{n=N+1}^{+\infty} Y_n W_n \right| \geq N^{-\delta} \right),$$

where $W_n = X_n \mathbf{1}_{|X_n| \leq M n^{1/r}}$. Since X_n, $n \geq 1$, are i.i.d. and symmetric, $(W_n)_{n\geq 1}$ is a sequence of independent symmetric random variables. Then, since $(W_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ are independent, by the contraction principle for symmetric random variables sequences, see [12] page 95,

$$\mathbb{P}\Big(\Omega_M \cap \Big\{ |R_N| \geq N^{-\delta} \Big\}\Big) \leq \mathbb{P}\left(C \left| \sum_{n=N+1}^{+\infty} T_n^{-\gamma} W_n \right| \geq N^{-\delta} \right).$$

Hence,

$$\mathbb{P}\Big(\Omega_M \cap \Big\{ |R_N| \geq N^{-\delta} \Big\}\Big) \leq \mathbb{P}\left(\sup_{n\geq N+1} \frac{n}{T_n} \geq 10 \right) + A_N \tag{38}$$

where

$$A_N = \mathbb{P}\left(\left\{ \sup_{n\geq N+1} \frac{n}{T_n} < 10 \right\} \cap \left\{ C \left| \sum_{n=N+1}^{+\infty} T_n^{-1/\gamma} W_n \right| \geq N^{-\delta} \right\} \right).$$

Step 1

$$\mathbb{P}\left(\sup_{n\geq N+1}\frac{n}{T_n}\geq 10\right) \leq \sum_{n=N+1}^{+\infty}\mathbb{P}(T_n\leq n/10),$$

$$\leq \sum_{n=N+1}^{+\infty}\frac{n^n}{10^n n!}.$$

Hence, by the Stirling Formula,

$$\mathbb{P}\left(\sup_{n\geq N+1}\frac{n}{T_n}\geq 10\right)\leq C_1\exp\left(-C_2N\right), \tag{39}$$

with $C_1>0$ and $C_2>0$.

Step 2 By the assumptions of independence, $(T_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ are independent. Therefore, by the contraction principle for symmetric random variables sequences,

$$A_N\leq 2\mathbb{P}\left(C\left|\sum_{n=N+1}^{+\infty}n^{-1/\gamma}W_n\right|\geq 10^{-1/\gamma}N^{-\delta}\right).$$

Furthermore, by independence and symmetry,

$$\begin{aligned}A_N &\leq 4\mathbb{P}\left(C\sum_{n=N+1}^{+\infty}n^{-1/\gamma}W_n\geq 10^{-1/\gamma}N^{-\delta}\right)\\ &\leq 4\exp\left(-\frac{10^{-1/\gamma}\lambda N^{-\delta}}{C}\right)\prod_{n=N+1}^{+\infty}\mathbb{E}\Big(\exp\Big(\lambda n^{-1/\gamma}W_n\Big)\Big),\end{aligned}$$

where $\lambda>0$. Moreover, since W_n is a symmetric random variable,

$$\mathbb{E}\Big(\exp\Big(\lambda n^{-1/\gamma}W_n\Big)\Big)=1+\sum_{j=1}^{+\infty}\frac{\lambda^{2j}}{(2j)!}n^{-2j/\gamma}\mathbb{E}\big(W_n^{2j}\big).$$

Then let $a=1/\gamma-1/r$ and $n\geq N+1$. Note that for $j\geq 1$, $2j\geq r$ and

$$\mathbb{E}\big(W_n^{2j}\big)\leq\mathbb{E}(|X_1|^r)\Big(Mn^{1/r}\Big)^{2j-r}.$$

Therefore,

$$\begin{aligned}\mathbb{E}\big(\exp\big(\lambda n^{-1/\gamma}W_n\big)\big) &\leq 1+\frac{\mathbb{E}(|X_1|^r)\lambda^2M^{2-r}\exp\left(\lambda^2M^2n^{-2a}\right)}{2n^{1+2a}}\\ &\leq \exp\left(\frac{\mathbb{E}(|X_1|^r)\lambda^2M^{2-r}\exp\left(\lambda^2M^2N^{-2a}\right)}{2n^{1+2a}}\right)\end{aligned}$$

As a consequence, taking $\lambda=10^{1/\gamma}N^a$, there exist $C_3>0$ and $C_4>0$, which do not depend on N, such that

$$A_N\leq C_3\exp\Big(-C_4N^{a-\delta}\Big). \tag{40}$$

Step 3 In view of (38), (39) and (40), for every $M>0$ and every $\varepsilon\in(0,1/\gamma-1/r)$,

$$\sum_{N=1}^{+\infty}\mathbb{P}\big(\Omega_M\cap\big\{|R_N|\geq N^{-\varepsilon}\big\}\big)<+\infty.$$

Hence, by the Borel Cantelli lemma, for almost $\omega \in \Omega_M$,

$$\sup_{N\geq 1} N^{\varepsilon}|R_N| < +\infty.$$

Furthermore, since $X_n \in L^r$,

$$\lim_{M\to+\infty} \mathbb{P}(\Omega_M) = \lim_{M\to+\infty} \mathbb{P}\Big(\sup_{n\geq 1} |X_n| n^{-1/r} \leq M\Big) = 1.$$

Then, for every $\varepsilon \in (0, 1/\gamma - 1/r)$, almost surely,

$$\sup_{N\geq 1} N^{\varepsilon}|R_N| < +\infty,$$

which concludes the proof.

B Proof of Lemma 3.8

It is a simple modification of the proof of Lemma 3.5.

Let $M > 0$, $\Omega_M = \Big\{\sup_{n\geq 1} \Big|n^{-1/r} X_n\Big| \leq M\Big\}$, $W_n = X_n \mathbf{1}_{|X_n|\leq n^{1/r} M}$ and

$$R_N = \sum_{n=N+1}^{+\infty} T_n^{-1/\gamma} |X_n|.$$

As in proof of Lemma 3.5,

$$\mathbb{P}\Big(\Omega_M \cap \Big\{|R_N| \geq N^{-\delta}\Big\}\Big) \leq \mathbb{P}\left(\sup_{n\geq N+1} \frac{n}{T_n} \geq 10\right) + A_N \tag{41}$$

where

$$A_N = \mathbb{P}\left(\left\{\sup_{n\geq N+1} \frac{n}{T_n} < 10\right\} \cap \left\{\sum_{n=N+1}^{+\infty} T_n^{-1/\gamma} |W_n| \geq N^{-\delta}\right\}\right).$$

Remark now that the contraction principle used in the proof of Lemma 3.5 can not be applied since $|W_n|$ is not a symmetric random variable. However, since $|W_n| \geq 0$,

$$\begin{aligned} A_N &\leq \mathbb{P}\left(\sum_{n=N+1}^{+\infty} n^{-1/\gamma} |W_n| \geq 10^{-1/\gamma} N^{-\delta}\right) \\ &\leq \exp\left(-10^{-1/\gamma}\lambda N^{-\delta}\right) \prod_{n=N+1}^{+\infty} \mathbb{E}\Big(\exp\left(\lambda n^{-1/r}|W_n|\right)\Big), \end{aligned}$$

where $\lambda > 0$. Furthermore,

$$\begin{aligned} \mathbb{E}\Big(\exp\left(\lambda n^{-1/r}|W_n|\right)\Big) &= 1 + \sum_{j=1}^{+\infty} \frac{\lambda^j}{j!} \mathbb{E}\Big(|W_n|^j\Big) \\ &\leq 1 + \mathbb{E}(|X_1|^r) \sum_{j=1}^{+\infty} \frac{\lambda^j}{j!} \Big(M n^{1/r}\Big)^{j-r} \text{ since } r \leq 1 \\ &\leq \exp\left(\lambda n^{-1-a} M^{1-r} \exp\left(\lambda M N^{-a}\right)\right), \end{aligned}$$

where $a = 1/\gamma - 1/r$ and $n \geq N+1$. Hence, choosing $\lambda = 10^{1/\gamma} N^a$, there exists C, which does not depend on N, such that

$$A_N \leq C \exp\left(-N^{a-\delta}\right).$$

Consequently, the arguments used in step 3 of the proof of Lemma 3.5 lead to the conclusion.

Acknowledgments

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